Truncated variation, upward truncated variation and downward truncated variation of Brownian motion with drift - their characteristics and applications

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Abstract

In [6] for $c > 0$ we defined truncated variation, TV_μ^c, of Brownian motion with drift, $W_t = B_t + \mu t, t \geq 0$, where (B_t) is a standard Brownian motion. In this article we define two related quantities - upward truncated variation

$$UTV_\mu^c[a,b] = \sup_n \sup_{a \leq t_1 < s_1 < \ldots < t_n < s_n \leq b} \sum_{i=1}^n \max\{W_{s_i} - W_{t_i} - c, 0\}$$

and, analogously, downward truncated variation

$$DTV_\mu^c[a,b] = \sup_n \sup_{a \leq t_1 < s_1 < \ldots < t_n < s_n \leq b} \sum_{i=1}^n \max\{W_{t_i} - W_{s_i} - c, 0\}.$$

We prove that exponential moments of the above quantities are finite (in opposite to the regular variation, corresponding to $c = 0$, which is infinite almost surely). We present estimates of the expected value of UTV_μ^c up to universal constants.

As an application we give some estimates of the maximal possible gain from trading a financial asset in the presence of flat commission (proportional to the value of the transaction) when the dynamics of the prices of the asset follows a geometric Browniam motion process. In the presented estimates upward truncated variation appears naturally.

1 Introduction

Let $(B_t, t \geq 0)$ be a standard Brownian motion, and $W_t = B_t + \mu t$ be a Brownian motion with drift μ.

In [6] truncated variation at the level $c > 0$ of Brownian motion with drift μ on the interval $[a, b]$ was defined as

$$TV_\mu^c[a,b] := \sup_n \sup_{a \le t_1 \le \cdots \le t_n \le b} \sum_{i=1}^{n-1} \max\left\{\left|W_{t_{i+1}} - W_{t_i}\right| - c, 0\right\}.$$

(Technical remark: for $a > b$ we set $TV_\mu^c[a,b] = 0$.)

There were also proved estimates of $\mathbf{E}TV_\mu^c[0,T]$ up to universal constants. Using similar techniques as in [6] we will prove existence of finite exponential moments of $TV_\mu^c[0,T]$, $\mathbf{E}\exp\left(\alpha TV_\mu^c[0,T]\right)$, for any α, $T > 0$.

Further we will consider two related quantities

- upward truncated variation, defined as

$$UTV_\mu^c[a,b] := \sup_n \sup_{a \le t_1 < s_1 < \cdots < t_n < s_n \le b} \sum_{i=1}^{n} \max\left\{W_{s_i} - W_{t_i} - c, 0\right\}$$

- and, analogously, downward truncated variation, defined as

$$DTV_\mu^c[a,b] := \sup_n \sup_{a \le t_1 < s_1 < \cdots < t_n < s_n \le b} \sum_{i=1}^{n} \max\left\{W_{t_i} - W_{s_i} - c, 0\right\}.$$

It is easy to see that all three above defined quantities have the following properties, which we state only for the truncated variation

- shift invariance property in distributions:

$$\mathcal{L}\left(TV_\mu^c[a,b]\right) = \mathcal{L}\left(TV_\mu^c[a+\Delta, b+\Delta]\right)$$

- superadditivity property: for any numbers $a \le a_1 < a_2 < \cdots < a_n \le b$

$$TV_\mu^c[a,b] \ge \sum_{i=1}^{n-1} TV_\mu^c[a_i, a_{i+1}].$$

It is also easy to see that the following relations hold

$$\begin{aligned} TV_\mu^c[0,T] &\ge UTV_\mu^c[0,T], & (1)\\ TV_\mu^c[0,T] &\ge DTV_\mu^c[0,T], & (2)\\ TV_\mu^c[0,T] &\le UTV_\mu^c[0,T] + DTV_\mu^c[0,T], & \\ UTV_\mu^c[0,T] &= DTV_{-\mu}^c[0,T]. & (3) \end{aligned}$$

By (3) all estimates proved for upward truncated variation have analogs for downward truncated variation.

Analogously as in [6] we will prove some estimates of $\mathbf{E}UTV_\mu^c[0,T]$ (and thus for $\mathbf{E}DTV_\mu^c[0,T]$) up to universal constants. Unfortunatelly, the presented estimates involve expected values of some other related variables.

Remark 1.1. *In order to shorten the proofs we did not put much stress on obtaining the best possible constants in the presented estimates.*

Remark 1.2. *K. Oleszkiewicz pointed out that it would be also interesting to have estimates for higher moments of the defined quantities. However, the author presumes that there are other methods than these used in this paper needed to obtain such estimates.*

Remark 1.3. *A. N. Chuprunov pointed to the author that it would be also interesting to have estimates of quadratic truncated variation, which one may define as*

$$QTV_{\mu}^{c}[a,b] := \sup_{n} \sup_{a \le t_1 \le \cdots \le t_n \le b} \sum_{i=1}^{n-1} \max\left\{ \left| W_{t_{i+1}} - W_{t_i} \right|^2 - c^2, 0 \right\}.$$

Remark 1.4. *Similar concept of truncation (or* shirinking*) of random variables on Hilbert spaces investigated Z. Jurek in series of his papers beginning with [2], [3], which now evolved in the theory of selfdecomposable distriutions (see e.g. [4]).*

2 Existence of exponential moments of truncated variation

Let us start with the existence of finite exponential moments of $TV_{\mu}^{c}[0,T]$. To prove this let us define

$$T_c = \inf\left\{ t \ge 0 : \sup_{0 \le s \le t} W_s \ge W_t + c \right\},$$

further let $T_c^{\sup}$ be the last instant when the maximum of W_t on $[0, T_c]$ is attained, and let $T_c^{\inf} \le T_c^{\sup}$ be such that $W_{T_c^{\inf}} = \inf_{0 \le s \le T_c^{\sup}} W_s$.

Let us fix $\alpha > 0$ and let $\delta > 0$ be such a small number that

$$1 - \mathbf{E} \exp\left(\alpha \sup_{0 \le t \le T} W_t + \alpha c \right) P\left(T_c < \delta \right) > 0.$$

By definition of T_c and $T_c^{\inf}$ we have $W_{T_c^{\inf}} > -c$ and $W_{T_c^{\sup}} - W_{T_c^{\inf}} - c \le W_{T_c^{\sup}}$. Now, by Lemma 1, Lemma 2 in [6] and independence of $W_t - W_{T_c}, t \ge T_c$, and T_c (strong Markov property of Brownian motion) for any $M > 0$ we have

$$\begin{aligned}
\mathbf{E}\exp\left(\alpha TV^c_\mu[0,T]\wedge M\right) &\le \mathbf{E}\exp\left(\alpha W_{T^{\sup}_c}+\alpha c+\alpha TV^c_\mu[T_c,T]\wedge M\right)\\
&\le \mathbf{E}\exp\left(\alpha W_{T^{\sup}_c}+\alpha c\right)\mathbf{E}\exp\left[\alpha TV^c_\mu[T_c,T]\wedge M; T_c<\delta\right]\\
&+ \mathbf{E}\exp\left(\alpha W_{T^{\sup}_c}+\alpha c\right)\mathbf{E}\exp\left[\alpha TV^c_\mu[T_c,T]\wedge M; T_c\ge\delta\right]\\
&\le \mathbf{E}\exp\left(\alpha W_{T^{\sup}_c}+\alpha c\right)\mathbf{E}\exp\left[\alpha TV^c_\mu[T_c,T+T_c]\wedge M; T_c<\delta\right]\\
&+ \mathbf{E}\exp\left(\alpha W_{T^{\sup}_c}+\alpha c\right)\mathbf{E}\exp\left[\alpha TV^c_\mu[T_c,T+T_c-\delta]\wedge M; T_c\ge\delta\right]\\
&\le \mathbf{E}\exp\left(\alpha\sup_{0\le t\le T}W_t+\alpha c\right)\mathbf{E}\exp\left(\alpha TV^c_\mu[0,T]\wedge M\right)P\left(T_c<\delta\right)\\
&+ \mathbf{E}\exp\left(\alpha\sup_{0\le t\le T}W_t+\alpha c\right)\mathbf{E}\exp\left(\alpha TV^c_\mu[0,T-\delta]\wedge M\right)P\left(T_c\ge\delta\right).
\end{aligned}$$

