# VACUUM STRUCTURE OF YANG-MILLS THEORY IN CURVED SPACETIME

by

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# ABSTRACT

The stability of the chromomagnetic Savvidy vacuum in QCD under the influence of positive Riemannian curvature is studied. The heat traces of the operators relevant to SO(2) gauge-invariant Yang-Mills fields and Faddeev-Popov ghosts are calculated on product spaces of  $S^2$  and  $S^1 \times S^1$ . It is shown that the chromomagnetic vacuum with covariantly constant chromomagnetic field is stable in a certain set of radii and field strengths.

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### CHAPTER 1

## INTRODUCTION

Quantum Chromodynamics is the highly successful theory of the strong interaction of elementary particles. It is based on SU(3) Yang-Mills theory, which is a non-Abelian gauge theory invariant under the gauge group SU(3), and a set of spin-1/2 quarks, which form the fundamental representation of the gauge group. The SU(3) degrees of freedom are referred to as "color."

The high-energy behavior of Yang-Mills theory is well-understood and leads to a renormalizable quantum field theory. Gross, Wilczek[12], and Politzer [17] discovered that at high energies (or short distance scales), the interaction strength of Yang-Mills fields decreases to zero, a property known as asymptotic freedom.

However, QCD is not well understood at low energies or large distance scales, i.e., those comparable to  $\Lambda_{QCD}$  (~ 10<sup>-13</sup> cm), at which point perturbation theory breaks down. The interaction strength increases, and the energy required to separate quarks becomes infinite. This leads to the property of confinement, which is exhibited by the experimental absence of free quarks, but has not been demonstrated theoretically. At energy scales less than  $\Lambda_{QCD}$ , QCD has to be replaced by an effective theory of composites of quarks in the form of color-neutral hadrons. The low-energy behavior of QCD can be examined through the effective potential. The effective potential is a function of the background Yang-Mills field which is minimized by the absolute lowest energy state of the physical system. This minimum defines the physical vacuum.

The physical vacuum is considered trivial when the effective potential takes a minimum with a zero background Yang-Mills field. If the minimum of the effective potential occurs when the Yang-Mills field is non-zero, the vacuum state will consist of a non-zero background field. The first attempt to study the Yang-Mills vacuum in this manner was by Savvidy[18] in 1977.

Savvidy introduced a background chromomagnetic field of constant field strength lying in the Cartan algebra of the Lie group SU(2). He found that the minimum of the one-loop effective action occurs at a non-zero value of the background field, which causes the vacuum to be infrared unstable.

It was later pointed out by H.B. Nielsen and P. Olesen [16] that the chromomagnetic vacuum discovered by Savvidy has an energy density with an imaginary part, which implies that it has a tachyonic mode, leading to instability of the vacuum. Further corrections[15] were made to the chromomagnetic vacuum to show that the energy is lower when the chromomagnetic vacuum consists of tube-like domain structures, with the chromomagnetic field pointed along the axis of each tube. The finite width of the tubes serves as an infrared cutoff, which destroys the low-energy instability. The minimum energy density of this type of state has been found by [14] to be a superposition of domains separated by a fixed distance. This model is known as the "spaghetti vacuum."

It is important to note that these calculations pertain to a chromo-

magnetic field in flat space. In this paper, we will consider the related problem of the stability of the Yang-Mills vacuum on a curved space. The addition of the curvature will make the operator corresponding to the second variation of the action positive definite for large enough values of curvature, which will change the tachyonic mode into a physical state and cause the vacuum to stabilize. In particular, we will consider non-zero covariantly constant SO(3) Yang-Mills fields on product spaces of spheres, valued in the sub-algebra SO(2). By analyzing the spectrum of the second variation of the action and computing the effective action, it will be shown that the vacuum will stabilize on spaces that have sufficiently strong curvature.

This paper is organized as follows. Chapter Two of this paper discusses the effective action approach to quantum field theory and how it applies to gauge theories, as well as the technique of zeta-function regularization. In order to compute the effective action, we need the spectrum of the Yang-Mills and Faddeev-Popov ghost operators on spheres, which are calculated in Chapter Three. In Chapter Four, we apply the results of Chapter Three to find the heat trace of Yang-Mills and ghost operators on  $S^2$  and  $S^1 \times S^1$ . Chapter Five is devoted to finding the total heat kernel on the products of spheres and determining in what cases the vacuum is stable.

## CHAPTER 2

# QUANTIZATION OF NON-ABELIAN GAUGE THEORIES

In this chapter, we will use the proper time method of Schwinger and DeWitt, and so the notation will follow that of DeWitt[7].

### 2.1 Kinematics

### Spacetime Geometry

The spacetime under consideration is an *n*-dimensional pseudo-Riemannian manifold M endowed with a globally hyperbolic metric g with signature  $(- + \cdots +)$ . We will assume that the spacetime manifold has a global time-like Killing vector so that  $M = \mathbb{R} \times \Sigma$ , where  $\Sigma$  is an (n - 1)-dimensional compact oriented spin manifold without boundary. Local coordinates  $x^{\mu}$  on Mare labeled by Greek indices that run over  $0, \ldots, n - 1$ . The coordinate basis  $\partial_{\mu}$  for the tangent space  $T_x M$  at the point  $x \in M$  has dual basis  $dx^{\mu}$  in  $T_x^*M$ .

The Christoffel symbols can be found from the metric

$$\Gamma^{\alpha}{}_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta} \left( \partial_{\gamma} g_{\delta\beta} + \partial_{\gamma} g_{\delta\gamma} - \partial_{\delta} g_{\beta\gamma} \right) \,. \tag{2.1}$$

The curvature of the metric  $g_{\mu\nu}$  is described by the Riemann curvature tensor

$$R^{\alpha}{}_{\mu\beta\nu} = \partial_{\beta}\Gamma^{\alpha}{}_{\nu\mu} - \partial_{\nu}\Gamma^{\alpha}{}_{\beta\mu} + \Gamma^{\eta}{}_{\nu\mu}\Gamma^{\alpha}{}_{\beta\eta} - \Gamma^{\eta}{}_{\beta\mu}\Gamma^{\alpha}{}_{\nu\eta}, \qquad (2.2)$$

and its contractions: the Ricci tensor,

$$R_{\mu\nu} = R^{\alpha}_{\ \mu\alpha\nu} \,, \tag{2.3}$$

and the Ricci scalar,

$$R = g^{\mu\nu} R_{\mu\nu} \,. \tag{2.4}$$

#### **Orthonormal Frame**

An orthonormal frame  $e_{(\alpha)}$  can be constructed at every point on the manifold and is labeled by lower case Greek indices in parentheses. The orthonormal frame of  $T_x M$  can be constructed as a set of vector fields over M, where  $\alpha = 1, \ldots, n$  so that

$$\langle e_{(\alpha)}, e_{(\beta)} \rangle = \eta_{(\alpha)(\beta)},$$
 (2.5)

where  $\eta_{(\alpha)(\beta)} = \text{diag}(-1, 1, ..., 1).$ 

The orthonormal basis  $e_{(\alpha)}$  may be expanded in the coordinate basis  $\partial_{\mu}$ ,

$$e_{(\alpha)} = e_{(\alpha)}{}^{\mu}\partial_{\mu} \,. \tag{2.6}$$

The inverse matrix  $e^{(\alpha)}{}_{\mu}$  of  $e_{(\alpha)}{}^{\mu}$  defines the dual basis

$$e^{(\alpha)} = e^{(\alpha)}{}_{\mu}dx^{\mu} \tag{2.7}$$

in the cotangent space  $T_x^*M$ . Then

$$g^{\mu\nu}e^{(\alpha)}{}_{\mu}e^{(\beta)}{}_{\nu} = \eta^{(\alpha)(\beta)}, \qquad g_{\mu\nu}e_{(\alpha)}{}^{\mu}e_{(\beta)}{}^{\nu} = \eta_{(\alpha)(\beta)}.$$
 (2.8)

#### 2.1.1 Gauge Group

Yang-Mills theory describes the dynamics of a vector bundle over M. To say that the theory is gauge invariant is to impose the restriction that the action does not change under transformations by a gauge group. In particular, consider a compact simple Lie group G attached to every point of M so that any neighborhood in a fiber bundle has the local structure  $M \times G$ . In other words let  $k^a$  be coordinates on G, so that for any coordinate patch on M, a point in the fiber bundle is described by the set of coordinates  $(x^{\mu}, k^a)$ . Group indices are labeled by lower case Latin letters, which run over  $1, \ldots, \dim G$ .

An element  $U \in G$  of a compact simple gauge group may be written in the form

$$U = \exp\left(k^a T_a\right),\tag{2.9}$$

where  $k^a$  are parameters and  $T_a$  are the generators of the Lie group G, which lie in the Lie algebra. It is clear that the expression

$$T_a = \frac{d}{dk^a} U|_{k^a = 0} \tag{2.10}$$

is an equivalent definition of the generators of G. The generators of a simple compact Lie algebra satisfy the relation

$$[T_a, T_b] = C^c_{\ ab} T_c \,, \tag{2.11}$$

where  $C^c_{ab}$  are the structure constants of the Lie algebra G.

The adjoint representation of the Lie algebra is defined by taking the generators to be

$$(T_a)^b{}_c = C^b{}_{ac} \,. \tag{2.12}$$

To form inner products between algebra-valued tensors, we must introduce an inner product on the Lie algebra. We define the Cartan-Killing metric

$$E_{ab} = -\frac{1}{2} C^c_{\ ad} C^d_{\ bc} = -\frac{1}{2} \text{tr} \left( T_a T_b \right)$$
(2.13)

to raise and lower group indices. In the case of compact simple Lie groups, this metric can be normalized by

$$E_{ab} = \delta_{ad} \,. \tag{2.14}$$

To make the action invariant under gauge transformations, the covariant derivative  $\nabla_{\mu}\varphi^{\nu}$  of a field must satisfy the condition

$$\nabla'_{\mu}\varphi'^{\nu} = U(\nabla_{\mu}\varphi^{\nu}). \qquad (2.15)$$

The gauge matrix U is local, i.e., depends on the coordinates, and primed quantities denote the transformed quantities. To do this, we let

$$\nabla_{\mu}\varphi^{\nu} = (\nabla^{LC}_{\mu} + \mathcal{A}_{\mu})\varphi^{\nu}, \qquad (2.16)$$

where  $\mathcal{A}_{\mu} = A^{a}_{\mu}T_{a}$  is an algebra-valued vector field that transforms as

$$\mathcal{A}'_{\mu} = U \mathcal{A}_{\mu} U^{-1} - (\partial_{\mu} U) U^{-1}$$
(2.17)

for any gauge transformation  $U \in G$ . In general,  $\mathcal{A}_{\mu}$  depends on the representation of the group G.

