

Nonlinear electrodynamics in 3D gravity with torsion

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Abstract

We study exact solutions of nonlinear electrodynamics coupled to three-dimensional gravity with torsion. We show that in any static and spherically symmetric configuration, at least one component of the electromagnetic field has to vanish. In the electric sector of the theory, we construct an exact solution, characterized by the azimuthal electric field. When the electromagnetic action is modified by a topological mass term, we find two types of the self-dual solutions.

1 Introduction

Three-dimensional (3D) gravity has played an important role in our attempts to properly understand basic features of the gravitational dynamics [1]. The study of 3D gravity led to a number of outstanding results, among which the discovery of the Bañados-Teitelboim-Zanelli (BTZ) black hole was of particular importance [2]. While the geometric structure of the theory is traditionally treated as being Riemannian, Mielke and Baekler proposed a new approach to 3D gravity [3], based on the more general, Riemann-Cartan geometry [4]. In this approach, the geometric structure of spacetime is described by both the curvature and the torsion, which offers an opportunity to explore the influence of geometry on the gravitational dynamics.

Recent developments showed that 3D gravity with torsion has a respectable dynamical structure [5, 6, 7, 8]. In particular, it possesses a BTZ-like black hole solution, the black hole with torsion [5], which is electrically neutral. An extension of this result to the electrically charged sector has been studied in Refs. [9, 10]. The treatment is based on Maxwell electrodynamics, and it reveals new dynamical aspects of 3D gravity with torsion. In particular, the structure of the new solutions, representing the azimuthal electric and the self-dual Maxwell field, is to a large extent influenced by the values of two different central charges that characterize the asymptotic dynamics of the BTZ black hole with torsion.

In the 1930s, in an attempt to construct a classical theory of charged particles with a finite self-energy, Born and Infeld proposed a nonlinear generalization of Maxwell electrodynamics [11]. More recently, it was found that the nonlinear Born-Infeld electrodynamics arises naturally as an effective low-energy action in open string theory [12], which led to

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an increased interest for Born-Infeld-like theories. In general relativity with a cosmological constant, asymptotically (anti-)de Sitter black holes were found in any dimension, and their thermodynamic properties were discussed in a number of papers [13]. Here, we study exact solutions for the system of *nonlinear electrodynamics* of the Born-Infeld type coupled to 3D gravity with torsion, generalizing thereby the analysis of [9, 10], based on *Maxwell electrodynamics*.

The layout of the paper is as follows. In section 2, we derive general field equations describing the nonlinear electrodynamics coupled to 3D gravity with torsion. In section 3, we use these equations to study static and spherically symmetric field configurations. After proving a general no-go theorem, valid for an arbitrary effective cosmological constant Λ_{eff} , which states that at least one component of the electromagnetic field has to vanish, we restrict our attention to the electric sector of the theory. As in the Maxwell-Mielke-Baekler system [9, 10], there is no nontrivial solution corresponding to the radial electric field. In section 4, we construct a solution generated by the azimuthal electric field. In section 5, we consider the sector of stationary and spherically symmetric configurations. After modifying the electromagnetic action by a topological mass term, we find two types of the self-dual solutions. All the solutions are defined in the AdS sector, with $\Lambda_{\text{eff}} < 0$. Finally, appendices contain some technical details.

Our conventions are the same as in [9, 10]: the Latin indices (i, j, k, \dots) refer to the local Lorentz frame, the Greek indices $(\mu, \nu, \lambda, \dots)$ refer to the coordinate frame, and both run over 0,1,2; the metric components in the local Lorentz frame are $\eta_{ij} = (+, -, -)$; totally antisymmetric tensor ε^{ijk} and the related tensor density $\varepsilon^{\mu\nu\rho}$ are both normalized by $\varepsilon^{012} = 1$.

2 Nonlinear electrodynamics coupled to 3D gravity with torsion

2.1 Geometric structure in brief

Theory of gravity with torsion can be naturally described as a Poincaré gauge theory (PGT), with an underlying spacetime structure corresponding to Riemann-Cartan geometry [4].

Basic gravitational variables in PGT are the triad field b^i and the Lorentz connection $A^{ij} = -A^{ji}$ (1-forms). The corresponding field strengths are the torsion and the curvature: $T^i := db^i + A^i_m \wedge b^m$, $R^{ij} := dA^{ij} + A^i_m \wedge A^{mj}$ (2-forms). In 3D, we can use the simplifying notation $A^{ij} =: -\varepsilon^{ij}_k \omega^k$ and $R^{ij} =: -\varepsilon^{ij}_k R^k$, which leads to

$$T^i = db^i + \varepsilon^i_{jk} \omega^j \wedge b^k, \quad R^i = d\omega^i + \frac{1}{2} \varepsilon^i_{jk} \omega^j \wedge \omega^k. \quad (2.1)$$

The covariant derivative $\nabla(\omega)$ acts on a general tangent-frame spinor/tensor in accordance with its spinorial/tensorial structure; when X is a form, $\nabla X := \nabla \wedge X$. PGT is characterized by a useful identity:

$$\omega^i \equiv \tilde{\omega}^i + K^i, \quad (2.2a)$$

where $\tilde{\omega}^i$ is the Levi-Civita (Riemannian) connection, and K^i is the contortion 1-form, defined implicitly by

$$T^i =: \varepsilon^i_{mn} K^m \wedge b^n. \quad (2.2b)$$

Using this identity, one can express the curvature $R_i = R_i(\omega)$ in terms of its *Riemannian* piece $\tilde{R}_i = R_i(\tilde{\omega})$ and the contortion K_i :

$$2R_i \equiv 2\tilde{R}_i + 2\tilde{\nabla}K_i + \varepsilon_{imn}K^m \wedge K^n. \quad (2.2c)$$

The antisymmetry of the Lorentz connection A^{ij} implies that the geometric structure of PGT corresponds to Riemann-Cartan geometry, in which b^i is an orthonormal coframe, $g := \eta_{ij}b^i \otimes b^j$ is the metric of spacetime, and ω^i is the Cartan connection.