From the above we have

$$\begin{aligned}
&\mathbf{E}\exp\left(\alpha TV^c_\mu[0,T]\wedge M\right)\\
&\le \frac{\mathbf{E}\exp\left(\alpha\sup_{0\le t\le T}W_t+\alpha c\right)P\left(T_c\ge\delta\right)}{1-\mathbf{E}\exp\left(\alpha\sup_{0\le t\le T}W_t+\alpha c\right)P\left(T_c<\delta\right)}\mathbf{E}\exp\left(\alpha TV^c_\mu[0,T-\delta]\wedge M\right).
\end{aligned}$$

Similarly

$$\begin{aligned}
&\mathbf{E}\exp\left(\alpha TV^c_\mu[0,T-\delta]\wedge M\right)\\
&\le \frac{\mathbf{E}\exp\left(\alpha\sup_{0\le t\le T}W_t+\alpha c\right)P\left(T_c\ge\delta\right)}{1-\mathbf{E}\exp\left(\alpha\sup_{0\le t\le T}W_t+\alpha c\right)P\left(T_c<\delta\right)}\mathbf{E}\exp\left(\alpha TV^c_\mu[0,T-2\delta]\wedge M\right).
\end{aligned}$$

Iterating and putting together the above inequalities we finally obtain

$$\mathbf{E}\exp\left(\alpha TV^c_\mu[0,T]\wedge M\right)\le\left(\frac{\mathbf{E}\exp\left(\alpha\sup_{0\le t\le T}W_t+\alpha c\right)P\left(T_c\ge\delta\right)}{1-\mathbf{E}\exp\left(\alpha\sup_{0\le t\le T}W_t+\alpha c\right)P\left(T_c<\delta\right)}\right)^{\lceil T/\delta\rceil}.$$

Letting $M\to\infty$ we get $\mathbf{E}\exp\left(\alpha TV^c_\mu[0,T]\right)<+\infty$.

By (1) and (2) we obtain the finiteness of exponential moments of $UTV^c_\mu[0,T]$ and $DTV^c_\mu[0,T]$ as well.

3 Estimates of expected value of upward and downward truncated variation

3.1 Preparatory lemmas

In order to obtain estimates of $\mathbf{E}UTV^c_\mu[0,T]$ (and analogously $\mathbf{E}DTV^c_\mu[0,T]$) we will use similar techniques as in [6]. Due to typographical reasons let us introduce notation $\max\{x,0\}=:(x)_+$.

We will need the following analogon of Lemma 2 from [6]:

Lemma 3.1. *We have the following identity*

$$UTV^c_\mu[0,T] = \sup_{0\le t<s\le T_c\wedge T} (W_s - W_t - c)_+ + UTV^c_\mu[T_c,T]. \tag{4}$$

Proof. Let $0 \le t_1 < s_1 < t_2 < s_2 ... < t_n < s_n \le T$ be numbers from the interval $[0,T]$.

We will prove that

$$\sum_{i=1}^n (W_{s_i} - W_{t_i} - c)_+ \le \sup_{0\le t<s\le T_c\wedge T} (W_s - W_t - c)_+ + UTV^c_\mu[T_c,T]. \tag{5}$$

Let n_0 be the greatest number such that $s_{n_0} < T_c$ and let us assume that $n_0 < n$ and $t_{n_0+1} < T_c$.

Let us consider several cases.

- $W_{t_{n_0+1}} \ge W_{T_c}$. In this case

$$\left(W_{s_{n_0+1}} - W_{t_{n_0+1}} - c\right)_+ \le \left(W_{s_{n_0+1}} - W_{T_c} - c\right)_+.$$

and

$$\sum_{i=1}^n (W_{s_i} - W_{t_i} - c)_+ \le \sum_{i=1}^{n_0} (W_{s_i} - W_{t_i} - c)_+ + \left(W_{s_{n_0+1}} - W_{T_c} - c\right)_+ + \sum_{i=n_0+2}^{n} (W_{s_i} - W_{t_i} - c)_+. \tag{6}$$

- $W_{t_{n_0+1}} < W_{T_c}$ and $W_{s_{n_0+1}} \le W_{T_c^{\sup}}$. In this case $t_{n_0+1} < T_c^{\sup}$ (since for $T_c^{\sup} < t < T_c$, $W_t > W_{T_c}$) so

$$\left(W_{s_{n_0+1}} - W_{t_{n_0+1}} - c\right)_+ \le \left(W_{T_c^{\sup}} - W_{t_{n_0+1}} - c\right)_+$$

and

$$\sum_{i=1}^n (W_{s_i} - W_{t_i} - c)_+ \le \sum_{i=1}^{n_0} (W_{s_i} - W_{t_i} - c)_+ + \left(W_{T_c^{\sup}} - W_{t_{n_0+1}} - c\right)_+ + \sum_{i=n_0+2}^{n} (W_{s_i} - W_{t_i} - c)_+. \tag{7}$$

- $W_{t_{n_0+1}} < W_{T_c}$ and $W_{s_{n_0+1}} > W_{T_c^{\sup}} = W_{T_c} + c$. In this case

$$\begin{aligned}\left(W_{s_{n_0+1}} - W_{t_{n_0+1}} - c\right)_+ &= W_{s_{n_0+1}} - W_{t_{n_0+1}} - c\\ &= W_{T_c^{\sup}} - W_{t_{n_0+1}} - c + W_{s_{n_0+1}} - W_{T_c^{\sup}}\\ &= W_{T_c^{\sup}} - W_{t_{n_0+1}} - c + W_{s_{n_0+1}} - W_{T_c} - c\\ &= \left(W_{T_c^{\sup}} - W_{t_{n_0+1}} - c\right)_+ + \left(W_{s_{n_0+1}} - W_{T_c} - c\right)_+\end{aligned}$$

and

$$\sum_{i=1}^{n}(W_{s_i}-W_{t_i}-c)_+ \le \sum_{i=1}^{n_0}(W_{s_i}-W_{t_i}-c)_+ + \left(W_{T_c^{\sup}}-W_{t_{n_0+1}}-c\right)_+ \\ + \left(W_{s_{n_0+1}}-W_{T_c}-c\right)_+ + \sum_{i=n_0+2}^{n}(W_{s_i}-W_{t_i}-c)_+ . \tag{8}$$

Thus for $t_{n_0+1} < T_c$ inequality (6), (7) or (8) holds and we may assume, adding in the case $t_{n_0+1} < T_c$ new terms in the partition and renaming the old ones, that

$$\begin{aligned} 0 &\le t_1 < s_1 < ... < t_{n_0} < s_{n_0} \le T_c, \\ T_c &\le t_{n_0+1} < s_{n_0+1} < ... < t_n < s_n \le T. \end{aligned}$$

In order to prove (5) without loss of generality we may assume that for any $1 \le i \le n_0$, $(W_{s_i}-W_{t_i}-c)_+ > 0$ (otherwise we may omit the summand $(W_{s_i}-W_{t_i}-c)_+$). From definition of T_c we have that for any $1 \le i \le n_0-1$, $W_{s_i}-W_{t_{i+1}} < c$, so

$$\begin{aligned} &(W_{s_i}-W_{t_i}-c)_+ + \left(W_{s_{i+1}}-W_{t_{i+1}}-c\right)_+ \\ &= W_{s_i}-W_{t_i}-c+W_{s_{i+1}}-W_{t_{i+1}}-c \\ &= W_{s_{i+1}}-W_{t_i}-c+\left(W_{s_i}-W_{t_{i+1}}-c\right) < W_{s_{i+1}}-W_{t_i}-c. \end{aligned}$$

Iterating the above inequality, we obtain

$$\sum_{i=1}^{n_0}(W_{s_i}-W_{t_i}-c)_+ \le W_{s_{n_0}}-W_{t_1}-c \le \sup_{0\le t<s\le T_c\wedge T}(W_s-W_t-c)_+ .$$

This, together with the obvious inequality

$$\sum_{i=n_0+1}^{n}(W_{s_i}-W_{t_i}-c)_+ \le \ UTV_{\mu}^{c}[T_c,T]$$

proves (5). Taking supremum over all partitions $0 \le t_1 < s_1 < t_2 < s_2 < ... < t_n < s_n \le T$ we finally get

$$UTV_{\mu}^{c}[0,T] \le \sup_{0\le t<s\le T_c\wedge T}(W_s-W_t-c)_+ + UTV_{\mu}^{c}[T_c,T] .$$

