## Decoherence of a Single Qubit from Quantum Anomaly

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We show via an explicit example that quantum mechanical anomalies can lead to decoherence of a single quantum qubit through phase relaxation. The anomaly causes the Hamiltonian to develop an non-self-adjoint piece due to the non-invariance of the domain of the Hamiltonian under symmetry transformation. The resulting decoherence originates completely from the dynamics of the system itself and not from interactions with the environment.

A physically realizable quantum computer must satisfy some delicate requirements [1]. One of these requirements is that coherence must be maintained within a single qubit and also among entangled qubits. Coherence within a single qubit requires dynamics of the two-level quantum state to be controlled by unitary evolution. This in turn is guaranteed by the self-adjointness of the Hamiltonian in the Schrödinger equation. Up to now, attention has been given mostly to decoherence that originates from the interaction of the quantum system with its external environment [2]. The purpose of this letter is to point out that decoherence can also come from anomalous symmetry breaking of the quantum mechanical system. The novelty of this phenomenon is that the decoherence originates from the system itself and not via interactions do with an external environment. This anomalous decoherence, which we make explicit in the following via a toy model, is potentially significant for quantum information theory and should in principle be taken into account in the construction of quantum computing models.

The model we consider is an electron in a magnetic field produced by the Dirac monopole. It is described by the following Hamiltonian,

$$H = \frac{[\sigma \cdot (\mathbf{p} - e\mathbf{A})]^2}{2m} = \frac{(\mathbf{p} - e\mathbf{A})^2}{2m} - \frac{e}{2m}\sigma \cdot \mathbf{B}$$
(1)

where **A** is the singularity-free vector potential of the Dirac magnetic monopole [3] and  $B = \nabla \times \mathbf{A} = g\mathbf{r}/r^3$ , the corresponding magnetic field. Further, the single-valueness of the wave function requires that 2eg should be an integer.

The model (1) possess a rotational symmetry SO(3) as well as a dynamical superconformal symmetry OSP(1,1) [4]. The SO(3) symmetry is generated by the angular momentum of the

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electron-monopole system,  $\mathbf{J} = \mathbf{r} \times (\mathbf{p} - e\mathbf{A}) - eg\mathbf{r}/r + \sigma/2$ . The OSP(1, 1) consists of two parts: one is the conformal symmetry SO(2, 1) generated by the Hamiltonian H, the dilatation operator D and the conformal generator K, and the other part is the N = 1/2 conformal supersymmetry generated by the supercharge Q and conformal supersymmetry generator S. The SO(2, 1) conformal symmetry is a generic feature of physical systems with  $1/r^2$  potential [5], whose algebra is realized as [H, D] = iH, [H, K] = 2iD, [D, K] = iK.

The large symmetry described above allows the model (1) to be solved exactly with a suitable representation of  $SO(3) \times OSP(1,1)$  [4]. The quantum states are characterized by the eigenstates  $|j, m, \alpha, E\rangle$  of a complete set of compatible operators  $J^2$ ,  $J_z$ , sign A and H, respectively. The dynamical operator  $A = \sigma \cdot (\mathbf{J} + eg\mathbf{r}/r) - 1/2$  is related to the Casimir of OSP(1,1), and the eigenvalues  $\alpha = \pm 1$  of sign A describe the two helicity states of the electron related by the superconformal transformation. Hence the state of the system is given by [4]

$$H|j, m, \alpha, E\rangle = E|j, m, \alpha, E\rangle$$

$$J^{2}|j, m, \alpha, E\rangle = j(j+1)|j, m, \alpha, E\rangle, \quad j = eg - \frac{1}{2}, eg + \frac{1}{2}, \cdots$$

$$J_{z}|j, m, \alpha, E\rangle = m|j, m, \alpha, E\rangle, \quad m = -j, -j + 1, \cdots, j - 1, j$$

$$A|j, m, \alpha, E\rangle = \alpha d_{j}|j, m, \alpha, E\rangle, \quad \alpha = \pm 1, \quad d_{j} = \left[\left(j + \frac{1}{2}\right)^{2} - e^{2}g^{2}\right]^{1/2}$$
(2)

The wave function in spherical coordinate and the Pauli two-component representation is  $\Psi_E(r,\theta,\phi) = \langle r,\theta,\phi,\sigma|j,m,\alpha,E\rangle = \Phi_E(r)\eta_{j,m,\alpha}(\theta,\phi)$  [4]. The angular part  $\eta_{j,m,\alpha}(\theta,\phi)$  can be expressed explicitly in terms of the monopole harmonics [3].

