

Test of weak measurement on a two- or three-qubit computer

Todd A. Brun

*Communication Sciences Institute,
University of Southern California,
Los Angeles, California 90089-2565, USA*

Lajos Diósi

*Centro Internacional de Ciencias, 62131 Cuernavaca, Mexico and
Research Institute for Particle and Nuclear Physics
H-1525 Budapest 114, P.O.Box 49, Hungary**

Walter T. Strunz

*Centro Internacional de Ciencias, 62131 Cuernavaca, Mexico and
Institut für Theoretische Physik, Technische Universität, Dresden, D-01062 Dresden, Germany†*
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Current quantum computer technology is sufficient to realize weak measurements and the corresponding concept of weak values. We demonstrate how the weak value anomaly can be tested, along with consistency and simultaneity of weak values, using only discrete degrees of freedom. All you need is a quantum computer with two—or better, three—qubits. We also give an interpretation of the weak value as an effective field strength in a postselected spin measurement.

I. INTRODUCTION

Of the many seeming paradoxes of quantum mechanics, one of the most interesting and bizarre is the idea of a *weak value*. First proposed by Aharonov, Albert and Vaidman [1], this uses a combination of weak measurements and postselection to derive measurement “results” which are far outside the normal range of values for the measured observable. While a remarkable theoretical result, direct experiments that actually demonstrate it are difficult—see, however the interpretation of correlation functions as “weak values” by Wiseman [2] and the quantum optical experiments described in [3] and [4]. With the rapid experimental progress from the surge of interest in quantum information processing, it may be possible to do such experiments in a highly controlled, repeatable fashion, using only discrete degrees of freedom.

In quantum measurements, the act of acquiring information about a quantum system is always accompanied by a complementary disturbance of the system. This is the content of the famous uncertainty principle of Heisenberg. A measurement which does not change the state of the system must also yield no information.

It is possible in principle, however, to make the disturbance as small as one likes, so long as one is content to acquire correspondingly little information. This is the idea of a weak measurement. To perform such a measurement in practice, one must generally cause the system to interact weakly with a second system—an ancillary system, or *ancilla*, sometimes called the “meter”—which is under one’s experimental control, and has been prepared

in a known initial state. This ancilla then undergoes a strong measurement of its own. In the limit where the system and ancilla do not interact at all, clearly this measurement will yield no information. As we gradually increase the strength of the interaction, the measurement outcome will contain more and more information about the system, until eventually the effect is the same as performing a strong measurement directly on the system.

The idea of postselection supposes that instead of performing repeated measurements on a single system, one prepares many copies of the system by repeating the same preparation procedure over and over. These copies will all have the same initial state. These copies then undergo some standard operation—some sequence of unitary transformations and measurements—followed by a final measurement. One then keeps the data only from those systems whose final measurement gave a particular outcome, and averages results over this sub-ensemble.

In this paper we will review the Aharonov, Albert, Vaidman definition of weak values, and then describe how experimental systems designed for quantum computation can lead to an immediate experimental implementation using existing quantum computers, for example in ion trap quantum computers [5, 6, 7].

II. TWO QUBIT INDIRECT MEASUREMENT DEVICE

A qubit is a two-dimensional quantum system, with a standard (“computational”) basis which we denote $\{|0\rangle, |1\rangle\}$. There can be many different physical embodiments of such a system: the spin of an electron, the polarization of a single photon, a two-level subspace of the electronic states of an atom or ion, etc. For quantum algorithms, much work has been devoted to the per-

*permanent address

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formance of *quantum gates*, analogous to classical logic gates, which effect a unitary transformation of one or two qubits at a time. The canonical two-qubit gate is the *controlled-NOT* (CNOT):

$$|i\rangle \otimes |j\rangle \longrightarrow U_{SA} [|i\rangle \otimes |j\rangle] = |i\rangle \otimes |j \oplus i\rangle, \quad (1)$$

where $j \oplus i$ is the exclusive-OR (XOR) of the bit values i and j , and U_{SA} is the unitary transformation which represents a CNOT between the system and the ancilla.