The strength  $\mathcal{F}_{\mu\nu} = F^a_{\mu\nu}T_a$  of the Yang-Mills field  $\mathcal{A}_{\mu}$  is defined by

$$\mathcal{F}_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} \mathcal{A}_{\mu} + [\mathcal{A}_{\mu}, \mathcal{A}_{\nu}]. \qquad (2.18)$$

The strength of the field can be used to define the action functional

$$S_{YM} = \frac{1}{8e^2} \int_M dx \operatorname{tr} \left( \mathcal{F}^{\mu\nu} \mathcal{F}_{\mu\nu} \right), \qquad (2.19)$$

where e is a coupling constant and tr denotes the trace over the Lie algebra.

#### 2.1.2 Scalar Fields

A scalar field  $\varphi$  is invariant under diffeomorphisms and has a covariant derivative of

$$\nabla_{\mu}\varphi = (\partial_{\mu} + \mathcal{A}_{\mu})\varphi, \qquad (2.20)$$

with  $\mathcal{A}$  in an appropriate representation. The action for a scalar field must be constructed out of a scalar potential term  $V(\varphi)$  and the quantity  $\varphi^T \Box \varphi$ , where T denotes transpose and  $\Box$  is the D'Alembert operator

$$\Box \equiv \nabla^{\mu} \nabla_{\mu} \,. \tag{2.21}$$

### 2.2 Effective Action

The effective action approach to quantum field theory is a highly useful approach that was developed by DeWitt and others [7, 8, 20, 5]. This section follows the method as developed for boson fields.

Consider two causally connected in- and out- regions of spacetime that lie in the past and future of a region  $\Omega$  in which physical dynamics will take place. The goal of quantum field theory is to compute the amplitude  $\langle in|out \rangle$  for some initial state  $|in\rangle$  in the in- region to evolve into some final state  $|out\rangle$  in the out region. To calculate this, consider a change in the action  $\delta S$ . The Schwinger variational principle states that the amplitude  $\langle in|out\rangle$  will change according to

$$\delta \langle \text{in} | \text{out} \rangle = \frac{i}{\hbar} \langle \text{in} | \delta S | \text{out} \rangle .$$
 (2.22)

Let  $\varphi^i$  be the boson fields relevant to the problem, where *i* is taken to run over both continuous (i.e. spacetime) and discrete (i.e. spinor, tensor, field) indices. Change the action by adding a linear interaction of  $\varphi^i$  with some classical sources  $J_i$  that vanish in the in- and out- regions  $\delta S = J_i \varphi^i$ , where the contraction over *i* is taken as both a summation over discrete indices and integration over spacetime

$$\varphi^i J_i = \int_M dx \,\sqrt{g} \,\varphi_{(A)} J^{(A)} \,, \qquad (2.23)$$

where  $g = \det g_{\mu\nu}$ .

With this variation, the solution to the Schwinger variational principle is expressed in terms of the Feynman path integral

$$\langle \text{out} | \text{in} \rangle = \int \mathcal{D}\varphi \exp\left\{\frac{i}{\hbar}[S(\varphi) + J^k \varphi_k]\right\},$$
 (2.24)

where  $\mathcal{D}\varphi$  represents the Feynman measure.

The generating functional for connected diagrams W(J) is defined in terms of the in-out transition amplitude by

$$\langle \text{out}|\text{in} \rangle \equiv \exp\left(\frac{i}{\hbar}W(J)\right)$$
 (2.25)

The first functional derivative of W gives the background field  $\Phi^i$ 

$$\Phi^{i}(J) = \frac{\delta}{\delta J_{i}} W(J), \qquad (2.26)$$

the second functional derivative produces the propagator

$$\mathcal{G}^{i_1 i_2}(J) = \frac{\delta^2}{\delta J_{i_1} \delta J_{i_2}} W(J) , \qquad (2.27)$$

and the higher derivatives produce the many-point Green functions

$$\mathcal{G}^{i_1\dots i_k}(J) = \frac{\delta^k}{\delta J_{i_1}\dots\delta J_{i_k}} W(J) \,. \tag{2.28}$$

In order to calculate vertex functions, we define the effective action  $\Gamma(\Phi)$  by the functional Legendre transform

$$\Gamma(\Phi) = W(J(\Phi)) - J_i(\Phi)\Phi^i, \qquad (2.29)$$

where the sources  $J(\Phi)$  are expressed in terms of the background fields. The first functional derivative of  $\Gamma$  is equal to the sources

$$\frac{\delta}{\delta\Phi^i}\Gamma(\Phi) = -J_i(\Phi), \qquad (2.30)$$

the second derivative defines the inverse propagator

$$\frac{\delta^2}{\delta \Phi^i \delta \Phi^j} \Gamma(\Phi) = D_{ij}(\Phi) , \qquad (2.31)$$

$$D_{ij}\mathcal{G}^{jk} = -\delta_i^{\ k}\delta(x,x')\,,\qquad(2.32)$$

and the higher derivatives determine the vertex functions

$$\Gamma_{i_1\dots i_k}(\Phi) \equiv \frac{\delta^k}{\delta \Phi_{i_1} \cdots \delta \Phi_{i_k}} \Gamma(\Phi) \,. \tag{2.33}$$

We see from (2.24), (2.25), and (2.29) that the effective action satisfies

$$\exp\left\{\frac{i}{\hbar}\Gamma(\Phi)\right\} = \int \mathcal{D}\varphi \exp\left\{\frac{i}{\hbar}\left[S(\varphi) - \frac{\delta\Gamma(\Phi)}{\delta\Phi^{i}}(\varphi^{i} - \Phi^{i})\right]\right\}.$$
 (2.34)

#### 2.2.1 Gauge Theory

In gauge field theories, the above formalism can not be applied immediately. Instead, there will be problems arising from the fact that the measure will include an integral over non-physical fields. To be more explicit, consider an action functional  $S(\varphi)$  that is invariant under some vector fields  $\mathbf{R}_A = R^i{}_A(\varphi)\delta/\delta\varphi^i$  on the configuration space. Transformations of the fields

$$\delta_{\xi}\varphi^{i} = R^{i}{}_{A}\xi^{A} \,, \tag{2.35}$$

with  $\xi^A$  being some parameters, that do not affect any real physics are called gauge transformations.

In the case of gauge theories, the Feynman path integral (2.24) will be carried over both physical and non-physical degrees of freedom. This adds divergences to the path integral that can only be removed by the DeWitt-Fadeev-Popov method. We can separate the field variables  $\varphi$  into physical variables  $I^{(A)}$  and gauge variables  $\chi^B$ , so that the action  $\bar{S}(I) = S(\varphi(I,\chi))$ does not depend on the group variables  $\chi$ , that is, it is invariant under the variations with respect to  $\chi^B$ , but not under the variations with respect to  $I^{(A)}$ .

To get rid of the excess degrees of freedom, we change the variables  $\varphi = \varphi(I, \chi)$  in the path integral and omit the integration over the group variables  $\chi$  (in other words, we divide out the volume of the gauge group), so that the integral is over the physical variables  $I^{(A)}(\varphi)$  only. If the inverse change of variables is given by  $\chi = \chi(\varphi)$ , then the Jacobian of the change of variables is Det  $Q(\varphi)$ , where

$$Q^{A}{}_{B}(\varphi) = R^{i}{}_{B}(\varphi) \frac{\delta \chi^{A}(\varphi)}{\delta \varphi^{i}}$$
(2.36)

is the Faddeev-Popov operator. The gauge condition is defined by a surface in the configuration space

$$\chi^A(\varphi) = \theta^A \tag{2.37}$$

where  $\theta^B$  are some constants. This surface intersects all orbits of the gauge group transversally, and thus has a one-to-one correspondence with the set of all physical states. With these changes, we get the Feynman measure

$$\mathcal{D}I = \mathcal{D}\varphi \text{Det} Q(\varphi)\delta(\chi^B(\varphi) - \theta^B), \qquad (2.38)$$

where  $\delta(\chi^B - \theta^B)$  is the functional delta function. The path integral becomes

$$\exp\left(\frac{i}{\hbar}\Gamma(\Phi)\right) = \int \mathcal{D}\varphi \,\delta(\chi^B(\varphi) - \theta^B) \text{Det}\,Q(\varphi) \\ \times \exp\left\{\frac{i}{\hbar}\left[S(\varphi) - \frac{\delta\Gamma(\Phi)}{\delta\Phi^i}(\varphi^i - \Phi^i)\right]\right\}.$$
(2.39)

In order to write the effective action in terms of gauge-invariant quantities, we integrate over  $\theta^B$  with Gaussian weight to get the expression

$$\exp\left(\frac{i}{\hbar}\Gamma(\Phi)\right) = \int \mathcal{D}\varphi \operatorname{Det} Q(\varphi) (\operatorname{Det} H(\Phi))^{\frac{1}{2}} \\ \times \exp\left\{\frac{i}{\hbar}\left[S(\varphi) - \frac{1}{2}\chi_A(\varphi)H^{AB}(\Phi)\chi_B(\varphi) - \frac{\delta\Gamma(\Phi)}{\delta\Phi^i}(\varphi^i - \Phi^i)\right]\right\},$$
(2.40)

where  $H^{AB}$  is some non-degenerate operator that does not depend on  $\varphi$ . The determinant Det Q is typically calculated in terms of Faddeev-Popov ghost fields, and Det H is calculated in terms of the Nielsen-Kallosh ghost. In the case that H does not depend on the background field  $\Phi$ , det H is simply an infinite constant that can be factored into the measure.

#### 2.3 Perturbation Theory

In order to calculate the effective action, we introduce perturbation theory. In this section, we follow [6]. Perturbation theory is based on the idea that the largest contributions to the effective action come from fields  $\varphi^i$  close to the background field  $\Phi^i$  in the sense that they can be split into a background part and a quantum part

$$\varphi^i = \Phi^i + \sqrt{\hbar} h^i \,, \tag{2.41}$$

with  $h^i$  being the quantum fluctuations of the background field. With this change of variables, the effective action will have an expansion of the form

$$\Gamma(\Phi) = S_{YM} + \hbar\Gamma_{(1)} + O(\hbar^2), \qquad (2.42)$$

which is known as the loop expansion.

Splitting the fields as in (2.41), the path integral (2.40) may be expanded using (2.42) to give the first order correction

$$\Gamma_{(1)} = \frac{i}{2} \ln \operatorname{Det} L_{YM} - \frac{i}{2} \ln \operatorname{Det} H - i \ln \operatorname{Det} L_{FP}, \qquad (2.43)$$

where

$$(L_{YM})_{ik} = -S_{,ik}(\Phi) + \chi_{A,i}(\Phi)H^{AB}\chi_{B,k}(\Phi), \qquad (2.44)$$

and

$$L_{FP} = Q(\Phi) \,. \tag{2.45}$$

The quantity  $\Gamma_{(1)}$  is the one-loop effective action.