Once the angular part of the wave function has been fixed, the Hamiltonian (1) reduces to

$$H = -\frac{1}{2m} \frac{1}{r} \frac{d^2}{dr^2} r + \frac{1}{2mr^2} (-A)(-A+1).$$
(3)

The radial eigenfunction  $\Phi_{\rm E}(r)$  is the solution to the eigenvalue equation  $H\Phi_E(r) = E\Phi_E(r)$ .

We are interested only in bound states (E < 0) since the goal is to describe a system which can be used for quantum computing. The bound state radial eigenfunction reads

$$\Phi_E(r) = Nr^{-1/2} K_{2\delta_{j,\alpha}-1}(\beta r),$$
  

$$\beta = (-2mE)^{1/2}, \quad \delta_{j,\alpha} = \frac{1}{2} - \frac{1}{4}\alpha + \frac{1}{2}d_j$$
(4)

where  $K_{\nu}$  is the modified Bessel function of the second kind and N is a normalization constant. It is easy to see that only when  $\nu < 1$ ,  $K_{\nu}$  is normalizable over the region containing the origin. It turns out that there exists only one such bound state,  $\Phi_E(r) = 2\beta K_{1/2}(\beta r)/\sqrt{\pi r}$ , which arises when j = |eg| - 1/2. In particular, the orbit angular momentum  $L^2$  is diagonal in the basis  $|j = |eg| - 1/2, m, \alpha, E\rangle$  and there exists  $\sigma \cdot \mathbf{r}/r |j = |eg| - 1/2, m\rangle = \pm |j = |eg| - 1/2, m\rangle$  [4].

However, in this case  $\Phi_E(r)$  is singular at the origin, and we need to perform regularization and renormalization operations on the Hamiltonian (1) to make it regular. It should be noted that the regularization of the model (1) is considerably complicated. First, the solvability of the theory depends on the larger dynamical symmetry OSP(1,1), which should be preserved as much as possible by the regularization scheme. Second, the theory has a U(1) gauge symmetry encoded in the angular wave function. The regularization should keep the angular part  $\eta_{j,m,\alpha}(\theta,\phi)$  intact so that the Hamiltonian can reduce to the form (3). Otherwise, the exact solvability of theory will be ruined. The regularization scheme we use is described as follows. First, we observe that the reduced Hamiltonian (3) at j = |eg| - 1/2 becomes

$$H = -\frac{1}{2m} \frac{1}{r} \frac{d^2}{dr^2} r + \frac{1}{2m} \frac{\mathbf{L}^2 - e^2 g^2 - eg\sigma \cdot \mathbf{e}_r}{r^2}$$
  
=  $-\frac{1}{2m} \nabla^2 - \frac{e^2 g^2}{2m} \left( 1 + \frac{1}{|eg|} \right) \frac{1}{r^2}$  (5)

The Hamiltonian (5) implies that at j = |eg| - 1/2 the radial sector of the spinning particle is equivalently described by a spinless particle in a spherically symmetric potential  $V(r) = -\lambda/r^2$ ,  $\lambda \equiv (1 + 1/|eg|) e^2 g^2/(2m)$ .