Since the opposite inequality is obvious, we finally get (4). □

Let us now define some auxiliary variables. Let $T_c^{(0)} \equiv 0$ and let $T_c^{(i)}, i = 1, 2, ...$ be defined recursively as

$$T_c^{(i)} = \inf\left\{ t > T_c^{(i-1)} : \sup_{T_c^{(i-1)}\le s\le t} W_s \ge W_t + c \right\}.$$

(notice that $T_c^{(1)} = T_c$). We define a new variable

$$UTV_\mu^c(T) := \sum_{i=1}^{\infty} e^{-T_c^{(i-1)}/T} \sup_{T_c^{(i-1)} \le t < s \le T_c^{(i)} \wedge \left(T_c^{(i-1)} + T\right)} (W_s - W_t - c)_+ .$$

We have the following

Lemma 3.2. *The variables $UTV_\mu^c[0,T]$ and $UTV_\mu^c(T)$ are related by the following relations*

$$UTV_\mu^c[0,T] \le e UTV_\mu^c(T) \tag{9}$$

$$UTV_\mu^c[0,T] \succeq \frac{1-e^{-1}}{2} UTV_\mu^c(T) \tag{10}$$

where the first relation holds almost surely and the second holds in the sense of stochastic domination i.e. for every $y \ge 0$, $P\left(UTV_\mu^c[0,T] \ge y\right) \ge P\left(\frac{1-e^{-1}}{2} UTV_\mu^c(T) \ge y\right)$.

Proof. By the previous lemma, we have

$$\begin{aligned} UTV_\mu^c[0,T] &= \sup_{0 \le t < s \le T_c^{(1)} \wedge T} (W_s - W_t - c)_+ + UTV_\mu^c\left[T_c^{(1)}, T\right] \\ &= \sup_{0 \le t < s \le T_c^{(1)} \wedge T} (W_s - W_t - c)_+ + \sup_{T_c^{(1)} \le t < s \le T_c^{(2)} \wedge T} (W_s - W_t - c)_+ \\ &\quad + UTV_\mu^c\left[T_c^{(2)}, T\right] \\ &= \ldots = \sum_{i \ge 1 : T_c^{(i-1)} \le T} \sup_{T_c^{(i-1)} \le t < s \le T_c^{(i)} \wedge T} (W_s - W_t - c)_+ . \end{aligned} \tag{11}$$

From (11) we almost immediately get (9)

$$\begin{aligned} UTV_\mu^c[0,T] &= \sum_{i \ge 1 : T_c^{(i-1)} \le T} \sup_{T_c^{(i-1)} \le t < s \le T_c^{(i)} \wedge T} (W_s - W_t - c)_+ \\ &\le \sum_{i=1}^{\infty} e^{1 - T_c^{(i-1)}/T} \sup_{T_c^{(i-1)} \le t < s \le T_c^{(i)} \wedge \left(T_c^{(i-1)} + T\right)} (W_s - W_t - c)_+ \\ &= e UTV_\mu^c(T) . \end{aligned}$$

In order to prove the second relation let $i_0 \ge 1$ be the greatest indice such that $T_c^{(i_0 - 1)} < T$ and let us consider the term

$$A = \sup_{T_c^{(i_0-1)} \le t < s \le T_c^{(i_0)} \wedge \left(T_c^{(i_0-1)} + T\right)} (W_s - W_t - c)_+ .$$

If $i_0 = 1$ then $A = \sup_{0 \le t < s \le T_c^{(1)} \wedge T} (W_s - W_t - c, 0)_+$, otherwise A is independent from $B = \sup_{0 \le t < s \le T_c^{(1)} \wedge T} (W_s - W_t - c, 0)_+$ but has the same distribution as B.

By (11) we have

$$UTV_{\mu}^{c}[0,T] = \sum_{i\geq 1: T_c^{(i-1)}\leq T} \sup_{T_c^{(i-1)}\leq t<s\leq T_c^{(i)}\wedge T} (W_s - W_t - c)_+ \qquad (12)$$

$$= \sum_{i=1}^{i_0-1} \sup_{T_c^{(i-1)}\leq t<s\leq T_c^{(i)}} (W_s - W_t - c)_+ + \sup_{T_c^{(i_0-1)}\leq t<s\leq T} (W_s - W_t - c)_+ .$$

In both cases ($i_0 = 1$ and $i_0 > 1$) $2UTV_{\mu}^{c}[0,T]$ stochastically dominates the sum

$$S_1 = \sum_{i=1}^{i_0} e^{-T_c^{(i-1)}/T} \sup_{T_c^{(i-1)}\leq t<s\leq T_c^{(i)}\wedge\left(T_c^{(i-1)}+T\right)} (W_s - W_t - c)_+ .$$

($\sum_{i=1}^{i_0-1}\sup_{T_c^{(i-1)}\leq t<s\leq T_c^{(i)}} (W_s - W_t - c)_+$ dominates the first $i_0 - 1$ terms in the above sum and B, which appears in the sum (12) dominates A.) Similarly, define i_k recursively as the greatest integer such that $T_c^{(i_k-1)} < T_c^{(i_{k-1})} + T$ and

$$S_k = \sum_{i=i_{k-1}+1}^{i_k} \exp\left(-\frac{T_c^{(i-1)} - T_c^{(i_{k-1})}}{T}\right) \sup_{T_c^{(i-1)}\leq t<s\leq T_c^{(i)}\wedge\left(T_c^{(i-1)}+T\right)} (W_s - W_t - c)_+ .$$

S_k is independent from $S_1, ..., S_{k-1}$, moreover it has the same distribution as S_1 and

$$UTV_{\mu}^{c}(T) = \sum_{k=1}^{\infty} e^{-T_c^{(i_{k-1})}/T} S_k.$$

By definition of $i_k, T_c^{(i_k)} \geq T_c^{(i_{k-1})} + T$, thus we have $T_c^{(i_k)} \geq (k-1)T$. Now, since $2UTV_{\mu}^{c}[0,T] \succeq S_k$, $k = 1, 2, ...$, we have that

$$\frac{2}{1-e^{-1}} UTV_{\mu}^{c}[0,T] = \sum_{k=1}^{\infty} e^{-(k-1)} 2UTV_{\mu}^{c}[0,T]$$

$$\succeq \sum_{k=1}^{\infty} e^{-T_c^{(i_{k-1})}/T} 2UTV_{\mu}^{c}[0,T]$$

$$\succeq \sum_{k=1}^{\infty} e^{-T_c^{(i_{k-1})}/T} S_k = UTV_{\mu}^{c}(T).$$

which proves (10). □

Next, let us state a refinement of Lemma 3 from [6]:

Lemma 3.3. *For any μ and $c > 0$*

$$P\left(T_c < \frac{1}{3}\mathbf{E}T_c\right) \leq \frac{7}{9}.$$

Proof. The proof follows exactly as in [6], since one can show that for any real μ

$$\frac{(\mathbf{E}T_c)^2}{\mathbf{E}T_c^2} = \frac{1}{2}\frac{\left(e^{2\mu c} - 1 - 2\mu c\right)^2}{e^{4\mu c} - 6e^{2\mu c}\mu c + e^{2\mu c} + 2\mu^2 c^2 - 2} \geq \frac{1}{2}$$

and, by the Paley-Zygmund inequality we obtain

$$P\left(T_c \geq \frac{1}{3}\mathbf{E}T_c\right) \geq \left(1 - \frac{1}{3}\right)^2 \frac{(\mathbf{E}T_c)^2}{\mathbf{E}T_c^2} \geq \frac{4}{9}\frac{1}{2} = \frac{2}{9}$$

and

$$P\left(T_c < \frac{1}{3}\mathbf{E}T_c\right) = 1 - P\left(T_c \geq \frac{1}{3}\mathbf{E}T_c\right) \leq \frac{7}{9}.$$

□

3.2 Estimates for long and short time intervals

Now we are ready to prove estimates of expected value of $UTV_\mu^c[0,T]$ for long and short time intervals ($T \geq \frac{1}{3}\mathbf{E}T_c$ and $T < \frac{1}{3}\mathbf{E}T_c$ respectively). We have