A quantum circuit with a single CNOT gate makes a perfectly controllable indirect measurement of a qubit in the computational basis, by storing the value of the qubit in a second (ancilla) qubit:

$$|i\rangle_S \otimes |0\rangle_A \longrightarrow |i\rangle_S \otimes |i\rangle_A, \quad (i = 0, 1),$$

where S and A label the system and ancilla, respectively, the system is initially in the computational state $|i\rangle$, and the ancilla is initially in the state $|0\rangle$. (We will suppress the labels S, A where there is no possibility of confusion.) The ancilla can then be measured by a strong measuring device, which will simultaneously “collapse the wavefunction” of the system qubit. This type of indirect measurement can be very useful when the only direct measurements are destructive (for example, a photodetector which absorbs the photon it is measuring). If the system is initially in a superposition of computational basis states, it will become entangled with the ancilla:

$$(\alpha|0\rangle + \beta|1\rangle) \otimes |0\rangle \longrightarrow \alpha|0\rangle \otimes |0\rangle + \beta|1\rangle \otimes |1\rangle.$$

When the ancilla is measured, one of these two terms will be selected with probability $|\alpha|^2$ or $|\beta|^2$.

Suppose now that instead of $|0\rangle$ we prepare the ancilla in the initial superposition

$$|\psi_A\rangle = \cos \frac{\vartheta}{2} |0\rangle + \sin \frac{\vartheta}{2} |1\rangle.$$

Let the system qubit be in the state $|\phi_i\rangle = \alpha|0\rangle + \beta|1\rangle$, and have the two qubits interact via the CNOT. Then we measure the ancilla in its computational basis in order to obtain information about $|\phi_i\rangle$. After the CNOT, the system and ancilla are in the state

$$|\phi_i\rangle \otimes |\psi_A\rangle \longrightarrow |\Psi\rangle = U_{SA} [|\phi_i\rangle \otimes |\psi_A\rangle] \quad (2)$$

with

$$\begin{aligned} |\Psi\rangle = & \alpha|0\rangle \otimes (\cos \frac{\vartheta}{2} |0\rangle + \sin \frac{\vartheta}{2} |1\rangle) \\ & + \beta|1\rangle \otimes (\sin \frac{\vartheta}{2} |0\rangle + \cos \frac{\vartheta}{2} |1\rangle). \end{aligned} \quad (3)$$

If $\vartheta = 0$ then $|\psi_A\rangle = |0\rangle$ and this is the case we have just considered: the indirect measurement is perfectly equivalent with a direct measurement of the first qubit. If $\vartheta = \pi/2$ then the indirect measurement does not give any information on the first qubit, whose state $|\phi_i\rangle$ will just survive the procedure unchanged, without being entangled with the state of the ancilla. Hence, the parameter

ϑ offers full control of the strength of the indirect measurement. We shall be interested in weak measurements, which are realized by $\vartheta = (\pi/2) - \epsilon$ where $0 < \epsilon \ll 1$. This will be discussed later.

Let us determine the expectation value of the operator $\hat{\sigma}_z \equiv |0\rangle\langle 0| - |1\rangle\langle 1|$ of the ancilla in the state $|\Psi\rangle$ given by Eq. (3):

$$\begin{aligned} \langle \hat{\sigma}_z^{\text{ancilla}} \rangle &= |\alpha|^2 \cos^2 \frac{\vartheta}{2} + |\beta|^2 \sin^2 \frac{\vartheta}{2} \\ &\quad - |\alpha|^2 \sin^2 \frac{\vartheta}{2} - |\beta|^2 \cos^2 \frac{\vartheta}{2} \\ &= (|\alpha|^2 - |\beta|^2) \left(\cos^2 \frac{\vartheta}{2} - \sin^2 \frac{\vartheta}{2} \right) \\ &= (|\alpha|^2 - |\beta|^2) \cos \vartheta. \end{aligned} \quad (4)$$

Since $|\alpha|^2 - |\beta|^2 = \langle \phi_i | \hat{\sigma}_z | \phi_i \rangle$ (which we simply denote by $\langle \hat{\sigma}_z \rangle$), it follows that

$$\langle \hat{\sigma}_z \rangle = \frac{1}{\cos \vartheta} \langle \hat{\sigma}_z^{\text{ancilla}} \rangle \quad (5)$$

where the expectation value on the l.h.s. is the expectation value of $\hat{\sigma}_z$ in the system initial state $|\phi_i\rangle$, while the expectation value on the r.h.s. stands for the post-interaction expectation value of $\hat{\sigma}_z^{\text{ancilla}}$.