In local coordinates x^μ , we can write $b^i = b^i_\mu dx^\mu$, the frame $h_i = h_i^\mu \partial_\mu$ dual to b^i is defined by the property $h_i \lrcorner b^j = h_i^\mu b^j_\mu = \delta_i^j$, where \lrcorner is the interior product. In what follows, we will omit the wedge product sign \wedge for simplicity.

2.2 Lagrangian

General gravitational dynamics in Riemann-Cartan spacetime is determined by Lagrangians which are at most quadratic in field strengths. Omitting the quadratic terms, we arrive at the topological *Mielke-Baekler model* for 3D gravity [3]:

$$I_0 = \int 2ab^i R_i - \frac{A}{3} \varepsilon_{ijk} b^i b^j b^k + \alpha_3 L_{CS}(\omega) + \alpha_4 b^i T_i. \quad (2.3a)$$

Here, $a = 1/16\pi G$ and $L_{CS}(\omega)$ is the Chern-Simons Lagrangian for the Lorentz connection, $L_{CS}(\omega) = \omega^i d\omega_i + \frac{1}{3} \varepsilon_{ijk} \omega^i \omega^j \omega^k$. The Mielke-Baekler (MB) model is a natural generalization of general relativity with a cosmological constant (GR_Λ).

The complete dynamics includes also the contribution of matter fields, minimally coupled to gravity. In this paper, we focus our attention to the case when matter is represented by the *nonlinear electrodynamics*:

$$I = I_0 + I_M, \quad I_M \equiv \int L_M = \int d^3x b \mathcal{L}_M(F^2). \quad (2.3b)$$

Here, the Lagrangian density \mathcal{L}_M is a function of the invariant $F^2 := F_{mn}F^{mn}$, where $F = dA$ and A is the electromagnetic potential, while $b = \det(b^i_\mu)$. We also assume that in the weak field limit, i. e. for $F^2 \rightarrow 0$, \mathcal{L}_M reduces to the Maxwell form:

$$\mathcal{L}_M(F^2) = -\frac{1}{4}F^2 + \mathcal{O}(F^4). \quad (2.3c)$$

A particularly important case of such a nonlinear electrodynamics is the Born-Infeld theory:

$$\mathcal{L}_M = -\frac{k^2}{4} \left(\sqrt{1 + \frac{2F^2}{k^2}} - 1 \right) =: \mathcal{L}_{BI}, \quad (2.4)$$

where k^2 is the Born-Infeld coupling. Any nonlinear electrodynamics that satisfies the weak field limit (2.3c) is said to be of the Born-Infeld type.

In our exposition, we shall maximally postpone using the explicit Born-Infeld form of \mathcal{L}_M for two reasons: first, the field equations are more compact, and second, some of the conclusions are independent of the explicit form of \mathcal{L}_M , and consequently, more general.

2.3 General field equations

The variation of I with respect to b^i and ω^i yields the gravitational field equations:

$$\begin{aligned} 2aR_i + 2\alpha_4 T_i - \Lambda \varepsilon_{ijk} b^j b^k &= \Theta_i, \\ 2\alpha_3 R_i + 2aT_i + \alpha_4 \varepsilon_{ijk} b^j b^k &= 0, \end{aligned} \quad (2.5)$$

where $\Theta_i := -\delta L_M / \delta b^i$ is the energy-momentum current (2-form) of matter. In the nondegenerate sector with $\Delta := \alpha_3 \alpha_4 - a^2 \neq 0$, these equations can be rewritten as

$$2T_i - p \varepsilon_{ijk} b^j b^k = u \Theta_i, \quad (2.6a)$$

$$2R_i - q \varepsilon_{ijk} b^j b^k = -v \Theta_i, \quad (2.6b)$$

where

$$\begin{aligned} p &:= \frac{\alpha_3 \Lambda + \alpha_4 a}{\Delta}, & u &:= \frac{\alpha_3}{\Delta}, \\ q &:= -\frac{(\alpha_4)^2 + a \Lambda}{\Delta}, & v &:= \frac{a}{\Delta}. \end{aligned}$$

Introducing the *energy-momentum tensor* of matter by $\mathcal{T}^k{}_i := *(b^k \Theta_i)$, the matter current Θ_i can be expressed as follows:

$$\Theta_i = \frac{1}{2} (\mathcal{T}^k{}_i \varepsilon_{kmn}) b^m b^n = \varepsilon_{imn} t^m b^n, \quad (2.7a)$$

$$t^m := - \left(\mathcal{T}^m{}_k - \frac{1}{2} \delta_k^m \mathcal{T} \right) b^k, \quad (2.7b)$$

where $\mathcal{T} = \mathcal{T}^k{}_k$.