We choose a real-space cut-off regularization by introducing a length scale L as the regulator and re-defining the effective potential as  $V_R(r) = -\lambda/r^2\theta(r-L)$ ,  $\theta$  denoting the Heaviside function. The regularized energy eigenvalue equation  $\tilde{H}\tilde{\Phi}_E(r) = E\tilde{\Phi}_E(r)$  reads

$$\left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{|eg|(|eg|+1)}{r^2} + \frac{2m\lambda}{r^2}\theta(r-L) + 2mE\right]\tilde{\Phi}_E(r) = 0$$
(6)

The normalizable bound state solution expressed in the modified Bessel functions is

$$\widetilde{\Phi}_{E}(r) = Ar^{-1/2}I_{|eg|+1/2}(\beta r), \quad r < L,$$
  
=  $Br^{-1/2}K_{1/2}(\beta r), \quad r > L$  (7)

The continuity of  $\tilde{\Phi}_E(r)$  at r = L yields  $A = BK_{1/2}(\beta L)/I_{|eg|+1/2}(\beta L)$ , and the normalization condition  $\int_0^\infty dr r^2 |\tilde{\phi}_E(r)|^2 = 1$  fixes  $B = 2\beta/\sqrt{\pi}$  as  $L \to 0$ . Finally, the continuity of  $d\tilde{\phi}_E(r)/dr$ at r = L leads to

$$1 + 2\beta L = -\beta L \frac{I_{|eg|-1/2}(\beta L) + I_{|eg|+3/2}(\beta L)}{I_{|eg|+1/2}(\beta L)}$$
(8)

Further, the expansion of  $I_{\nu}(x)$  near x = 0 gives the lowest order reduction of (8) at  $L \to 0$ ,  $\beta L = -(|eg| + 1)$ . Hence we get the regularized bound state energy  $E = -(|eg| + 1)^2/(2mL^2)$ . Obviously, the regulator dependent E is divergent as  $L \to 0$ , and the spectrum is unbounded from below in this limit. There are two ways to cure this pathology. The first one is to adopt the viewpoint of the Wilsonian effective field theory [6]. We directly take the regulator L as the cut-off length scale  $\Lambda$  and consider the regularized Hamiltonian  $\tilde{H} \equiv -\nabla^2/(2m) + V_R$  as an effective Hamiltonian above the length scale  $\Lambda$ . The bound state energy at  $r = \Lambda$  is

$$E_B = -\frac{(|eg|+1)^2}{2m\Lambda^2}$$
(9)

The second one is the traditional approach of calculating the one-particle irreducible (1PI) effective action and performing a renormalization procedure as advocated in Ref. [7]. At the renormalization scale  $r = \Lambda$ , we make the subtraction by splitting  $E = E_B + E_{\text{div}}$ . In order to enforce the physical requirement that the wave function should vanish at the origin, which is needed for the self-adjointness of the Hamiltonian, we introduce a counterterm to the  $1/r^2$  potential. This counterterm cancels the short-distance divergence  $E_{\text{div}} = (1/\Lambda^2 - 1/L^2) (|eg| + 1)^2/(2m)$  in the regularized energy E. Furthermore, as in field theory, the counterterm should be absorbed into the redefinition of the coupling constant  $\lambda$ . One particular challenge in the present context is that the condition  $2eg \in Z$  must be preserved for quantum mechanical consistency. A detailed analysis of this procedure will be presented elsewhere [8].

It is clear that both of the above approaches break the SO(2, 1) conformal symmetry due to the unavoidable presence of a length scale  $\Lambda$ . This is a direct manifestation of the conformal anomaly in this system, which has been shown in the modification of the SO(2, 1) commutator algebra through a deformation of the Hamiltonian by the anomaly operator:

$$H \to H + \widehat{\mathcal{A}}, \quad \widehat{\mathcal{A}} \equiv -i[H, D]_A$$

$$\tag{10}$$

The resulting anomalous conformal algebra [7, 9] is composed of  $[H, D] = iH + [H, D]_A$ ,  $[H, K] = 2iD + 2t[H, D]_A$  and  $[D, K] = iK + t^2[H, D]_A$ . The Heisenberg equation further reveals the conformal anomaly as the non-conservation of the conformal charges,  $dD/dt = \hat{A}$ ,  $dK/dt = 2t\hat{A}$ .