The simple relationship (5) suggests that we can still measure the system expectation value of $\hat{\sigma}_z$ if we measure the ancilla expectation value of $\hat{\sigma}_z^{\text{ancilla}}$ instead, and rescale the result by $1/\cos \vartheta$. Of course, the statistical error of the indirect measurement is larger than the statistical error of the direct measurement. Suppose that many copies of the system and ancilla are prepared in the same initial state. For each copy, the CNOT interaction is performed, and then the ancilla is measured in the computational basis. These measurements are used to estimate the expectation value of the operator $\hat{\sigma}_z^{\text{ancilla}}$. This latter quantity is what we estimate from the measurement statistics:

$$\langle \hat{\sigma}_z^{\text{ancilla}} \rangle \approx \frac{N_0 - N_1}{N_0 + N_1} \quad (6)$$

where N_0, N_1 are the measurement counts corresponding to the outcomes $|0\rangle$ and $|1\rangle$ when measuring $\hat{\sigma}_z^{\text{ancilla}}$, respectively, obtained from a total number of measurements $N = N_0 + N_1$. Let us determine the statistical error of the quantity (6) for large N :

$$\Delta \langle \hat{\sigma}_z^{\text{ancilla}} \rangle \approx \sqrt{\frac{2(1 + \langle \hat{\sigma}_z^{\text{ancilla}} \rangle)}{N}}, \quad (7)$$

yielding the following statistical error of the indirect measurement of $\langle \hat{\sigma}_z \rangle$:

$$\Delta \langle \hat{\sigma}_z \rangle \approx \frac{1}{\cos \vartheta} \sqrt{\frac{2(1 + \cos \vartheta \langle \hat{\sigma}_z \rangle)}{N}}, \quad (8)$$

which increases with ϑ . Observe that the value $\vartheta = 0$ would formally correspond to the direct measurement.

We are interested in the weak measurement limit $\vartheta = (\pi/2) - \epsilon$. To leading order in the small parameter ϵ we have:

$$\langle \hat{\sigma}_z \rangle = \frac{1}{\epsilon} \langle \hat{\sigma}_z^{\text{ancilla}} \rangle, \quad (9)$$

and

$$\Delta \langle \hat{\sigma}_z \rangle \approx \frac{1}{\epsilon} \sqrt{\frac{2}{N}}, \quad (10)$$

since $\epsilon \ll 1$. The latter equation also means that the statistical error of a *single* measurement is $\sim \sqrt{2}/\epsilon$. The two equations (9,10) assure, that our indirect measurement is a weak measurement of the system's $\hat{\sigma}_z$, cf. the general definitions in [11]: i) our measurement yields the unbiased mean of $\langle \hat{\sigma}_z \rangle$ and ii) the statistical error (regarding $\langle \hat{\sigma}_z \rangle$) of a single measurement is much larger than total range of all possible values of the measured quantity $\hat{\sigma}_z$. For an arbitrary small ϵ we need to have suitably large statistics $N \sim \epsilon^{-2}$ to yield an estimate of $\langle \hat{\sigma}_z \rangle$ with any desired precision. Accordingly, our weak measurement reproduces all basic features of the AAV weak measurement, that we are going to show by detailed proofs in the forthcoming sections.

Other than convenience, there is no particular reason to chose $\hat{\sigma}_z$ as the observable. For later reference we mention that such an indirect measurement of, for instance, $\hat{\sigma}_x$ is best formulated in terms of its eigenstates $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$. Accordingly, the ancilla qubit should be prepared in the state

$$|\psi_A^x\rangle = \cos \frac{\vartheta}{2} |+\rangle + \sin \frac{\vartheta}{2} |-\rangle \quad (11)$$

and for the corresponding CNOT^x operation one has to replace the computational basis states $\{|0\rangle, |1\rangle\}$ by $\{|+\rangle, |-\rangle\}$ in expression (1). (This interaction is the same as the usual CNOT with the control and target qubits interchanged.)

III. TWO QUBIT INDIRECT MEASUREMENT WITH POSTSELECTION AND THE WEAK VALUE ANOMALY

Up to this point, we have assumed that after the interaction with the ancilla and the ancilla's subsequent measurement we make no further use of the original system. It is possible, however, to measure the system as well as the ancilla. Then, instead of the usual statistics including all measurement outcomes as described above, we keep only results where the additional system measurement confirms the system qubit to be in a certain final state $|\phi_f\rangle$. This is the idea of *postselection*, described by Aharonov, Albert and Vaidman in [1].