Following [9], we can now simplify the field equations (2.7). If we substitute the above Θ_i into (2.6a) and compare the result with (2.2b), we find the following form of the contortion:

$$K^m = \frac{1}{2} (p b^m + u t^m). \quad (2.8a)$$

After that, we can rewrite the second field equation (2.6b) as

$$2R_i = q \varepsilon_{imn} b^m b^n - v \varepsilon_{imn} t^m b^n, \quad (2.8b)$$

where the Cartan curvature R_i can be conveniently calculated using the identity (2.2c):

$$2R_i = 2\tilde{R}_i + u \tilde{\nabla} t_i + \varepsilon_{imn} \left(\frac{p^2}{4} b^m b^n + \frac{up}{2} t^m b^n + \frac{u^2}{4} t^m t^n \right). \quad (2.8c)$$

In this form of the gravitational field equations, the role of matter field as a source of gravity is clearly described by the 1-form t^i .

In the case of nonlinear electrodynamics, the energy-momentum tensor is given by

$$\mathcal{T}^k{}_i = 4\mathcal{L}_M^{(1)} F^{km} F_{im} - \delta_i^k \mathcal{L}_M, \quad (2.9)$$

where $\mathcal{L}_M^{(1)} := \partial \mathcal{L}_M / \partial F^2$.

Varying the action with respect to A , one obtains the electromagnetic field equations:

$$d(\mathcal{L}_M^{(1)*} F) = 0. \quad (2.10)$$

Equations (2.8) and (2.10), together with a suitable set of boundary conditions, define the complete dynamics of both the gravitational and the electromagnetic field.

3 Static and spherically symmetric configurations

In order to explore basic dynamical features of the nonlinear electrodynamics coupled to 3D gravity with torsion, we begin by looking at *static* and *spherically symmetric* field configurations. Using the Schwarzschild-like coordinates $x^\mu = (t, r, \varphi)$, we make the following ansatz for the triad field,

$$b^0 = N dt, \quad b^1 = L^{-1} dr, \quad b^2 = K d\varphi, \quad (3.1)$$

and for the Maxwell field:

$$F = E_r b^0 b^1 - H b^1 b^2 + E_\varphi b^2 b^0. \quad (3.2)$$

Here, N, L, K and E_r, H, E_φ are the unknown functions of the radial coordinate r , and we have $F^2 = (-E_r^2 + E_\varphi^2 + H^2)/2$.

3.1 The field equations

Using the components of Riemannian connection α, β and γ , defined in appendix A, the electromagnetic field equations (2.10) take the form

$$\begin{aligned} \left(E_r \mathcal{L}_M^{(1)} \right)' L + \gamma E_r \mathcal{L}_M^{(1)} &= 0, \\ \left(H \mathcal{L}_M^{(1)} \right)' L + \alpha H \mathcal{L}_M^{(1)} &= 0, \end{aligned} \quad (3.3)$$

where prime denotes the radial derivative. Note that equations (3.3) do not determine E_φ , it remains an arbitrary function of r . The first integrals of these equations read:

$$-4\mathcal{L}_M^{(1)} E_r K = Q_1, \quad -4\mathcal{L}_M^{(1)} H N = Q_3, \quad (3.4)$$

where Q_1 and Q_3 are constants.

To find the form of the gravitational field equations, we first calculate the energy-momentum tensor,

$$\mathcal{T}^i_j = 4\mathcal{L}_M^{(1)} \begin{pmatrix} -\left(E_r^2 + E_\varphi^2 \right) - \frac{\mathcal{L}_M}{4\mathcal{L}_M^{(1)}} & -E_\varphi H & -E_r H \\ E_\varphi H & \left(H^2 - E_r^2 \right) - \frac{\mathcal{L}_M}{4\mathcal{L}_M^{(1)}} & E_\varphi E_r \\ E_r H & E_\varphi E_r & \left(H^2 - E_\varphi^2 \right) - \frac{\mathcal{L}_M}{4\mathcal{L}_M^{(1)}} \end{pmatrix}, \quad (3.5)$$

and the corresponding expression for t^i :

$$\begin{aligned} t^0 &= -\left(\frac{1}{2} \mathcal{L}_M - 4\mathcal{L}_M^{(1)} H^2 \right) b^0 + 4\mathcal{L}_M^{(1)} E_\varphi H b^1 + 4\mathcal{L}_M^{(1)} E_r H b^2, \\ t^1 &= -4\mathcal{L}_M^{(1)} E_\varphi H b^0 - \left(\frac{1}{2} \mathcal{L}_M + 4\mathcal{L}_M^{(1)} E_\varphi^2 \right) b^1 - 4\mathcal{L}_M^{(1)} E_r E_\varphi b^2, \\ t^2 &= -4\mathcal{L}_M^{(1)} E_r H b^0 - 4\mathcal{L}_M^{(1)} E_r E_\varphi b^1 - \left(\frac{1}{2} \mathcal{L}_M + 4\mathcal{L}_M^{(1)} E_r^2 \right) b^2. \end{aligned} \quad (3.6)$$

Technical details leading to the explicit form of the gravitational field equations (2.8b) are summarized in appendix A. All the components of these equations can be conveniently divided in two sets: those with $(i, m, n) = (0, 1, 2), (2, 0, 1), (1, 2, 0)$ are called diagonal, all the others are nondiagonal. Introducing

$$V := v + up/2$$

to simplify the notation, the nondiagonal equations read:

$$E_r H \left(V - \frac{u^2}{4} \mathcal{L}_M \right) = u \left[\frac{L}{2} (E_r E_r' + E_\varphi E_\varphi' + HH') + \alpha E_\varphi^2 \right], \quad (3.7a)$$

$$E_\varphi H \left(V - \frac{u^2}{4} \mathcal{L}_M \right) = -u\alpha E_r E_\varphi, \quad (3.7b)$$

$$E_r E_\varphi \left(V - \frac{u^2}{4} \mathcal{L}_M \right) = uH (LE_\varphi' + \alpha E_\varphi), \quad (3.7c)$$

