

Averaging in cosmological models using scalars

A.A. COLEY

Department of Mathematics and Statistics

Dalhousie University, Halifax, NS B3H 3J5, Canada

aac@mathstat.dal.ca

Abstract

The averaging problem in cosmology is of considerable importance for the correct interpretation of cosmological data. A rigorous mathematical definition of averaging in a cosmological model is necessary. In general, a spacetime is completely characterized by its scalar curvature invariants, and this suggests a particular spacetime averaging scheme based entirely on scalars. We clearly identify the problems of averaging in a cosmological model. We then present a precise definition of a cosmological model, and based upon this definition, we propose an averaging scheme in terms of scalar curvature invariants. This scheme is illustrated in a simple static spherically symmetric perfect fluid cosmological spacetime, where the averaging scales are clearly identified.

1 The averaging problem in cosmology

Cosmological observations [1, 2], based on the assumption of a spatially homogeneous and isotropic Friedmann-Lemaître-Robertson-Walker (FLRW) model plus small perturbations, are usually interpreted as implying that there exists dark energy, the spatial geometry is flat, and that there is currently an accelerated expansion giving rise to the so-called Λ CDM-concordance model. Although the concordance model is quite remarkable, it does not convincingly fit all data [3]. Unfortunately, if the underlying cosmological model is not a perturbation of an exact flat FLRW solution, the conventional data analysis and their interpretation is not necessarily valid. For example, the standard analysis of type Ia supernovae (SNIa) and CMB data in FLRW models cannot be applied directly when backreaction effects are present, because of the different dynamical behaviour of the spatial curvature [4]. Indeed, supernovae data can be explained without dark energy in inhomogeneous models, where the full effects of general relativity (GR) come into play. For example, it has been indeed shown that the Lemaître-Tolman-Bondi (LTB) solution can be used to fit the observed data without the need of dark energy [5], although it is necessary to place the observer at the center of a rather large-scale underdensity. Therefore, the averaging problem in cosmology is of considerable importance for the correct interpretation of cosmological data. The correct governing equations on cosmological scales are obtained by averaging the Einstein field equations (EFE) of GR (plus a theory of photon propagation; i.e., information on what trajectories actual particles follow). By assuming spatial homogeneity and isotropy on the largest scales, the inhomogeneities affect the dynamics through correction (backreaction) terms, which can lead to behaviour qualitatively and quantitatively different from the FLRW models.

1.1 General Approaches

The gravitational field equations on large scales are obtained by averaging the EFE of GR. It is necessary to use an exact covariant approach which gives a prescription for the correlation functions that emerge in an averaging of the full tensorial EFE. The Universe is not isotropic or spatially homogeneous on local scales. An averaging of inhomogeneous spacetimes on large scales can lead to important effects. For example, on cosmological scales the dynamical behavior can differ from that in the spatially homogeneous and isotropic FLRW model [6]; in particular, the expansion rate may be significantly affected. Consequently, a solution of the averaging problem is of considerable importance for the correct interpretation of cosmological data. The averaging problem in GR and cosmology is of fundamental importance.

There are a number of approaches to the averaging problem [6, 7, 8]. In the approach of Buchert [8] a 3+1 cosmological space-time splitting is employed and only scalar quantities are averaged. The perturbative approach involves averaging the perturbed Einstein equations; however, a perturbation analysis cannot provide any information about an averaged geometry. In the space-time or space volume averaging approach tensors, and in some cases only scalar quantities, are averaged; this procedure is not generally covariant hence the results are somewhat limited and the conclusions unreliable. In all of these approaches, in analogy with Lorentz's approach to electrodynamics, an averaging of the Einstein equations is performed to obtain the averaged field equations.

However, to date the macroscopic gravity (MG) approach is the only approach to the averaging problem in GR [7] which gives a prescription for the correlation functions which emerge in an averaging of the non-linear field equations (without which the averaging of the Einstein equations simply amount to definitions of the new averaged terms). The MG approach is a fully covariant, gauge independent and exact method. The space-time averaging procedure adopted in MG for any differentiable manifold is based on the concept of Lie-dragging of averaging regions, and it has been proven to exist on an arbitrary Rie-

mannian space-times with well-defined local averaged properties. Averaging of the structure equations for the geometry of GR brings about the structure equations for the averaged (macroscopic) geometry and the definitions and the properties of the correlation tensors. The averaged Einstein equations can always be written in the form of the Einstein equations for the macroscopic metric tensor when the correlation terms are moved to the right-hand side of the averaged Einstein equations to serve as the geometric modification to the averaged (macroscopic) matter energy-momentum tensor [7].

The formal mathematical issues of averaging tensors on a differential manifold has recently been revisited [9]. However, we note that integrating scalars on spacetime regions is always well-defined.

1.2 Scales

In any theory of physics, the scales over which the physical theory is applicable must be specified [10]. There is a hierarchy of cosmological scales of physical interest. Consequently, we must specify the cosmological scale over which averaging occurs (i.e., we must specify the averaging scale ℓ or averaging region, which then determines the type of averaging or smoothing that occurs). In particular, a given geometry may be inhomogeneous on both small scales ℓ_s (i.e., local scales such as local density inhomogeneities) and large scales ℓ_l with $\ell_s \ll \ell_l$ (where $\ell_l \sim \ell_H$, and ℓ_H is the Hubble scale). The averaging scale ℓ will satisfy $\ell_s \ll \ell$, but it is possible that $\ell \ll \ell_l$, so that the geometry is still inhomogeneous (but ‘smooth’) on large scales. In cosmology it is assumed that the averaging scale ℓ is bigger than scale of largest observed structures (clusters of galaxies) and voids and that $\ell < \ell_H$ [11]. There is also the homogeneity scale, ℓ_{hom} , the largest scale any inhomogeneities are observed. It is usually assumed that $\ell_{hom} < \ell (< \ell_H)$.

The physical description of a cosmological model depends on the averaging scale [12].

The scales ℓ_s , ℓ_l , ℓ in the cosmological model and the range of validity (i.e., $\ell_s \ll \ell < \ell_l$) must be specified. Of course, the range of scales relevant to cosmology are the largest scales of averaging, larger than the largest scale of cosmological structures and comparable to a fraction of ℓ_H . In the context of perturbation theory, particular attention is paid to both the scale of inhomogeneities in the background and the scale of inhomogeneities of the perturbations [13].

1.3 Spatial curvature

In [14] the MG equations were explicitly solved in a FLRW background geometry and it was found that the correlation tensor (backreaction) is of the form of a spatial curvature. Thus, the averaged Einstein equations for a flat spatially homogeneous, isotropic macroscopic space-time geometry has the form of the Einstein equations of GR for a non-flat spatially homogeneous, isotropic space-time geometry.

The relevance of spatial curvature in realistic models of the universe that describe the dynamics of structure formation since the epoch of last scattering was discussed in [13]; in particular, in arguments about spatial curvature in perturbation theory, the quasi-Newtonian approximation must be used with care since spatial curvature is an inherently relativistic phenomenon (that does not occur in Newtonian physics). We note that a spatially dependent constant spatial curvature k can alleviate the tension in observational data [15]. Indeed, if the spatial curvature parameter k is allowed to be a function of position, then considerable spatial curvature (locally) is permissible (consistent with CMB observations) [16].

1.4 Discussion

Clearly, backreaction (averaging) effects are real, but their relative importance must be determined [17]. Observational data suggests a normalized spatial curvature $|\Omega_k| \approx 0.01 - 0.02$ (i.e., of about a percent). Combining these observations with large scale structure observations then puts stringent limits on the curvature parameter in the context of adiabatic Λ CDM models; however, these data analyses are very model- and prior-dependent [4], and care is needed in the proper interpretation of the data. There is a heuristic argument that $\Omega_k \sim 10^{-3} - 10^{-2}$ [18, 15], which is consistent with CMB observations [1, 2, 3, 4] and agrees with estimates for intrinsic curvature fluctuations using realistically modelled clusters and voids in a Swiss-cheese model. It must be appreciated that such a value for Ω_k , at the 1% level, is relatively large and may have a significant dynamical effect on the evolution of the universe and the interpretation of cosmological observations.