### Wick rotation

The determinants of differential operators  $\ln \text{Det } L$  are problematic for two reasons. One reason is that  $\ln \text{Det } L$  is divergent and must be regularized in order to make sense. This will be dealt with in the next section. Before that can be accomplished we must first deal with the fact that L is, in general, not a positive definite operator. To get rid of this latter problem, we Wick rotate time in the complex plane

$$t = -i\tau \,. \tag{2.46}$$

Under this transformation, the metric is changed to have the Riemannian signature  $(+, \ldots, +)$ , so the operator  $-\Box$  becomes an elliptic operator rather than a hyperbolic one. In addition, the measure picks up an additional factor of -i:

$$\sqrt{|g|} \operatorname{d} t \operatorname{d}^{n-1} x \to -i\sqrt{|g|} \operatorname{d} \tau \operatorname{d}^{n-1} x, \qquad (2.47)$$

which in turn causes the action to pick up the same factor. For the remainder of the paper, all quantities will be assumed to be Wick rotated. To describe finite temperature effects, the Euclidean time is compactified to  $S^1$ , with radius given by the inverse temperature  $\beta = 1/T$ .

### 2.4 Heat Kernel Method for Computing the One-Loop Effective Action

The quantity  $\Gamma_{(1)}$  in (2.43) is the functional determinant of an elliptic differential operator. This quantity will always be infinite and therefore must be regularized in order for any physical calculation to make sense. Following [19, 5], this section will show that using the heat kernel representation, the effective action can be expressed in terms of a zeta function. Then by analytic continuation, the zeta function can be regularized and made to yield finite physical results.

#### Green Functions

For a bosonic field, the second functional derivative of the action may be brought by choice of gauge to the form

$$L + m^2 = -\Box + Q + m^2 \mathbb{I}, \qquad (2.48)$$

where Q is a matrix-valued function acting on the fields  $\varphi^i$ , m is the mass (which may be zero), and  $\mathbb{I}$  is the identity matrix.

Green functions are solutions  $\mathcal{G}^{\boldsymbol{j}}{}_k$  of the equation

$$(L+m^2)\mathcal{G}(x,x') = \mathbb{I}\delta(x,x'), \qquad (2.49)$$

with

$$\delta(x, x') = g^{-1/2}(x)\delta(x - x').$$
(2.50)

They can be constructed in terms of a contour integral of the heat kernel  $U(t) \equiv U(t|x,x')$ 

$$\mathcal{G}(x, x') = \int_0^\infty dt \exp(-tm^2) U(t|x, x') \,. \tag{2.51}$$

The heat kernel satisfies the heat equation

$$\left(\frac{\partial}{\partial t} + L\right)U(t) = 0 \tag{2.52}$$

with the boundary condition

$$U(t|x, x')|_{t=0} = \mathbb{I}\delta(x, x').$$
(2.53)

### From Green functions to Effective Action

The heat equation (2.52) has the formal solution

$$U(t) = \exp\left(-tL\right) \,. \tag{2.54}$$

In terms of the eigenfunctions  $\phi_n$  corresponding to eigenvalues  $\lambda_n$  of the operator L, i.e.

$$L\phi_n = \lambda_n \phi_n \,, \tag{2.55}$$

the heat kernel can be written

$$U(t|x,x') = \sum_{n=1}^{\infty} \phi_n(x) \otimes \phi_n^{\dagger}(x') e^{-t\lambda_n} \,. \tag{2.56}$$

The heat kernel diagonal is defined by taking the coincidence limit of this expression

$$U(t|x,x) = \sum_{n=1}^{\infty} \phi_n(x)\phi_n^{\dagger}(x)e^{-t\lambda_n}, \qquad (2.57)$$

and the functional  $L^2$  heat trace is the trace of the diagonal over all indices

$$\operatorname{Tr} \exp(-tL) = \int_{M} dx \operatorname{tr} U(t|x, x)$$
$$= \sum_{n=1}^{\infty} e^{-t\lambda_{n}}, \qquad (2.58)$$

In the case of Yang-Mills theory, the trace tr is over both group indices and tangent space indices. If the eigenvalues are degenerate, we may express the heat kernel in terms of the eigenvalues  $\{\lambda_n\}$  and degeneracies  $\{d_n\}$ 

$$\operatorname{Tr} \exp(-tL) = \sum_{n=1}^{\infty} d_n e^{-t\lambda_n}$$
(2.59)

If the mass is sufficiently large so that  $\lambda_n + m^2 > 0$ , the quantity ln Det  $(L + m^2)$  can similarly be expressed

$$\ln \text{Det} [L + m^2] = \sum_{n=0}^{\infty} \ln(\lambda_n + m^2). \qquad (2.60)$$

We can use the identity

$$\ln \lambda = -\int_0^\infty \frac{dt}{t} e^{-t\lambda} + C \tag{2.61}$$

with C an infinite constant, and the expression for the heat trace (2.58) to find

$$\ln \text{Det} (L+m^2) = -\int_0^\infty \frac{dt}{t} \exp(-tm^2) \int_M dx \, \text{tr} \, U(t|x,x) + \text{const} \, . \quad (2.62)$$

The infinite constant has no effect on the dynamics, and can be dropped. The one-loop effective action is expressed completely in terms of the logarithms of determinants of operators, so calculation of the heat kernel U(t|x,x) for various operators gives all of the information needed to calculate  $\Gamma_{(1)}$ . This reduces the task of calculating the one-loop effective action to that of finding the eigenvalues of the second variation of the action.

#### Zeta-Function Regularization

The quantity  $\ln \text{Det} (L + m^2)$  (2.62) is infinite. In order to make it finite, it must be regularized in terms of the  $\zeta$ -function

ln Det 
$$(L + m^2) = -\zeta'(0)$$
, (2.63)

$$\zeta'(0) = \frac{\mathrm{d}}{\mathrm{d}p} \zeta(p)|_{p=0} \,. \tag{2.64}$$

The  $\zeta$ -function of a differential operator M is defined in terms of the heat kernel by

$$\zeta_M(p) = \mu^{2p} \operatorname{Tr} M^{-p} = \frac{\mu^{2p}}{\Gamma(p)} \int_0^\infty dt \, t^{p-1} \operatorname{Tr} \, \exp(-tM) \,, \qquad (2.65)$$

where  $\mu$  is a renormalization parameter with dimension of inverse length. The  $\zeta$ -function is analytic at p = 0, so the expression (2.63) is finite and well-defined.

#### 2.4.1 Yang-Mills One-Loop Effective Action

With this method of regularization in mind, we may return to Yang-Mills theory. The one-loop effective action for Yang-Mills theory (2.43) in a general covariant gauge is the sum of contributions from Yang-Mills fields and ghosts. In terms of functional determinants in Euclidean space,

$$\Gamma_{(1)} = \ln \operatorname{Det} L_{YM}(\lambda) - 2 \ln \operatorname{Det} L_{FP}(\lambda), \qquad (2.66)$$

where  $\lambda$  is a gauge-fixing parameter, and  $L_{YM}(\lambda)$  is the operator acting on gauge fields as found in (2.44)

$$L_{YM}(\lambda) = L_{YM} + \lambda H , \qquad (2.67)$$

$$(L_{YM}\varphi)^{\mu} = -\Box\varphi^{\mu} - 2\mathcal{F}^{\mu}{}_{\nu}\varphi^{\nu} + R^{\mu}{}_{\nu}\varphi^{\nu}$$
(2.68)

$$(H\varphi)^{\mu} = \nabla^{\mu}\nabla_{\nu}\varphi^{\nu}, \qquad (2.69)$$

 ${\cal L}_{FP}$  is the Faddeev-Popov ghost operator acting on anti-commuting scalar fields

$$L_{FP}(\lambda) = \sqrt{1 - \lambda} L_{FP}, \qquad (2.70)$$

$$L_{FP}\eta = -\Box\eta. \tag{2.71}$$

We can regularize the gauge-fixed  $\Gamma_{(1)}$  by expressing it in terms of the  $\zeta-{\rm function}$ 

$$\Gamma_{(1)} = -\frac{1}{2}\zeta_{\rm tot}'(0), \qquad (2.72)$$

where

$$\zeta_{\text{tot}}(p) = \zeta_{L_{YM}}(p) - 2\zeta_{L_{FP}}(p) \tag{2.73}$$

is the total  $\zeta$ -function. The zeta-function can be analytically continued to give a renormalized expression for the effective action.

The factor  $\sqrt{1-\lambda}$  guarantees gauge independence of the regularized effective action on the mass shell[3]. It can also be proven [3] that  $\zeta_{\text{tot}}$  is independent of gauge, and so we may choose  $\lambda = 0$  so that we are left with minimal differential operators as in (2.48).

#### 2.5 Chromomagnetic Vacuum

It was shown by Savvidy that the one-loop effective action for SU(2)Yang-Mills in flat space takes a minimum for a non-zero field. Consider a covariantly constant Yang-Mills background in flat space

$$\nabla_{\mu} \mathcal{F}_{\alpha\beta} = 0. \qquad (2.74)$$

One flat space solution to this equation is

$$\mathcal{A}^{a}{}_{\alpha} = -\frac{1}{2} F_{\alpha\beta} x^{\beta} n^{a}, \qquad (2.75)$$

so that  $F^a_{\mu\nu}$  takes the form  $F^a{}_{\alpha\beta} = F_{\alpha\beta}n^a$ , where  $n_b$  is a unit vector in the Cartan subalgebra of the Lie algebra of G,  $n^b n_b = 1$ . To make this a "magnetic" background, conditions on group invariants are imposed

$$F_{\alpha\beta}F^{\alpha\beta} = \frac{1}{2}(H^2 - E^2) > 0, \qquad (2.76)$$

$$\epsilon^{\alpha\beta\gamma\delta}F_{\alpha\beta}F_{\gamma\delta} = H \cdot E = 0. \qquad (2.77)$$

Expanding the one-loop effective action in terms of momenta for a magnetictype field and taking the first term[18], the one-loop correction is  $\Gamma_{(1)H} = \int dx \mathbf{L}_{(1)H}$ 

$$\mathcal{L}_{(1)H} = \frac{1}{8\pi^2 e^2} \int_0^\infty \frac{\mathrm{d}s}{s^3} \left( \frac{Hs}{\sinh(Hs)} + 2Hs\,\sin(Hs) \right) \,. \tag{2.78}$$

Renormalization of this expression gives

$$\mathcal{L}_{(1)H} = -\frac{11H^2}{48\pi^2} \left[ \ln \frac{H}{\mu^2} - \frac{1}{2} \right]$$
(2.79)

to which the corresponding energy density is

$$\mathcal{H} = \frac{H^2}{2e^2} + \frac{11H^2}{48\pi^2} \left[ \ln \frac{H}{\mu^2} - \frac{1}{2} \right] \,. \tag{2.80}$$

It is easily seen [16] that the energy density has a minimum at

$$H_{\min} = \mu^2 \exp\left(-\frac{24\pi^2}{11e^2}\right)$$
 (2.81)

It was later pointed out by Nielsen and Olesen [16] that the energy density for this model has an imaginary part, which leads to instability of this model.