$$HE_r \left(V - \frac{u^2}{4} \mathcal{L}_M \right) = u \left[\frac{L}{2} (E_r E_r' - E_\varphi E_\varphi' + HH') - \gamma E_\varphi^2 \right], \quad (3.7d)$$

$$E_\varphi E_r \left(V - \frac{u^2}{4} \mathcal{L}_M \right) = -u\gamma HE_\varphi, \quad (3.7e)$$

$$HE_\varphi \left(V - \frac{u^2}{4} \mathcal{L}_M \right) = uE_r (LE_\varphi' + \gamma E_\varphi), \quad (3.7f)$$

while the diagonal ones are:

$$\begin{aligned} -2(\gamma'L + \gamma^2) &= 2\Lambda_{\text{eff}} + V \left[\mathcal{L}_M + 4\mathcal{L}_M^{(1)} (E_r^2 + E_\varphi^2) \right] \\ &\quad - \frac{u^2}{4} \mathcal{L}_M \left[\frac{1}{2} \mathcal{L}_M + 4\mathcal{L}_M^{(1)} (E_r^2 + E_\varphi^2) \right] - 4uLE_r H' \mathcal{L}_M^{(1)}, \end{aligned} \quad (3.8a)$$

$$\begin{aligned} -2\alpha\gamma &= 2\Lambda_{\text{eff}} + V \left[\mathcal{L}_M + 4\mathcal{L}_M^{(1)} (E_r^2 - H^2) \right] \\ &\quad - \frac{u^2}{4} \mathcal{L}_M \left[\frac{1}{2} \mathcal{L}_M + 4\mathcal{L}_M^{(1)} (E_r^2 - H^2) \right] + 4u(\gamma - \alpha) E_r H \mathcal{L}_M^{(1)}, \end{aligned} \quad (3.8b)$$

$$\begin{aligned} -2(\alpha'L + \alpha^2) &= 2\Lambda_{\text{eff}} + V \left[\mathcal{L}_M + 4\mathcal{L}_M^{(1)} (E_\varphi^2 - H^2) \right] \\ &\quad - \frac{u^2}{4} \mathcal{L}_M \left[\frac{1}{2} \mathcal{L}_M + 4\mathcal{L}_M^{(1)} (E_\varphi^2 - H^2) \right] + 4uLHE_r' \mathcal{L}_M^{(1)}, \end{aligned} \quad (3.8c)$$

where $\Lambda_{\text{eff}} := q - p^2/4$ is the effective cosmological constant. These equations are invariant under the *duality mapping*

$$\begin{aligned} \alpha &\rightarrow \gamma, & \gamma &\rightarrow \alpha, \\ E_r &\rightarrow iH, & H &\rightarrow iE_r, \end{aligned} \quad (3.9)$$

which has the same form as in GR_Λ [14]. Consequently, the set of all solutions is also invariant under (3.9).

3.2 A no-go theorem

In the case $u = 0$, the nondiagonal field equations imply

$$E_r E_\varphi = 0, \quad E_r H = 0, \quad E_\varphi H = 0.$$

This is a specific result, valid also in GR_Λ [14], which states that configurations with two nonvanishing components of the electromagnetic field are dynamically not allowed.

In the general case $u \neq 0$, the analysis of the field equations leads to the general no-go theorem (appendix B):

$$E_r E_\varphi H = 0. \quad (3.10)$$

- Static and spherically symmetric configurations with three nonvanishing components of the electromagnetic field are dynamically forbidden.

The theorem is valid for any form of $\mathcal{L}_M(F^2)$, and for any value of Λ_{eff} . Thus, it generalizes the no-go theorem found in [14], based on Maxwell electrodynamics.

Motivated by this result, we now restrict our considerations to the electric sector of the theory, specified by $H = 0$.

3.3 Dynamics in the electric sector

The condition $H = 0$, which defines the electric sector, leads to a significant simplification of the field equations. The nondiagonal equations take the form

$$\frac{L}{2} (E_r E'_r + E_\varphi E'_\varphi) + \alpha E_\varphi^2 = 0, \quad (3.11a)$$

$$\alpha E_r E_\varphi = 0, \quad (3.11b)$$

$$E_r E_\varphi \left(V - \frac{u^2}{4} \mathcal{L}_M \right) = 0, \quad (3.11c)$$

$$E_r (L E'_\varphi + \gamma E_\varphi) = 0, \quad (3.11d)$$

$$\frac{L}{2} (E_r E'_r - E_\varphi E'_\varphi) - \gamma E_\varphi^2 = 0, \quad (3.11e)$$

while the diagonal ones are

$$\begin{aligned} -2(\gamma' L + \gamma^2) &= 2\Lambda_{\text{eff}} + V \left[\mathcal{L}_M + 4(E_r^2 + E_\varphi^2) \mathcal{L}_M^{(1)} \right] \\ &\quad - \frac{u^2}{4} \mathcal{L}_M \left[\frac{1}{2} \mathcal{L}_M + 4(E_r^2 + E_\varphi^2) \mathcal{L}_M^{(1)} \right], \end{aligned} \quad (3.12a)$$

$$-2\alpha\gamma = 2\Lambda_{\text{eff}} + V \left[\mathcal{L}_M + 4E_r^2 \mathcal{L}_M^{(1)} \right] - \frac{u^2}{4} \mathcal{L}_M \left[\frac{1}{2} \mathcal{L}_M + 4E_r^2 \mathcal{L}_M^{(1)} \right], \quad (3.12b)$$

$$-2(\alpha' L + \alpha^2) = 2\Lambda_{\text{eff}} + V \left[\mathcal{L}_M + 4E_\varphi^2 \mathcal{L}_M^{(1)} \right] - \frac{u^2}{4} \mathcal{L}_M \left[\frac{1}{2} \mathcal{L}_M + 4E_\varphi^2 \mathcal{L}_M^{(1)} \right]. \quad (3.12c)$$

The analysis of these equations leads to the conclusion that the only interesting configuration is the one defined by the azimuthal electric field E_φ (appendix C).