Theorem 3.4. *For any $T \geq \frac{1}{3}\mathbf{E}T_c$ we have*

$$\begin{aligned} 0.3\frac{T}{\mathbf{E}T_c}\mathbf{E}\sup_{0\leq t<s\leq T_c\wedge T}(W_s - W_t - c)_+ &\leq \mathbf{E}UTV_\mu^c[0,T] \\ &\leq 27\frac{T}{\mathbf{E}T_c}\mathbf{E}\sup_{0\leq t<s\leq T_c\wedge T}(W_s - W_t - c)_+ . \end{aligned}$$

Proof. By Lemma 3.1 and independence of $W_t - W_{T_c}, t \geq T_c$, and T_c (strong Markov property of Brownian motion) we calculate

$$\begin{aligned} \mathbf{E}UTV_\mu^c[0,T] &= \mathbf{E}\sup_{0\leq t\leq s\leq T_c\wedge T}(W_s - W_t - c)_+ + \mathbf{E}UTV_\mu^c[T_c\wedge T, T] \\ &\leq \mathbf{E}\sup_{0\leq t<s\leq T_c\wedge T}(W_s - W_t - c)_+ + \mathbf{E}\left[UTV_\mu^c[T_c,T]; T_c < \frac{1}{3}\mathbf{E}T_c\right] \\ &\quad + \mathbf{E}\left[UTV_\mu^c[T_c,T]; \frac{1}{3}\mathbf{E}T_c \leq T_c \leq T\right] \\ &\leq \mathbf{E}\sup_{0\leq t<s\leq T_c\wedge T}(W_s - W_t - c)_+ + \mathbf{E}\left[UTV_\mu^c[T_c,T+T_c]; T_c < \frac{1}{3}\mathbf{E}T_c\right] \\ &\quad + \mathbf{E}\left[UTV_\mu^c\left[T_c, T + T_c - \frac{1}{3}\mathbf{E}T_c\right]; \frac{1}{3}\mathbf{E}T_c \leq T_c \leq T\right] \\ &\leq \mathbf{E}\sup_{0\leq t<s\leq T_c\wedge T}(W_s - W_t - c)_+ + \mathbf{E}UTV_\mu^c[0,T]\,P\left(T_c < \frac{1}{3}\mathbf{E}T_c\right) \\ &\quad + \mathbf{E}UTV_\mu^c\left[0, T - \frac{1}{3}\mathbf{E}T_c\right]P\left(T_c \geq \frac{1}{3}\mathbf{E}T_c\right). \end{aligned}$$

Now, by the above inequality and Lemma 3.3

$$\begin{aligned}
\mathbf{E}UTV_\mu^c[0,T] &\le \frac{\mathbf{E}\sup_{0\le t<s\le T_c\wedge T}(W_s-W_t-c)_+}{P\left(T_c\ge\frac{1}{3}\mathbf{E}T_c\right)}+\mathbf{E}UTV_\mu^c\left[0,T-\frac{1}{3}\mathbf{E}T_c\right]\\
&\le \frac{9}{2}\mathbf{E}\sup_{0\le t<s\le T_c\wedge T}(W_s-W_t-c)_+ +\mathbf{E}UTV_\mu^c\left[0,T-\frac{1}{3}\mathbf{E}T_c\right].
\end{aligned}$$

Similarly

$$\mathbf{E}UTV_\mu^c\left[0,T-\frac{1}{3}\mathbf{E}T_c\right]\le\frac{9}{2}\mathbf{E}\sup_{0\le t<s\le T_c\wedge T}(W_s-W_t-c)_+ +\mathbf{E}UTV_\mu^c\left[0,T-\frac{2}{3}\mathbf{E}T_c\right].$$

Iterating and putting together the above inequalities we obtain the estimate from above

$$\begin{aligned}
\mathbf{E}UTV_\mu^c[0,T] &\le \left\lceil\frac{T}{\frac{1}{3}\mathbf{E}T_c}\right\rceil\frac{9}{2}\mathbf{E}\sup_{0\le t\le s\le T_c\wedge T}(W_s-W_t-c)_+\\
&\le \left(\frac{3T}{\mathbf{E}T_c}+1\right)\frac{9}{2}\mathbf{E}\sup_{0\le t<s\le T_c\wedge T}(W_s-W_t-c)_+\\
&\le \frac{6T}{\mathbf{E}T_c}\frac{9}{2}\mathbf{E}\sup_{0\le t\le s\le T_c\wedge T}(W_s-W_t-c)_+\\
&\le 27\frac{T}{\mathbf{E}T_c}\mathbf{E}\sup_{0\le t\le s\le T_c\wedge T}(W_s-W_t-c)_+.
\end{aligned}$$

The estimate from below is obtained from Lemma 3.2 (see also the comment after the calculation):

$$\begin{aligned}
\mathbf{E}UTV_\mu^c[0,T] &\ge \frac{1-e^{-1}}{2}\mathbf{E}UTV_\mu^c(T)\ge 0.3\mathbf{E}UTV_\mu^c(T)\\
&= 0.3\sum_{i=1}^{\infty}\mathbf{E}e^{-T_c^{(i-1)}/T}\sup_{T_c^{(i-1)}\le t<s\le T_c^{(i)}\wedge\left(T_c^{(i-1)}+T\right)}(W_s-W_t-c)_+\\
&= 0.3\sum_{i=1}^{\infty}\mathbf{E}e^{-T_c^{(i-1)}/T}\mathbf{E}\sup_{T_c^{(i-1)}\le t<s\le T_c^{(i)}\wedge\left(T_c^{(i-1)}+T\right)}(W_s-W_t-c)_+\\
&= 0.3\left(\sum_{i=1}^{\infty}\left(\mathbf{E}e^{-T_c^{(1)}/T}\right)^{i-1}\right)\mathbf{E}\sup_{0\le t\le s\le T_c\wedge T}(W_s-W_t-c)_+\\
&= 0.3\frac{1}{1-\mathbf{E}e^{-T_c^{(1)}/T}}\mathbf{E}\sup_{0\le t<s\le T_c\wedge T}(W_s-W_t-c)_+\\
&\ge 0.3\frac{1}{1-\mathbf{E}\left(1-T_c^{(1)}/T\right)}\mathbf{E}\sup_{0\le t<s\le T_c\wedge T}(W_s-W_t-c)_+\\
&= 0.3\frac{T}{\mathbf{E}T_c}\mathbf{E}\sup_{0\le t\le s\le T_c\wedge T}(W_s-W_t-c)_+.
\end{aligned}$$

In the above calculations we used consecutively: independence of $T_c^{(i-1)}$ and $W_s - W_{T_c^{(i-1)}}$, $s \geq T_c^{(i-1)}$, equality of distributions of every term

$$\sup_{T_c^{(i-1)} \leq t < s \leq T_c^{(i)} \wedge \left(T_c^{(i-1)}+T\right)} (W_s - W_t - c)_+$$

for $i = 1, 2, ...$, definition of $T_c^{(i-1)}$, which implies the equality

$$\mathbf{E}e^{-T_c^{(i-1)}/T} = \left(\mathbf{E}e^{-T_c^{(1)}/T}\right)^{i-1}$$

and finally we used the inequality $e^x \geq 1 + x$. □

The estimates in Theorem 3.4 involve expected value of the variable

$$\sup_{0 \leq t < s \leq T_c \wedge T} (W_s - W_t - c)_+$$

distribution of which, as far as author knows, is not known, but it may be simulated numerically. We also have

Corollary 3.5. *For any $T \geq \frac{1}{3}\mathbf{E}T_c$ we have*

$$\begin{aligned} 3\frac{T}{\mathbf{E}T_c}\mathbf{E} \sup_{0 \leq t \leq s \leq \frac{1}{3}\mathbf{E}T_c} (W_s - W_t - c)_+ &\leq \mathbf{E}UTV_\mu^c[0,T] \\ &\leq 27\frac{T}{\mathbf{E}T_c}\mathbf{E} \sup_{0 \leq t \leq s \leq T_c} (W_s - W_t - c)_+ \end{aligned} \tag{13}$$