# CHAPTER 3

## HEAT TRACE ON SPHERES

#### 3.1 Existence of covariantly constant Yang-Mills fields on spheres

In the spirit of Savvidy [18], we consider a covariantly constant field strength tensor  $\mathcal{F}_{\mu\nu}$  that takes values in the center of the gauge Lie algebra. The condition that the field is constant gives rise to the equation

$$[\nabla_{\mu}, \nabla_{\nu}]\mathcal{F}_{\alpha\beta} - [\nabla_{\alpha}, \nabla_{\beta}]\mathcal{F}_{\mu\nu} = 0.$$
(3.1)

It can be shown[2] that this yields the integrability condition

$$[\mathcal{F}_{\mu\nu}, \mathcal{F}_{\alpha\beta}] + R_{\mu\nu\lambda[\alpha} \mathcal{F}^{\lambda}{}_{\beta]} - R_{\alpha\beta\lambda[\mu} \mathcal{F}^{\lambda}{}_{\nu]} = 0.$$
(3.2)

By taking  $\mathcal{F}_{\mu\nu}$  to be in the center of the Cartan algebra,  $[\mathcal{F}_{\mu\nu}, \mathcal{F}_{\alpha\beta}] = 0$ , writing the Riemann tensor of the *N*-sphere as

$$R^{\mu\nu}{}_{\lambda\alpha} = \rho(\delta^{\mu}_{\lambda}\delta^{\nu}_{\alpha} - \delta^{\nu}_{\lambda}\delta^{\mu}_{\alpha}) \tag{3.3}$$

for  $N \ge 2$ , and contracting over  $\mu$  and  $\alpha$  in (3.2), we find

$$\rho(N-2)F_{\nu\beta} = 0. (3.4)$$

Therefore, non-zero covariantly constant magnetic fields in the center of the algebra of the Lie group can only exist on  $S^2$  or  $\mathbb{R}^N$ .

#### 3.2 Product Manifolds

Each of the manifolds that we consider has the product manifold structure  $M = M_1 \times \cdots \times M_n$ , where  $M_i$  are submanifolds of M. In this case, we have the decomposition of the operator L

$$L = L_1 \otimes \mathbb{I}_2 \otimes \cdots \otimes \mathbb{I}_n + \cdots + \mathbb{I}_1 \otimes \dots \mathbb{I}_{n-1} \otimes L_n, \qquad (3.5)$$

where  $L_i$  is the projection of L onto the submanifold  $M_i$ . In this case, the heat kernel has the form

$$\exp(-tL) = \exp(-tL_1)\dots\exp(-tL_n).$$
(3.6)

To calculate the heat kernel on a general product manifold, we only need to calculate the heat kernel on each submanifold and multiply the results

$$\operatorname{Tr} e^{-tL} = \operatorname{Tr}_{M_1} \exp(-tL_1) \dots \operatorname{Tr}_{M_n} \exp(-tL_n), \qquad (3.7)$$

where Tr denotes the functional trace, which is also taken to be the trace over group and coordinate indices.

### **3.3** Heat Trace of Laplacian on $S^1$

On  $S^1$ , a circle of radius r, the Laplacian acting on any function is simply the operator

$$L = -\Delta = -\frac{1}{r^2} \partial_{\phi}^2, \qquad (3.8)$$

where  $\phi$  is the coordinate along the circle,  $0 \leq \phi < 2\pi$ . The eigenvalues for this operator are

$$\lambda_n = \frac{n^2}{r^2}, \qquad n = 0, \pm 1, \dots$$
 (3.9)

The multiplicities are

$$d_0 = 1$$
 (3.10)

$$d_n = 2, \qquad n = 1, 2, \dots$$
 (3.11)

(3.12)

The heat kernel trace for a function on  $S^1$  can then be calculated using the formula for the heat kernel (2.59)

Tr 
$$\exp(-tL) = 1 + 2\sum_{n=1}^{\infty} e^{-tn^2/r^2}$$
. (3.13)

We define this to be the function

$$S\left(\frac{t}{r^2}\right) = 1 + 2\sum_{n=1}^{\infty} e^{-tn^2/r^2}$$
 (3.14)

There is no difference between a scalar and a p-form on  $S^1$ , so this heat trace applies to all geometric objects on  $S^1$ .

### **3.4 Heat Trace on** $S^2$

The heat trace for  $S^2$  is non-trivial and its calculation is a significantly more complicated problem. On  $S^2$ , scalars and one-forms will be distinct objects and will form different representations of both the gauge group and the isotropy group, which will determine the covariant derivative and thus the form of the Laplacian. In addition, the existence of non-zero chromomagnetic fields will cause the the eigenvalues to split, leading to a much more complicated spectrum.

## **3.4.1** Geometry of $S^2$

Consider the 2-sphere  $S^2$  of radius R endowed with the standard Riemannian metric

$$ds^{2} = e^{(\alpha)}{}_{\mu}e^{(\alpha)}{}_{\nu}dx^{\mu}dx^{\nu} = R^{2}(d\theta^{2} + \sin^{2}\theta d\phi^{2}), \qquad (3.15)$$

where  $0 \leq \theta < \pi$ , and  $0 \leq \phi < 2\pi$ . Greek letters without parentheses denote the coordinate indices, which range over the two values  $\theta$  and  $\phi$ . The orthonormal basis one-forms  $e^{(\alpha)}$  are given by

$$e^{(1)} = Rd\theta, \qquad e^{(2)} = R\sin\theta d\phi. \qquad (3.16)$$

Greek indices with parentheses range over 1, 2 and denote indices of the orthonormal basis. The volume form is given by

$$d\text{vol} = R^2 \sin\theta d\theta \wedge d\phi \,. \tag{3.17}$$

The components of the spin connection 1-form can be found by the Cartan method to be

$$\omega_{(\alpha)(\beta)} = -\epsilon_{(\alpha)(\beta)} \cos \theta d\phi \,. \tag{3.18}$$

The curvature tensor components are

$$R^{(\alpha)(\beta)}{}_{(\gamma)(\delta)} = \frac{1}{R^2} \left( \delta^{(\alpha)}_{(\gamma)} \delta^{(\beta)}_{(\delta)} - \delta^{(\beta)}_{(\gamma)} \delta^{(\alpha)}_{(\delta)} \right) \,. \tag{3.19}$$

### 3.4.2 Isometries

The sphere  $S^2$  is diffeomorphic to the quotient space SO(3)/SO(2). Here, SO(3) is the isometry group of  $S^2$  and SO(2) is the isotropy group of  $S^2$ . The rotation group SO(3) maps  $S^2$  to itself and SO(2) rotations centered around a point will leave the point unmoved.

Let  $\varphi = (\varphi^A)$  be a field which transforms under a representation of the group SO(2). Let  $\Sigma_{(\alpha)(\beta)} = (\Sigma_{(\alpha)(\beta)}{}^B{}_A)$  be the generators of the group SO(2) in the representation acting on  $\varphi$ . Because SO(2) is a one-dimensional, Abelian group, there is only one generator  $\Sigma = \Sigma_{(1)(2)}$ . Then the covariant derivative of  $\varphi$  is

$$\nabla_{\mu}\varphi = \left(\partial_{\mu} + \omega_{\mu}\Sigma\right)\varphi, \qquad (3.20)$$

where

$$\omega_1 = 0 \qquad \omega_2 = -\cos\theta \,. \tag{3.21}$$

#### 3.4.3 Gauge Curvature

Now assume that the field  $\varphi$  also transforms under another representation T of the group SO(2), which we call a gauge representation. Then  $\varphi$ transforms under the product of two representations of the group SO(2). Let  $\mathcal{A}$  be the corresponding gauge connection and  $\mathcal{F} = d\mathcal{A}$  be the curvature of this connection.

The gauge curvature  $\mathcal{F}$  is a 2-form on a 2-dimensional space, and so  $\mathcal{F}$  must be proportional to the volume form dvol

$$\mathcal{F} = \frac{TH}{2} \sin \theta d\phi \wedge d\theta \,, \tag{3.22}$$

where, in general, H is some function of the coordinates. By expressing  $\mathcal{F}$  in components

$$\mathcal{F}^{\mu}{}_{\nu} = \frac{TH}{2R^2} E^{\mu}{}_{\nu} \,, \tag{3.23}$$

where  $E_{\mu\nu} = \sqrt{g} \epsilon_{\mu\nu}$  is the invariant volume form, we see

$$\nabla_{\alpha} \mathcal{F}_{\mu\nu} = \frac{1}{2} T E_{\mu\nu} \nabla_{\alpha} H \,. \tag{3.24}$$

Requiring the curvature to be covariantly constant yields the condition that H is a constant.

Physically, H can be interpreted as the charge of a monopole at the center of the sphere. Because  $\mathcal{F}$  is precisely the Chern form of a line bundle over  $S^2$  [11], we have

$$\int_{S^2} \mathcal{F} = 2\pi nT \,, \quad n \in \mathbb{Z} \,. \tag{3.25}$$

Therefore, the monopole charge H can only take integer values

$$H = n, \quad n \in \mathbb{Z}. \tag{3.26}$$

The corresponding gauge connection is found by solving the equation

$$\mathcal{F} = d\mathcal{A} \,, \tag{3.27}$$

which results in

$$\mathcal{A} = \frac{TH}{2}\cos\theta d\phi \tag{3.28}$$

We see that  $\mathcal{A}$  is proportional to the spin connection  $\omega$ . The covariant derivative invariant under both space rotations and gauge transformations is then

$$\nabla_{\mu}\varphi = (\partial_{\mu} + \mathcal{T}\omega_{\mu})\varphi, \qquad (3.29)$$

where

$$\mathcal{T} = \mathbb{I} \otimes \Sigma + \frac{TH}{2} \otimes \mathbb{I}, \qquad (3.30)$$

and T is the generator of the gauge group SO(2).

### **3.5** Spectrum of the Laplacian on $S^2$

In order to calculate the heat traces (2.59) on  $S^2$ , we need to analyze the spectrum of the operator  $L_{YM}$ . We know from (2.68) that this operator is equal to the negative Laplacian plus Yang-Mills strength and Ricci curvature terms. The field strength and curvature tensors are covariantly constant. Therefore, the eigenfunctions of  $L_{YM}$  are proportional to the eigenfunctions of the Laplacian. This causes the eigenvalues of  $L_{YM}$  to be the eigenvalues of the Laplacian, shifted by the eigenvalues of the sum of the other two operators.