An algebraic calculation of the first anomalous commutator shows that the anomaly operator  $\widehat{\mathcal{A}}$  is directly related to the scaling behavior of the  $1/r^2$  potential at the quantum level [7],

$$\widehat{\mathcal{A}} \equiv -i[H,D]_A \equiv i[H,D] + H = \left(1 + \frac{1}{2}\mathbf{r} \cdot \nabla\right) V(r)$$
(11)

We use the regularized wave function (7) and the regularized potential  $V_R$  to explicitly evaluate expectation value of the anomaly operator (11),

$$\mathcal{A} = \langle \widehat{\mathcal{A}} \rangle = \langle V_{\text{eff}}(r) \rangle + \frac{1}{2} \langle \mathbf{r} \cdot \nabla V_R(r) \rangle$$

$$= \lim_{L \to 0} \int_0^\infty dr r^2 \left( 1 + \frac{1}{2} r \frac{\partial}{\partial r} \right) V_R(r) |\tilde{\Phi}_E(r)|^2 \tag{12}$$

A straightforward calculation gives

$$\mathcal{A} = \frac{e^2 g^2 \beta^2}{m} \left( 1 + \frac{1}{|eg|} \right) = -2E_B e^2 g^2 \left( 1 + \frac{1}{|eg|} \right) \tag{13}$$

On the other hand, an alternative and elegant interpretation on the origin of the anomaly in the Hamiltonian formalism has been presented in [9], where it was demonstrated that the anomaly is due to the fact that the symmetry generator does not leave the domain of definition of the Hamiltonian invariant. By a careful observation on the Heisenberg equation, it had been shown [9] that the anomaly arises as  $\mathcal{A} = i\langle \Psi(t) | (H^{\dagger} - H) G | \Psi(t) \rangle$ , G denoting a certain symmetry generator operator which is D for the scale symmetry. This means that the anomaly operator can formally written as

$$\widehat{\mathcal{A}} = \left(H^{\dagger} - H\right)G = -i\mathcal{A}|\Psi(t)\rangle\langle\Psi(t)| \tag{14}$$

According to the rigorous definition of a self-adjoint operator [10], Eq. (14) implies that the Hamiltonian has always acquired a non-self-adjoint piece once its domain of definition cannot be preserved by the symmetry transformation.

The non-self-adjointness induced by the anomaly greatly modifies the quantum dynamics of the system. In the Heisenberg picture, the generator G satisfies a generalized Heisenberg equation [9],

$$\frac{dG}{dt} = \frac{\partial G}{\partial t} + i \left[ H, G \right] + i \left( H^{\dagger} - H \right) G = \frac{\partial G}{\partial t} + i \left[ H, G \right] + i \widehat{\mathcal{A}}$$
(15)

and it implies the following time-evolution of G,

$$G(t) = \exp\left[i\int_0^t ds\left(H^{\dagger}(s) - H(s)\right)\right] \exp\left[i\int_0^t dsH(s)\right]G(0)\exp\left[-i\int_0^t dsH(s)\right]$$
(16)

In the Schrödinger picture, we have the time-evolution in terms of the modified Hamiltonian shown in Eq. (10)

$$|\Psi(t)\rangle = \exp\left[-i\int_0^t ds H(s) - \int_0^t ds \mathcal{A}|\Psi(s)\rangle\langle\Psi(s)|\right]|\Psi(0)\rangle$$
(17)

The formal integration solution (17) for  $|\Psi(t)\rangle$  shows that in the presence of the anomaly the quantum system undergoes a non-unitary evolution resultant from the anomaly. This is consistent with the fact that anomalous effects in a quantum theory contribute only to the imaginary part of the quantum effective action [11].

Turning to the model at hand, we take G to be the generator D of the scale symmetry. The conformal anomaly arises only for the normalizable bound state  $\Psi_E(r, \theta, \phi, \sigma)$  in the s-wave sector,

and originates from its radial part  $\Phi_E(r)$ . Therefore, Eq. (17) tells that the time-evolution for this specific stationary state should be

$$|\Phi_E(t)\rangle = e^{-i(E-i\mathcal{A})t}|\Phi_E(0)\rangle \tag{18}$$

where the energy and anomaly are provided by Eqs. (9) and (13), respectively. Note that the completeness condition  $\sum_{E} |\Phi_E\rangle \langle \Phi_E| = \mathbf{1}$  is used in deriving Eq. (18). Although all the energy eigenstates, including the scattering states, should be taken into account in the the completeness condition, the anomaly only pertains to the bound state and vanishes for all other eigenstates. Thus we effectively take  $|\Phi_E\rangle \langle \Phi_E| = \mathbf{1}$  and get Eq. (18).