Note that in a scenario in which $|\Omega_k| \sim 0.01 - 0.02$, the current contribution from the spatial curvature is much greater than the energy density of radiation, and is comparable to the energy density in luminous matter. In addition, such a value cannot be naturally explained by inflation. From standard analysis, depending on the initial conditions and the details of a specific model of inflation, $|\Omega - 1|$ would be extremely small. Indeed, any value for Ω_k at the 1 % level would be very difficult to explain within the theory of inflation; therefore, any non-zero residual curvature at this level can only be naturally explained in terms of an averaging effect.

1.5 Null geodesics

Ultimately we wish to determine the effects of averaging or backreaction on the evolution of the universe and the interpretation of cosmological observations. All deductions about cosmology are based on light paths. Only the redshift and the energy flux of light arriving

from a distant source are observed, rather than the expansion rate or the matter density. It is often assumed that intervening inhomogeneities average out. However, inhomogeneities affect curved null geodesics [18, 8] and can drastically alter observed distances when they are a sizable fraction of the curvature radius. In the real universe, voids occupy a much larger region as compared to structures, hence light preferentially travels much more through underdense regions and the effects of inhomogeneities on luminosity distance are likely to be significant.

The effect of averaging null geodesics in inhomogeneous models was discussed in [17]. GR is treated as a microscopic (classical) theory. Real photons travel on null geodesics in the microscopic geometry. However, because all observations are of finite resolution, observations necessarily involve averages of measured quantities. Therefore, in interpreting real observations, it is necessary to model properties of (not only a single photon but of) a ‘narrow’ beam or bundle of photons (i.e., a local congruence of null geodesics). From the geometric optics approximation we can obtain the optical scalar (Dyer-Roeder) equations that govern the propagation of the local shearing and expansion (of the cross-sectional area of the beam) with respect to the affine parameter along the congruence due to Ricci focussing and Weyl tidal focussing [19]. Since the *nonlinear* optical scalar equations require integration along the beam, the optics for a lumpy distribution does not average and there may be important resulting effects. Similar issues have been discussed recently from a different point of view [18, 20]. The formulation of the EFE on the null cone was discussed in [21]. Clearly averaging can have an important effect on photon propagation and hence observations.

In particular, assuming GR to be a microscopic theory on small scales with local metric field \mathbf{g} (the micro-geometry) and matter fields, a photon follows a null geodesic \mathbf{k} in the local geometry, and after averaging we obtain [17] a smoothed out macroscopic geometry (with macroscopic metric $\langle \mathbf{g} \rangle$) and macroscopic matter fields, valid on larger scales. But, in general, the “averaged” vector $\langle \mathbf{k} \rangle$ need not be null, need not be geodesic (and even if it is, need not be affinely parametrized) in the macro-geometry.

2 Scalar curvature invariants and averaging

In [22], it was shown that the class of four-dimensional (4D) Lorentzian manifolds that cannot be completely characterized by the scalar polynomial curvature invariants constructed from the Riemann tensor and its covariant derivatives must be of Kundt form. This implies that, in general, a spacetime is completely characterized by its scalar curvature invariants and this suggests a particular spacetime averaging scheme based entirely on scalars. Let us first review the main mathematical background.

For any given spacetime (\mathbf{M}, \mathbf{g}) we define the set of all scalar invariants

$$\mathcal{I} \equiv \{R, R_{\mu\nu}R^{\mu\nu}, C_{\mu\nu\alpha\beta}C^{\mu\nu\alpha\beta}, R_{\mu\nu\alpha\beta;\gamma}R^{\mu\nu\alpha\beta;\gamma}, R_{\mu\nu\alpha\beta;\gamma\delta}R^{\mu\nu\alpha\beta;\gamma\delta}, \dots\}. \quad (2.1)$$

Consider a spacetime (\mathbf{M}, \mathbf{g}) with a set of invariants \mathcal{I} . Then, if there does not exist a continuous metric deformation of g having the same set of invariants as g , we will call the set of invariants *non-degenerate*. Furthermore, the spacetime metric, g , will be called *\mathcal{I} -non-degenerate*. This implies that for a metric which is \mathcal{I} -non-degenerate the invariants characterize the spacetime uniquely, at least locally, in the space of (Lorentzian) metrics. This means that these metrics are characterized by their curvature invariants and therefore we can distinguish such metrics using their invariants.

It was proven, on a case-by-case (depending on the algebraic type; characterized by their Petrov and Segre types or, equivalently, in terms of their Ricci, Weyl (and Riemann) types [23]), that a 4D spacetime is either \mathcal{I} -non-degenerate, which implies that the spacetime metric is determined locally by its curvature invariants, or the metric is a Kundt metric. This is a striking result because it tells us that metrics not determined by their curvature invariants must be of Kundt form. These Kundt metrics therefore correspond to degenerate metrics in the sense that many such spacetimes can have identical invariants. The Kundt class is defined by those metrics admitting a null vector ℓ that is geodesic, expansion-free,

shear-free and twist-free ^{1 2}.

Let us also consider the ‘inverse’ question: given a set of scalar polynomial invariants, what can we say about the underlying spacetime? In practice, it is somewhat tedious and a lengthy ordeal to determine the spacetime from the set of invariants. In 4D we can partially characterize the Petrov type in terms of scalar curvature invariants. In most circumstances we only need some partial results or necessary conditions or we are dealing with special cases. For example, we found that if $27J^2 \neq I^3$, or if $27J^2 = I^3$ but the differential invariants $S_1 \neq 0$ or $S_2 \neq 0$, then the spacetime is \mathcal{I} -non-degenerate. Having determined when a spacetime is completely characterized by its scalar curvature invariants, it is also of interest to determine the minimal set of such invariants needed for this classification. It is also of interest to study when a spacetime can be explicitly *constructed* from scalar curvature invariants.

Let us return to the question of averaging. We have noted that a general spacetime is completely characterized by its scalar curvature invariants. Since we know how to average scalar quantities, we can average all of the scalar curvature invariants that can then represent an averaged spacetime (with that set of averaged scalar invariants). We shall return to this in Section 4. However, we note that cosmological models belong to the set of spacetimes completely characterized by their scalar curvature invariants, suggesting that we can average a cosmological model using scalar invariants. We must first define a cosmological model rigorously, which we do next.

¹ We recall that in the Riemannian case a manifold is always locally characterized by its scalar polynomial invariants.

² We note that this exceptional property of the the degenerate Kundt metrics essentially follows from the fact that they do not define a unique timelike curvature operator.

3 Cosmological models and averaging

There are a number of technical problems with averaging of tensor fields on a differentiable manifold \mathbf{M} . Clearly we need a covariant approach. Is the average of the metric the metric? ³ There is the question of averaging verses smoothing. We want to avoid issues with respect to coordinates; i.e., if we choose more smooth coordinates, we are not averaging but just using different coordinates to represent the same geometry. It may be possible to avoid several of these technical problems by adopting an approach based on scalar curvature invariants. Note that although such an approach may work for any differentiable manifold which is *\mathcal{I} -non-degenerate*, we shall focus on the cosmological problem for the most part here.

Therefore, we wish to discuss the averaging problem in the context of cosmology. Although this context may simplify the issue of averaging in technical terms, there are some new problems of principle that are then introduced. First, the cosmological model is a *mixed* model, in that the matter is already assumed to be averaged, but the geometry is not (necessarily) ⁴. Therefore, we need a consistent model for the matter, represented on the characteristic averaging scale, and its appropriate (averaged) physical properties ⁵. It is known that the separation between the gravitational field and the matter is not scale invariant and the notion of a perfect fluid is not scale invariant [25]; averaging (in the presence of a gravitational field) modifies the equation of state of the matter [26]. In addition, since averaging does not conserve geodesics (in fact, averaging need not even conserve the metric signature), we need further assumptions in order to be able to compare the models with observational data.