Naively, we calculate the same quantity as before,

$$\frac{N_{f0} - N_{f1}}{N_{f0} + N_{f1}}, \quad (12)$$

and we call it the *postselected estimate* of $\hat{\sigma}_z^{\text{ancilla}}$, with respect to the final system state $|\phi_f\rangle$. As before, we rescale the above quantity by $1/\cos\vartheta$ and expect that in the large N limit we obtain something sensible in terms of the system's $\hat{\sigma}_z$ and of the initial as well of the final states $|\phi_i\rangle, \langle\phi_f|$. While this expectation fails in general, it becomes true in the weak measurement limit. Then, surprisingly, the postselection rate is just $|\langle\phi_f|\phi_i\rangle|^2$, independent of the (weak) interaction with the ancilla. As first defined by AAV [1], the so-called *weak value* of $\hat{\sigma}_z$ is

$${}_f\langle \hat{\sigma}_z \rangle_i \equiv \text{Re} \frac{\langle \phi_f | \hat{\sigma}_z | \phi_i \rangle}{\langle \phi_f | \phi_i \rangle}, \quad (13)$$

(once again expressed in terms of the system qubit state). We will now show that in the large N limit

$$\frac{1}{\epsilon} \frac{N_{f0} - N_{f1}}{N_{f0} + N_{f1}} \approx {}_f\langle \hat{\sigma}_z \rangle_i. \quad (14)$$

In other words, our indirect device *with* postselection measures the weak value of the system qubit, in the very same way that it gave us the ordinary value $\langle \hat{\sigma}_z \rangle$ *without* postselection, c.f., Eq. (9).

Let us begin with the post-interaction state $|\Psi\rangle$ from (3). The probabilities of finding the system in the state $|\phi_f\rangle$ and the ancilla in the state $|0\rangle$ or $|1\rangle$, respectively, are

$$\begin{aligned} p_{f0} &= \left| \langle \phi_f | \left(\alpha \cos \frac{\vartheta}{2} |0\rangle + \beta \sin \frac{\vartheta}{2} |1\rangle \right) \right|^2, \\ p_{f1} &= \left| \langle \phi_f | \left(\alpha \sin \frac{\vartheta}{2} |0\rangle + \beta \cos \frac{\vartheta}{2} |1\rangle \right) \right|^2. \end{aligned} \quad (15)$$

In the limit of large N , the postselected estimate of $\hat{\sigma}_z^{\text{ancilla}}$ conditioned on the outcome of the system measurement being $|\phi_f\rangle$ is $(p_{f0} - p_{f1})/(p_{f0} + p_{f1})$. Unlike the case we considered in Sec. II, in general these quantities have nothing universal to do with $\hat{\sigma}_z$ of the system qubit.

However, the case becomes positive in the weak measurement limit $\vartheta = \pi/2 - \epsilon$. In this limit there is indeed a relationship between the outcomes of the postselected measurement and $\hat{\sigma}_z$ of the system qubit, which we can see by expanding Eqs. (15) to first order in ϵ :

$$\begin{aligned} p_{f0} &\approx \frac{1}{2} (|\langle \phi_f | \phi_i \rangle|^2 + \epsilon \cdot \text{Re}[\langle \phi_f | \hat{\sigma}_z | \phi_i \rangle \langle \phi_i | \phi_f \rangle]), \\ p_{f1} &\approx \frac{1}{2} (|\langle \phi_f | \phi_i \rangle|^2 - \epsilon \cdot \text{Re}[\langle \phi_f | \hat{\sigma}_z | \phi_i \rangle \langle \phi_i | \phi_f \rangle]) \end{aligned} \quad (16)$$

To lowest order in ϵ we thus find

$$\frac{p_{f0} - p_{f1}}{p_{f0} + p_{f1}} \approx \epsilon \cdot \text{Re} \frac{\langle \phi_f | \hat{\sigma}_z | \phi_i \rangle}{\langle \phi_f | \phi_i \rangle} = \epsilon \cdot {}_f\langle \hat{\sigma}_z \rangle_i. \quad (17)$$

If we repeat this procedure N times and get N_{f0} results $|\phi_f\rangle|0\rangle$ and N_{f1} results $|\phi_f\rangle|1\rangle$, where $N_{f0}/N_{f1} \approx p_{f0}/p_{f1}$ holds, the postselected indirect estimate of $\hat{\sigma}_z$ (i.e. the weak value of $\hat{\sigma}_z$) is just given by (14) in the limit of large N , as claimed above.

Let us emphasize that we are extending the original AAV theory, which makes use of a von Neumann measuring device where the ancilla is a fictitious particle, whose position serves as the meter. In the case presented here, the ancilla is just a qubit, and all of the degrees of freedom are discrete. Such an extended theory of weak measurement was given recently in [11], and a simple version of this was earlier used in [12].

