

Effect of Minimal lengths on Electron Magnetism

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Abstract

We study the magnetic properties of electron in a constant magnetic field and confined by a isotropic two dimensional harmonic oscillator on a space where the coordinates and momenta operators obey generalized commutation relations leading to the appearance of a minimal length. Using the momentum space representation we determine exactly the energy eigenvalues and eigenfunctions. We prove that the usual degeneracy of Landau levels is removed by the presence of the minimal length in the limits of weak and strong magnetic field. The thermodynamical properties of the system, at high temperature, are also investigated showing a new magnetic behavior in terms of the minimal length.

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1 Introduction

In a series of papers Kempf et al. [1, 2, 3, 4] introduced a deformed quantum mechanics based on modified commutation relation between position and momentum operators. These commutation relation lead to generalized Heisenberg uncertainty principle (GUP) which define non zero minimum length in position or minimal length. The concepts of GUP and minimal length originate from several studies in string theory [9], loop quantum gravity [10] and non-commutative field theories [11]. Other similar constructions leading to the concept of GUP have been also initiated by some authors [5, 6, 7, 8]. The fundamental outcome of the GUP is the appearance of an UV/IR "bootstrap". This mixing between UV and IR divergences, first noticed in the AdS/CFT correspondence [12], is also a feature of non-commutative quantum field theory [13]. On the other some scenarios have been

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proposed where the minimal length is related to large extra dimensions [5], to the running coupling constant [6] and to the physics of black holes production [7].

Recently, a great interest has been devoted to studies of quantum systems in the presence of minimal lengths. The solution of Schrodinger equation in momentum space for the harmonic oscillator in D-dimensions [2, 3, 14] and the cosmological constant problem with minimal lengths have been investigated in [15, 16]. Furthermore, the effect of the minimal length on the energy spectrum and momentum wave functions of the Coulomb potential in one dimension and three dimensions has been studied respectively in [17] and [18, 19, 20], the high temperature properties of the one dimensional Dirac oscillator has been investigated by the author in [21], the solution of the three dimensional Dirac oscillator using supersymmetric quantum mechanics [22] and the Casimir force for the electromagnetic field in the presence of the minimal length has been also computed [23, 24].

In this paper we are interested by the effect of the minimal length on the electron magnetism confined by a harmonic potential in the regime of high temperatures. The electron magnetism under confining potential has been considered in the ordinary case in [25, 26] and recently in the context of canonical non-commutative quantum mechanics in [27, 28]. The rest of the paper is organized as follow. In section II, we give a brief review of quantum mechanics with generalized commutation relations and solve exactly the stationary Schrodinger equation, in the momentum space representation, for a spinless electron under the action of a constant magnetic field and a isotropic harmonic oscillator. In section III, the magnetic moment and susceptibility of the system are examined in the regime of high temperature. Section V is left for concluding remarks.

2 Electron in the presence of minimal lengths

Let us start with the following generalized D-dimensional commutation relations [2]

$$[X_i, P_j] = i\hbar (\delta_{ij} + \delta_{ij}\beta P^2 + \beta' P_i P_j), \quad (1)$$

$$[X_i, X_j] = -i\hbar [(2\beta - \beta') + (2\beta + \beta')\beta P^2] \epsilon_{ijk} L_k, \quad (2)$$

$$[P_i, P_j] = 0, \quad (3)$$

with β and β' two very small non negative parameters and D the space dimension. The components of the angular momentum given by

$$L_i = \frac{1}{1 + \beta P^2} \epsilon_{ijk} X_j P_k, \quad (4)$$

satisfy the usual commutation relations

$$[L_i, X_j] = i\hbar \epsilon_{ijk} X_k, \quad [L_i, P_j] = i\hbar \epsilon_{ijk} P_k. \quad (5)$$

Using the fact that $\langle P_i \rangle = 0$, and that (ΔP_i) is isotropic we easily obtain the generalized uncertainty principle (GUP)

$$(\Delta X_i) (\Delta P_i) \geq \frac{\hbar}{2} (1 + \beta D (\Delta P_i)^2 + \beta' (\Delta P_i)^2). \quad (6)$$

A minimization of the saturate GUP with respect to ΔP_i gives an isotropic minimal length

$$(\Delta X_i)_{min} = \hbar \sqrt{D\beta + \beta'}, \quad i = 1, 2, 3, \dots D. \quad (7)$$

This relation implies a lost of the notion of localization in the position space. Since we are going to work in momentum space we use the following representation of the position and momentum operator

$$X_i = i\hbar \left[(1 + \beta p^2) \frac{\partial}{\partial p_i} + \beta' p_i p_j \frac{\partial}{\partial p_j} + \gamma p_i \right], \quad P_i = p_i, \quad L_i = -i\hbar \varepsilon_{ijk} p_j \frac{\partial}{\partial p_k}. \quad (8)$$

The parameter γ does not affect the commutation relations and only modify the squeezing factor of the momentum space measure. In fact the inner product is now defined as

$$\int \frac{d^D p}{(1 + (\beta + \beta') p^2)^{1-\alpha}} |p\rangle \langle p|, \quad \alpha = \frac{\gamma - \beta' \frac{D-1}{2}}{\beta + \beta'}, \quad (9)$$

In the following we use the simple algebra with $\beta' = 0$ and $\gamma = 0$.