³ Note that if the average of the metric is not the metric itself, then the conformal structure of the macro- and micro- geometries are not the same and photons will not follow null geodesics in the macrogeometry in general.

⁴ Since the time dependence is smooth, perhaps some smoothing of the microgeometry has already been assumed.

⁵ Note that a discrete (non-continuous fluid) model for the cosmological matter has been discussed recently [24].

First, a precise definition of a cosmological model is necessary; i.e., a framework in which to do averaging. This definition includes an appropriate way to do averaging and how to deal with photons. In particular, it is necessary to make all of the assumptions in the model clear.

The precise definition of a cosmological model we shall adopt, based in part on [15, 17, 21], is given by the following conditions C1 – C5 (cf. [10]):

3.1 Definition of a cosmological model

C1. Spacetime Geometry:

The spacetime geometry (\mathbf{M}, \mathbf{g}) is defined by a smooth Lorentzian metric \mathbf{g} (characterizing the macroscopic gravitational field) defined on a smooth differentiable manifold \mathbf{M} .

The macroscopic metric geometry is obtained by an appropriate spacetime averaging of the microgeometry; thus part of the definition of a cosmological model consists of specifying the averaging scheme (which must be consistent with the physical assumptions of the model encapsulated in the conditions C3 and C4 below) and the cosmological scale over which averaging or the smoothing occurs (i.e., we must specify the averaging scale ℓ or averaging region). In particular, a given microgeometry may be inhomogeneous on both small scales ℓ_s and large scales ℓ_l . The averaging scale ℓ will satisfy $\ell_s \ll \ell$, but it is possible that $\ell \ll \ell_l$, so that the macrogeometry is still inhomogeneous (but ‘smooth’) on large scales. As noted earlier, the scales for which the cosmological model is defined must be explicitly specified.

C2. Timelike Congruence:

There exists a timelike congruence (\mathbf{u}) (in principle locally, but by definition we can extend this to the whole manifold), representing a family of fundamental observers.

This congruence is associated with the 4-velocity of the averaged matter in the model; i.e., the matter admits a formulation in terms of an averaged matter content which defines an average (macroscopic) timelike congruence. If there is more than one matter component giving rise to more than one macroscopic timelike congruence we identify a fundamental macroscopic timelike congruence. There is always one, which has a physical meaning. This leads to a covariant $3 + 1$ split of spacetime [10]. Mathematically this means that the spacetime is \mathcal{I} -non-degenerate and hence the spacetime is uniquely characterized by its scalar curvature invariants.

In addition to the formal parts C1 and C2 of the definition of a cosmological model $(\mathbf{M}, \mathbf{g}, \ell, \mathbf{u})$, we must also specify the physical relationship (interaction) between the macroscopic geometry and the matter fields, including how the matter responds to the macroscopic geometry.

C3. Macroscopic field equations:

There exists an appropriate set of macroscopic field equations relating the averaged matter and appropriately averaged (or macroscopic) geometry. This is based on an underlying microscopic theory of gravity (such as, for example, GR), and an appropriate formalism to average the geometry and find corrections (correlations) due to averaging to the Einstein tensor in the resulting field equations:

$$\tilde{G}^a_b + C^a_b = T^a_b, \tag{3.2}$$

in the usual geometrized units in which $c = 1 = 8\pi G/c^2$, where $\tilde{G}^a_b \equiv \tilde{R}^a_b - \frac{1}{2}\delta^a_b\tilde{R}$ and \tilde{R}^a_b is the Ricci tensor of the averaged macrogeometry, C^a_b is the correlation tensor, and T^a_b is the energy momentum tensor (already assumed averaged).

This is the emphasis of the analysis here. Note that in this context only the Ricci tensor needs to be averaged.

Discussion

The energy-momentum tensor that appears in the EFE (defined formally by the variation of the matter action with respect to the metric) depends on the gravitational field. Indeed, in GR the gravitational energy contributes to the total energy-momentum. So in this sense some aspects of the averaged gravitational field are already included in the model.⁶

In general, it does not follow from the contracted Bianchi identities that the energy-momentum is conserved with respect to the macrometric: $T^c_{b;c} \neq 0$. Is it possible to define a new effective energy-momentum tensor \hat{T}^a_b which is conserved? For example, $\hat{T}^a_b \equiv \tilde{G}^a_b = T^a_b - C^a_b$ (which satisfies the macro-conservation equations with respect to the macrogeometry by definition). However, it must be ensured that the relationship between the effective matter and the (averaged) macrogeometry is consistent with the underlying microphysical model and the averaging scheme. At a heuristic level, this could be (within the cosmological model under consideration) modelled qualitatively through an appropriate (new type of) effective equation of state.

In particular, the separation between the gravitational field and the matter is not scale invariant. Indeed, the notion of a perfect fluid is not scale invariant [25], and in the presence of a gravitational field the equation of state of the matter is consequently in general scale dependent [26]. Thus, averaging affects the effective equation of state. In the simplest cosmological models we can write $T^a_b = \text{diag}[-\rho, p, p, p]$ and $C^a_b \equiv \text{diag}[-\rho_c, p_c, p_c, p_c]$, and

⁶ To minimize this effect, the mixed components of T^a_b are usually considered.

we have that $\hat{T}_b^a = \text{diag}[\rho - \rho_c, p - p_c, p - p_c, p - p_c]$. A physical equation of state relates ρ and p , whereas the effective equation state is a relationship between $\hat{\rho} \equiv \rho - \rho_c$ and $\hat{p} \equiv p - p_c$.⁷

In this heuristic framework all of the effects of averaging go into the redefined energy-momentum tensor \hat{T}_b^a and the effective equation state of the macro-matter (subject to any changes in the equations of motion of the macro-matter, although we again note that \hat{T}_b^a is conserved relative to the macrogeometry). Therefore, in this reinterpretation, any non-standard equation of state is not due to any exotic matter, but is the effect of averaging (cf. [28]). In particular, an equation of state renormalization at the level of about 1% $\approx \mathcal{O}(\ell/\ell_H)$ naturally fits into this new averaging interpretation (and especially an additional C_b^a corresponding to spatial curvature arising from averaging [23]).

C4. Equations of motion:

We also need to know the trajectories along which the cosmological matter moves (and also the light trajectories, which determine observational relations).⁸ In principle, averaging the 4-velocities of the microscopic matter particles does not necessarily give the macroscopic 4-velocity of the cosmological matter⁹ (and the average motion of a photon need not be a null geodesic in the averaged geometry).

Discussion

⁷ If we split up \hat{T} into a standard matter part plus an ‘additional’ matter term, in general there exists an interaction between them (i.e., each matter term is not necessarily conserved [27]).

⁸ This may add additional geometrical quantities (other than the curvature) that need to be considered (and need to be averaged). Perhaps this can be done using appropriate scalars, but this is the subject of future investigation.

⁹ In GR a free point particle moves on a timelike or null geodesic. But a system may behave as point particles on small scales but *not* on larger scales (alternatively, the definition of a point particle is also *not* scale invariant). Therefore, macroscopic (averaged) matter need not move on geodesics of the macrogeometry.

This needs to be consistent with the averaged field equations: if the matter satisfies the conservation equations in the macroscopic geometry (which follows from the averaged field equations), then the equations of motion must be consistent with this (i.e., cosmological dust follows macroscopic timelike geodesics). Note that \tilde{G}^a_b satisfies the contracted Bianchi identities (with respect to the macrogeometry).¹⁰

The fundamental congruence is essentially the average of the timelike congruences along which particles move in the microgeometry. This implies a relationship between matter and macrogeometry. If the particles move on timelike geodesics in the microgeometry, does the fundamental congruence consist of timelike geodesics (of the macrogeometry). Does (a beam of) photons move on null geodesics of the macrogeometry. This raises the question of whether it is possible to perform an appropriate averaging so that on average the macroscopic matter moves on timelike/null geodesics of the macrogeometry.