Proof. The estimate from above is a straighforward consequence of Theorem 3.4 and the estimate from below is obtained immediately by the superadditivity property

$$\begin{aligned} \mathbf{E}UTV_\mu^c[0,T] &\geq \sum_{i=1}^{\lfloor 3T/\mathbf{E}T_c \rfloor} \mathbf{E}UTV_\mu^c\left[\frac{i-1}{3}\mathbf{E}T_c, \frac{i}{3}\mathbf{E}T_c\right] \\ &\geq \lfloor 3T/\mathbf{E}T_c \rfloor \mathbf{E}UTV_\mu^c\left[0, \frac{1}{3}\mathbf{E}T_c\right] \\ &\geq 3\frac{T}{\mathbf{E}T_c}\mathbf{E} \sup_{0 \leq t \leq s \leq \frac{1}{3}\mathbf{E}T_c} (W_s - W_t - c)_+ . \end{aligned}$$

□

Remark 3.6. *Using results of of Hadjiliadis and Vecer [1] we are able to calculate exactly the estimate from above appearing in (13). Using the notation from [1], for $z > 0$ we have*

$$\begin{aligned} P\left(\sup_{0 \leq t \leq s \leq T_c} (W_s - W_t - c)_+ \geq z\right) &= P\left(\sup_{0 \leq t \leq s \leq T_c} (W_s - W_t) \geq z + c\right) \\ &= P\left(T(c, z+c) = T_2(z+c)\right) \end{aligned}$$

and by Theorem 2.1 from [1], for $y > c$ we have

$$P\left(\sup_{0\le t\le s\le T_c}(W_s - W_t) \ge y\right) = \frac{e^{2\mu c} - 2\mu c - 1}{e^{2\mu c} + e^{-2\mu c} - 2}\exp\left(-\frac{2\mu}{e^{2\mu c} - 1}(y - c)\right).$$

Hence

$$\begin{aligned}\mathbf{E}\sup_{0\le t\le s\le T_c}(W_s - W_t - c)_+ &= \int_c^\infty P\left(\sup_{0\le t\le s\le T_c}(W_s - W_t) \ge y\right)dy \\ &= \frac{e^{2\mu c} - 2\mu c - 1}{e^{2\mu c} + e^{-2\mu c} - 2}\int_c^\infty \exp\left(-\frac{2\mu}{e^{2\mu c} - 1}(y - c)\right)dy \\ &= \frac{e^{2\mu c} - 2\mu c - 1}{e^{2\mu c} + e^{-2\mu c} - 2}\frac{e^{2\mu c} - 1}{2\mu}.\end{aligned}$$

Estimates of $\mathbf{E}UTV_\mu^c[0,T]$ for short time intervals ($T < \frac{1}{2}\mathbf{E}T_c$) are the subject of the next theorem.

Theorem 3.7. *For any $T < \frac{1}{3}\mathbf{E}T_c$ we have*

$$\begin{aligned}\mathbf{E}\sup_{0\le t\le s\le T}(W_s - W_t - c)_+ &\le \mathbf{E}UTV_\mu^c[0,T] \\ &\le 5\mathbf{E}\sup_{0\le t\le s\le T}(W_s - W_t - c)_+.\end{aligned}$$

Proof. Applying Lemma 3.1 and independence of $W_t - W_{T_c}, t \ge T_c$, and T_c we again calculate

$$\begin{aligned}\mathbf{E}UTV_\mu^c[0,T] &\le \mathbf{E}\sup_{0\le t\le s\le T_c\wedge T}(W_s - W_t - c)_+ + \mathbf{E}UTV_\mu^c[T_c\wedge T, T] \\ &\le \mathbf{E}\sup_{0\le t\le s\le T}(W_s - W_t - c)_+ + \mathbf{E}\left[UTV_\mu^c[T_c, T]; T_c < T\right] \\ &\le \mathbf{E}\sup_{0\le t\le s\le T}(W_s - W_t - c)_+ + \mathbf{E}UTV_\mu^c[0,T]\,P\left(T_c < \frac{1}{3}\mathbf{E}T_c\right) \\ &\le \mathbf{E}\sup_{0\le t\le s\le T}(W_s - W_t - c)_+ + \mathbf{E}UTV_\mu^c[0,T]\,\frac{7}{9}.\end{aligned}$$

Thus we got

$$\mathbf{E}UTV_\mu^c[0,T] \le \frac{9}{2}\mathbf{E}\sup_{0\le t\le s\le T}(W_s - W_t - c)_+.$$

The estimate from above is self-evident

$$\mathbf{E}UTV_\mu^c[0,T] \ge \mathbf{E}\sup_{0\le t\le s\le T}(W_s - W_t - c)_+.$$

□

Remark 3.8. *In order to calculate the quantity* $\mathbf{E}\sup_{0\leq t\leq s\leq T}(W_s - W_t - c)_+$ *for* $T \leq \frac{1}{3}\mathbf{E}T_c$*, which appears in Corollary 3.5 and in Theorem 3.7, one may use results of [5]. Let*

$$G_{\bar{D}}(y) = 2e^{\mu y}\left\{L + \sum_{n=1}^{\infty}\frac{\theta_n \sin\theta_n}{\theta_n^2 + \mu^2 y^2 + \mu y}\left(1 - \exp\left(-\frac{\theta_n^2 T}{2y^2} - \frac{\mu^2 T}{2}\right)\right)\right\},$$

where θ_n *are positive solutions of the eigenvalue condition* $\tan\theta_n = -\frac{\theta_n}{\mu y}$,

$$L = \begin{cases} 0, 0 < y < -\frac{1}{\mu}; \\ \frac{3}{2}\left(1 - e^{-\mu^2 T/2}\right), y = -\frac{1}{\mu}; \\ \frac{2\eta\sinh\eta}{\eta^2 - \mu^2 y^2 - \mu y}\left(1 - \exp\left(\frac{\eta^2 T}{2y^2} - \frac{\mu^2 T}{2}\right)\right), y > -\frac{1}{\mu}; \end{cases}$$

and η *is the unique positive solution of* $\tanh\eta = -\frac{\eta}{\mu y}$*. In the notation used in [5] for* $z > 0$ *we have*

$$\begin{aligned} P\left(\sup_{0\leq t\leq s\leq T}(W_s - W_t - c)_+ \geq z\right) &= P\left(\sup_{0\leq t\leq s\leq T}(W_s - W_t) \geq z + c\right) \\ &= P\left(\bar{D}(T; -\mu, 1) \geq z + c\right) = G_{\bar{D}}(z + c) \end{aligned}$$

and thus

$$\mathbf{E}\sup_{0\leq t\leq s\leq T}(W_s - W_t - c)_+ = \int_0^{\infty} G_{\bar{D}}(z + c)\,dz = \int_c^{\infty} G_{\bar{D}}(z)\,dz.$$

However, the above formula is very numerically unstable and it seems not to be a straightforward task to obtain using it good numerical or analytical estimates of expected value of the variable $\sup_{0\leq t\leq s\leq T}(W_s - W_t - c)_+$.

4 Example of application

As it was mentioned earlier, upward truncated variation appears naturally in the expression for **the least upper bound** for the rate of return from any trading of a financial asset, dynamics of which follows geometric Brownian motion, in the presence of flat commission. Similar result was proved in [6] for truncated variation, however, truncated variation is not the least upper bound.

Indeed, similarly as in [6], let us assume that the dynamics of the prices P_t of some financial asset (e.g. stock) is the following $P_t = \exp(\mu t + \sigma B_t)$. We are interested in the maximal possible profit coming from trading this single instrument during time interval $[0, T]$. We buy the instrument at the moments $0 \leq t_1 < ... < t_n < T$ and sell it at the moments $s_1 < ... < s_n \leq T$, such that $t_1 < s_1 < t_2 < s_2 < ... < t_n < s_n$, in order to obtain the maximal possible profit. Furthermore we assume that for every transaction we have to pay a flat commission and γ is the ratio of the transaction value paid for the commission.