Let us consider a spinless electron under the action of a constant magnetic field and confined by a two dimensional isotropic harmonic oscillator of frequency ω_0 . This system is described by the following Hamiltonian

$$H = \frac{1}{2m} \left(\mathbf{P} - \frac{q}{c} \mathbf{A} \right)^2 + \frac{m\omega_0^2}{2} (X^2 + Y^2), \quad (10)$$

where \mathbf{A} is the vector potential. In the symmetric gauge \mathbf{A} is given by

$$\mathbf{A} = \frac{B_0}{2} (-y\mathbf{i} + x\mathbf{j}), \quad (11)$$

where B_0 is the magnitude of the magnetic field.

Using the commutation relation (1), the Hamiltonian is then written as

$$H = \frac{P_z^2}{2m} + \frac{P_x^2 + P_y^2}{2m} + \frac{m\tilde{\omega}^2}{2} (X^2 + Y^2) + \frac{\omega}{2} L_z, \quad (12)$$

with $\tilde{\omega} = \sqrt{\omega^2 + \omega_0^2}$, $\omega = \frac{qB_0}{2mc}$ the cyclotron frequency and $L_z = -i\hbar (p_x \partial_{p_y} - p_y \partial_{p_x})$.

In the appendix the solutions to the eigenvalue equation $H\Psi_{nl}(\mathbf{p}) = E_{nl}\Psi_{nl}(\mathbf{p})$ are found and the radial momentum wave functions given by

$$R_{nl}(p) = \mathcal{N} (1 + \beta p^2)^{-\frac{\lambda+|l|}{2}} (\beta p^2)^{\frac{|l|}{2}} P_n^{(\lambda-1, |l|)} \left(\frac{\beta p^2 - 1}{\beta p^2 + 1} \right), \quad (13)$$

with λ given by (A.8). The constant \mathcal{N} is calculated by employing the normalization condition $\int \frac{d^3 p}{(1+\beta p^2)} |R_{nl}(p)|^2 = 1$ and the Jacobi polynomials orthogonality relation [29]. Finally the normalized radial momentum wave functions are given by

$$R_{nl}(p) = \sqrt{\beta} \sqrt{\frac{2(n!)(2n + \lambda + |l|)\Gamma(n + \lambda + |l|)}{\Gamma(n + \lambda)\Gamma(n + |l| + 1)}} (1 + \beta p^2)^{-\frac{\lambda+|l|}{2}} (\beta p^2)^{\frac{|l|}{2}} P_n^{(\lambda-1, |l|)} \left(\frac{\beta p^2 - 1}{\beta p^2 + 1} \right). \quad (14)$$

However, as pointed in [2], the normalization condition alone does not guaranteed physically relevant wavefunctions but the latter must be in the domain of p , which physically means that it should have a finite uncertainty in momentum. This leads to the condition

$$\langle p^2 \rangle = \int_0^\infty \frac{p^3 dp}{1 + \beta p^2} |R_{nl}(p)|^2 < \infty. \quad (15)$$

In our case the integrand in (15) behaves like $p^{-2\lambda+1}$ which requires $\lambda > \frac{1}{2}$. Then we choose the upper sign in the expression of λ . However the condition $\lambda > \frac{1}{2}$ can be also obtained from physical considerations. Let us take λ with the minus and work with $l = 0$

$$\lambda = 1 - \frac{1}{m\tilde{\omega}\hbar\beta}. \quad (16)$$

Using the fact that $\hbar\sqrt{2\beta} \leq l_c$ where $l_c = \sqrt{\frac{2\hbar}{m\tilde{\omega}}}$ is the characteristic length of the oscillator, we get

$$m\hbar\tilde{\omega}\beta = \frac{(\Delta X)_{\min}^2}{l_c^2} < 1, \quad (17)$$

and then the condition $\lambda > \frac{1}{2}$ is not satisfied.