The weak value anomaly is reflected in the fact that if we invariably trust in our weak measurement device on a post-selected ensemble as well as on the whole ensemble then the postselected indirect estimate of $\hat{\sigma}_z$ falls well outside the range $[-1, 1]$ of “normal” expectation values for $\hat{\sigma}_z$. For a concrete example, consider the initial and final states

$$\begin{aligned} |\phi_i\rangle &= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \\ |\phi_f\rangle &= \frac{1}{\sqrt{2(z^2+1)}}((z+1)|0\rangle - (z-1)|1\rangle) \end{aligned} \quad (18)$$

with arbitrary real parameter z . We find ${}_f\langle\hat{\sigma}_z\rangle_i = z$, and therefore the postselected estimate of $\langle\hat{\sigma}_z\rangle$ would be z , which can take any (arbitrarily large) value. The effect of the anomalous large mean value in the postselected states is real: a probe will sense it as a large mean field, cf. Sec. V. Clearly, the more “anomalous” the postselected estimate, i.e. the larger z , the less likely the postselection criterion will be met: the initial and final states $|\phi_i\rangle$ and $|\phi_f\rangle$ are almost orthogonal, and the probability for a successful run of the experiment is $p_f = p_{f0} + p_{f1} \approx 1/(z^2+1)$. For large z , most runs of the experiment will have to be discarded. To infer the weak value with a mean square precision $\Delta_w < 1$ will require on the order of $(z^2+1)/\Delta_w^2$ experimental runs.

Since the postselected estimate of $\hat{\sigma}_z^{\text{ancilla}}$ given by (12) must obviously be in the range $[-1, 1]$, we see that it is necessary that $|\epsilon z| < 1$. A choice of parameters such that $|\epsilon z| > 1$ implies that the expansion in ϵ given by (16) can no longer be a valid approximation. In fact, the AAV equation (14) is always meant to hold in the asymptotic weak measurement limit $\epsilon \rightarrow 0$.

IV. THREE QUBIT CONSISTENCY AND SIMULTANEITY TEST OF WEAK VALUES

In order to test the consistency of weak values, we need a three-qubit quantum computer. We use the third qubit as a second ancilla, prepared in the initial state $|\psi_{A_2}\rangle = \cos(\vartheta_2/2)|0\rangle + \sin(\vartheta_2/2)|1\rangle$, and perform another indirect weak measurement of $\hat{\sigma}_z$ on the first qubit. The question is, are both weak measurements *consistent*—that is, do both measurements give the same weak value ${}_f\langle\hat{\sigma}_z\rangle_i$?

To answer this question, we perform an additional CNOT operation between the system and the second ancilla, to obtain the three qubit state

$$|\Psi_0\rangle = |\phi_i\rangle |\psi_{A_1}\rangle |\psi_{A_2}\rangle \quad (19)$$

$$\longrightarrow |\Psi_{zz}\rangle = U_{SA_2} U_{SA_1} |\Psi_0\rangle$$

(similar to expression (3)), with

$$\begin{aligned} |\Psi_{zz}\rangle &= \alpha |0\rangle \otimes \left(\cos \frac{\vartheta_1}{2} |0\rangle + \sin \frac{\vartheta_1}{2} |1\rangle \right) \\ &\otimes \left(\cos \frac{\vartheta_2}{2} |0\rangle + \sin \frac{\vartheta_2}{2} |1\rangle \right) \\ &+ \beta |1\rangle \otimes \left(\sin \frac{\vartheta_1}{2} |0\rangle + \cos \frac{\vartheta_1}{2} |1\rangle \right) \\ &\otimes \left(\sin \frac{\vartheta_2}{2} |0\rangle + \cos \frac{\vartheta_2}{2} |1\rangle \right). \end{aligned} \quad (20)$$