The maximal possible rate of return from our strategy reads as (cf. [6])

$$\sup_n \sup_{0\le t_1<s_1<...<t_n<s_n\le T} \frac{P_{s_1}}{P_{t_1}}\frac{1-\gamma}{1+\gamma}...\frac{P_{s_n}}{P_{t_n}}\frac{1-\gamma}{1+\gamma}-1.$$

Let M_n be the set of all partitions

$$\pi = \{0 \le t_1 < s_1 < ... < t_n < s_n \le T\}.$$

To see that $\exp\left(\sigma UTV^{c/\sigma}_{\mu/\sigma}[0,T]\right)-1$ with $c=\ln\frac{1+\gamma}{1-\gamma}$ is the least upper bound for maximal possible rate of return let us substitute

$$\begin{aligned}
\sup_n \sup_{M_n} \prod_{i=1}^n \left\{\frac{P_{s_i}}{P_{t_i}}\frac{1-\gamma}{1+\gamma}\right\} &= \sup_n \sup_{M_n} \prod_{i=1}^n \left\{\frac{\exp\left(\mu s_i+\sigma B_{s_i}\right)}{\exp\left(\mu t_i+\sigma B_{t_i}\right)}e^{-c}\right\} \\
&= \sup_n \sup_{M_n} \exp\left(\sigma \sum_{i=1}^n \left\{\left(\frac{\mu}{\sigma}s_i+B_{s_i}\right)-\left(\frac{\mu}{\sigma}t_i+B_{t_i}\right)-\frac{c}{\sigma}\right\}\right) \\
&= \exp\left(\sigma \sup_n \sup_{M_n} \sum_{i=1}^n \left\{\left(\frac{\mu}{\sigma}s_i+B_{s_i}\right)-\left(\frac{\mu}{\sigma}t_i+B_{t_i}\right)-\frac{c}{\sigma}\right\}\right) \\
&= \exp\left(\sigma UTV^{c/\sigma}_{\mu/\sigma}[0,T]\right).
\end{aligned}$$

This gives the claimed bound.

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Truncated variation, upward truncated variation and downward truncated variation of Brownian motion with drift - their characteristics and applications

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Abstract

In Łochowski [2008] we defined truncated variation of Brownian motion with drift, $W_t = B_t + \mu t, t \geq 0$, where (B_t) is a standard Brownian motion. Truncated variation differs from regular variation by neglecting jumps smaller than some fixed $c > 0$. We prove that truncated variation is a random variable with finite moment-generating function for any complex argument.

We also define two closely related quantities - upward truncated variation and downward truncated variation.

The defined quantities may have some interpretation in financial mathematics. Exponential moment of upward truncated variation may be interpreted as the maximal possible return from trading a financial asset in the presence of flat commission when the dynamics of the prices of the asset follows a geometric Brownian motion process.

We calculate the Laplace transform with respect to time parameter of the moment-generating functions of the upward and downward truncated variations.

As an application of the obtained formula we give an exact formula for expected value of upward and downward truncated variations. We give also exact (up to universal constants) estimates of the expected values of the mentioned quantities.

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1. Introduction

Let $(W_t, t \geq 0)$ be a Brownian motion with drift, $W_t = B_t + \mu t$, where $(B_t, t \geq 0)$ is a standard Brownian motion.

The well known result of Paul Lévy (cf. Lévy [1940]) states that for any $0 \leq a < b$ and any $p \leq 2$ the p-variation of the process W_t on the interval $[a, b]$ is almost surely infinite:

$$\sup_n \sup_{a \leq t_1 < t_2 < \ldots < t_n \leq b} \sum_{i=1}^{n-1} \left| W_{t_{i+1}} - W_{t_i} \right|^p = +\infty$$

and if $a \leq t_{1,k} < t_{2,k} < \ldots < t_{n_k,k} \leq b$ is a descending sequence of partitions of the interval $[a, b]$ such that $\lim_{k\to\infty} \max_{1 \leq i \leq n_k - 1}(t_{i+1,k} - t_{i,k}) = 0$, then

$$\lim_{k\to\infty} \sum_{i=1}^{n_k-1} \left| W_{t_{i+1,k}} - W_{t_{i,k}} \right|^2 = b - a \text{ a.s.} \tag{1}$$

The further results of this type state that if $n_k \to \infty$ and $\max_{1 \leq i \leq n_k - 1}(t_{i+1,k} - t_{i,k}) = o(1/\ln(n_k))$ then equality (1) also holds (Dudley [1973]), but if it is not true, then (1) may not be true as well (de la Vega [1974]).

In 1972 S. J. Taylor proved (Taylor [1972]) that the function $\psi(x) = x^2 / \ln \max\{\ln(1/x), e\}$ is a function with the smallest order around 0 and such that

$$\sup_n \sup_{a \leq t_1 < t_2 < \ldots < t_n \leq b} \sum_{i=1}^{n-1} \psi\left(\left| W_{t_{i+1}} - W_{t_i} \right| \right) < +\infty \text{ a.s.}$$

In the paper Łochowski [2008] we started to investigate another type of variation of Brownian paths, which neglects small jumps (smaller than some $c > 0$) and defined *truncated variation* of W_t at the level $c > 0$ on the interval $[a, b]$ as

$$TV_\mu^c[a, b] := \sup_n \sup_{a \leq t_1 < t_2 < \ldots < t_n \leq b} \sum_{i=1}^{n-1} \phi_c\left(\left| W_{t_{i+1}} - W_{t_i} \right| \right),$$

where $\phi_c(x) = \max\{x - c, 0\}$. We will prove that the truncated variation is not only finite almost surely, but has finite moment-generating function for any complex number.

Remark 1. *A. N. Chuprunov pointed to the author that it would be also interesting to have estimates of quadratic truncated variation, which one may define as*

$$QTV_\mu^c[a,b] := \sup_n \sup_{a \le t_1 < \cdots < t_n \le b} \sum_{i=1}^{n-1} \phi_{c^2}\left(\left|W_{t_{i+1}} - W_{t_i}\right|^2\right).$$

Remark 2. *Similar concept of truncation (or* shrinking*) of random variables on Hilbert spaces investigated Z. Jurek in series of his papers beginning with Jurek [1975], Jurek [1985], which now evolved into the theory of s-selfdecomposable distributions (see e.g. Iksanov, Jurek and Schreiber [2004]).*

Let us define two quantities closely related to truncated variation - *upward truncated variation* of W_t on the interval $[a,b]$

$$UTV_\mu^c[a,b] := \sup_n \sup_{a \le t_1 < s_1 < t_2 < s_2 < \ldots < t_n < s_n \le b} \sum_{i=1}^{n} \phi_c\left(W_{s_i} - W_{t_i}\right)$$

and, analogously, *downward truncated variation*

$$DTV_\mu^c[a,b] := \sup_n \sup_{a \le t_1 < s_1 < t_2 < s_2 < \ldots < t_n < s_n \le b} \sum_{i=1}^{n} \phi_c\left(W_{t_i} - W_{s_i}\right).$$

The defined quantities are related in the following way

$$\begin{aligned} \max\left\{UTV_\mu^c[a,b], DTV_\mu^c[a,b]\right\} &\le TV_\mu^c[a,b] \\ &\le UTV_\mu^c[a,b] + DTV_\mu^c[a,b]. \end{aligned} \quad (2)$$

It is easy to see that the three above defined quantities have the following properties, which we state only for the truncated variation

1. Shift invariance property in distributions: for any stopping time Δ relative to the natural filtration of $(W_t, t \ge 0)$

$$\mathcal{L}\left(TV_\mu^c[a,b]\right) = \mathcal{L}\left(TV_\mu^c[a+\Delta, b+\Delta]\right).$$

2. Superadditivity property: for any numbers $a \leq a_1 < a_2 < \cdots < a_n \leq b$

$$TV_{\mu}^{c}[a,b] \geq \sum_{i=1}^{n-1} TV_{\mu}^{c}[a_i, a_{i+1}].$$

Upward truncated variation has also some interpretation in financial mathematics. We will prove that $\exp UTV_{\mu}^{c}[a,b] - 1$ is the least upper bound for the maximum possible rate of return from any trading a single asset on time interval $[a,b]$ in the presence of flat commission (proportional to the value of the transaction) when asset's prices follow the geometric motion process $\exp(W_t)$.

Due to this fact and (2) we will be interested in calculating the moment-generating function of the variables $UTV_{\mu}^{c}[a,b]$ and $DTV_{\mu}^{c}[a,b]$.

Since the distribution of $DTV_{\mu}^{c}[a,b]$ is the same as the distribution of $UTV_{-\mu}^{c}[a,b]$, we will deal with the moment-generating function of upward truncated variation only.