The energy spectrum is now derived from equation (A.12)

$$\frac{\mathcal{E}_{nl} - \Omega}{\beta} = 2(N + 1)(\lambda - 1) + (N^2 + l^2 + 2N + 2), \quad (18)$$

where $N = 2n + |l|$ is the principal quantum number. Using the expressions of κ , \mathcal{E}_{nl} and Ω we finally obtain

$$E_{nl} = \frac{p_z^2}{2m} + \hbar\tilde{\omega} \left[(N + 1)\sqrt{1 + (m\hbar\tilde{\omega}\beta)^2 (1 + l^2)} + \frac{m\hbar\tilde{\omega}\beta}{2}(N^2 + l^2 + 2N + 2) + \frac{\omega}{2\tilde{\omega}}l \right]. \quad (19)$$

Ignoring the contribution of $\frac{p_z^2}{2m}$ and setting $\omega = \omega_0$ we reproduce exactly the energy spectrum of the two dimensional harmonic oscillator with minimal length [14]. A Taylor expansion to first order in $m\hbar\tilde{\omega}\beta$ gives

$$E_{nl} = \frac{p_z^2}{2m} + \hbar\tilde{\omega} \left[(N + 1) + \frac{\beta m\hbar\tilde{\omega}}{2}(N^2 + l^2 + 2N + 2) + \frac{\omega l}{2\tilde{\omega}} \right]. \quad (20)$$

Introducing the following quantum numbers

$$n_d = n + \frac{|l| + l}{2}, \quad n_g = n + \frac{|l| - l}{2}, \quad (21)$$

we obtain

$$\begin{aligned} E_{n_d, n_g} &= \frac{p_z^2}{2m} + \hbar\tilde{\omega} (1 + \beta m\hbar\tilde{\omega}) + \hbar\tilde{\omega} \left[\left(1 + \beta m\hbar\tilde{\omega} + \frac{\omega}{2\tilde{\omega}}\right) n_d + \beta m\hbar\tilde{\omega} n_d^2 \right] \\ &\quad + \hbar\tilde{\omega} \left[\left(1 + \beta m\hbar\tilde{\omega} - \frac{\omega}{2\tilde{\omega}}\right) n_g + \beta m\hbar\tilde{\omega} n_g^2 \right] \end{aligned} \quad (22)$$

Let us in the following ignore the term $\frac{p_z^2}{2m}$ and examine the degeneracy of Landau levels in some limiting cases of the magnetic field.

- Weak magnetic field.

This case corresponds to $\omega \ll \omega_0$ such that we have $\tilde{\omega} \approx \omega_0$. The energy spectrum is then approximated by

$$E_{\gamma,\rho} \approx 2\hbar\omega_0 \left[\left(\gamma + \frac{1}{2} \right) + \beta m \hbar \omega_0 \left[\gamma(\gamma+1) + \rho^2 + \frac{1}{2} \right] \right], \quad (23)$$

where $\gamma = \frac{n_d+n_g}{2}$, $\rho = \frac{n_d-n_g}{2}$. We observe that the usual degeneracy of the Landau levels is removed by the presence of the minimal length.

- Strong magnetic field.

In this case we have $\omega \gg \omega_0$, $\tilde{\omega} \approx \omega$ and the energy spectrum is given by

$$E_{n_d, n_g} \approx 2\hbar\omega \left[\left(\gamma + \frac{1+\rho}{2} \right) + \beta m \hbar \omega \left[\gamma(\gamma+1) + \rho^2 \right] \right]. \quad (24)$$

In this case also we observe that the degeneracy is removed. This latter result shows a difference with the commutative and the canonical non-commutative cases respectively, where the degeneracy of the Landau levels in the strong magnetic field limit is still present [26, 28].

3 Thermodynamical properties

In this section we are interested by the thermodynamical properties of the system at high temperatures. We set $z = e^{\tilde{\beta}\mu}$ and $\tilde{\beta} = 1/kT$, where μ is the chemical potential and use the following assumption $\tilde{\beta} |\mu - \hbar\tilde{\omega}| \ll 1$.

Let us start by computing the one particle states density $g(n_d, n_g)$ given by

$$g(n_d, n_g) = \frac{V^{\frac{2}{3}}}{4\pi\hbar^2} \int_{E_n \leq E < E_{n+1}} \frac{dp_x dp_y}{(1 + \beta p^2)}, \quad (25)$$

with $V^{\frac{2}{3}}$ a surface term. In polar coordinates we obtain

$$g(n_d, n_g) = \frac{V^{\frac{2}{3}}}{2\pi\hbar^2} \int_{p_n}^{p_{n+1}} \frac{p dp}{(1 + \beta p^2)} = \frac{V^{\frac{2}{3}}}{4\pi\beta\hbar^2} \ln \frac{1 + \beta p^2(n_d + 1, n_g + 1)}{1 + \beta p^2(n_d, n_g)}. \quad (26)$$