The Laplacian acting on general spin-tensor is given by the expression

$$\Delta = g^{\mu\nu} \nabla_{\mu} \nabla_{\nu} = |g|^{-1/2} (\partial_{\mu} + \mathcal{T}\omega_{\mu}) |g|^{1/2} g^{\mu\nu} (\partial_{\nu} + \mathcal{T}\omega_{\nu}) .$$
(3.31)

In the case of  $S^2$ , the Laplacian becomes

$$\Delta = \frac{1}{R^2} \left[ \partial_{\theta}^2 + \cot \theta \partial_{\theta} + \frac{1}{\sin^2 \theta} (\partial_{\phi} - \mathcal{T} \cos \theta)^2 \right]$$
(3.32)

It can be noted that this will yield the standard Laplacian for a particle in a magnetic field [13] in the limit  $R \to \infty$ . Using polar coordinates near  $\theta = 0$  with  $\theta = \rho/R$ , along with using the choice of gauge  $\mathcal{A} = \frac{tH}{2} (\cos \theta - 1) d\phi$ and denoting the generator of SO(2) by *i*, the connection becomes

$$\mathcal{A}_{(1)} = 0, \qquad \mathcal{A}_{(2)} = \frac{iMR^2}{2}(\cos\theta - 1), \qquad (3.33)$$

where M is the magnetic field  $H = MR^2$ . Taking the limit  $R \to \infty$  then gives the standard Laplacian on  $\mathbb{R}^2$ 

$$\Delta = \partial_{\rho}^{2} + \frac{1}{\rho}\partial_{\rho} + \frac{1}{\rho^{2}}\partial_{\phi}^{2} + \frac{iH}{2}\partial_{\phi} - \frac{H^{2}}{16}\rho^{2}.$$
 (3.34)

#### 3.5.1 Action of Laplacian on one-forms

For 1-forms, the generator  $\mathcal{T}$  is the matrix with components

$$\mathcal{T}^{(\alpha)a}{}_{(\beta)b} = \epsilon^{(\alpha)}{}_{(\beta)}\delta^{a}{}_{b} + \frac{H}{2}\epsilon^{a}{}_{b}\delta^{(\alpha)}{}_{(\beta)}.$$
(3.35)

The eigenvalues of the matrix  $\mathcal{T}$  are  $ik_j$ , j = 1, 2, 3, 4

$$k_1 = 1 + \frac{H}{2}, \quad k_2 = 1 - \frac{H}{2}, \quad k_3 = -1 + \frac{H}{2}, \quad k_4 = -1 - \frac{H}{2}.$$
 (3.36)

In the same basis, the field strength tensor can be diagonalized with corresponding eigenvalues

$$f_1 = -\frac{H}{2}, \quad f_2 = \frac{H}{2}, \quad f_3 = \frac{H}{2}, \quad f_4 = -\frac{H}{2}, \quad (3.37)$$

and the Ricci tensor will be proportional to the identity, with all eigenvalues  $r_i$  given by

$$r_i = \frac{1}{R^2} \tag{3.38}$$

Diagonalizing the matrix  $\mathcal{T}$  will cause the Yang-Mills operator  $L_{YM}$  to break into four separate operators of the form

$$L^{(i)} = -\frac{1}{R^2} \left[ \partial_\theta^2 + \cot\theta \partial_\theta + \frac{1}{\sin^2\theta} (\partial_\phi - ik_i \cos\theta)^2 \right] - 2\frac{f_i}{R^2} + \frac{1}{R^2}, \quad (3.39)$$

where k is a half-integer that takes one of the four values  $k_1, k_2, k_3, k_4$ . The values  $f_i$  are the corresponding eigenvalues of the matrix  $\mathcal{F}$  and are given by the values

$$f_1 = -\frac{H}{2}, \qquad f_2 = \frac{H}{2}, \qquad f_3 = \frac{H}{2}, \qquad f_4 = -\frac{H}{2}.$$
 (3.40)

It is clear that the spectrum of the Laplacian will be invariant under change of sign of the field H. For the remainder of the paper, we will then assume without loss of generality that H is positive. The spectrum of the operator  $L^{(i)}$  is defined by regular normalized solutions of the equation

$$L^{(i)}u = \lambda u \,, \tag{3.41}$$

where  $\lambda \in \mathbb{C}$  is a complex spectral parameter. This equation has regular solutions only for certain real discrete values of  $\lambda$ , which determine the spectrum of  $L^{(i)}$ .

The operators  $\mathcal{F}$  and R have no dependence on the coordinates, and so the eigenfunctions of the operator  $L^{(i)}$  are the same as for the Laplacian. The eigenvalues of  $L^{(i)}$  are obtained from the eigenvalues of the Laplacian  $(-\Delta)$ by shifting

$$\lambda(L^{(i)}) = \lambda(-\Delta^{(i)}) + \frac{1}{R^2} - \frac{2f_i}{R^2}, \qquad (3.42)$$

so we find the spectrum of the Laplacian first.

Separating variables with the substitution

$$u = e^{im\phi} h_m(\theta), \quad m \in \mathbb{Z}, \qquad (3.43)$$

we obtain an ordinary differential equation for  $h(\theta)$ 

$$\left\{\partial_{\theta}^{2} + \cot\theta\partial_{\theta} - \frac{1}{\sin^{2}\theta}(m - k_{i}\cos\theta)^{2} + R^{2}\lambda\right\}h_{m}(\theta) = 0.$$
(3.44)

Let us introduce the notation

$$a_{ml}^{\pm} = \frac{1}{2} + \left|\frac{m-k}{2}\right| + \left|\frac{m+k}{2}\right| \pm \frac{1}{2}(1+4R^2\lambda)^{1/2}.$$
 (3.45)

The index l labels the eigenvalues, as will be described below. As explained in

Appendix A, this equation has regular normalized solutions given by

$$h_{mk}^{l}(\theta) = (1 - \cos \theta)^{\left|\frac{m-k}{2}\right|} (1 + \cos \theta)^{\left|\frac{m+k}{2}\right|} \times F\left(a_{ml}^{+}, a_{ml}^{-}; 1 + |m-k|; \frac{1 - \cos \theta}{2}\right),$$
(3.46)

where F(a, b; c; z) is the hypergeometric function. In the case of integer k, these solutions exist for the following values of  $\lambda_l$  and m

$$\lambda_l = \frac{1}{R^2} \left( |k| + l \right) \left( |k| + l + 1 \right), \tag{3.47}$$

$$-l \le m \le l, \tag{3.48}$$

where l is an integer greater than or equal to 0:

$$l \ge 0. \tag{3.49}$$

By counting all possible values of m we obtain the multiplicities of the eigenvalues  $\lambda_l$  for integer k

$$d_l = 2(l+|k|) + 1. (3.50)$$

In the case of half-integer k, there are two series of solutions. The first series is given by the following values of  $\lambda_l$  and m:

$$\lambda_l = \frac{1}{R^2} \left( |k| + l \right) \left( |k| + l + 1 \right), \tag{3.51}$$

$$-|k| + \frac{1}{2} \le m \le |k| - \frac{1}{2}, \qquad (3.52)$$

where l is an integer greater than or equal to 0

$$l \ge 0, \qquad (3.53)$$

giving degeneracies

$$d_l = 2|k|. (3.54)$$

The second series is given by

$$\lambda_{n} = \frac{1}{R^{2}} \left( |k| + \frac{1}{2} + n \right) \left( |k| + \frac{3}{2} + n \right) , \qquad (3.55)$$
$$- \left( |k| + \frac{1}{2} + n \right) \le m \le - \left( |k| + \frac{1}{2} \right) \text{ or }$$
$$\left( |k| + \frac{1}{2} \right) \le m \le \left( |k| + \frac{1}{2} + n \right) , \qquad (3.56)$$

which gives the degeneracies

$$d_n = 2n + 2, \quad n \ge 0. \tag{3.57}$$

Therefore, the eigenvalues of the operator  $(-\Delta_j)$  are  $\lambda_l$  and the corresponding eigenfunctions are

$$u_m^l(\theta,\phi) = e^{im\phi} h_m^l(\theta) \,. \tag{3.58}$$

It should be noted here that the eigenvalues of the Laplacian here are very different from those of the Laplacian for flat space (3.34). In the flat space, the eigenvalues are that of a harmonic oscillator [13]

$$\lambda = HR^{-2} \left( n + \frac{1}{2} \right) \,, \tag{3.59}$$

whereas in our case, the eigenvalues are quadratic in |k|, and so will increase quadratically in H as H becomes large.

The spectrum of the operator  $L_{YM}$  is then obtained by the shift of the Laplacian's eigenvalues (3.42). The heat kernel of the operator  $L_{YM}$  acting

on one-forms is found using the eigenvalues of the Laplacian, which are the values of  $\lambda$  in (3.47)- (3.55) plus the value  $1/R^2 - 2f_i/R^2$ .

The eigenvalues for even magnetic field are given by

$$\lambda_{il} = \frac{1}{R^2} (|k_i| + l)(|k_i| + l + 1) + \frac{1}{R^2} - \frac{2f_i}{R^2}, \qquad (3.60)$$

where *i* runs over the tangent space and group indices  $i = 1, 2, 3, 4, l \ge 0$ , and the degeneracies are given by

$$d_{il} = 2(l + |k_i|) + 1. (3.61)$$

The eigenvalues for odd magnetic field are in two series for each value of i. The first series is given by

$$\lambda_{il} = \frac{1}{R^2} (|k_i| + l)(|k_i| + l + 1) + \frac{1}{R^2} - \frac{2f_i}{R^2}$$
(3.62)

with  $i = 1, 2, 3, 4, l \ge 0$ , and degeneracies given by

$$d_{il} = 2|k_i| \,. \tag{3.63}$$

The second series is given by

$$\lambda_{in} = \frac{1}{R^2} \left( |k_i| + \frac{1}{2} + n \right) \left( |k_i| + \frac{3}{2} + n \right) , \qquad (3.64)$$

with  $i = 1, 2, 3, 4, n \ge 0$ , and degeneracies

$$d_{in} = 2n + 2. (3.65)$$

#### 3.5.2 Action of Laplacian on Ghosts

In addition to the Yang-Mills field, we must consider the scalar Faddeev-Popov ghost field to compute the effective action. In this case, the relevant operator is  $L_{FP} = -\Delta$ . Scalar fields are invariant under coordinate transformations, so the generator  $\mathcal{T}$  as given in (3.30) will only transform under the gauge group SO(2), which means that the total generator for the ghosts will be

$$X = \frac{H}{2} \epsilon^a{}_b \,, \tag{3.66}$$

which has the two eigenvalues  $i\kappa_j$  with

$$\kappa_1 = \frac{H}{2}, \qquad \kappa_2 = -\frac{H}{2}. \tag{3.67}$$

The values that the parameter  $\lambda$  takes on will be exactly the same as in (3.47), (3.51), and (3.55), except with these values of  $\kappa$  to replace the values of k. The eigenvalues are again characteristically different for even and odd magnetic field. For even values of H, the eigenvalues of  $L_{FP}$  are given by

$$\lambda_l = \frac{1}{R^2} \left( l + \frac{H}{2} \right) \left( l + \frac{H}{2} + 1 \right) \tag{3.68}$$

The eigenvalues do not depend on the sign of  $\kappa$ , so the degeneracies are just doubled to account for the two values:

$$d_l = 4\left(l + \frac{H}{2}\right) + 2. \tag{3.69}$$

For odd values of H, the eigenvalues are in two series. The first series is given by

$$\lambda_l = \frac{1}{R^2} \left( l + \frac{H}{2} \right) \left( l + \frac{H}{2} + 1 \right) , \qquad (3.70)$$

with the degeneracies doubled to account for positive and negative kappa:

$$d_l = 2H. (3.71)$$

The second series is given by

$$\lambda_n = \frac{1}{R^2} \left( n + \frac{H}{2} + \frac{1}{2} \right) \left( n + \frac{H}{2} + \frac{3}{2} \right) , \qquad (3.72)$$

with degeneracies

$$d_n = 4n + 4. (3.73)$$

By using these values, we will be able to calculate the heat kernel in the next chapter.