We now consider the electron-monopole system as a physical model for quantum computing. The quantum state we manipulate is just the normalizable bound state  $\Psi_E(r, \theta, \phi, \sigma)$  at j = |eg| - 1/2, its two-level spin degrees of freedom playing the role of a qubit:

$$\Psi_E(r,\theta,\phi,\sigma) = \Phi_E(r)\eta_{j,m,\alpha}(\theta,\phi,\alpha) \equiv f_1(r,\theta,\phi) \begin{pmatrix} 1\\ 0 \end{pmatrix} + f_2(r,\theta,\phi) \begin{pmatrix} 0\\ 1 \end{pmatrix}$$
(19)

The spatial amplitudes  $f_i(r, \theta, \phi)$  (i = 1, 2) can be obtained with some algebraic operations [4].

As we will show below, it is the time-evolution of  $f_i(r, \theta, \phi)$  related to  $\Phi_E(r)$  that brings about the decoherence between two spin states during a quantum computing due to the presence of anomaly. Roughly speaking, the two spin states constitutes a qubit, and one must control their dynamical evolution to carry out information processing. We therefore switch on a time-dependent Hamiltonian to make the spin flips that can ultimately be used in a quantum algorithm. However, the spatial sector  $f_i(r, \theta, \phi)$  of the wave function will evolves in time controlled by the quantum effective Hamiltonian of the system itself along with the spin flipping dominated by the external Hamiltonian. According to Eq. (18), the anomaly will cause  $f_i(r, \theta, \phi, t)$  to have a damping factor which in turn will lead to decoherence. In the following we show the details of how this phenomenon happens.

Let us first analyze the quantum effective Hamiltonian provided by the system itself. Obviously, the time-evolution (18) of  $\Phi_E(r)$  of the bound state wave function gives the spatial part,  $H_{\rm spa} = E - i\mathcal{A}$ . As for the spin sector, we use the fact that at j = |eg| - 1/2 the orbit- and spin- angular momenta decouple, and the spin part of the wave function is the eigenstate of the operator  $\sigma \cdot \mathbf{r}/r$ . Specifically, the form of the radial Hamiltonian (5) shapes only when the eigenvalue equation of the operator  $\sigma \cdot \mathbf{r}/r$  has been applied. So we can simply choose  $H_{\rm spin} = \sigma \cdot \mathbf{r}/r$ . A combination of the spatial and spin sectors determines that the effective Hamiltonian with resect to the bound

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state (19) should take the following form:

$$H_{\rm sys} = H_{\rm spa} \otimes H_{\rm spin} = (E - i\mathcal{A}) \frac{\sigma \cdot \mathbf{r}}{r}$$
$$= (E - i\mathcal{A}) (\sigma_x \sin \vartheta \cos \varphi + \sigma_y \sin \vartheta \sin \varphi + \sigma_z \cos \vartheta)$$
(20)

where  $(\vartheta, \varphi)$  represents the spin orientation in three-dimensional space.

Eq. (20) is the effective Hamiltonian realized on the bound sate of the system. We now switch on a time-dependent external Hamiltonian to make the spin flip. A typical choice is the interaction of the spin with an oscillating external magnetic field in two-dimensional x - y plane,  $\mathbf{B}_{\text{ext}} = B_0 (\cos \omega t \mathbf{e}_x + \sin \omega t \mathbf{e}_x)$ , and the Hamiltonian  $H_{\text{ext}} = e/2m\sigma \cdot \mathbf{B}_{\text{ext}}$ . The spin dynamics is dominated by the Schrödinger equation

$$i\frac{\partial\Psi_{E}(t)}{\partial t} = (H_{\text{sys}} + H_{\text{ext}})\Psi_{E}(t) = \left\{ \left[ \frac{eB_{0}}{2m}\cos\omega t + (E - i\mathcal{A})\sin\vartheta\cos\varphi \right] \sigma_{x} + \left[ \frac{eB_{0}}{2m}\sin\omega t + (E - i\mathcal{A})\sin\vartheta\sin\varphi \right] \sigma_{y} + (E - i\mathcal{A})\cos\vartheta\sigma_{z} \right\} \Psi_{E}(t)$$
(21)