Using

$$p(n_d, n_g) = \sqrt{2m\hbar\tilde{\omega}} \left[(1 + \beta m \hbar \tilde{\omega})(n_d + n_g) + \frac{\omega}{2\tilde{\omega}}(n_d - n_g) + \beta m \hbar \tilde{\omega}(n_d^2 + n_g^2) \right]^{\frac{1}{2}}$$

and the fact that $1 + \beta m \hbar \tilde{\omega} \approx 1$, by virtue of the GUP, we obtain

$$g(n_d, n_g) \approx g(\gamma) = \frac{V^{\frac{2}{3}}}{4\pi\beta\hbar^2} \ln [1 + 4m\beta\hbar\tilde{\omega} [1 + \beta m\hbar\tilde{\omega} (2\gamma + 1)]] . \quad (27)$$

The expression of the one particle states density in the standard situation is obtained by taking the limit $\beta \rightarrow 0$

$$g = \frac{m\tilde{\omega}V^{\frac{2}{3}}}{\pi\hbar} . \quad (28)$$

The thermodynamical potential is defined by the following expression

$$\Phi = -\frac{V^{\frac{1}{3}}}{2\pi\tilde{\beta}\hbar} \int_{-\infty}^{+\infty} \frac{dp_z}{1 + \beta p_z^2} \sum_{\gamma=0}^{\infty} g(\gamma) \ln [1 + z \exp(-\tilde{\beta}E)] . \quad (29)$$

Using $g(\gamma)$ and E given respectively by (27) and (22), we obtain

$$\begin{aligned} \Phi &\approx -\frac{V}{8\pi^2\tilde{\beta}\beta\hbar^3} \exp\left(-\tilde{\beta}\hbar\tilde{\omega}(1 + \beta m\hbar\tilde{\omega})\right) \int_{-\infty}^{+\infty} \frac{dp_z}{1 + \beta p_z^2} \exp\left(-\tilde{\beta}\frac{p_z^2}{2m}\right) \\ &\times \sum_{n_d, n_g=0}^{\infty} \ln [1 + 4m\beta\hbar\tilde{\omega} [1 + \beta m\hbar\tilde{\omega} (n_d + n_g + 1)]] \\ &\times \exp\left(-\tilde{\beta}\hbar\tilde{\omega} \left[(1 + \beta m\hbar\tilde{\omega}) (n_d + n_g) + \frac{\omega}{2\tilde{\omega}} (n_d - n_g) + \beta m\hbar\tilde{\omega} (n_d^2 + n_g^2) \right] \right) . \end{aligned} \quad (30)$$

Using the approximation $\ln(1 + ua) \approx ua$ we write Φ as

$$\begin{aligned} \Phi &\approx -\frac{4m\beta\hbar\tilde{\omega}V}{8\pi^2\tilde{\beta}\beta\hbar^3} \exp\left(-\tilde{\beta}\hbar\tilde{\omega}(1 + \beta m\hbar\tilde{\omega})\right) \int_{-\infty}^{+\infty} \frac{dp_z}{1 + \beta p_z^2} \exp\left(-\tilde{\beta}\frac{p_z^2}{2m}\right) \\ &[(1 + \beta m\hbar\tilde{\omega}) S_1^+ S_1^- + \beta m\hbar\tilde{\omega} (S_2^+ S_1^- + S_2^- S_1^+)] , \end{aligned} \quad (31)$$

with the sums S_1^\pm and S_2^\pm defined by

$$S_1^\pm = \sum_n \left[\exp\left(-\tilde{\beta}\hbar\tilde{\omega} \left[\left(1 + \beta m\hbar\tilde{\omega} \pm \frac{\omega}{2\tilde{\omega}}\right) n + \beta m\hbar\tilde{\omega} n^2 \right] \right) \right] , \quad (32)$$

$$S_2^\pm = \sum_n n \left[\exp\left(-\tilde{\beta}\hbar\tilde{\omega} \left[\left(1 + \beta m\hbar\tilde{\omega} \pm \frac{\omega}{2\tilde{\omega}}\right) n + \beta m\hbar\tilde{\omega} n^2 \right] \right) \right] . \quad (33)$$

These sums are computed using the Euler Formula given by

$$\sum_{n=0}^{\infty} f(n) = \frac{f(0)}{2} + \int_0^{\infty} f(x) dx - \sum_{p=1}^{\infty} \frac{1}{(2p)!} B_{2p} f^{(2p-1)}(0) , \quad (34)$$

where B_{2p} are Bernoulli's numbers and $f^{(2p-1)}(0)$ derivatives of the function $f(x)$ at $x = 0$. In the high temperatures regime the contribution of the third term in (34) is negligible.