Now we count the number of events corresponding to the $|0\rangle$ and $|1\rangle$ states of both ancillas, and perform the postselection with respect to the final system state $|\phi_f\rangle$. As we will now show, in the weak measurement limit ($\vartheta_i = (\pi/2) - \epsilon_i, i = 1, 2$) the estimates for the postselected $\hat{\sigma}_z$ are entirely consistent. For the probabilities, we find to leading order in ϵ_1, ϵ_2

$$\begin{aligned} p_{f00} &\approx \frac{|\langle\phi_f|\phi_i\rangle|^2 + (\epsilon_1 + \epsilon_2)\text{Re}[\langle\phi_f|\hat{\sigma}_z|\phi_i\rangle\langle\phi_i|\phi_f\rangle]}{4}, \\ p_{f01} &\approx \frac{|\langle\phi_f|\phi_i\rangle|^2 + (\epsilon_1 - \epsilon_2)\text{Re}[\langle\phi_f|\hat{\sigma}_z|\phi_i\rangle\langle\phi_i|\phi_f\rangle]}{4}, \\ p_{f10} &\approx \frac{|\langle\phi_f|\phi_i\rangle|^2 - (\epsilon_1 - \epsilon_2)\text{Re}[\langle\phi_f|\hat{\sigma}_z|\phi_i\rangle\langle\phi_i|\phi_f\rangle]}{4}, \\ p_{f11} &\approx \frac{|\langle\phi_f|\phi_i\rangle|^2 - (\epsilon_1 + \epsilon_2)\text{Re}[\langle\phi_f|\hat{\sigma}_z|\phi_i\rangle\langle\phi_i|\phi_f\rangle]}{4}. \end{aligned}$$

The postselected estimates of each ancilla can be determined by averaging over the results for the other. So we get $p_{f0*} = p_{f00} + p_{f01}$ and $p_{f1*} = p_{f10} + p_{f11}$, and similar expressions for p_{f*0} and p_{f*1} . Putting these expressions together, we get postselected expectations that are entirely consistent with our first result (17):

$$\begin{aligned} \frac{p_{f0*} - p_{f1*}}{p_{f0*} + p_{f1*}} &\approx \epsilon_1 \cdot {}_f\langle\hat{\sigma}_z\rangle_i \\ \frac{p_{f*0} - p_{f*1}}{p_{f*0} + p_{f*1}} &\approx \epsilon_2 \cdot {}_f\langle\hat{\sigma}_z\rangle_i. \end{aligned} \quad (21)$$

We conclude that an identical, second weak measurement of the same observable gives the same weak value, and hence that the weak value measurements are consistent. Given that the correct measurement outcome is observed for the system, all of the ancillas which interacted weakly with the system will yield the same weak value.

A possibly even more intriguing property of weak measurement is that it is possible to simultaneously measure consistent weak values of *non-commuting* observables. To demonstrate this on a three-qubit quantum computer, we choose to weakly measure $\hat{\sigma}_z$ with the help of the first ancilla, as before. The second ancilla, however, will now be used to weakly measure $\hat{\sigma}_x$, as briefly described at the end of the Sec. II.

As before, we start with a three-qubit product state; but now the third qubit is prepared in state $|\psi_{A_2}^x\rangle$, as

given in (11). The usual CNOT=CNOT^z operation is performed between the system and first ancilla qubit, followed by a CNOT^x operation between the system and second ancilla qubit. Let us now denote the unitary operation between the system and first ancilla by $U_{SA_1}^z$ and the unitary operation between the system and the second ancilla by $U_{SA_2}^x$. Similar to the previous double operation (20), we obtain the three qubit state

$$\begin{aligned} |\Psi_0\rangle &= |\phi_i\rangle |\psi_{A_1}^z\rangle |\psi_{A_2}^x\rangle \\ &\longrightarrow |\Psi_{xz}\rangle = U_{SA_2}^x U_{SA_1}^z |\Psi_0\rangle, \end{aligned} \quad (22)$$

with a somewhat lengthy expression for $|\Psi_{xz}\rangle$ (which we omit for the sake of brevity). One can think of this as weakly measuring $\hat{\sigma}_z$ with the first ancilla and $\hat{\sigma}_x$ with the second ancilla. Because these are weak measurements, this does not violate the usual restriction against simultaneously measuring noncommuting observables, because each individual weak measurement yields only a small amount of information. To find the expectations, we repeat this procedure many times.