More precisely, we will find the Laplace transform with respect to time parameter T of the ... moment-generating function of the variable $UTV_{\mu}^{c}[0,T]$. Let us explain that here we use term "Laplace transform" in a broad sense. For a measurable (with respect to the Lebesgue measure dt) complex function f, defined on a positive half-line, by the Laplace transform of f we will mean the value of the integral $\int_0^\infty e^{\nu t} f(t) dt$ for any complex ν, for which this integral exists. Similarly, by the moment-generating function of a complex random variable X we will mean the expected value $\mathbf{E}\exp(\lambda X)$ for any complex λ, for which this value is well defined.

As an application of the obtained formula we will give an exact formula for expected value of upward and downward truncated variations. We give also exact (up to universal constants) estimates of the expected values of the mentioned quantities.

The obtained formula may be also used in order to obtain exact formulas for higher moments.

Let us comment on the organization of the paper. In the next section we introduce some notation and prove the existence of moment-generating functions of truncated variation, upward truncated variation and downward truncated variation for any complex argument. In the third section we calculate formula for the Laplace transform with respect to time parameter of the moment-generating function of upward truncated variation. In the fourth section we give examples of applications of the derived formula. In the last section we give possible interpretation of upward truncated variation in financial mathematics.

2. Existence of moment-generating functions for any complex argument

Let us start with some definitions and notation. The drawdown and drawup processes of W_t are defined respectively as

$$\begin{aligned} DD_s &= \sup_{0\le t\le s} W_t - W_s, \\ DU_s &= W_s - \inf_{0\le t\le s} W_t. \end{aligned}$$

The times of drawdown of c units and drawup of c units are defined respectively as

$$\begin{aligned} T_D(c) &= \inf\{s\ge 0 | DD_s = c\}, \\ T_U(c) &= \inf\{s\ge 0 | DU_s = c\}. \end{aligned}$$

Further let $T_D^{\sup}(c)$ be the last instant when the maximum of W_t on the interval $[0, T_D(c)]$ is attained and let $T_D^{\inf}(c) \le T_D^{\sup}(c)$ be such that $W_{T_D^{\inf}(c)} = \inf_{0\le s\le T_D^{\sup}(c)} W_s$.

Let us fix $\alpha > 0$. We will prove the existence of moment-generating function of truncated variation, upward truncated variation and downward truncated variation for argument α. Since the truncated variation and two other variables are non-negative, this will prove the existence of moment-generating function of those variables for any complex argument.

Proof. Let $\delta > 0$ be such a small number that

$$1 - \mathbf{E}\exp\left(\alpha \sup_{0\le t\le T} W_t + \alpha c\right)\mathbf{P}\left(T_D(c) < \delta\right) > 0.$$

By definition of $T_D(c)$ and $T_D^{\inf}(c)$ we have $W_{T_D^{\inf}(c)} > -c$ and hence, $W_{T_D^{\sup}(c)} - W_{T_D^{\inf}(c)} - c \le W_{T_D^{\sup}(c)}$. Let us fix $M > 0$. By Lemma 1 and Lemma 2 in Łochowski [2008], by independence of $W_t - W_{T_D(c)}, t \ge T_D(c)$, and

$T_D(c)$ (strong Markov property of Brownian motion) and by shift invariance property of truncated variation for stopping time $T_D(c)$ we have

$$\begin{aligned}
&\mathbf{E}\exp\left(\alpha TV^c_\mu[0,T]\wedge M\right) \leq \mathbf{E}\exp\left(\alpha W_{T_D^{\sup}(c)}+\alpha c+\alpha TV^c_\mu[T_D(c),T]\wedge M\right)\\
&\quad\leq \mathbf{E}\exp\left(\alpha W_{T_D^{\sup}(c)}+\alpha c\right)\mathbf{E}\exp\left[\alpha TV^c_\mu[T_D(c),T]\wedge M;T_D(c)<\delta\right]\\
&\quad+\mathbf{E}\exp\left(\alpha W_{T_D^{\sup}(c)}+\alpha c\right)\mathbf{E}\exp\left[\alpha TV^c_\mu[T_D(c),T]\wedge M;T_D(c)\geq\delta\right]\\
&\leq \mathbf{E}\exp\left(\alpha W_{T_D^{\sup}(c)}+\alpha c\right)\mathbf{E}\exp\left[\alpha TV^c_\mu[T_D(c),T+T_D(c)]\wedge M;T_D(c)<\delta\right]\\
&+\mathbf{E}\exp\left(\alpha W_{T_D^{\sup}(c)}+\alpha c\right)\mathbf{E}\exp\left[\alpha TV^c_\mu[T_D(c),T+T_D(c)-\delta]\wedge M;T_D(c)\geq\delta\right]\\
&\quad\leq \mathbf{E}\exp\left(\alpha\sup_{0\leq t\leq T}W_t+\alpha c\right)\mathbf{E}\exp\left(\alpha TV^c_\mu[0,T]\wedge M\right)\mathbf{P}\left(T_D(c)<\delta\right)\\
&\quad+\mathbf{E}\exp\left(\alpha\sup_{0\leq t\leq T}W_t+\alpha c\right)\mathbf{E}\exp\left(\alpha TV^c_\mu[0,T-\delta]\wedge M\right)\mathbf{P}\left(T_D(c)\geq\delta\right).
\end{aligned}$$

From the above we have

$$\begin{aligned}
&\mathbf{E}\exp\left(\alpha TV^c_\mu[0,T]\wedge M\right)\\
&\leq \frac{\mathbf{E}\exp\left(\alpha\sup_{0\leq t\leq T}W_t+\alpha c\right)\mathbf{P}\left(T_D(c)\geq\delta\right)}{1-\mathbf{E}\exp\left(\alpha\sup_{0\leq t\leq T}W_t+\alpha c\right)\mathbf{P}\left(T_D(c)<\delta\right)}\mathbf{E}\exp\left(\alpha TV^c_\mu[0,T-\delta]\wedge M\right).
\end{aligned}$$

Similarly

$$\begin{aligned}
&\mathbf{E}\exp\left(\alpha TV^c_\mu[0,T-\delta]\wedge M\right)\\
&\leq \frac{\mathbf{E}\exp\left(\alpha\sup_{0\leq t\leq T}W_t+\alpha c\right)\mathbf{P}\left(T_D(c)\geq\delta\right)}{1-\mathbf{E}\exp\left(\alpha\sup_{0\leq t\leq T}W_t+\alpha c\right)\mathbf{P}\left(T_D(c)<\delta\right)}\mathbf{E}\exp\left(\alpha TV^c_\mu[0,T-2\delta]\wedge M\right).
\end{aligned}$$

Iterating and putting together the above inequalities we finally obtain

$$\mathbf{E}\exp\left(\alpha TV^c_\mu[0,T]\wedge M\right)\leq\left(\frac{\mathbf{E}\exp\left(\alpha\sup_{0\leq t\leq T}W_t+\alpha c\right)\mathbf{P}\left(T_D(c)\geq\delta\right)}{1-\mathbf{E}\exp\left(\alpha\sup_{0\leq t\leq T}W_t+\alpha c\right)\mathbf{P}\left(T_D(c)<\delta\right)}\right)^{\lceil T/\delta\rceil}.$$

Letting $M\to\infty$ we get $\mathbf{E}\exp\left(\alpha TV^c_\mu[0,T]\right)<+\infty$.

By (2) we obtain the finiteness of moment-generating functions of $UTV^c_\mu[0,T]$ and $DTV^c_\mu[0,T]$ as well. □

3. Calculation of the Laplace transform of the moment-generating function

Due to typographical reasons let us introduce notation $\max\{x,0\} =: (x)_+$.

The main difference between truncated variation and upward as well as downward truncated variation is such that for the latter quantities we have the following analog of Lemma 2 from Łochowski [2008], where instead of inequality we have equality.

Lemma 3. *We have the following identities*

$$UTV_\mu^c[0,T] = \sup_{0\le t<s\le T_D(c)\wedge T}(W_s - W_t - c)_+ + UTV_\mu^c[T_D(c)\wedge T, T]. \quad (3)$$

and

$$DTV_\mu^c[0,T] = \sup_{0\le t<s\le T_U(c)\wedge T}(W_t - W_s - c)_+ + DTV_\mu^c[T_U(c)\wedge T, T]. \quad (4)$$

Proof. We will only prove the first formula (3), since the proof of the second one is identical.