With the aid of the formula [29]

$$\int_0^\infty x^{\nu-1} e^{-px^2-qx} dx = (2p)^{-\frac{\nu}{2}} \Gamma(\nu) \exp\left(\frac{q^2}{8p}\right) D_{-\nu}\left(\frac{q}{\sqrt{2p}}\right), \quad (35)$$

where $D_\mu(u)$ is the cylindrical function, performing the integration over p_z and using the approximation $1 + \beta m \hbar \tilde{\omega} \approx 1$ we obtain

$$\begin{aligned} \Phi \approx & -\frac{4m\tilde{\omega}V}{8\pi\sqrt{\beta}\tilde{\beta}\hbar^2} \exp(-\tilde{\beta}\hbar\tilde{\omega}) \exp\left(\frac{\tilde{\beta}}{2\beta m}\right) \left[1 - \operatorname{erf}\left(\sqrt{\frac{\tilde{\beta}}{2\beta m}}\right)\right] \\ & \times \left[\left(\frac{1}{2} + A_1^+\right) \left(\frac{1}{2} + A_1^-\right) + \beta m \hbar \tilde{\omega} \left(A_2^+ \left(\frac{1}{2} + A_1^-\right) + A_2^- \left(\frac{1}{2} + A_1^+\right)\right)\right], \quad (36) \end{aligned}$$

where we have set

$$A_1^\pm = \frac{e^{\frac{u_\pm^2}{4}}}{\hbar\tilde{\omega}\sqrt{2\tilde{\beta}\beta m}} D_{-1}(u_\pm), \quad A_2^\pm = \frac{e^{\frac{u_\pm^2}{4}}}{2\hbar^2\tilde{\omega}^2\tilde{\beta}\beta m} D_{-2}(u_\pm), \quad u_\pm = \frac{(1 \pm \frac{\omega}{2\tilde{\omega}})}{\sqrt{2}} \sqrt{\frac{\tilde{\beta}}{m\beta}}. \quad (37)$$

At this stage we make the physical assumption that u_\pm is a large parameter. In fact we rewrite u_\pm as

$$u_\pm = \frac{(1 \pm \frac{\omega}{2\tilde{\omega}})}{\sqrt{2\pi}} \frac{\lambda}{(\Delta X)_{\min}}, \quad (38)$$

where $\frac{\lambda}{(\Delta X)_{\min}}$ is the ratio between the thermal wave length $\lambda = \sqrt{\frac{2\pi\tilde{\beta}\hbar^2}{m}}$ and the minimal length $(\Delta X)_{\min} = \hbar\sqrt{2\beta}$. The thermal wave length is a physical characteristic length of the system and, in order to be experimentally probed, must be larger than the minimal length. The latter assertion is the physical content of the GUP and is expressed by

$$\frac{\lambda}{(\Delta X)_{\min}} > 1. \quad (39)$$

Then, for large values of $\frac{\tilde{\beta}}{m\beta}$ we use the following approximations

$$e^{\frac{\tilde{\beta}}{2\beta m}} \left[1 - \operatorname{erf}\left(\sqrt{\frac{\tilde{\beta}}{2\beta m}}\right)\right] \approx \sqrt{\frac{2\beta m}{\pi\tilde{\beta}}} \left[1 - \frac{\beta m}{\tilde{\beta}}\right], \quad (40)$$

and

$$e^{\frac{u_\pm^2}{4}} D_{-1}(u_\pm) \approx \frac{\sqrt{2m\beta/\tilde{\beta}}}{(1 \pm \frac{\omega}{2\tilde{\omega}})}, \quad e^{\frac{u_\pm^2}{4}} D_{-2}(u_\pm) \approx \frac{2m\beta/\tilde{\beta}}{(1 \pm \frac{\omega}{2\tilde{\omega}})^2}, \quad (41)$$

which leads to

$$A_1^\pm = \frac{1}{\hbar\tilde{\omega}\tilde{\beta}(1 \pm \frac{\omega}{2\tilde{\omega}})}, \quad A_2^\pm = \frac{1}{\hbar^2\tilde{\omega}^2\tilde{\beta}^2(1 \pm \frac{\omega}{2\tilde{\omega}})^2}. \quad (42)$$

Using $\frac{1}{2} + A_1^\pm \approx A_1^\pm$, we finally obtain the thermodynamical potential at high temperatures

$$\Phi \approx -\frac{2V}{\tilde{\beta}\hbar\lambda^3} \left[\left[1 - \frac{m\beta}{\tilde{\beta}} \right] \frac{1}{\tilde{\omega} \left(1 - \left(\frac{\omega}{2\tilde{\omega}} \right)^2 \right)} + \frac{2\beta m}{\tilde{\omega}\tilde{\beta}} \frac{1}{\left(1 - \left(\frac{\omega}{2\tilde{\omega}} \right)^2 \right)^2} \right]. \quad (43)$$