4 Solution with azimuthal electric field

Since the electromagnetic field equations do not impose any restriction on the azimuthal electric field, it is completely determined by the gravitational field equations.

The nondiagonal gravitational field equations (3.11) are very simple:

$$LE'_\varphi + 2\alpha E_\varphi = 0, \quad LE'_\varphi + 2\gamma E_\varphi = 0. \quad (4.1a)$$

They imply $\alpha = \gamma$ and consequently

$$N = C_1 K, \quad E_\varphi K^2 = Q_2, \quad (4.1b)$$

where C_1 and Q_2 are the integration constants.

With $\alpha = \gamma$, the set of the diagonal field equations (3.12) reduces to

$$-\alpha' L = 2E_\varphi^2 \left(V - \frac{u^2}{4} \mathcal{L}_M \right) \mathcal{L}_M^{(1)}, \quad (4.2a)$$

$$-2\alpha^2 = 2\Lambda_{\text{eff}} + \left(V - \frac{u^2}{8} \mathcal{L}_M \right) \mathcal{L}_M. \quad (4.2b)$$

Moreover, since equation (4.2a) is obtained as the derivative with respect to r of (4.2b), only the last equation is relevant. It is convenient to fix the radial coordinate by choosing $K = r$, which yields

$$E_\varphi = \frac{Q_2}{r^2}, \quad F^2 = -\frac{2Q_2^2}{r^4},$$

$$L^2 = -r^2 \Lambda_{\text{eff}} - \frac{r^2}{2} \bar{\mathcal{L}}_M \left(V - \frac{u^2}{8} \bar{\mathcal{L}}_M \right),$$

where

$$\bar{\mathcal{L}}_M = \mathcal{L}_M \left(\frac{-2Q_2^2}{r^4} \right).$$

The expression for L^2 is obtained using (4.2b).

The above expressions for N, L, K and E_φ represent a complete solution for any value of Λ_{eff} . In what follows, we shall restrict our considerations to the case of negative Λ_{eff} ,

$$\Lambda_{\text{eff}} = -\frac{1}{\ell^2},$$

which corresponds asymptotically to an AdS configuration. Using $C_1 = 1/\ell$, the complete solution takes the form

$$b^0 = \frac{r}{\ell} dt, \quad b^1 = L^{-1} dr, \quad b^2 = r d\varphi,$$

$$\tilde{\omega}^0 = -L d\varphi, \quad \tilde{\omega}^1 = 0, \quad \tilde{\omega}^2 = -\frac{L}{\ell} dt,$$

$$\omega^i = \tilde{\omega}^i + \frac{1}{2} (pb^i + ut^i),$$

$$F = \frac{Q_2}{\ell} d\varphi dt, \quad (4.3)$$

where

$$t^0 = -\frac{r \bar{\mathcal{L}}_M}{2\ell} dt, \quad t^1 = -\frac{d(r \bar{\mathcal{L}}_M)}{2L}, \quad t^2 = -\frac{r \bar{\mathcal{L}}_M}{2} d\varphi.$$

4.1 Geometric structure

The metric of the solution (4.3) has the form:

$$ds^2 = \frac{r^2}{\ell^2} dt^2 - \frac{dr^2}{\frac{r^2}{\ell^2} - \frac{r^2}{2} \bar{\mathcal{L}}_M \left(V - \frac{u^2}{8} \bar{\mathcal{L}}_M \right)} - r^2 d\varphi^2. \quad (4.4)$$

The expressions for the scalar Cartan curvature and the pseudoscalar torsion read:

$$R = -\varepsilon^{imn} R_{imn} = -6q - \frac{v(r^3 \bar{\mathcal{L}}_M)'}{r^2},$$

$$T = -\varepsilon^{imn} T_{imn} = -6p + \frac{u(r^3 \bar{\mathcal{L}}_M)'}{r^2}.$$

The metric is free of coordinate singularities in the range of r for which $L^2(r) > 0$. To examine this range, we rewrite $L^2(r)$ in the form

$$L^2(r) = r^2 \frac{u^2}{16} (\bar{\mathcal{L}}_M - y^-) (\bar{\mathcal{L}}_M - y^+),$$

$$y^\pm := \frac{4}{u^2} \left(V \pm \sqrt{V^2 - \frac{u^2}{\ell^2}} \right).$$

As shown in section 5 of Ref. [9], the positivity of the classical central charges c^\mp ensures the conditions $V < 0$ and $V^2 - u^2/\ell^2 > 0$. Thus, both y^+ and y^- are real and negative, and the metric (4.4) is well-defined for those r , for which:

$$\bar{\mathcal{L}}_M(r) < y^- \quad \text{or} \quad y^+ < \bar{\mathcal{L}}_M(r).$$

In further analysis, we need to know the explicit form of $\bar{\mathcal{L}}_M$. In the Born-Infeld theory, we have

$$\bar{\mathcal{L}}_{\text{BI}} = -\frac{k^2}{4} \left(\sqrt{1 - \frac{r_0^4}{r^4}} - 1 \right),$$

where $r_0^4 = 4Q_2^2/k^2$, the scalar curvature is given by

$$R_{\text{BI}} = -6q + \frac{3vk^2}{4} \left(\frac{1 - r_0^4/3r^4}{\sqrt{1 - r_0^4/r^4}} - 1 \right),$$

and similarly for T_{BI} . Thus, the metric is well-defined for $r \geq r_0$, and both the scalar curvature and the pseudoscalar torsion are singular at $r = r_0$.