## CHAPTER 4

## HEAT KERNEL TRACE AND EFFECTIVE ACTION

The heat trace of the total Yang-Mills and ghost operators can be computed on products of spheres using the factorization property of the heat kernel. In this chapter, we calculate the heat traces of each operator on the spaces  $T^2$  and  $S^2$ , from which we will be able to compute the total heat trace for the product manifolds in the next chapter. The heat kernel can readily be calculated from the eigenvalues and degeneracies that have been found, as well as using the defining formula for the heat trace (2.58), written in terms of eigenvalues  $\lambda_l(L)$  and their degeneracies,

$$\operatorname{Tr}\left(e^{-tL}\right) = \sum_{l} d_{l} e^{-t\lambda_{l}(L)} \,. \tag{4.1}$$

## 4.1 Yang-Mills on $T^2$

The simplest space to consider is the two-torus,  $T^2 = S^1 \times S^1$ , with each copy of  $S^1$  having a different radius  $(r_1, r_2)$ . The two-torus has no curvature and can support no covariantly constant chromomagnetic field due to topological constraints. Thus, the operator  $L_{YM}$  is just the Laplacian

$$L_{YM} = -\Delta \,. \tag{4.2}$$

It is then straightforward to find that the heat trace is

Tr 
$$\exp(-tL_{YM}) = 4S\left(\frac{t}{r_1^2}\right)S\left(\frac{t}{r_2^2}\right)$$
. (4.3)

The operator for ghosts in this case is simply the scalar Laplacian  $-\Delta$ ,

Tr 
$$\exp(-tL_{FP}) = 2S\left(\frac{t}{r_1}\right)S\left(\frac{t}{r_2}\right)$$
, (4.4)

with the factor of 2 coming from the trace over group indices.

## 4.2 Yang-Mills on $\mathbb{R}^2$

The heat trace for Yang-Mills theory on  $\mathbb{R}^2$  has been found  $[1,\,2]$  to be

$$\operatorname{Tr} e^{-tL_{YM}} = \int_{\mathbb{R}^2} dx (4\pi t)^{-1} \left[ 2 + \frac{tHR^{-2}}{\sinh(tHR^{-2}/2)} \left( 2 + 4\sinh^2(tHR^{-2}/2) \right) \right]$$
(4.5)

and the heat trace for the corresponding ghost operator is

$$\operatorname{Tr} e^{-tL_{FP}} = \int_{\mathbb{R}^2} dx (4\pi t)^{-1} \left[ 1 + \frac{tHR^{-2}}{\sinh(tHR^{-2}/2)} \right].$$
(4.6)

## 4.3 Yang-Mills on $S^2$

# 4.3.1 Yang-Mills Operator on $S^2$

The heat kernel for the Yang-Mills field can be found by performing the spectral sum (4.1), using the eigenvalues and degeneracies for the operator  $L_{YM}$ .

For the case of even magnetic charge H, the heat kernel for the operator  $L_{YM}$  acting on one-forms is

$$Tr(e^{-tL_{YM}}) = \sum_{j=1}^{4} \sum_{l=0}^{\infty} (2l+2|k_j|+1) \\ \times \exp\left\{-\frac{t}{R^2} \left[(|k_j|+l)(|k_j|+l+1) - 2f_j+1\right]\right\}. \quad (4.7)$$

For odd H,

$$Tr(e^{-tL_{YM}}) = \sum_{j=1}^{4} \left[ \sum_{l=0}^{\infty} 2|k_j| \exp\left\{ -\frac{t}{R^2} \left[ (|k_j|+l)(|k_j|+l+1) - 2f_j + 1 \right] \right\} + \sum_{l=0}^{\infty} (2l+2) \exp\left( -\frac{t}{R^2} \left( |k_j|+l+\frac{1}{2} \right) \left( |k_j|+l+\frac{3}{2} \right) - 2f_i + \frac{1}{R^2} \right) \right]$$

$$(4.8)$$

These sums can be expressed in terms of the functions

$$\Theta_j(t) = \sum_{l=1}^{\infty} l^j e^{-tl(l+1)} , \qquad (4.9)$$

$$\Phi_j(t) = \sum_{l=1}^{\infty} l^j e^{-tl^2} \,. \tag{4.10}$$

These functions are regular in the limit  $t \to \infty$ .

For H = 0, the heat trace is given by

$$\operatorname{Tr}\left(e^{-tL_{YM}}\right) = 4e^{-t/R^2} \left[2\Theta_1\left(\frac{t}{R^2}\right) + \Theta_0\left(\frac{t}{R^2}\right)\right].$$
(4.11)

For H = 1,

For H = 3,

$$\operatorname{Tr} \left( e^{-tL_{YM}} \right) = \left( 4e^{-2t/R^2} + 4 \right) \Theta_1 \left( \frac{t}{R^2} \right) - 4e^{-2t/R^2} \Theta_0 \left( \frac{t}{R^2} \right) + \left( 6e^{-9t/4R^2} + 2e^{-t/4R^2} \right) \Phi_0 \left( \frac{t}{R^2} \right) - 6e^{-13t/4R^2} \quad (4.12)$$

For 
$$H = 2$$
,  

$$\operatorname{Tr} \left( e^{-tL_{YM}} \right) = 2\left( e^{t/R^2} + e^{-3t/R^2} \right) \left[ 2\Theta_1 \left( \frac{t}{R^2} \right) + \Theta_0 \left( \frac{t}{R^2} \right) \right] + 2e^{t/R^2} - 6e^{-5t/R^2}.$$
(4.13)

$$\operatorname{Tr} \left( e^{-tL_{YM}} \right) = \left( 2e^{7t/4R^2} + 10e^{-19t/4R^2} \right) \Phi_0 \left( \frac{t}{R^2} \right) + \left( 4e^{2t/R^2} + 4e^{-4t/R^2} \right) \Theta_1 \left( \frac{t}{R^2} \right) \\ - 10e^{-23t/4R^2} - 4e^{-6t/R^2} - 10e^{-35t/4R^2} - 4e^{-10t/R^2} \,.$$
(4.14)

For |H| = 4,  $\operatorname{Tr}\left(e^{-tL_{YM}}\right) = 2\left(e^{3t/R^2} + e^{-5t/R^2}\right) \left[2\Theta_1\left(\frac{t}{R^2}\right) + \Theta_0\left(\frac{t}{R^2}\right)\right] - 6e^{-7t/R^2} - 10e^{-11t/R^2}.$ (4.15)

For  $|H| \ge 5$  with |H| odd,

$$\operatorname{Tr} \left( e^{-tL_{YM}} \right) = 8 \cosh(tH/R^2) e^{-t/R^2} \Theta_1 \left( \frac{t}{R^2} \right) \\ + \left[ (6 - 2H) e^{t(H-1)/R^2} - 2(H+1) e^{-t(H+1)/R^2} \right] \Theta_0 \left( \frac{t}{R^2} \right) \\ + \left[ (2H+4) e^{t(H-3/4)/R^2} + (2H+4) e^{-t(H+3/4)} - 8 e^{t(H-5/4)/R^2} \right] \Phi_0 \left( \frac{t}{R^2} \right) \\ - 2 e^{-t(H+3/4)/R^2} (H+2) \sum_{l=1}^{\frac{H}{2} + \frac{1}{2}} e^{-tl^2/R^2} \\ - 2 e^{t(H-3/4)/R^2} (H-2) \sum_{l=1}^{\frac{H}{2} - \frac{3}{2}} e^{-tl^2/R^2} \\ - 2 e^{-t(H+1)/R^2} \sum_{l=1}^{\frac{H}{2} + \frac{1}{2}} (2l - H - 1) e^{-tl(l+1)/R^2} \\ - 2 e^{t(H-1)/R^2} \sum_{l=1}^{\frac{H}{2} - \frac{3}{2}} (2l + 3 - H) e^{-tl(l+1)/R^2}$$

$$(4.16)$$

For  $|H| \ge 6$  with |H| even,

$$\operatorname{Tr} \left( e^{-tL_{YM}} \right) = 4 \cosh(tH/R^2) e^{-t/R^2} \left[ 2\Theta_1 \left( \frac{t}{R^2} \right) - \Theta_0 \left( \frac{t}{R^2} \right) \right] -2e^{-t(H+1)/R^2} \sum_{l=1}^{\frac{H}{2}} (2l+1) e^{-tl(l+1)/R^2} -2e^{t(H-1)/R^2} \sum_{l=1}^{\frac{H}{2}-2} (2l+1) e^{-tl(l+1)/R^2}.$$
(4.17)

# **4.3.2** Ghost Operator on $S^2$

In addition to the Yang-Mills field itself, there are ghost fields to eliminate the extra degrees of freedom caused by gauge invariance. The ghost fields on  $S^2$  are scalar fields, which means that the eigenvalues and degeneracies are given by (3.47)-(3.57) with the values  $k = \pm |H|/2$ . With these values, it is straightforward to calculate the heat trace.

For H = 0,

$$\operatorname{Tr}\left(e^{-tL_{FP}}\right) = 4\Theta_1\left(\frac{t}{R^2}\right) + 2\Theta_0\left(\frac{t}{R^2}\right) + 2.$$
(4.18)

For H = 1,

$$\operatorname{Tr}\left(e^{-tL_{FP}}\right) = 2e^{-t/4R^{2}}\Phi_{0}\left(\frac{t}{R^{2}}\right) + 4\Theta_{1}\left(\frac{t}{R^{2}}\right).$$
(4.19)

For H = 2,

$$\operatorname{Tr}\left(e^{-tL_{FP}}\right) = 4\Theta_1\left(\frac{t}{R^2}\right) + 2\Theta_0\left(\frac{t}{R^2}\right).$$
(4.20)

For H odd, with  $|H| \ge 3$ ,

$$\operatorname{Tr} \left( e^{-tL_{FP}} \right) = 2He^{t/4R^2} \Phi_0 \left( \frac{t}{R^2} \right) + 4\Theta_1 \left( \frac{t}{R^2} \right) + 2(1-H)\Theta_0 \left( \frac{t}{R^2} \right) -2He^{t/4R^2} \sum_{l=1}^{\frac{H}{2} - \frac{1}{2}} e^{-tl^2/R^2} - \sum_{l=1}^{\frac{H}{2} - \frac{1}{2}} (4l + 2 - 2H)e^{-tl(l+1)/R^2}$$

$$(4.21)$$

For even H,  $|H| \ge 4$ , we have

$$\operatorname{Tr} e^{t\Delta_0} = 4\Theta_1\left(\frac{t}{R^2}\right) + 2\Theta_0\left(\frac{t}{R^2}\right) - 2\sum_{l=1}^{\lfloor\frac{|H|}{2}-1} (2l+1)e^{-\frac{t}{R^2}l(l+1)}.$$
 (4.22)

#### 4.4 Heat Trace on Product Spaces

With the heat trace of both the Yang-Mills and ghost operators calculated on all relevant submanifolds, it is possible to assemble the total heat trace the gauge-fixed Yang-Mills field on four-dimensional manifolds by using the factorization property of the heat kernel and calculating the total heat trace  $\operatorname{Tr} \exp(-tL_{YM}) - 2\operatorname{Tr} \exp(-tL_{FP})$ .