The magnetic moment of the system defined by $M_\beta = -\frac{\partial}{\partial B}\Phi$ is then easily obtained

$$M_\beta = -\frac{8V}{\tilde{\beta}\hbar\lambda^3} \frac{q}{2mc} \frac{\omega}{\tilde{\omega} (3\tilde{\omega}^2 + \omega_0^2)^2} \left[(3\tilde{\omega}^2 - \omega_0^2) - \frac{m\beta}{\tilde{\beta}} \left[(3\tilde{\omega}^2 - \omega_0^2) - \frac{24\tilde{\omega}^2\omega^2}{(3\tilde{\omega}^2 + \omega_0^2)} \right] \right]. \quad (44)$$

This expression is always negative otherwise we have $\beta < 0$. This case is discarded since the parameter β defines the minimal length which is a physical scale. On the other hand the dependence of the magnetic moment on the applied external magnetic field is not trivial and in order to extract useful magnetic properties of the system we have shown, in figure 1, the behavior of the susceptibility $\chi_\beta = \frac{\partial M}{\partial B}$ as a function of the magnetic field. We first observe that we have two critical values B_1 and B_2 of the magnetic field for which the effect of the minimal length is undetectable. For values smaller than B_1 , corresponding to weak magnetic fields, the Landau diamagnetism is less pronounced than in the standard situation without the minimal length. In the strong magnetic field case corresponding to values larger than B_2 , the system exhibits a stronger paramagnetic behavior with increasing values of the minimal length. A third regime, corresponding to intermediate values of the magnetic field, the situation is inverted since the diamagnetism and the paramagnetism behaviors are respectively stronger and weaker with increasing values of the minimal length.

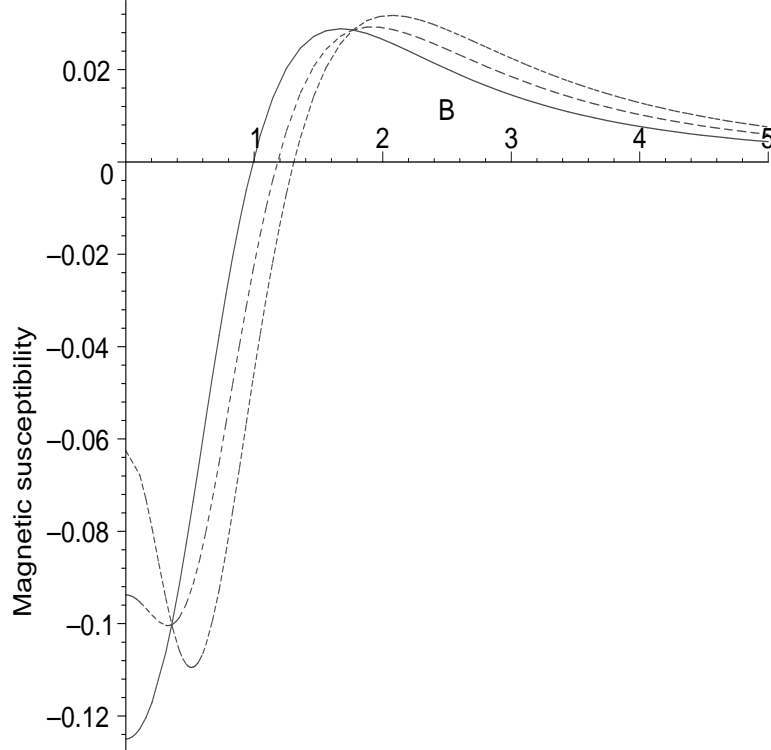


Figure 1: Magnetic susceptibility vs magnetic field for $\beta = 0$ (solid line), $\beta = 0.5$ (dotted line) and $\beta = 1$ (dashed line).

Let us examine in details the behavior of the susceptibility in the following limiting cases:

- Weak magnetic field $\omega < \omega_0$:

Deriving (44) with respect to the magnetic field, the susceptibility is given by

$$\chi_\beta = -\frac{V\mu_B^2}{\tilde{\beta}\hbar^3\lambda^3\omega_0^3} \left[1 - \frac{m\beta}{\tilde{\beta}} \left(1 - 9\frac{\omega^2}{\omega_0^2} \right) \right], \quad (45)$$

which shows that the system exhibits the usual Landau diamagnetism in the regime of high temperatures since we have always $1 > \frac{m\beta}{\tilde{\beta}}$ by virtue of the GUP. Here we note that we have the value of B_1 given by $\omega = \omega_0/3$. For zero magnetic field we have

$$\chi_\beta = -\frac{V\mu_B^2}{\tilde{\beta}\hbar^3\lambda^3\omega_0^3} \left[1 - \frac{m\beta}{\tilde{\beta}} \right]. \quad (46)$$

Switching off the minimal length we obtain

$$\chi_\beta = -\frac{V\mu_B^2}{\tilde{\beta}\hbar^3\lambda^3\omega_0^3}. \quad (47)$$

Let us mention that the susceptibility obtained in [28] in the context of θ -non-commutative quantum mechanics shows an unusual behavior since it vanishes for $\theta = 0$.