- In the Born-Infeld theory, the solution (4.4) is regular for $r > r_0$, but it does not have the black hole structure. For $Q_2 = 0$, it coincides with the black hole vacuum.

It is interesting to note that the replacement $k^2 \rightarrow -k^2$ in the Born-Infeld Lagrangian makes the solution (4.4) regular in the whole region $r > 0$.

4.2 Conserved charges

Following the standard canonical procedure described in Ref. [9], the conserved charges defined by the Born-Infeld electrodynamics (energy, angular momentum and the electric charge) are found to vanish:

$$E = 0, \quad M = 0, \quad Q = 0. \quad (4.5)$$

The vanishing of Q is related to the fact that the azimuthal electric field does not produce the radial flux, while the vanishing of E and M is a consequence of the rapid asymptotic fall-off of the solution (4.3).

5 Solutions with self-dual electromagnetic field

The solution with azimuthal electric field is the only interesting configuration in the sector of *static* and spherically symmetric configurations (Appendix C). In this section, we discuss another exact solution, the solution with self-dual electromagnetic field, which belongs to the sector of *stationary* and spherically symmetric configurations.

5.1 The field equations

We consider an action for the nonlinear electrodynamics, modified by a topological (Chern-Simons) mass term:

$$I_M = \int d^3x (b\mathcal{L}_M - \mu\varepsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho), \quad (5.1a)$$

where μ is the mass parameter. The presence of the mass term modifies the field equations for the electromagnetic field:

$$d \left(\mathcal{L}_M^{(1)\star} F - \frac{1}{4} \mu F \right) = 0, \quad (5.1b)$$

Looking for *stationary and spherically symmetric* field configurations, we adopt the following ansatz for the triad field,

$$b^0 = N dt, \quad b^1 = L^{-1} dr, \quad b^2 = K(d\varphi + C dt), \quad (5.2a)$$

and for the electromagnetic field strength:

$$F = E b^0 b^1 - H b^1 b^2. \quad (5.2b)$$

Here, N, L, C, K and E, H are six unknown functions of the radial coordinate r .

Now, we assume the generalized *self-duality* condition for the electromagnetic field:

$$E = \epsilon H, \quad \epsilon^2 = 1. \quad (5.3)$$

Combining the self-duality condition with the adopted behavior of \mathcal{L}_M in the weak field limit (2.3c), we obtain the following relations:

$$F^2 = 0, \quad \mathcal{L}_M^{(1)} = -\frac{1}{4}. \quad (5.4)$$

After that, the electromagnetic field equations (5.1b) reduce to the simple form

$$d(*F + \mu F) = 0,$$

which coincides with the Maxwell field equations [10].

Now, we focus our attention on the gravitational field equations. Using again (5.4), one finds the simplified form of the electromagnetic energy-momentum tensor:

$$\mathcal{T}^i_j = \begin{pmatrix} E^2 & 0 & EH \\ 0 & 0 & 0 \\ -EH & 0 & -H^2 \end{pmatrix}, \quad (5.5)$$

which turns out to be identical to the corresponding Maxwell energy-momentum tensor. Consequently, the gravitational field equations take the same form as in Maxwell theory coupled to 3D gravity with torsion [10]:

$$2\tilde{R}_i = \Lambda_{\text{eff}} \varepsilon_{imn} b^m b^n - \tilde{V}(\epsilon) \varepsilon_{imn} t^m b^n, \quad (5.6)$$

$$\tilde{V}(\epsilon) := v + \frac{pu}{2} + \epsilon \frac{u}{\ell} \equiv \frac{1}{\Delta} \frac{1}{24\pi\ell} c(\epsilon),$$

where $c(\epsilon)$ are *classical central charges*, characterizing the asymptotic conformal structure of 3D gravity with torsion [6]. Hence, we arrive at the following conclusion:

- Consider the nonlinear electrodynamics modified by a topological mass term, and the corresponding Maxwell theory; in the self-dual sector of stationary and spherically symmetric configurations, these two theories in interaction with 3D gravity with torsion have identical field equations.

Studying the Maxwell-Mielke-Baekler system, the authors of [10] found two self-dual solutions, one for $\mu = 0$ and the other for $\mu \neq 0$. They represent natural generalizations of the Riemannian Kamata-Koikawa [15] and Fernando-Mansouri [16] solutions, respectively. It is now simple to conclude that these solutions are also solutions of the nonlinear electrodynamics in interaction with 3D gravity with torsion. Their explicit form can be found in sections 4 and 5 of Ref. [10].

5.2 Conserved charges

Since the energy-momentum tensor takes the same form as in [10], the corresponding expressions for the energy and angular momentum of the self-dual solutions remain unchanged.

The electric charge can be calculated using the Noether or the canonical approach, with the result

$$Q = - \int_0^{2\pi} (4b\mathcal{L}_M^{(1)} F^{tr} - \mu A_\varphi) d\varphi, \quad (5.7)$$

where the integral is taken over the boundary at spatial infinity. For the specific case of the Born-Infeld theory, where the self-duality implies $4\mathcal{L}_M^{(1)} = -1$, the electric charge takes the same value as in Maxwell electrodynamics.