We can characterize the stability of the Yang-Mills vacuum by examining the large t behavior if this function. A negative eigenvalue corresponds to an unstable mode, which would indicate that the vacuum is unstable. However, we find that with a sufficiently strong positive curvature on  $S^2$ , we can make all eigenvalues positive, and the vacuum becomes stable.

# 4.5 Yang-Mills on $S^1 \times S^1 \times \mathbb{R}^2$ with non-zero chromomagnetic field on $\mathbb{R}^2$

This case is the analog of the problem studied by Savvidy. Spacetime has zero curvature and a constant chromomagnetic field exists. The total heat kernel is given by

$$U_{\text{tot}}(t) = \text{Tr} \exp(-tL_{YM})_{\mathbb{R}^2} \times \text{Tr} \exp(-tL_{YM})_{S^1 \times S^1}$$
$$-2 \text{Tr} \exp(-tL_{FP})_{\mathbb{R}^2} \times \text{Tr} \exp(-tL_{FP})_{S^1 \times S^1}.$$
$$(4.23)$$

This is evaluated using the heat traces (4.5) and (4.6):

$$U_{\rm tot}(t) = \int_{\mathbb{R}^2} dx (4\pi t)^{-1} \left[ \frac{tHR^{-2}}{\sinh(tHR^{-2}/2)} \left( 4\sinh^2(tHR^{-2}/2) \right) \right] S\left(\frac{t}{r_1^2}\right) S\left(\frac{t}{r_2^2}\right) ,$$
(4.24)

where  $r_1$  and  $r_2$  are the radii of the two copies of  $S^1$ .

## 4.6 Yang-Mills on $S^1 \times S^1 \times S^2$ with non-zero chromomagnetic field on $\mathbb{R}^2$

Superficially, it would seem that there are two different configurations of chromomagnetic field that can exist on the manifold  $S^1 \times S^1 \times S^2$ – either with the chromomagnetic field polarized along the torus  $S^1 \times S^1$  or along the sphere  $S^2$ . The first case cannot be realized because a covariantly constant chromomagnetic field can not exist on  $S^1 \times S^1$ . However, we can consider the related problem of having the non-zero field polarized along  $\mathbb{R}^2$  on the manifold  $\mathbb{R}^2 \times S^2$ . This case is not physical because it leaves the time direction to be incorporated into  $S^2$ , which means that spacetime can no longer be deforemed to have the structure  $\mathbb{R} \times \Sigma$ . This case has been investigated by Elizalde, et. al. [9]. In the limit that the curvature is small, it has been determined that the vacuum stabilizes for some radius of  $S^2$ . We can also analyze this problem from our standpoint.

In the case of a chromomagnetic field directed along  $\mathbb{R}^2$ , the operator  $L_{YM}$  will have the block diagonal form

$$L_{YM} = \left(-\Delta_{1(1)} - \Delta_{1(2)}\right) \begin{pmatrix} \mathbb{I} & 0\\ 0 & \mathbb{I} \end{pmatrix} + \begin{pmatrix} 0 & 0\\ 0 & \mathcal{R}_2 \end{pmatrix} - 2 \begin{pmatrix} \mathcal{F}_1 & 0\\ 0 & 0 \end{pmatrix} , \quad (4.25)$$

where  $\Delta_{1(1)}$  is the Laplacian acting on one-forms on  $\mathbb{R}^2$ ,  $\Delta_{1(2)}$  is the Laplacian acting on one-forms on  $S^2$ ,  $\mathcal{R}_2$  is the Ricci tensor on  $S^2$ , and  $\mathcal{F}_1$  is the chromomagnetic field restricted to  $\mathbb{R}^2$ . The operator  $L_{FP}$  is simply the Laplacian acting on scalars

$$L_{FP} = -\Delta_{0(1)} - \Delta_{0(2)} , \qquad (4.26)$$

where  $\Delta_{0(1)}$  is the Laplacian acting on scalars on  $\mathbb{R}^2$  and  $\Delta_{1(2)}$  is the Laplacian acting on scalars on  $S^2$ .

The total heat kernel  $U_{\text{tot}}(t)$  is then

$$U_{\text{tot}}(t) = \text{Tr} \exp(-tL_{YM})_{S^2} \times \text{Tr} \exp(-tL_{YM})_{\mathbb{R}^2}$$
$$-2\text{Tr} \exp(-tL_{FP})_{S^2} \times \text{Tr} \exp(-tL_{FP})_{\mathbb{R}^2} \qquad (4.27)$$

Using the heat trace expressions (4.5), (4.6), (4.11), (4.18), this gives the result

$$U_{\text{tot}}(t) = \int_{\mathbb{R}^{2}} dx (4\pi t)^{-1} \left\{ \left[ 2\Theta_{1}\left(\frac{t}{R^{2}}\right) + \Theta_{0}\left(\frac{t}{R^{2}}\right) \right] \\ \times \left( \frac{tHR^{-2}}{\sinh(tHR^{-2}/2)} \left\{ 4e^{-t/R^{2}} \left[ 2 + 4\sinh^{2}\left(\frac{tHR^{-2}}{2}\right) \right] - 4 \right\} + 8e^{-t/R^{2}} - 4 \right) \\ -4 \left[ 1 + \frac{tHR^{-2}}{\sinh(tHR^{-2}/2)} \right] \right\}$$
(4.28)

# 4.7 Yang-Mills on $S^1 \times S^1 \times S^2$ with non-zero chromomagnetic field on $S^2$

Another allowable configuration is to let the chromomagnetic field lie along  $S^2$ . In this case, the total heat kernel is given by

$$U_{\text{tot}}(t) = \text{Tr} \exp(-tL_{YM})_{S^2} \times \text{Tr} \exp(-tL_{YM})_{S^1 \times S^1}$$
$$-2\text{Tr} \exp(-tL_{FP})_{S^2} \times \text{Tr} \exp(-tL_{FP})_{S^1 \times S^1}$$
(4.29)

Using the  $T^2$  heat trace expressions (4.3),(4.4) with H = 0, and the  $S^2$  heat trace expressions (4.11)-(4.17) calculated in Chapter 4, this gives us the following results:

For H = 0,

$$U_{\text{tot}}(t) = 8S\left(\frac{t}{r_1^2}\right)S\left(\frac{t}{r_2^2}\right) \times \left\{ \left[2e^{-t/R^2} - 1\right] \left[2\Theta_1\left(\frac{t}{R^2}\right) + \Theta_0\left(\frac{t}{R^2}\right)\right] - 1 \right\}. \quad (4.30)$$

For H = 1,

$$U_{\text{tot}}(t) = 8S\left(\frac{t}{r_1^2}\right)S\left(\frac{t}{r_2^2}\right) \\ \times \left\{3e^{-(9/4)t/R^2}\Phi_0\left(\frac{t}{R^2}\right) + (4\cosh(t/R^2)e^{-t/R^2} - 2)\Theta_1\left(\frac{t}{R^2}\right) \\ -2e^{-2t/R^2}\Theta_0\left(\frac{t}{R^2}\right) - 6e^{-9t/4R^2}\right\}.$$
(4.31)

For H = 2,

$$U_{\text{tot}}(t) = 4S\left(\frac{t}{r_1^2}\right)S\left(\frac{t}{r_2^2}\right)$$
$$\times \left\{ \left(4e^{-t/R^2}\cosh(2t/R^2) - 4\right)\left[2\Theta_1\left(\frac{t}{R^2}\right) + \Theta_0\left(\frac{t}{R^2}\right)\right] + 2e^{t/R^2} - 6e^{-3t/R^2} \right\}.$$
(4.32)

For H = 3,

$$U_{\text{tot}}(t) = 4S\left(\frac{t}{r_1^2}\right)S\left(\frac{t}{r_2^2}\right)$$

$$\times \left[ (2e^{7t/4R^2} + 10e^{-19t/4R^2} - 6e^{t/4R^2})\Phi_0\left(\frac{t}{R^2}\right) \right]$$

$$\left(8\cosh(3t/R^2)e^{-t/R^2} - 4\Theta_1\left(\frac{t}{R^2}\right) + 4\Theta_0\left(\frac{t}{R^2}\right) \right]$$

$$-20e^{-8t/R^2} - 20e^{-11t/R^2} - 8e^{-6t/R^2} - 8e^{-10t/R^2}) + 6e^{-3t/4R^2} \left] \quad (4.33)$$

For H = 4,

$$U_{\text{tot}}(t) = 4S\left(\frac{t}{r_1^2}\right)S\left(\frac{t}{r_2^2}\right) \\ \times \left\{ \left[ 4e^{-t/R^2}\cosh(4t/R^2) - 2 \right] \left[ 4\Theta_1\left(\frac{t}{R^2}\right) + \Theta_0\left(\frac{t}{R^2}\right) \right] \right. \\ \left. - 6e^{-7t/R^2} + 10e^{-11t/R^2} + 6e^{-2t/R^2} \right\}.$$
(4.34)

For H odd,  $H \ge 5$ ,

$$U_{\text{tot}}(t) = 4S\left(\frac{t}{r_1^2}\right) S\left(\frac{t}{r_2^2}\right) \\ \times \left\{ (8\cosh(tH/R^2)e^{-t/R^2} - 4)\Theta_1\left(\frac{t}{R^2}\right) + \left[ (2H-2) - (4H+4)\cosh(tH/R^2)e^{-t/R^2} \right] \Theta_0\left(\frac{t}{R^2}\right) \right. \\ \left. \left[ (4H+8)\cosh(tH/R^2)e^{-t/R^2} - 8e^{-tH/R^2}e^{-t/R^2} - 2H \right] e^{t/4R^2} \Phi_0\left(\frac{t}{R^2}\right) - 2e^{-t(H+3/4)/R^2} (H+2) \sum_{l=1}^{\frac{H}{2}+\frac{1}{2}} e^{-tl^2/R^2} \\ \left. - 2e^{t(H-3/4)/R^2} (H-2) \sum_{l=1}^{\frac{H}{2}-\frac{3}{2}} e^{-tl^2/R^2} - 2e^{-t(H+1)/R^2} \sum_{l=1}^{\frac{H}{2}+\frac{1}{2}} (2l+H-1)e^{-tl(l+1)/R^2} \\ \left. - 2e^{t(H-1)/R^2} \sum_{l=1}^{\frac{H}{2}-\frac{3}{2}} (2l+3-H)e^{-tl(l+1)/R^2} + 2He^{t/4R^2} \sum_{l=1}^{\frac{H}{2}-\frac{1}{2}} e^{-tl^2/R^2} + \sum_{l=1}^{\frac{H}{2}-\frac{1}{2}} (4l+2-2H)e^{-tl(l+1)/R^2} \right\}$$

$$(4.35)$$

For H even,  $H \ge 6$ ,

$$U_{\text{tot}}(t) = 4S\left(\frac{t}{r_1^2}\right)S\left(\frac{t}{r_2^2}\right) \\ \times \left\{ \left(4e^{-t/R^2}\cosh(tH/R^2) - 2\right)\left[2\Theta_1\left(\frac{t}{R^2}\right) - \Theta_0\left(\frac{t}{R^2}\right)\right] \right. \\ \left. -2e^{t(H-1)/R^2}\sum_{l=1}^{\frac{H}{2}-2}(2l+1)e^{-tl(l+1)} - 2e^{-t(H+1)/R^2}\sum_{l=1}^{\frac{H}{2}}(2l+1)e^{-tl(l+1)} \right. \\ \left. +2\sum_{l=1}^{\frac{H}{2}-1}(2l+1)e^{-tl(l+1)}\right\}$$
(4.36)

### 4.8 Stability

When the heat traces above contain an exponential that grows or stays constant with t, then the Yang-Mills vacuum will be unstable, causing the configuration with constant chromomagnetic field to decay into another state. However, if all exponentials are decreasing, then the vacuum will be stable.