We neglect the  $E - i\mathcal{A}$  term in the  $\sigma_x$  and  $\sigma_y$  components since usually the microscopic values of the energy E and the anomaly are much smaller than the macroscopic magnetic field,  $|E|, |\mathcal{A}| \ll$  $|e|B_0/2m$ . In this approximation the time-evolution of the spin state reads

$$\Psi_E(t) = \exp\left\{-\mathcal{A}\cos\vartheta\sigma_z t - i\left[\left(E\cos\vartheta - \frac{\omega}{2}\right)\sigma_z + \frac{eB_0}{2m}\sigma_x\right]t\right\}\Psi_E(0)$$
(22)

To show explicitly the occurrence of the decoherence implied from  $\Psi_E(t)$ , we take  $E \cos \vartheta = \omega/2$ as in nuclear magnetic resonance and use again  $|\mathcal{A}| \ll |e|B_0/2m$ . Assume that the initial state is spin-up,  $\Psi_E(0) = f_1(r, \theta, \phi) (1, 0)^T$ , Eq. (22) yields

$$\Psi_{E}(t) = c_{1}(t) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_{2}(t) \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$c_{1}(t) = \cos \left[ \left( \frac{e^{2}B_{0}^{2}}{4m^{2}} - \mathcal{A}^{2}\cos^{2}\vartheta \right)^{1/2} t \right] - \mathcal{A}\cos\vartheta \frac{\sin \left[ \left( \frac{e^{2}B_{0}^{2}}{4m^{2}} - \mathcal{A}^{2}\cos^{2}\vartheta \right)^{1/2} t \right]}{\left( \frac{e^{2}B_{0}^{2}}{4m^{2}} - \mathcal{A}^{2}\cos^{2}\vartheta \right)^{1/2}}$$

$$c_{2}(t) = e^{i\pi/2} \frac{eB_{0}}{2m} \frac{\sin \left[ \left( \frac{e^{2}B_{0}^{2}}{4m^{2}} - \mathcal{A}^{2}\cos^{2}\vartheta \right)^{1/2} t \right]}{\left( \frac{e^{2}B_{0}^{2}}{4m^{2}} - \mathcal{A}^{2}\cos^{2}\vartheta \right)^{1/2}}$$
(23)

Clearly, the non-vanishing  $\mathcal{A}$  leads to  $|c_1(t)|^2 + |c_2(t)|^2 \neq 1$ , and hence the decoherence between the two helicity states occurs and the qubit is destroyed. To summarize, we have used an electron-monopole system to reveal a phenomenon not previously discussed in the quantum computing literature: a quantum mechanical anomaly can result in decoherence. Note that anomaly is a quantum dynamical phenomenon rooted within the system itself. It reflects how quantum effects can render a classically feasible symmetry unrealizable. One typical example is the case where the configuration space has non-trivial topology so that the Hilbert space constructed via the quantization procedure from the classical phase space cannot sustain all the classical symmetries. In the case we have just considered, the source of the anomaly is the singular behaviors of the interaction potential near the magnetic monopole. The classical conformal symmetry does not preserve the Hilbert space as the domain of definition of the Hamiltonian due to the singular behavior of the wave function in the s-wave sector.

Until now the search for a physically realizable quantum computer has been concerned only with decoherence that arises due to interactions with the external environment. It is important to emphasize that decoherence can also in principle be induced by quantum anomalies. Since this dissipation originates from the dynamics of the quantum system itself, it seems that it has the potential of being more destructive than the standard mechanisms for decoherence.

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