Ultimately we need a theory for the propagation of photons. The motion of light in the macroscopic geometry must be consistent with the limiting motion of timelike particles in the macrogeometry. Alternatively, we could average the Einstein-Maxwell equations and take the geometric optics limit [19]; i.e., we assume that the actual photons satisfy the Einstein-Maxwell equations in the optical limit, and thus follows a null geodesic in the macrogeometry. We must then average the Einstein-Maxwell equations [26] and obtain a suitable form for the macroscopic Einstein-Maxwell equations and an appropriate (corresponding) optical limit in the macroscopic regime.

C5. Observations:

Finally, we need to be able to relate averaged quantities with physical observables, which ultimately must be consistent with cosmological data.

¹⁰ Since \hat{T} is conserved the motion of matter is restricted; we still need a model for the motion of photons.

3.2 Covariance

We have advocated an approach to averaging a cosmological model utilizing scalar invariants. Thus far we have focussed attention on averaging the EFE (e.g., the Ricci tensor). We shall briefly address the more general question of averaging or smoothing a differentiable manifold in the next section. However, the second aspect of averaging in a cosmological model involves the equations of motion of matter and the geodesic structure.

Since there exists a global timelike congruence \mathbf{u} , global covariance is broken (and the spacetime admits of a $1 + 3$ split). This timelike congruence is defined physically, and in principle is measurable. The resulting theory need only be invariant with respect to restricted covariance [29]. Perhaps, for describing the equations of motion of particles and null rays, we only need to consider scalars invariant with respect to restricted sets of coordinate transformations (this would correspond to the kinematic variables such as the shear and expansion scalars associated with the macroscopic covariant derivative of \mathbf{u} [10]).

This suggests that a description of the geodesic structure might be possible utilizing averaged the restricted (kinematical) scalar quantities. We shall not pursue this further here (cf. [8]).

3.3 Example: FLRW models

In the standard FLRW model there are a number of simplifications and assumptions. However, the cosmological model is not fully defined in the sense above and hence has a lack of physical predictability.

The past approaches to averaging have been ideally suited to the FLRW models (with small, vanishing in the limit, perturbations). Therefore, an extension of these approaches to cosmological models other than FLRW models is not possible [26]. Also, some of these

approaches deal exclusively with dust. Therefore, in these approaches to averaging and cosmology it is impossible to obtain the potentially most important effects.

Let us examine the assumptions C1 – C5 in the usual FLRW plus perturbations model. The macrometric \mathbf{g} is the FLRW metric (C1) and \mathbf{u} also has a geometric meaning (C2). In the usual point of view there are no correlations due to averaging (i.e., the correlation tensor is zero; $C^a_b = 0$) or, more precisely, they are negligible (C3). In this case it follows from the contracted Bianchi identities that energy-momentum is conserved: $T^c_{b;c} = 0$, which relates the matter to the averaged geometry. All other effects are assumed negligible (C4).

However, there is no formal proof that such assumptions arise from a rigorous averaging scheme of some appropriate (physically motivated) microgeometry. In addition, there are some important effects which are not necessarily small perturbations. In [23] it was argued that important effects at 1% level cannot be neglected.

Since there is no scale included in the model, in a sense the model is incomplete. Indeed, the model does not even have the ability to determine whether there is a scale above which the geometry is exactly FLRW or whether at all scales the geometry is only approximately FLRW (with a given perturbation scale). Indeed, in cosmological perturbation theory, both the scale of the background and the scale of perturbations need to be specified, neither of which are given since no notion of order of approximation is included [10, 13, 12]. In addition, the motion of photons must be independently postulated.

Regarding C5 we ask the question: does this model agree with observations? If it does not, then even if the model agrees in some approximate sense with most observations, there is no structure within which to discuss the potential small discrepancies with observed data, which is a deficiency of the model. If the model does, then it would be remarkable, although there is still the need for a physical explanation of the dark energy.

In the standard approach, there might be a spatial curvature term on the left-hand side

of the macroscopic field equations with intrinsic curvature parameter k , and an effective spatial curvature term on the right-hand side of the macroscopic field equations (due to an effective equation of state arising from averaging) with curvature parameter \hat{k} (which could be thought of as a ‘renormalization’ to the intrinsic curvature parameter). If observations indicate that $\Omega_k \approx 1 - 2\%$ [1, 2], then there is no physical mechanism within the model (particularly if there is an inflationary period) to produce an intrinsic curvature parameter k of this magnitude, whereas an effective curvature parameter \hat{k} of about a percent arises naturally from averaging [15].

4 An approach to averaging cosmological models using scalar invariants

Since for a cosmological model the geometry is completely characterized by its scalar curvature invariants [22], we shall adopt the approach that we shall only average the scalar curvature invariants (thereby avoiding the problems mentioned earlier). Therefore, we have a microgeometry completely characterized by its set of scalar curvature invariants \mathcal{I} . We then average these microgeometry scalar curvature invariants to obtain a new set of macrogeometry scalar curvature invariants $\tilde{\mathcal{I}}$, which now completely characterizes the macrogeometry. In general, this macrogeometry is unique (for a given averaging region). Let us consider first the more general mathematical question of averaging the geometry; we shall then focus attention on the physical case of a cosmological model.

4.1 Averaging the geometry

In the general mathematical context we want to describe the averaged geometry (represented by the Riemann tensor and its covariant derivatives) and interpret the results. Let us consider, $\mathcal{I} \equiv \{R, R^\mu{}_\nu R^\nu{}_\mu, \dots\}$, which is an ordered set of functions on \mathbf{M} . $\tilde{\mathcal{I}}$ is then also an ordered set of functions. If we write $\tilde{\mathcal{I}} \equiv \{\tilde{R}, \widetilde{R^\mu{}_\nu R^\nu{}_\mu}, \dots\}$, then of course it does not follow that, for example, $\widetilde{R^\mu{}_\nu R^\nu{}_\mu} = \tilde{R}^\mu{}_\nu \tilde{R}^\nu{}_\mu$ (under averaging). But the question is: does the ordered set of functions $\tilde{\mathcal{I}}$ correspond to the associated scalar curvature invariants for some metric \tilde{g} (which will then serve to *define* the macrometric \tilde{g}). This raises some interesting mathematical questions (see below). In principle, this is not necessarily true in general.

Indeed, we note that it is not generally true that the set of averaged scalar invariants $\tilde{\mathcal{I}}$ actually determines a Riemannian geometry. In addition, if $A = B$, then $\langle A \rangle = \langle B \rangle$.¹¹ If

¹¹ For convenience we use the simpler notation $\langle A \rangle \equiv \tilde{A}$ here and in the last section.

$A^2 = B^2$, then $\langle A^2 \rangle = \langle B^2 \rangle$, but this does not imply $\langle A \rangle^2 = \langle B \rangle^2$. In this particular case it is true because if A, B are scalars, then $A^2 = B^2$ implies $A = \pm B$ and $\langle A \rangle = \pm \langle B \rangle$; but if (as in the static example below) $(R2)^2 = (R1)^3$, while it is true that $\langle (R2)^2 \rangle = \langle (R1)^3 \rangle$, it does not follow in general that $\langle R2 \rangle^2 = \langle R1 \rangle^3$. However, relationships of this form will be true for many of the applications of interest (see the next section).