If the chromomagnetic field is polarized along  $S^2$  on the manifold, the stability of the the constant chromomagnetic state will depend on the strength H of the chromomagnetic field and the radius R of the sphere  $S^2$ . The case H = 1 will always be stable, but H = 2, and H = 3 will not be stable for any radii. In the case of H = 1, the lowest eigenvalue is given by

$$\lambda_{\min} = \frac{1}{R^2} \left(\frac{5}{4}\right) \,, \tag{4.37}$$

which implies that the vacuum stabilizes for H = 1. Similarly, for H = 2, the minimum eigenvalue is

$$\lambda_{\min} = -\frac{1}{R^2} \,, \tag{4.38}$$

and the vacuum is unstable. For H = 3, the minimum eigenvalue is

$$\lambda_{\min} = -\frac{3}{4R^2} \tag{4.39}$$

and the vacuum is unstable. For  $H \ge 4$ , the lowest mode will correspond to the eigenvalue

$$\lambda_{\min} = \frac{1}{R^2} \left[ \left( \frac{H}{2} - 1 \right) \frac{H}{2} - H + 1 \right] \,. \tag{4.40}$$

The vacuum will be stable when this eigenvalue is positive, which occurs when the condition

$$\frac{H^2}{4} - \frac{3H}{2} + 1 \ge 0 \tag{4.41}$$

is satisfied. This occurs for

$$H \ge 6. \tag{4.42}$$

Thus, the vacuum is unstable for H = 2, 3, 4, 5 and stable for H = 0, 1 and  $H \ge 6$ . Because H is the dimensionless parameter relating to the magnetic field M,  $H = MR^2$ , this implies that we can make a configuration stable by increasing either the magnetic field or the radius. A large radius would intuitively return us to the Saviddy flat-space case, but we instead see that it actually increases the lowest eigenvalue. The local behavior of these cases is the same, so this must be a topological phenomenon.

#### 4.9 Effective Action

If we consider the radii of the spheres to be variable, then the lowest energy state is state that minimizes the effective action, which will be a function of both the chromomagnetic field and the radius of the spheres. The one-loop effective action is written in terms of the heat kernel as

$$\Gamma_{(1)} = -\frac{1}{2} \frac{d}{dp} \left[ \frac{\mu^{2p}}{\Gamma(p)} \int_0^\infty dt \, t^{p-1} U_{\text{tot}}(t) \right]_{p=0}$$
(4.43)

The effective action, then, to first order is

$$\Gamma = S + \hbar \Gamma_{(1)} \tag{4.44}$$

where S is the classical action

$$S = -\frac{1}{8e^2} \int_M dx \operatorname{tr} \mathcal{F}^{\mu\nu} \mathcal{F}_{\mu\nu} , \qquad (4.45)$$

which in our case can be integrated over the manifold to give

$$S = \frac{H^2}{8e^2 R^2} \text{vol}(M) \,. \tag{4.46}$$

Calculating the effective action and finding a global minimum for all R and H will reveal the vacuum with minimum energy.

### CHAPTER 5

### CONCLUSION

We have calculated the heat traces for pure Yang-Mills on products of spheres with a covariantly constant chromomagnetic field, and have shown that for a space with certain values of curvature and magnetic field, the covariantly constant chromomagnetic vacuum forms a local minimum of the effective action. This lends creedence to the possibility that the Savvidy-type vacuum, with a covariantly constant magnetic field will form an absolute minimum on the relevant spaces.

Contrary to expectations, the limit in which the radius of the sphere becomes infinite does not yield the standard flat-space results for eigenvalues of the Laplacian. Instead of having eigenvalues that are linear in the magnetic field, our results show that on the two-sphere, the lowest eigenvalue of the Laplacian will increase quadratically with the magnetic field and quadratically with the radius of the sphere, leading to the counter-intuitive result that the space most closely approximating the flat-space Saviddy vacuum will have a minimum eigenvalue that will be farthest from being unstable. This effect is a topological phenomenon that requires further study.

The next step in examining this model should be to calculate the full effective potential as a function of both the strength of the chromomagnetic field and the radius of the sphere. The absolute minimum of the effective potential would yield the absolute vacuum state of Yang-Mills. In our case, we would find a state that would be at least a local minimum of the vacuum, and possibly the absolute minimum.

# APPENDIX A

# EIGENVALUES AND DEGENERACIES OF $-\Delta$

In this section, we find the eigenfunctions and the corresponding eigenvalues  $\lambda_l$  of the operator  $-\Delta$ . They are given by regular solutions of the equation

$$\left\{-\frac{1}{R^2\sin\theta}\left[\sin\theta\partial_\theta^2 + \cos\theta\partial_\theta + \frac{1}{\sin\theta}(im - ik_j\cos\theta)^2\right] - \lambda_l\right\}u(\theta) = 0.$$
(A.1)

The label l on  $\lambda_l$  is only a label here. It's allowed values will be found later. Introducing the change of variables  $x = \cos \theta$ , the equation becomes

$$\left[\partial_x (1-x^2)\partial_x - \frac{1}{1-x^2}(m-kx)^2 + R^2\lambda\right]u(x) = 0.$$
 (A.2)

We may make the substitution

$$u(x) = (1-x)^{\alpha} (1+x)^{\beta} f(x), \qquad (A.3)$$

where

$$\alpha = \left| \frac{m-k}{2} \right|, \quad \beta = \left| \frac{m+k}{2} \right|. \tag{A.4}$$

to get the equation

$$\begin{cases} (1-x^2)\frac{d^2}{dx^2} - [(2\beta - 2\alpha) + (-2 - 2\alpha - 2\beta)x]\frac{d}{dx} \\ + \left[-\alpha - \beta - (\alpha + \beta)^2 + R^2\lambda\right] \end{cases} f(x) = 0. \quad (A.5)$$

By switching variables to  $z = \frac{1-x}{2}$ , we obtain the hypergeometric equation

$$\left\{ z(1-z)\frac{d^2}{dz^2} + \left[ (1+2\alpha) - 2(1+\alpha+\beta)z \right] \frac{d}{dz} + \left[ -\alpha - \beta - (\alpha+\beta)^2 + R^2\lambda \right] \right\} f(z) = 0.$$
 (A.6)

Finally, introducing the notation

$$a^{\pm} = \frac{1}{2} + \alpha + \beta \pm \frac{1}{2} (1 + 4R^2 \lambda)^{1/2}, \qquad (A.7)$$

we obtain the solution

$$f(z) = F(a^+, a^-; 1 + |m - k|; z) , \qquad (A.8)$$

where the function F is the hypergeometric function [10]

$$F(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{n! (c)_n} z^n \,. \tag{A.9}$$

The notation  $(a)_n$  denotes the Pochhammer symbol

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}.$$
 (A.10)

In our case, it is easy to show that c > 0 and  $a + b - c \ge 0$ , so the expression (A.9) will diverge at |z| = 1 unless the series terminates [10]. Regular solutions will exist only in the degenerate case when the series terminates and the hypergeometric function becomes a polynomial. This happens when at least one of the first two arguments of F is a negative integer. Thus, the only regular solutions occur when

$$a^{-} = \frac{1}{2} + \alpha + \beta - \frac{1}{2}(1 + 4R^{2}\lambda)^{1/2} = -q, \quad q = 0, 1, 2, \dots$$
 (A.11)

where q is a non-negative integer. This yields the eigenvalues

$$\lambda = \frac{1}{R^2} (\alpha + \beta + q)(\alpha + \beta + q + 1).$$
 (A.12)

The quantity  $\alpha + \beta$  takes on the values

$$\alpha + \beta = \frac{1}{2} \{ |m+k| + |m-k| \} = \max \{ |m|, |k| \} .$$
 (A.13)

Then it can be directly seen from (A.12) that the lowest eigenvalue  $\lambda_0$  is

$$\lambda_0 = \frac{1}{R^2} |k| (|k|+1).$$
 (A.14)

For integer values of k, the quantity  $\alpha + \beta$  is always integer, and so the eigenvalues are

$$\lambda_l = \frac{1}{R^2} (|k| + l) (|k| + l + 1), \qquad l = 0, 1, 2, \dots$$
 (A.15)

Degeneracies of these eigenvalues can be counted using (A.12), (A.15), and the fact that m is an integer. For any given value of l, the 2|k| + 1 cases  $|m| \leq |k|$ correspond to q = l. The 2l cases  $m = \pm (|k| + l), \pm (|k| + l - 1), \ldots, \pm (|k| + 1)$ correspond to  $q = 0, 1, \ldots, l - 1$ , respectively. Counting these cases, the total degeneracy  $d_l$  of  $\lambda_l$  from (A.15) is

$$d_l = 2(|k|+l) + 1. (A.16)$$

Now consider the case where k is a half-integer. Then there are two cases: when the quantity  $\alpha + \beta$  is a half-integer, and when  $\alpha + \beta$  is an integer.

First consider the case when  $\alpha + \beta$  is half-integer. The eigenvalues are then

$$\lambda_l = \frac{1}{R^2} (|k| + l) (|k| + l + 1), \qquad l = 0, 1, 2, \dots$$
 (A.17)

In this case, we must have  $\alpha + \beta = |k|$ , which corresponds to the cases  $|m| \le |k|$ . The number of integer values of m that satisfy this inequality is

$$d_l = 2|k|. \tag{A.18}$$

In the second case,  $\alpha + \beta$  is integer, in which case  $\alpha + \beta = |m|$ . Because  $\alpha + \beta$  and q are integers, the eigenvalues written in terms of k are

$$\lambda_l = \frac{1}{R^2} \left( |k| + l + \frac{1}{2} \right) \left( |k| + l + \frac{3}{2} \right) , \qquad l = 0, 1, 2, \dots$$
 (A.19)

The 2l + 2 cases

$$m = \pm \left( |k| + \frac{1}{2} \right), \pm \left( |k| + \frac{1}{2} + 1 \right), \dots, \pm \left( |k| + \frac{1}{2} + 2 \right)$$
(A.20)

correspond to q = l, l - 1, ..., 0, respectively. Counting these gives the degeneracies

$$d_l = 2l + 2. \tag{A.21}$$

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