Notice that the susceptibility given by (46) is weaker than the one in the ordinary case since the contribution of the minimal length is of paramagnetic nature. This suggests that the perturbation of the space by the minimal length generates magnetic moments in the direction of the applied external magnetic field. We note also that the limit $T \rightarrow \infty$ is forbidden by the condition (39), and then the susceptibility is always finite. However, we note that this result is a consequence of the physical statement that, all physical lengths must be larger than the minimal length.

On the other we obtain $\chi_\beta = 0$ for a minimal thermal wavelength given by

$$\lambda_{\min} = \sqrt{\pi} (\Delta X)_{\min} \quad (48)$$

which in turns define a maximal temperature

$$kT_{\max} = \frac{2}{mc^2} \left(\frac{\hbar c}{(\Delta X)_{\min}} \right)^2. \quad (49)$$

The existence of a maximal temperature have been recently revealed in the context of the thermodynamics of black holes in the framework of canonical non-commutative theories [30] and with generalized uncertainty principle [31]. It seems that such a finding is a common feature of quantum theories on quantized spacetimes.

- Strong magnetic field $\omega \gg \omega_0$:

In this case the susceptibility is given by

$$\chi_\beta = \frac{16V\mu_B^2}{3\tilde{\beta}\hbar^3\lambda^3\omega^3} \left[1 + \frac{5}{3} \frac{m\beta}{\tilde{\beta}} \right]. \quad (50)$$

Here we observe that we have, at high temperatures, orbital paramagnetism and that $\chi_\beta \neq 0$ for finite magnetic field and minimal length.

4 Conclusion

In this paper we have investigated the electron magnetism on space where the coordinate and momentum operators obey generalized commutation relations. Using the momentum space representation, the eigenstates and the corresponding energy eigenvalues has been exactly calculated. In the limiting cases of weak and strong magnetic fields the usual degeneracy of the Landau levels is now removed by the minimal length. We have also investigated the magnetic behavior of the system at high temperatures. For strong magnetic field the contribution of the confining potential is negligible and we obtain a paramagnetic behavior of the Landau system. The latter result show, a tendency of the magnetic moments, generated by the minimal length, to be aligned in the direction of the applied external field giving then a paramagnetic contribution. For weak magnetic field the orbital diamagnetism is more pronounced in the standard situation without the minimal length. For intermediary values of the magnetic field the situation is inverted and the diamagnetic and paramagnetic behaviors are respectively stronger and weaker for increasing values of the minimal length. The main consequence of the minimal length is the existence of a maximal temperature which renders the susceptibility, in terms of the minimal length, finite. This important result reflects the regularizing effect of the minimal length.

Appendix: Radial momentum wave functions

In this appendix we calculate, with some details, the radial momentum wave functions.

To solve the eigenvalues equation $H\Psi_{nl}(\mathbf{p}) = E_{nl}\Psi_{nl}(\mathbf{p})$ with H given by (12), in the momentum space representation, we exploit the rotational invariance of the problem and write $\Psi_{nl}(\mathbf{p})$ as

$$\Psi_{nl}(\mathbf{p}) = \frac{e^{\frac{i}{\hbar}l\phi}}{\sqrt{2\pi\hbar}} R_{nl}(p), \quad (A.1)$$

where n is the radial quantum number and l the magnetic number.

Using the two dimensional representation of the position operators given by eq.(8) we have the following differential equation for the radial part of the wave function

$$\begin{aligned} & \left((1 + \beta p^2) \frac{\partial}{\partial p} \right)^2 R_{nl}(p) + (1 + \beta p^2)^2 \left(\frac{1}{p} \frac{\partial}{\partial p} - \frac{l^2}{p^2} \right) R_{nl}(p) \\ & - \left(\frac{\omega l}{2m\tilde{\omega}^2\hbar} + \frac{p^2}{(m\hbar\tilde{\omega})^2} + \mathcal{E}_{nl} \right) R_{nl}(p) = 0 \end{aligned} \quad (\text{A.2})$$

with \mathcal{E}_{nl} given by

$$\mathcal{E}_{nl} = \frac{2E_{nl}}{m\hbar^2\tilde{\omega}^2} - \frac{p_z^2}{(m\hbar\tilde{\omega})^2}. \quad (\text{A.3})$$