Indeed, it is plausible that the ordered set of functions $\tilde{\mathcal{I}}$ correspond to the associated scalar curvature invariants for some macrometric \tilde{g} for the class of \mathcal{I} -non-degenerate geometries that constitute the class of cosmological models defined here. Since the geometries are \mathcal{I} -non-degenerate and in 4D properties of the geometry can be represented in terms of scalars, and since relations between different terms (functions) in the set \mathcal{I} (e.g., R and R^2 are functionally dependent) and the corresponding terms in the set $\tilde{\mathcal{I}}$ (e.g., \tilde{R} and \tilde{R}^2) are functionally related in exactly the same way and syzygies (e.g., describing the algebraic type) are maintained under averaging, it follows that in general the set $\tilde{\mathcal{I}}$ uniquely gives rise to a macrometric \tilde{g} (which will have similar algebraic properties to the micrometric g).¹²

We note that a subset of 4D geometries are uniquely determined by their curvature only (i.e., no covariant derivatives are necessary), so that we only need to consider the scalar polynomial invariants constructed from the curvature tensor (i.e., an appropriate subset of $\tilde{\mathcal{I}}$)¹³ For algebraically general geometries this is indeed true, and it is certainly true for many of the important applications we are interested in. In addition, as noted above, we are primarily interested in the Ricci curvature (and the 4 Ricci scalar invariants) in applications in cosmology.

¹² This has not been proven here, but it is plausible; if there are any exotic counterexamples, then the definition of a cosmological model, and the rules of averaging, could be suitably amended to avoid this. It is true for the cosmological models of physical interest. An alternative approach is suggested in the scalar averaging procedure described below.

¹³ That is, we do not need to consider invariants like $diWeyl \equiv C_{abcd;e}C^{abcd;e}$ or $diRicci = R_{ab;e}R^{ab;e}$, which depend on derivatives of the metric greater than 2, which may add new problems regarding averaging.

If a spacetime is not \mathcal{I} -non-degenerate it can not be uniquely determined by its scalar polynomial invariants. There is one subclass of \mathcal{I} -non-degenerate spacetimes which, in a certain sense, can be determined by its scalar polynomial invariants; namely the type D^k spacetimes (i.e., spacetimes for which the curvature and all of its covariant derivatives are simultaneously of algebraic type D) [22]; this subclass includes the static spherically symmetric spacetimes of particular interest here.

4.1.1 Uniqueness

The macrogeometry is completely specified by the set $\tilde{\mathcal{I}}$. Suppose that $\tilde{\mathcal{I}}$ (or some appropriate subset of $\tilde{\mathcal{I}}$) is derivable from a macrometric \tilde{g} . Although this macrogeometry is unique, it is possible for two different (non-isomorphic) microgeometries to give rise to the same macrogeometry. However, the algebraic properties (i.e., the algebraic properties of the Riemann tensor and its covariant derivatives) of the underlying microgeometry and the resulting macrogeometry must be the same.

Since in the cosmological models under consideration the microgeometry is completely characterized by the set of scalar curvature invariants \mathcal{I} , the algebraic properties of the underlying microgeometry is generally determined through the syzygies of \mathcal{I} .¹⁴ Any syzygy \mathcal{S} is maintained under averaging (integration), giving rise to a corresponding syzygy $\tilde{\mathcal{S}}$ of the set $\tilde{\mathcal{I}}$ and then the averaged geometry will obey a similar set of (averaged versions) of these syzygies; that is, the averaged geometry will be of the same type as the microgeometry, and hence the macrogeometry is at least as algebraically special as its underlying microgeometry.¹⁵ In addition, if certain terms in \mathcal{I} satisfy certain algebraic conditions due to the differential Bianchi identities, then the corresponding terms in the set $\tilde{\mathcal{I}}$ would sat-

¹⁴ A particular cosmological spacetime (an exact solution or a subset of spacetimes with arbitrary functions in 4D) is often completely determined by syzygies involving scalar invariants.

¹⁵In principle, new syzygies may appear via averaging and so it is possible, in principle, for the macrogeometry to be more algebraically special than its underlying microgeometry.

isfy a corresponding set of algebraic conditions, consistent with the Bianchi identities of the macrogeometry. Therefore, in applications in cosmology this approach will produce a unique macrogeometry.

4.1.2 Proposal: Scalar Averaging Procedure

Let us consider the ordered set of functions \mathcal{I} on \mathbf{M} in the form:

$$\mathcal{I} \equiv \{R, R1, R2, R3, R^2, R^\mu R^\nu, \dots, C^2, \dots\}.$$

First, let us omit any scalars from this set that are not algebraically independent (e.g., $\{R^2, R^\mu R^\nu \dots\}$)¹⁶ to obtain an (the appropriate ‘independent’) subset \mathcal{I}_A . Second, for a particular spacetime, we omit any scalars from \mathcal{I}_A that can be obtained from syzygies defining that particular spacetime (e.g., defining the algebraic type of the spacetime, such as the Segre type or the Petrov type). For example, for a Ricci tensor corresponding to the algebraic form of a perfect fluid we could omit $\{R2, R3\}$ (relative to $\{R, R1\}$). We consequently obtain the subset \mathcal{I}_{SA} : e.g.,

$$\mathcal{I}_{SA} \equiv \{R, R1, \dots, C^2, \dots\}.$$

For the spacetimes under consideration the microgeometry is then completely characterized by the (sub)set of scalar curvature invariants \mathcal{I}_{SA} .¹⁷

We now construct the new ordered set of functions $\tilde{\mathcal{I}}_{SA}$ by averaging the various scalar invariants of \mathcal{I}_{SA} :

¹⁶ Including, for example, the differential Bianchi identities.

¹⁷ This subset is not unique; perhaps a scheme should be adopted whereby higher order scalars should always be omitted; e.g., R^2 should be omitted relative to R .

$$\tilde{\mathcal{I}}_{SA} \equiv \{\tilde{R}, \widetilde{R1}, \dots, \widetilde{C^2}, \dots\}$$

where all of the original scalar invariants omitted from the original set \mathcal{I} are replaced by a new set of functions obeying exactly the same algebraic properties (or syzygies) as \mathcal{I}_{SA} . Therefore, it is assumed that $\tilde{\mathcal{I}}_{SA}$ comes equipped with these syzygies, so that we could construct the corresponding set $\tilde{\mathcal{I}}$ consisting of the members of $\tilde{\mathcal{I}}_{SA}$ and all of the corresponding syzygies. Consequently, the set \mathcal{I}_{SA} is an ordered set of functions (scalar curvature invariants) on \mathbf{M} which uniquely determines the macrogeometry with exactly the same algebraic properties as the original microgeometry.

We shall refer to this proposal to obtain the set \mathcal{I}_{SA} and the associated averaged (macro) geometry as the **Scalar Averaging Procedure**.

Note that in the cosmological application, in which the Ricci tensor is the fundamental object, the simple set $\tilde{\mathcal{I}}_{SA} \equiv \{\tilde{R}, \widetilde{R1}\}$ completely characterizes the averaged Ricci tensor (cf. [8]). However, this set does not completely determine the macrogeometry. In addition, we still need to know the trajectories along which the cosmological matter and null rays move; that is, macrogeometric effects not arising from the Ricci tensor alone.

4.2 Cosmological model

In the case of a cosmological model, we only need to be able to average the Ricci tensor (or Einstein tensor) that appears in the EFE.

4.2.1 Ricci tensor

Since from C3 we have an effective set of field equations:

$$\tilde{R}^a_b + C^a_b = T^a_b, \quad (4.3)$$

we only need to consider the macrogeometric Ricci tensor \tilde{R}^a_b here (the correlation tensor is obtained from the averaging procedure). The microgeometric Ricci tensor R^a_b is completely characterized by a set of scalar curvature invariants \mathcal{I}_R . Averaging these scalar curvature invariants we obtain the set $\tilde{\mathcal{I}}_R$, which completely characterizes the macrogeometric Ricci tensor \tilde{R}^a_b . Since constructing the Ricci tensor from a set of scalar curvature invariants \mathcal{I}_R is relatively simple compared to the corresponding problem for the Riemann tensor, and since the reduced set of scalar curvature invariants \mathcal{I}_R is considerably smaller than \mathcal{I} , we have considerably reduced the complexity of the problem in this new averaging approach. Indeed, for a Ricci tensor of the algebraic form of a perfect fluid, there are effectively (only) two independent scalar invariants, the Ricci scalar and a single Ricci eigenvalue (corresponding to the effective energy density, ρ , and pressure, p , of the perfect fluid). Therefore, in the context of the scalar averaging procedure, we have the set $\{\tilde{R}, \widetilde{R1}\}$.