Consequently, the self-dual solutions with $\mu = 0$ and $\mu \neq 0$ for the Born-Infeld and Maxwell electrodynamics coupled to 3D gravity with torsion, have the same values of the energy, angular momentum and electric charge.

6 Concluding remarks

In this paper, we studied exact solutions of the nonlinear electrodynamics in interaction with 3D gravity with torsion.

(1) We proved a general no-go theorem, stating that in any static and spherically symmetric configuration, at least one component of the electromagnetic field has to vanish. The theorem holds for a general nonlinear electrodynamics, independently of its weak field limit, and for any value of Λ_{eff} .

(2) Restricting our attention to the electric sector with $\Lambda_{\text{eff}} < 0$, we constructed a solution with azimuthal electric field, which represents a generalization of the Cataldo solution found in GR_Λ [14], and of the related solution for the Maxwell-Mielke-Baekler system [9]. In the case of Born-Infeld electrodynamics, the geometric content of the solution is clarified.

(3) In the self-dual sector of stationary and spherically symmetric configurations, the field equations are seen to coincide with those for the Maxwell-Mielke-Baekler system [10]. These equations reveal a new dynamical role of the classical central charges. Relying on the results of [10], we found two types of the self-dual solutions, both for $\Lambda_{\text{eff}} < 0$, which are natural generalizations of the Kamata-Koikawa and Fernando-Mansouri solutions [15, 16], known from GR_Λ .

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A The Cartan curvature

In this appendix, we present some technical details regarding the structure of the field equations (2.8b) for static and spherically symmetric configurations.

In Schwarzschild-like coordinate (t, r, φ) , the form of the static and spherically symmetric triad field is defined in (3.1). After calculating the Levi-Civita connection,

$$\tilde{\omega}^0 = -\gamma b^2, \quad \tilde{\omega}^1 = 0, \quad \tilde{\omega}^2 = -\alpha b^0,$$

where $\alpha = LN'/N$, $\gamma = LK'/K$, we find the Riemannian curvature:

$$\begin{aligned} \tilde{R}_0 &= -(\gamma'L + \gamma^2)b^1b^2, & \tilde{R}_1 &= -\alpha\gamma b^2b^0, \\ \tilde{R}_2 &= -(\alpha'L + \alpha^2)b^0b^1. \end{aligned}$$

In the next step, we calculate the expressions $B_i := \varepsilon_{imn}t^m b^n$ and $C_i := \varepsilon_{imn}t^m t^n$:

$$\begin{aligned} B_0 &= 4E_r H \mathcal{L}_M^{(1)} b^0 b^1 - \left[\mathcal{L}_M + 4(E_\varphi^2 + E_r^2) \mathcal{L}_M^{(1)} \right] b^1 b^2 + 4E_\varphi H \mathcal{L}_M^{(1)} b^2 b^0, \\ B_1 &= 4E_\varphi E_r \mathcal{L}_M^{(1)} b^0 b^1 - 4E_\varphi H \mathcal{L}_M^{(1)} b^1 b^2 - \left[\mathcal{L}_M + 4(E_r^2 - H^2) \mathcal{L}_M^{(1)} \right] b^2 b^0, \\ B_2 &= - \left[\mathcal{L}_M + 4(E_\varphi^2 - H^2) \mathcal{L}_M^{(1)} \right] b^0 b^1 - 4E_r H \mathcal{L}_M^{(1)} b^1 b^2 + 4E_\varphi E_r \mathcal{L}_M^{(1)} b^2 b^0. \end{aligned}$$

$$\begin{aligned}
C_0 &= -4E_r H \mathcal{L}_M \mathcal{L}_M^{(1)} b^0 b^1 + \mathcal{L}_M \left[\frac{1}{2} \mathcal{L}_M + 4(E_\varphi^2 + E_r^2) \mathcal{L}_M^{(1)} \right] b^1 b^2 - 4H E_\varphi \mathcal{L}_M \mathcal{L}_M^{(1)} b^2 b^0, \\
C_1 &= -4E_\varphi E_r \mathcal{L}_M \mathcal{L}_M^{(1)} b^0 b^1 + 4E_\varphi H \mathcal{L}_M \mathcal{L}_M^{(1)} b^1 b^2 + \mathcal{L}_M \left[\frac{1}{2} \mathcal{L}_M + 4(E_r^2 - H^2) \mathcal{L}_M^{(1)} \right] b^2 b^0, \\
C_2 &= \mathcal{L}_M \left[\frac{1}{2} \mathcal{L}_M + 4(E_\varphi^2 - H^2) \mathcal{L}_M^{(1)} \right] b^0 b^1 + 4E_r H \mathcal{L}_M \mathcal{L}_M^{(1)} b^1 b^2 - 4E_\varphi E_r \mathcal{L}_M \mathcal{L}_M^{(1)} b^2 b^0.
\end{aligned}$$

After using the relations (3.3), we obtain the simplified form of $\tilde{\nabla} t_i$:

$$\begin{aligned}
\tilde{\nabla} t_0 &= -4 \left[\frac{L}{2} (E_r E_r' + E_\varphi E_\varphi' + H H') + \alpha E_\varphi^2 \right] \mathcal{L}_M^{(1)} b^0 b^1 \\
&\quad + 4L H' E_r \mathcal{L}_M^{(1)} b^1 b^2 + 4\alpha E_\varphi E_r \mathcal{L}_M^{(1)} b^2 b^0, \\
\tilde{\nabla} t_1 &= -4 (L E_\varphi' + \alpha E_\varphi) H \mathcal{L}_M^{(1)} b^0 b^1 + 4 (L E_\varphi' + \gamma E_\varphi) E_r \mathcal{L}_M^{(1)} b^1 b^2 \\
&\quad - 4(\gamma - \alpha) H E_r \mathcal{L}_M^{(1)} b^2 b^0, \\
\tilde{\nabla} t_2 &= -4L H E_r' \mathcal{L}_M^{(1)} b^0 b^1 + 4 \left[\frac{L}{2} (E_r E_r' - E_\varphi E_\varphi' + H H') - \gamma E_\varphi^2 \right] \mathcal{L}_M^{(1)} b^1 b^2 \\
&\quad + 4\gamma H E_\varphi \mathcal{L}_M^{(1)} b^2 b^0.
\end{aligned}$$

The above results completely determine the Cartan curvature (2.8c), and lead to the second gravitational field equation as displayed in (3.7) and (3.8).