In terms of the new variable $\xi = \frac{1}{\sqrt{\beta}} \arctan(p\sqrt{\beta})$ we write (A.2) as

$$\begin{aligned} & R_{nl}''(\xi) + \sqrt{\beta} \left(\cot\sqrt{\beta}\xi + \tan\sqrt{\beta}\xi \right) R_{nl}'(\xi) - \beta l^2 \left(\cot\sqrt{\beta}\xi + \tan\sqrt{\beta}\xi \right)^2 R_{nl}(\xi) \\ & - \left(\Omega + \frac{1}{\beta(m\tilde{\omega}\hbar)^2} \right) \tan^2\sqrt{\beta}\rho R_{nl}(\xi) + (\mathcal{E}_{nl} - \Omega) R_{nl}(\xi) = 0, \end{aligned} \quad (\text{A.4})$$

with $\Omega = \frac{l\omega}{2m\tilde{\omega}^2\hbar}$. We simplify (A.4) by setting $R_{nl}(\xi) = c^\lambda f(s)$ with c and s defined as

$$c = \cos\sqrt{\beta}\xi, \quad s = \sin\sqrt{\beta}\xi. \quad (\text{A.5})$$

A straightforward calculation gives the following differential equation for $f(s)$

$$\begin{aligned} & (1 - s^2) f''(s) + \left(\frac{1}{s} - (2\lambda + 1) \right) f'(s) \\ & + \left(\left(\lambda(\lambda - 2) - \frac{1}{\kappa^4} - l^2 \right) \frac{s^2}{c^2} + \left(\frac{\mathcal{E}_{nl} - \Omega}{\beta} - 2\lambda - l^2 \right) - \frac{l^2}{s^2} \right) f(s) = 0, \end{aligned} \quad (\text{A.6})$$

where we have set $\kappa = \sqrt{m\tilde{\omega}\hbar\beta}$. Then we cancel the term with $\frac{s^2}{c^2}$ by choosing λ such that

$$\lambda^2 - 2\lambda - l^2 - \frac{1}{\kappa^4} = 0. \quad (\text{A.7})$$

The solutions of this equation are given by

$$\lambda = 1 \pm \frac{1}{m\tilde{\omega}\hbar\beta} \sqrt{1 + (m\hbar\tilde{\omega}\beta)^2 (1 + l^2)} \quad (\text{A.8})$$

The next step is to cancel the centrifugal barrier in equation (A.6) by setting $f(s) = s^{|l|}g(s)$. Then we have

$$(1 - s^2)g''(s) + \left(\frac{2|l| + 1}{s} - (2\lambda + 2|l| + 1)s \right) g'(s) + \left(\frac{\mathcal{E}_{nl} - \Omega}{\beta} - 2l^2 - 2\lambda(|l| + 1) \right) g(s) = 0. \quad (\text{A.9})$$

At this stage we use the variable $z = 2s^2 - 1$ to obtain

$$(1-z^2)g''(z) + [(|l| - \lambda + 1) - (|l| + \lambda + 1)z]g'(z) + \frac{1}{4} \left(\frac{\mathcal{E}_{nl} - \Omega}{\beta} - 2l^2 - 2\lambda(|l| + 1) \right) g(z) = 0 \quad (\text{A.10})$$

Defining

$$a = \lambda - 1, \quad b = |l|, \quad (\text{A.11})$$

and imposing the following condition, to get a polynomial solution,

$$\frac{\mathcal{E}_{nl} - \Omega}{\beta} - 2L^2 - 2\lambda(|l| + 1) = 4n(n + a + b + 1), \quad (\text{A.12})$$

with n a non negative integer, we reduce (A.10) to the following form

$$(1 - z^2)g''(z) + [(b - a) - (a + b + 2)z]g'(z) + n(n + a + b + 1)g(z) = 0. \quad (\text{A.13})$$

The solutions of equation (A.13) are given by Jacobi polynomials

$$g(z) = P_n^{(a,b)}(z). \quad (\text{A.14})$$

Using the old variable p , the radial part of the wave function is then given by

$$R_{nl}(p) = \mathcal{N}(1 + \beta p^2)^{-\frac{\lambda+|l|}{2}} (\beta p^2)^{\frac{|l|}{2}} P_n^{(\lambda-1,|l|)} \left(\frac{\beta p^2 - 1}{\beta p^2 + 1} \right), \quad (\text{A.15})$$

where \mathcal{N} is a normalization constant.

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