We also note that the syzygies of the macrogeometry Ricci tensor must be consistent with the syzygies of the energy-momentum tensor through the macroscopic field equation. If the energy-momentum tensor is of the form of a perfect fluid; i.e., $T^a_b = \text{diag}[-\rho, p, p, p]$, then T^a_b obeys a number of syzygies. A relation (equation of state) between ρ and p would be represented by a further algebraic syzygy. For example, defining $A^a_b = T^a_b - \frac{1}{2}T\delta^a_b$, where T is the trace of T^a_b (compare with the form of the Ricci tensor in the EFE), the equation of state $\rho + 3p = 0$ corresponds to the syzygy $A^2 = \frac{1}{3}A^a_b A^b_a$, where A is the trace of A^a_b .^{18 19}

¹⁸ Since this particular syzygy can only be satisfied if $\rho + 3p = 0$ or $5\rho + 3p = 0$, if the cosmological matter satisfies appropriate energy conditions, this syzygy uniquely specifies the equation of state $\rho + 3p = 0$.

¹⁹ Therefore, note that in order to obtain a macroscopic ‘spatial curvature’ it is necessary to include a cosmological pressure.

Finally, we note that if every member of \mathcal{I}_R is zero (which uniquely characterizes a Ricci flat spacetime), then every member of $\tilde{\mathcal{I}}_R$ is also zero, and the macrogeometric Ricci tensor is also zero (flat). Therefore it follows that the average of a microvacuum $R^a_b = 0$ gives rise to a macrovacuum $\tilde{R}^a_b = 0$ (and the corresponding correlation tensor is zero, and the macroscopic field equations are trivial). Therefore, within this approach, no new effects due to averaging occur at the level of the macroscopic field equations when there is no cosmological (average) matter content (i.e., a non zero energy momentum tensor T^a_b is needed).²⁰ As we noted above, the primary focus of this paper is C3 and the effective macroscopic field equations.

However, this does not imply that there are no averaging affects in the case of vacuum. Since from C4 we also need to be able to relate averaged quantities with observables and we need to know the trajectories along which the cosmological matter and null rays move, macrogeometric effects (not arising from the Ricci tensor alone) will indeed play a role. However, this is beyond the scope of the present work. It may be possible to express the effects in terms of kinematic scalars which are invariant with respect to all coordinate changes that preserve the fundamental congruence.

4.2.2 Interpretation

Finally, it is necessary to determine whether the correlations due to averaging alter the geometry or affect the effective energy-momentum tensor. This is partly a question of interpretation. This must be done within the context of the underlying cosmological model. We shall address this in the simple example in the next section.

²⁰ Note that in this approach, if the microgeometry is Ricci flat then the macrogeometry is Ricci flat. In vacuum, there will be no local variations (local inhomogeneities) in the Ricci tensor since there is local source with local inhomogeneities with a physical intrinsic scale. Indeed, the only inhomogeneities in the geometry must therefore come from (not from the energy-momentum tensor through the EFE but) physical boundary conditions, which would consequently be large scale inhomogeneities. Therefore, there would be no physical mechanism to introduce local inhomogeneities whose average would lead to a non-Ricci flat macrogeometry.

In particular, in the cosmological application it may be appropriate to reinterpret the averaging correlations as corrections to the matter fields through the EFE. Thus, concentrating on the Ricci tensor again, writing $\tilde{G}^a_b = T^a_b$ and defining $\mathcal{G}^a_b = \tilde{G}^a_b - C^a_b$, we can rewrite this as $\mathcal{G}^a_b = \mathcal{T}^a_b$ (by absorbing the correlation tensor C^a_b into \mathcal{T}^a_b). We can now demand that the new macro-Ricci tensor \mathcal{R}^a_b (corresponding to \mathcal{G}^a_b) has exactly the same algebraic properties as the Ricci tensor of the microgeometry, so that the averaging correlations are interpreted in the cosmological context as appropriate corrections to the matter fields. That is, we have defined the new averaged Ricci tensor \mathcal{R}^a_b , derivable from some appropriate averaged geometry with an appropriate micro-metric, and some of the correction terms have been included in the new effective energy-momentum tensor \mathcal{T}^a_b .

5 Example: Static Spherically Symmetric Perfect Fluid Spacetimes

We shall consider the specific example of a static spherically symmetric perfect fluid spacetime in this section. This is an appropriate simple model for illustration since it can include an arbitrary function of one variable, there is a non-vanishing pressure, the averaging region does not change with time and there are no gravitational waves.

Let us write the static spherically symmetric perfect fluid line element in the form

$$ds^2 = -e^{f(r)} dt^2 + e^{g(r)} [dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2)]. \quad (5.4)$$

The two arbitrary functions $f(r)$ and $g(r)$ satisfy the differential constraint

$$\frac{d^2 g}{dr^2} - \frac{1}{r} \frac{dg}{dr} - \frac{1}{r} \frac{df}{dr} - \frac{df}{dr} \frac{dg}{dr} - \frac{1}{2} \left(\frac{dg}{dr} \right)^2 + \frac{1}{2} \left(\frac{df}{dr} \right)^2 + \frac{d^2 f}{dr^2} = 0, \quad (5.5)$$

in order for the Einstein tensor to be of the form of a perfect fluid.

The 16 CM polynomial scalar curvature invariants [30], as given in GRTensor [31], for the static spherically symmetric spacetime are then as follows (after the constraint has been applied). The 4 (linear, quadratic, cubic and quartic) Ricci tensor scalar invariants (which are related to the Ricci scalar and the three Ricci tensor eigenvalues) are:

$$R = \frac{1}{2} e^{-g} \left[2 \frac{d^2 f}{dr^2} - 5 \left(\frac{df}{dr} \right) \left(\frac{dg}{dr} \right) + \left(\frac{df}{dr} \right)^2 - 3 \left(\frac{dg}{dr} \right)^2 - \frac{8}{r} \frac{df}{dr} - \frac{12}{r} \frac{dg}{dr} \right] \quad (5.6)$$

$$R1 = \frac{3}{64} e^{-2g} \left[2 \frac{d^2 f}{dr^2} - \left(\frac{df}{dr} \right) \left(\frac{dg}{dr} \right) - \left(\frac{dg}{dr} \right)^2 + \left(\frac{df}{dr} \right)^2 - \frac{4}{r} \frac{dg}{dr} \right]^2 \quad (5.7)$$

and $R2$ and $R3$ are proportional to $(R1)^{3/2}$ and $(R1)^2$, respectively.

The 4 polynomial Weyl scalar invariants are:

$$W1R = \frac{1}{24} e^{-2g} \left[2 \frac{d^2 f}{dr^2} + \left(\frac{df}{dr} \right)^2 - 2 \left(\frac{df}{dr} \right) \left(\frac{dg}{dr} \right) - \frac{2}{r} \frac{df}{dr} \right]^2 \quad (5.8)$$

$W2R \propto (W1R)^{3/2}$ and $W1I = W2I = 0$, and the mixed invariants are $M3 = M2R \propto (R1)(W1R)$, $M4 \propto (R1)(W1R)^{3/2}$, $W5R \propto (R1)^{3/2}(W1R)$ and $M1R = M1I = M2I = M5I = 0$. Note that GRTensor is also able to compute the differential scalar invariants $diWeyl$ and $diRicci$.

In the case of a static spherically symmetric constant curvature spacetime with $f(r) = 1$, $g(r) = (1 + kr^2)^{-2}$ (k constant), we have that:

$$R = 24k, \quad R1 = 12k^2 \quad (5.9)$$

($R2 = 24k^3$, $R3 = 84k^2$), and all other scalar invariants (including the differential invariants) vanish.