B The proof of $E_r E_\varphi H = 0$

In this appendix, we prove the general no-go theorem formulated in section 3.

We begin by assuming $E_r E_\varphi H \neq 0$. Then, the consistency of the first three and the last three nondiagonal equations in (3.7) is ensured by a single relation:

$$E_r' L + E_\varphi (\alpha + \gamma) = 0 \quad \Rightarrow \quad E_\varphi A K = C_1,$$

where C_1 is a constant. After that, the set of equations (3.7) reduces to:

$$\begin{aligned}
E_r H \left(V - \frac{u^2}{4} \mathcal{L}_M \right) &= \frac{u}{2} [L(E_r E_r' + H H') + (\alpha - \gamma) E_\varphi^2], \\
E_\varphi H \left(V - \frac{u^2}{4} \mathcal{L}_M \right) &= -u \alpha E_r E_\varphi, \\
E_r E_\varphi \left(V - \frac{u^2}{4} \mathcal{L}_M \right) &= -u \gamma H E_\varphi.
\end{aligned}$$

Now, the last two equations imply:

$$\alpha E_r^2 = \gamma H^2, \quad V - \frac{u^2}{4} \mathcal{L}_M = -u \alpha \frac{E_r}{H},$$

whereupon (3.8b) reduces to

$$\Lambda_{\text{eff}} + \left(\frac{u}{4} \mathcal{L}_M - \alpha \frac{E_r}{H} \right)^2 = 0,$$

or equivalently:

$$\Lambda_{\text{eff}} + \left(\frac{V}{u}\right)^2 = 0. \quad (\text{B.1})$$

Using $V = v + up/2$ and the identity $ap + \alpha_3 q + \alpha_4 = 0$, the last relation leads to $\alpha_3 \alpha_4 - a^2 = 0$, which is in contradiction with the property $\Delta \neq 0$, adopted in section 2.

This completes the proof that the configuration $E_r E_\varphi H \neq 0$ is not allowed dynamically, which is exactly the content of the no-go theorem.

C Field configurations with $E_r \neq 0$

In this appendix, we show that static and spherically symmetric field configurations with the radial electric field $E_r \neq 0$ either do not exist or are physically irrelevant.

C.1 $H = 0$

Let us first assume the existence of a general electric field, $E_r E_\varphi \neq 0$. In this case, equations (3.11b) and (3.11c) imply

$$\alpha = 0, \quad V - \frac{u^2}{4} \mathcal{L}_M = 0,$$

whereupon equation (3.12b) leads to (B.1), which is a contradiction. Hence, our assumption cannot be true, i.e. at least one of the components E_r, E_φ must vanish.

Next, we assume $E_\varphi = 0$, whereupon equation (3.11a) implies $E_r = \text{const}$. We have here a solution with constant electric field, which is of no physical interest.

C.2 $H \neq 0$

In this sector, we have both electric and magnetic fields, and the duality symmetry (3.9) implies that the only nontrivial case is $E_\varphi = 0$. If we limit our attention to the AdS sector and choose $K = r$, the asymptotic behavior of N and L is given by

$$N \sim \frac{r}{\ell} + \frac{\eta}{r^{n-1}}, \quad \frac{L}{N} \sim 1 + \frac{\chi}{r^m},$$

with $n, m > 0$. Since the electromagnetic field is asymptotically weak, we have $\mathcal{L}_M^{(1)} \sim -1/4$.

The nondiagonal equations (3.7) reduce to a single relation,

$$E_r H \left(V - \frac{u^2}{4} \mathcal{L}_M \right) = \frac{uL}{2} (E_r E_r' + H H'), \quad (\text{C.1})$$

while the difference between (3.8a) and (3.8b), combined with (C.1) and (3.3), leads to:

$$\left(\frac{Q_3^2}{Q_1^2} - \frac{N^2}{r^2} \right) \frac{H'}{H} = \frac{4N^3}{uQ_1 r^2 H} \left(\frac{N}{r} \right)' - \frac{3}{2} \left(\frac{N^2}{r^2} \right)' H. \quad (\text{C.2})$$

The consistency of (C.2) to leading order requires $Q_1 = \epsilon \ell Q_3$, where $\epsilon^2 = 1$, and we have the asymptotic duality relation $E_r \sim \epsilon H$, as expected. However, the consistency of (C.2) to higher orders yields additionally one of the following two conditions:

$$\begin{aligned} 3n = 2, \quad \chi = 0, \\ m = n + 2, \quad \chi = \frac{ul^2 Q_1^2}{4} \frac{3n - 2}{n + 2} \eta. \end{aligned}$$

In both cases ($\chi = 0$ and $\chi \neq 0$), the diagonal equations imply $\Delta = 0$, which is not allowed. Hence, asymptotically AdS solutions we were looking for do not exist.

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