Now we count the number of events corresponding to the $|0\rangle$ and $|1\rangle$ states of the first ancilla, and the $|+\rangle$ and $|-\rangle$ state of the second ancilla. Again, we postselect with respect to the final system state $|\phi_f\rangle$. After some algebra, in the weak measurement limit ($\vartheta_i = (\pi/2) - \epsilon_i, i = 1, 2$) we find to leading order in ϵ_1, ϵ_2 the probabilities

$$\begin{aligned} p_{f0+} &\approx \frac{1}{4} \left(|\langle \phi_f | \phi_i \rangle|^2 + \epsilon_1 \text{Re}[\langle \phi_f | \hat{\sigma}_z | \phi_i \rangle \langle \phi_i | \phi_f \rangle] \right. \\ &\quad \left. + \epsilon_2 \text{Re}[\langle \phi_f | \hat{\sigma}_x | \phi_i \rangle \langle \phi_i | \phi_f \rangle] \right), \\ p_{f0-} &\approx \frac{1}{4} \left(|\langle \phi_f | \phi_i \rangle|^2 + \epsilon_1 \text{Re}[\langle \phi_f | \hat{\sigma}_z | \phi_i \rangle \langle \phi_i | \phi_f \rangle] \right. \\ &\quad \left. - \epsilon_2 \text{Re}[\langle \phi_f | \hat{\sigma}_x | \phi_i \rangle \langle \phi_i | \phi_f \rangle] \right), \\ p_{f1+} &\approx \frac{1}{4} \left(|\langle \phi_f | \phi_i \rangle|^2 - \epsilon_1 \text{Re}[\langle \phi_f | \hat{\sigma}_z | \phi_i \rangle \langle \phi_i | \phi_f \rangle] \right. \\ &\quad \left. + \epsilon_2 \text{Re}[\langle \phi_f | \hat{\sigma}_x | \phi_i \rangle \langle \phi_i | \phi_f \rangle] \right), \\ p_{f1-} &\approx \frac{1}{4} \left(|\langle \phi_f | \phi_i \rangle|^2 - \epsilon_1 \text{Re}[\langle \phi_f | \hat{\sigma}_z | \phi_i \rangle \langle \phi_i | \phi_f \rangle] \right. \\ &\quad \left. - \epsilon_2 \text{Re}[\langle \phi_f | \hat{\sigma}_x | \phi_i \rangle \langle \phi_i | \phi_f \rangle] \right). \end{aligned} \quad (23)$$

We expect that the counts of the first ancilla will give a postselected estimate of $\hat{\sigma}_z$, while the counts of the second ancilla will give a postselected estimate of $\hat{\sigma}_x$. With the notation $p_{f0*} = p_{f0+} + p_{f0-}$, $p_{f1*} = p_{f1+} + p_{f1-}$, $p_{f*+} = p_{f0+} + p_{f1+}$, and $p_{f*-} = p_{f0-} + p_{f1-}$, we find the *simultaneously* valid expressions

$$\begin{aligned} \frac{p_{f0*} - p_{f1*}}{p_{f0*} + p_{f1*}} &\approx \epsilon_1 \cdot f\langle \hat{\sigma}_z \rangle_i \\ \frac{p_{f*+} - p_{f*-}}{p_{f*+} + p_{f*-}} &\approx \epsilon_2 \cdot f\langle \hat{\sigma}_x \rangle_i. \end{aligned} \quad (24)$$

Again, these results are entirely consistent with our first result (17). We conclude that the simultaneous weak measurement of non-commuting observables gives consistent weak values for both observables.

Unsurprisingly, the order of the weak interactions is entirely irrelevant. If we choose instead to first interact with the third and then with the second qubit, we will obtain a (slightly) different state $|\Psi_{zx}\rangle$; however, despite this difference, the probabilities determined with $|\Psi_{zx}\rangle$ still coincide with the expressions (23) above, and yield the very same postselected estimates (24) as with $|\Psi_{xz}\rangle$.

V. TWO QUBIT DYNAMICAL TEST OF THE WEAK FIELD

The role of the weak value in the dynamic effect on the *probe* was already discussed in Ref. [10]. We will now show that, using just two qubits, we can get perfect quantitative evidence of the weak value as an objective dynamic quantity of the usual sense.

Suppose we prepare our first qubit in state $|\phi_i\rangle$ and postselect it in state $|\phi_f\rangle$. Between pre- and postselection we let it interact with a probe prepared in a certain state $|\psi\rangle$. Assume that their interaction Hamiltonian is $\hat{\sigma}_z \otimes \hat{\mu}$ where we can say that $\hat{\sigma}_z$ stands for the “magnetic field” of the qubit and $\hat{\mu}$ stands for the “magnetic dipole” of the probe. The interaction is switched on for a short period δt between pre- and postselection, and we assume that the effective coupling remains weak. (Its weakness will be specified later.) We can calculate the unnormalized final state of the probe on the postselected statistics:

$$|\psi\rangle \longrightarrow |\psi\rangle - i\delta t \frac{\langle \phi_f | \hat{\sigma}_z | \phi_i \rangle}{\langle \phi_f | \phi_i \rangle} \hat{\mu} |\psi\rangle, \quad (25)$$

which means the probe feels an effective “magnetic field”

$$\frac{\langle \phi_f | \hat{\sigma}_z | \phi_i \rangle}{\langle \phi_f | \phi_i \rangle}. \quad (26)$$