In the case of the static Schwartzchild-deSitter (Kottler [32]) spacetime with $f(r) = 1 - \frac{2m}{r} - \frac{\Lambda r^2}{3} = [g(r)]^{-1}$, we have that:

$$R = 4\Lambda, W1R = \frac{6m^2}{r^6}, W2R = \frac{-6m^3}{r^9}, diWeyl = -240(6m - 3r + \Lambda r^3)\frac{m^2}{r^9} \quad (5.10)$$

and all other scalar invariants (including diRicci) are zero. For the Schwarzschild vacuum solution $\Lambda = 0$. In the case of an Einstein space (with $m = 0$), the only non-vanishing scalar invariant is $R = 4\Lambda$.

The approach outlined in the previous section is then as follows. Averaging the scalar invariants we obtain, for example, $\langle R \rangle$ and $\langle R1 \rangle$ ($\langle W1R \rangle$ and so on, averaging eqns. (5.6)-(5.8), where $\langle R \rangle \equiv \tilde{R}$) together with an appropriately averaged constraint eqn. (5.5). These are then the scalar curvature invariants of the averaged static spherically symmetric geometry. As noted above, we will focus on $\langle R \rangle$ and $\langle R1 \rangle$ here.

5.1 Averaging Scales

Suppose that the averaging scale is $\ell \equiv L$. Suppose also that there are inhomogeneities with scales β^{-1} and λ^{-1} , where β^{-1} of order unity ($\beta \sim 1$, $L < \beta^{-1}$; i.e., global inhomogeneities)

and λ^{-1} is a small scale ($\lambda^{-1} \ll 1$, $\lambda = \frac{1}{\alpha}L$ where α is large (integer) so that $\lambda/L < 1$; i.e., local small scale inhomogeneities). Hence, the function f varies slowly with respect to βr , and varies quickly with respect to λr (on smaller scales).

Let us write the scale dependence explicitly as

$$f = f(\beta r, \lambda r), \quad (5.11)$$

where we effectively treat λr as a separate variable. Consider $r = r_0$, and a neighbourhood of r_0 , $I \equiv (r_0 - \frac{L}{2}, r_0 + \frac{L}{2})$ of length L (the averaging region). Define $r_0 + r' \in I$, so that $r' \in (-\frac{L}{2}, \frac{L}{2})$ which parameterizes points in I . We can write (set $\beta = 1$)

$$f(r) = f(r_0, \lambda(r_0 + r')) \quad (5.12)$$

in I , where $f(r_0) = f(r_0, \lambda r_0)$. Therefore, in I , $f(r)$ has small scale variations with respect to $\lambda r'$. We can write

$$f(r) \simeq f(r_0, \lambda r_0) + \mathcal{O}(\mu) \quad (5.13)$$

where μ is the small scale of the amplitude of inhomogeneities in I ($\mu \ll 1$).

We now average $f(r)$ over these small scale inhomogeneities in I , and define the average $\langle f(r_0) \rangle = \langle f(r, \lambda r) \rangle|_{r=r_0}$ by:

$$\langle f(r_0) \rangle = \frac{1}{L} \int_{-L/2}^{L/2} f(r_0, \lambda(r_0 + r')) dr' \quad (5.14)$$

(i.e., we are effectively ‘averaging over $\lambda r'$ ’). Notice that there are a number of scales in the problem: $\frac{1}{\lambda} \ll 1$, $\mu \ll 1$, $\frac{1}{\alpha} = \frac{\lambda}{L} < 1$. Note also that

$$\frac{\partial f}{\partial r} = \frac{\partial}{\partial(\beta r)} f(\beta r, \lambda r) + \frac{\partial}{\partial(\lambda r)} f(\beta r, \lambda r), \quad (5.15)$$

where the second term is a dominant fast varying term.

We now Fourier analyse the functions $f(r)$ and $g(r)$ in the averaging region I with respect to $(\lambda r')$ [$\lambda^{-1} = \alpha/L$]:

$$f(r) \equiv f(r_0) + \mu f_1(r_0) \sin\left(\frac{\alpha \pi r'}{L}\right) + \sum_{n=2} \mu^n f_n(r_0) \sin\left(\frac{n \alpha \pi r'}{L}\right)$$

$$g(r) \equiv g(r_0) + \mu g_1(r_0) \sin\left(\frac{\alpha\pi r'}{L}\right) + \sum_{n=2} \mu^n g_n(r_0) \sin\left(\frac{n\alpha\pi r'}{L}\right) \quad (5.16)$$

where $f_1(r_0)$, $f_n(r_0)$, $g_1(r_0)$, $g_n(r_0)$ are slowly varying functions of r (i.e., their derivatives are small). Note that

$$\begin{aligned} \langle f(r_0) \rangle &= f(r_0) \\ \langle g(r_0) \rangle &= g(r_0). \end{aligned} \quad (5.17)$$

Calculating, we obtain (with an abuse of notation)

$$\begin{aligned} g_r(r) &= g_r(r_0) + \frac{\alpha\mu}{L}\pi g_1 \cos\left(\frac{\alpha\pi r'}{L}\right) + \frac{\mu^2\alpha}{L}2\pi g_2 \cos\left(\frac{2\alpha\pi r'}{L}\right) \\ &\quad + \mu g_1' \sin\left(\frac{\alpha\pi r'}{L}\right) + \mu^2 g_2' \sin\left(\frac{2\alpha\pi r'}{L}\right) + \mathcal{O}(\mu^3) + \mathcal{O}\left(\frac{\mu^2\alpha^2}{L^2}\right) \\ g_{rr}(r) &= g_{rr}(r_0) - \frac{\mu\alpha^2}{L^2}\pi g_1 \sin\left(\frac{\alpha\pi r'}{L}\right) - \frac{\mu^2\alpha^2}{L^2}4\pi g_2 \sin\left(\frac{2\alpha\pi r'}{L}\right) \\ &\quad + \mathcal{O}(\mu^3) + \mathcal{O}\left(\frac{\mu\alpha^3}{L^3}\right) \end{aligned} \quad (5.18)$$

We can now average the constraint equation (5.5) [using various trigonometric formulae in evaluating the integrals]: To zeroth order we obtain the constraint in terms of $f(r_0)$ and $g(r_0)$ [effectively eqn. (5.5)]. To next leading order we obtain

$$f_1 g_1 + \frac{1}{2} g_1^2 - \frac{1}{2} f_1^2 = 0 \quad (5.19)$$

(and to higher orders: $f_1' g_1' + \frac{1}{2}(g_1')^2 - \frac{1}{2}(f_1')^2 = 0$, $f_2 g_2 + \frac{1}{2}(g_2^2) - \frac{1}{2} f_2^2 = 0$, etc.). For example, using the first of eqns. (5.18) to expand $g_r(r)$ and $f_r(r)$ and integrating we obtain

$$\begin{aligned} \left\langle \frac{df}{dr} \frac{dg}{dr} \right\rangle &= f_r(r_0) g_r(r_0) + \frac{\pi^2}{2} f_1 g_1 \left(\frac{\alpha}{L}\right)^2 \mu^2 + \frac{1}{2} f_1' g_1' \mu^2 \\ &\quad + \frac{\pi^2}{2} f_2 g_2 \left(\frac{\alpha}{L}\right)^2 \mu^4 + \frac{1}{2} f_2' g_2' \mu^4 + \mathcal{O}\left(\frac{\alpha^4 \mu^2}{L^4}\right) + \mathcal{O}(\mu^6), \end{aligned} \quad (5.20)$$

Therefore, by a direct calculation (and using the constraint (5.19), we find that

$$\langle R \rangle = R(r_0) \left(1 + \frac{1}{4L} g_1 \mu^2\right) - \frac{3\pi^2}{4} e^{-g(r_0)} \left(\frac{\alpha}{L}\right)^2 \mu^2 \frac{1}{L} \left(f_1 g_1 + \mathcal{O}\left(\frac{1}{L}\right)\right) + \mathcal{O}(\mu^4) \quad (5.21)$$

We also find that

$$\langle R1 \rangle = R1(r_0) \left\{ 1 + \mathcal{O}\left(\frac{\mu^2}{L}\right) \right\} + \frac{64\pi^2}{3} F(r_0) \left(\frac{\alpha}{L}\right)^2 \mu^2 \frac{1}{L} (f_1 g_1), \quad (5.22)$$

where $F(r_0)$ is a specific function of r_0 (not explicitly displayed here).