This quantity is complex, in general. Its real part is the weak value $f\langle \hat{\sigma}_z \rangle_i$ of the qubit “magnetic field” $\hat{\sigma}_z$, which we could infer by doing the corresponding weak measurements as in Secs. III and IV. Now we see an alternative approach: instead of inferring the weak value from a weak measurement, we can detect it dynamically, since the postselected qubit has effectively created a “magnetic field” which is equal to the weak value $f\langle \hat{\sigma}_z \rangle_i$.

The weak value anomaly is also persistent dynamically (in the same sense that it is robust under multiple weak measurements). The mechanism is a natural extension of the usual mean-field mechanism to the case of postselection. The surprising consequence in the use of postselected states is that the mean field of the qubit can be many times larger than the common (i.e., not postselected) mean field. Note, however, that the interpretation requires either a weak field or a short interaction time; that is, the condition $\delta t |f\langle \hat{\sigma}_z \rangle_i| \ll 1$. To produce

such a “multiplied field” over a longer time would require repeated postselected measurements, so that the probability of success quickly goes to zero.

We have restricted ourselves to the interpretation of the real part of the effective field (26). The imaginary part is a separate issue, perhaps corresponding to a non-dynamical irreversible effect, superimposed on the purely dynamical effect of the real part. The interpretation deserves further investigation, cf. e.g. [13].

Realizing such a dynamical test is straightforward. For the pre- and postselected qubit, we choose the example (18), which we have already shown to produce arbitrarily large weak values z . Let the probe be a second qubit of dipole moment $\hat{\mu} = \hat{\sigma}_x$ and initial state $|\psi\rangle = |0\rangle$. We must perform the following weak interaction:

$$|\phi\rangle \otimes |0\rangle \longrightarrow (1 - i\delta t \hat{\sigma}_z \otimes \hat{\sigma}_x) |\phi\rangle \otimes |0\rangle. \quad (27)$$

This is not a standard quantum-logical operation, but it should certainly be realizable by the hardware of a quantum computer. The effect (25) of this interaction on the probe is this:

$$|\psi\rangle \longrightarrow (1 - i\delta t z \hat{\sigma}_x) |\psi\rangle, \quad (28)$$

as if the first qubit creates a “mean field” z , and rotates the probe qubit from state $|0\rangle$ into $|0\rangle - i\delta t z \hat{\sigma}_x |0\rangle$. We could choose, e.g., $z = 100$ and $\delta t = 1/1000$, measure the state of the probe in the computational basis, and detect the rate of $|1\rangle$ outcomes. It must be $\sim (\delta t z)^2 = 1/100$, corresponding to a two-order-of-magnitude enhancement of the mean field. However, we recall that $(z^2 + 1)/\Delta_w^2 \sim 10,000$ experimental runs would be needed to confirm the anomalous value $z=100$ by weak measurements (see Sec. III). Since the dynamical effect of the post-selected qubit remains perturbative, we would need even higher statistics to confirm the enhanced value $z = 100$ of the post-selected “magnetic field”—approximately 10^6 runs in the case described above.

Of course, the coupling to the probe qubit could also be added to the weak measurement device of Secs. III and

IV, and the dynamical effect of the post-selected qubit will turn out to be consistent with the outcome of the weak measurement. Moreover, we could implement further probe qubits with different couplings $\hat{\mu}$, that would all “feel” the same mean field $i\langle \hat{\sigma}_z \rangle_f$.

VI. SUMMARY

We have presented a detailed analysis of a realistic scheme to probe the concept of “weak value” in quantum mechanics [1], based on a quantum computer of just two or three qubits. This seems to be in comparatively easy reach of current quantum technology [5, 6, 7]. We have also discussed the appearance of the weak value anomaly, which is measurable with just two qubits. It is possible, as we show, to test both the consistency of weak values and the simultaneity of weak values of non-commuting observables using three qubits. Finally, the dynamic implications of a weak value “mean field” were analyzed.

We strongly believe that the realization of a weak measurement in such a few-qubit quantum system will help to clarify the true meaning and relevance of the concepts surrounding the “weak value” in quantum mechanics. We look forward to seeing our proposal implemented in existing quantum computers.

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