5.2 Interpretation

The $\mathcal{O}\left(\frac{L}{\mu^2}\right)$ correction in the first term of $\langle R \rangle$ in eqn. (5.21) is a higher order renormalization term. To lowest order we have that

$$\langle R \rangle = R(r_0) - \frac{3\pi^2}{4} e^{-g(r_0)} \left(\frac{\alpha^2 \mu^2}{L^3}\right) (f_1 g_1)$$

and hence

$$\begin{aligned} \langle R \rangle^2 &\cong R^2(r_0) - \frac{3\pi^2}{2} e^{-g(r_0)} R(r_0) \left(\frac{\alpha^2 \mu^2}{L^3}\right) (f_1 g_1) \\ &\equiv R^2(r_0) - \frac{3}{2} G(r_0) \epsilon \end{aligned} \quad (5.23)$$

where $G(r_0) \equiv e^{-g(r_0)} R(r_0)$ and $\epsilon \equiv \frac{\pi^2 \alpha^2 \mu^2}{L^3} (f_1 g_1) \simeq \text{constant}$. Also, we have that

$$\langle R1 \rangle = R1(r_0) + \frac{64}{3} H(r_0) \epsilon, \quad (5.24)$$

where $H(r_0) = R1(r_0)F(r_0)$. For a space of constant curvature k with $R(r_0) = 24k$, $R1(r_0) = 12k^2 = \frac{1}{48}(24k)^2$, we have that $G(r_0) = 768\gamma_0 k^2$, $H(r_0) = -\frac{9}{16}k^2\gamma_0$ (where γ_0 is a constant). Therefore, we have

$$\begin{aligned} \langle R \rangle^2 &= R^2(r_0) + 2(24k)^2(-\gamma_0\epsilon) \\ \langle R1 \rangle &= R1(r_0) + \frac{1}{48}(24k)^2(-\gamma_0\epsilon). \end{aligned} \quad (5.25)$$

and we can interpret the average correlations as contributing a small, $\mathcal{O}(\alpha^2 \mu^2 L^{-3})$, constant curvature term, arising from the averaging of local inhomogeneities in the micro-Ricci tensor, to the smooth macro-Ricci tensor (consistent with the results of [14]).

Acknowledgements. I would like to thank Sigbjorn Hervik and Robert van den Hoogen for helpful comments. This work was supported by NSERC of Canada.

References

- [1] P. Astier *et al*, *Astron. Astrophys.* **447**, 31 (2006); T.M. Davis *et al*, *Ap. J.* **666**, 716 (2007).
- [2] J. Dunkley *et al*, *Ap. J. Suppl.* **180**, 306 (2009) ; D. N. Spergel *et al*, *Ap. J. Suppl.* **170**, 377 (2007).
- [3] A. Kashlinsky *et al.*, arXiv:0809.3733; R. Watkins *et al.*, arXiv:0809.4041; M. L. McClure and C. C. Dyer, *New. Astron.* **12** 533 (2007); S. Ho *et al*, *Phys. Rev. D* **78** 043519 (2008).
- [4] C. Shapiro and M. S. Turner, *Ap. J.* **649** 563 (2006).
- [5] H. Alnes *et al* *Phys. Rev. D.* **73** 083519 (2006); K. Enqvist and T. Mattsson, *JCAP* **0702** 19 (2007); J. Garcia-Bellido and T. Haugbolle, *JCAP* **0804** 003 (2008).
- [6] G. F. R. Ellis and W. Stoeger, *Class. Quant. Grav.* **4** (1987) 1697; S. Bildhauer and T. Futamase, *Gen. Rel. Grav.* **23** (1991) 1251; T. Futamase, *Phys. Rev. D* **53** (1993) 681; J. P. Boersma, *Phys. Rev. D* **57** (1998) 798.
- [7] R.M. Zalaletdinov, *Gen. Rel. Grav.* **24** (1992) 1015; *ibid. Gen. Rel. Grav.* **25** (1993) 673; M. Mars and R.M. Zalaletdinov, *J. Math. Phys.* **38** (1997) 4741.
- [8] T. Buchert. *Gen. Rel. Grav.*, **32** 105 (2000) & **33** 1381 (2001).
- [9] J. Behrend, arXiv:0812.2859; R. van den Hoogen *et al.*, preprint.
- [10] G. F. R. Ellis, *Relativistic Cosmology in General Relativity and Cosmology*, Proc. Int. School of Physics ‘Enrico Fermi’ (Varenna) Course XLVII pp104-179, ed. R. K. Sachs (Academic Press, 1971)
- [11] T. Buchert and M. Carfora, *Class. Quant. Grav.* **25** (2008) 195001 [arXiv:0803.1401].
- [12] G. F. R. Ellis and T. Buchert, *Phys. Lett.* **A347** 38 (2005).
- [13] T. Buchert, G. F. R. Ellis and H. van Elst, arXiv:0906.0134.
- [14] A. A. Coley *et al*, *Phys. Rev. Letts.* **595** 115102 (2005) [gr-qc/0504115]; A. A. Coley and N. Pelavas, *Phys. Rev. D* **75** 043506 (2006) & **74** 087301 (2006); R. van den Hoogen, *J. Math. Phys.*
- [15] A. Coley, arXiv:0704.1734.
- [16] T. Clifton, P. G. Ferreira and J. Zuntz, *JCAP*, **07** 029 (2009).

- [17] A. Coley, arXiv:0812.4566.
- [18] S. Räsänen, JCAP **0611** 003 (2006) & JCAP **0804** 026 (2008). S. Räsänen, JCAP **0902** 011 (2009).
- [19] R.K. Sachs, Proc. Roy. Soc. (London) **A264** 309 (1961); C.C. Dyer and R. C. Roeder, Ap. J. **174** L115 (1972), **180** L31 (1973), & **189** 167 (1974).
- [20] V. Marra *et al*, Phys. Rev. D **76**, 123004 (2007); V. Marra, E. W. Kolb and S. Matarrese, Phys. Rev. D **77**, 023033 (2008); E. W. Kolb, V. Marra and S. Matarrese [arXiv:0901.4566].
- [21] A. Coley, arXiv:0905.2442.
- [22] A. Coley, S. Hervik and N. Pelavas, 2009, Class. Quantum Grav. **26**, 025013 [arXiv:0904.4877]
- [23] A. Coley, 2008, Class. Quant. Grav. **25**, 033001 [arXiv:0710.1598].
- [24] T. Clifton and P. G. Ferreira, arXiv:0907.4169.
- [25] L. D. Landau and E. M. Lifshitz, *Fluid Mechanics* (Pergammon Press, Oxford, 1987).
- [26] F. Debbasch, Eur. Phys. J. **B37** 257 (2004) & **B43** 143 (2005).
- [27] A. Billyard and A. Coley, 2000, Phys. Rev. **D61** 083503 [astro-ph/9908224].
- [28] T. Buchert, J. Larena and J. Alimi, *Class. Quant. Grav.* **23** (2006) 6379.
- [29] G. F. R. Ellis and D. R. Matravers, *Gen. Rel. Grav.* **27** 777 (1995).
- [30] J. Carminati and R. G. McLenaghan, J. Math. Phys. **32** 3135 (1991).
- [31] This is a package which runs within Maple. It is entirely distinct from packages distributed with Maple and must be obtained independently. The GRTensorII software and documentation is distributed freely on the World-Wide-Web from the address <http://grtensor.org>
- [32] H. Stephani, D. Kramer, M. A. H. MacCallum, C. A. Hoenselaers, E. Herlt 2003 *Exact solutions of Einstein's field equations, second edition* (Cambridge University Press; Cambridge).