Let $0 \le t_1 < s_1 < t_2 < s_2 ... < t_n < s_n \le T$ be numbers from the interval $[0,T]$.

We will prove that

$$\sum_{i=1}^{n}(W_{s_i} - W_{t_i} - c)_+ \le \sup_{0\le t<s\le T_D(c)\wedge T}(W_s - W_t - c)_+ + UTV_\mu^c[T_D(c)\wedge T, T]. \quad (5)$$

Let n_0 be the greatest number such that $s_{n_0} < T_D(c)$ and let us assume that $n_0 < n$ and $t_{n_0+1} < T_D(c)$.

Let us consider several cases.

- $W_{t_{n_0+1}} \ge W_{T_D(c)}$. In this case

  $$\left(W_{s_{n_0+1}} - W_{t_{n_0+1}} - c\right)_+ \le \left(W_{s_{n_0+1}} - W_{T_D(c)} - c\right)_+.$$

  and

  $$\sum_{i=1}^{n}(W_{s_i} - W_{t_i} - c)_+ \le \sum_{i=1}^{n_0}(W_{s_i} - W_{t_i} - c)_+ + \left(W_{s_{n_0+1}} - W_{T_D(c)} - c\right)_+ + \sum_{i=n_0+2}^{n}(W_{s_i} - W_{t_i} - c)_+. \quad (6)$$

- $W_{t_{n_0+1}} < W_{T_D(c)}$ and $W_{s_{n_0+1}} \le W_{T_D(c)^{\sup}}$. In this case $t_{n_0+1} < T_D^{\sup}(c)$ (since for $T_D^{\sup}(c) < t < T_D(c)$, $W_t > W_{T_D(c)}$) so

$$\left(W_{s_{n_0+1}} - W_{t_{n_0+1}} - c\right)_+ \le \left(W_{T_D^{\sup}(c)} - W_{t_{n_0+1}} - c\right)_+$$

and

$$\begin{aligned}\sum_{i=1}^{n}(W_{s_i} - W_{t_i} - c)_+ \le \sum_{i=1}^{n_0}(W_{s_i} - W_{t_i} - c)_+ + \left(W_{T_D^{\sup}(c)} - W_{t_{n_0+1}} - c\right)_+ \\ + \sum_{i=n_0+2}^{n}(W_{s_i} - W_{t_i} - c)_+ . \qquad (7)\end{aligned}$$

- $W_{t_{n_0+1}} < W_{T_D(c)}$ and $W_{s_{n_0+1}} > W_{T_D^{\sup}(c)} = W_{T_D(c)} + c$. In this case

$$\begin{aligned}\left(W_{s_{n_0+1}} - W_{t_{n_0+1}} - c\right)_+ &= W_{s_{n_0+1}} - W_{t_{n_0+1}} - c \\ &= W_{T_D^{\sup}(c)} - W_{t_{n_0+1}} - c + W_{s_{n_0+1}} - W_{T_D^{\sup}(c)} \\ &= W_{T_D^{\sup}(c)} - W_{t_{n_0+1}} - c + W_{s_{n_0+1}} - W_{T_D(c)} - c \\ &= \left(W_{T_D^{\sup}(c)} - W_{t_{n_0+1}} - c\right)_+ + \left(W_{s_{n_0+1}} - W_{T_D(c)} - c\right)_+\end{aligned}$$

and

$$\begin{aligned}\sum_{i=1}^{n}(W_{s_i} - W_{t_i} - c)_+ \le \sum_{i=1}^{n_0}(W_{s_i} - W_{t_i} - c)_+ + \left(W_{T_D^{\sup}(c)} - W_{t_{n_0+1}} - c\right)_+ \\ + \left(W_{s_{n_0+1}} - W_{T_D(c)} - c\right)_+ + \sum_{i=n_0+2}^{n}(W_{s_i} - W_{t_i} - c)_+ . \qquad (8)\end{aligned}$$

Thus for $t_{n_0+1} < T_D(c)$ inequality (6), (7) or (8) holds and we may assume, adding in the case $t_{n_0+1} < T_D(c)$ new terms in the partition and renaming the old ones, that

$$\begin{aligned} 0 &\le t_1 < s_1 < ... < t_{n_0} < s_{n_0} \le T_D(c), \\ T_D(c) &\le t_{n_0+1} < s_{n_0+1} < ... < t_n < s_n \le T. \end{aligned}$$

In order to prove (5) without loss of generality we may assume that for any $1 \le i \le n_0$, $(W_{s_i} - W_{t_i} - c)_+ > 0$ (otherwise we may omit the summand $(W_{s_i} - W_{t_i} - c)_+$). From definition of $T_D(c)$ we have that for any $1 \le i \le n_0 - 1$, $W_{s_i} - W_{t_{i+1}} < c$, so

$$\begin{aligned}
&(W_{s_i} - W_{t_i} - c)_+ + \left(W_{s_{i+1}} - W_{t_{i+1}} - c\right)_+ \\
&= W_{s_i} - W_{t_i} - c + W_{s_{i+1}} - W_{t_{i+1}} - c \\
&= W_{s_{i+1}} - W_{t_i} - c + \left(W_{s_i} - W_{t_{i+1}} - c\right) < W_{s_{i+1}} - W_{t_i} - c.
\end{aligned}$$

Iterating the above inequality, we obtain

$$\sum_{i=1}^{n_0} (W_{s_i} - W_{t_i} - c)_+ \le W_{s_{n_0}} - W_{t_1} - c \le \sup_{0 \le t < s \le T_D(c) \wedge T} (W_s - W_t - c)_+ .$$

This, together with the obvious inequality

$$\sum_{i=n_0+1}^{n} (W_{s_i} - W_{t_i} - c)_+ \le \ UTV_\mu^c \left[T_D(c) \wedge T, T\right]$$

proves (5). Taking supremum over all partitions $0 \le t_1 < s_1 < t_2 < s_2 < \ldots < t_n < s_n \le T$ we finally get

$$UTV_\mu^c \left[0, T\right] \le \sup_{0 \le t < s \le T_D(c) \wedge T} (W_s - W_t - c)_+ + UTV_\mu^c \left[T_D(c) \wedge T, T\right].$$

Since the opposite inequality is obvious, we finally get (3). □

Now we are ready to state

Lemma 4. *Let λ be an arbitrary complex number and let*

$$L(\lambda, T) := \mathbf{E} \exp(\lambda UTV_\mu^c[0, T]),$$

$T > 0$, be a family of moment-generating functions of variables $UTV_\mu^c \left[0, T\right]$. This family satisfies the following integral equation

$$\begin{aligned}
L(\lambda, T) = &\int_0^T \int_c^\infty e^{\lambda(y-c)} L(\lambda, T-t) \mathbf{P}\left(T_D\left(c\right) \in dt, \sup_{0 \le s \le T_D(c)} DU_s \in dy\right) \\
&+ \int_0^T L(\lambda, T-t) \mathbf{P}\left(T_D\left(c\right) \in dt, \sup_{0 \le s \le T_D(c)} DU_s < c\right) \\
&+ \int_c^\infty e^{\lambda(y-c)} \mathbf{P}\left(T_D\left(c\right) > T, \sup_{0 \le s \le T} DU_s \in dy\right) \\
&+ \mathbf{P}\left(T_D\left(c\right) > T, \sup_{0 \le s \le T} DU_s < c\right). \quad (9)
\end{aligned}$$

Proof. By Lemma 3 we have that for any $T > 0$

$$UTV_{\mu}^{c}\left[0,T\right]=\sup_{0\leq s\leq T_{D}(c)\wedge T}\left(DU_{s}-c\right)_{+}+UTV_{\mu}^{c}\left[T_{D}\left(c\right)\wedge T,T\right].$$

From dependence of $W_t, t \in [0, T_D(c) \wedge T]$ and $W_t - W_{T_D(c)\wedge T}, t \in [T_D(c) \wedge T, T]$, only through $T_D(c)$, and by equality of distribution of $UTV_{\mu}^{c}[T_D(c) \wedge T, T]$ and $UTV_{\mu}^{c}[0, T - T_D(c) \wedge T]$ we have

$$\begin{aligned}
&\mathbf{E}\exp\left(\lambda UTV_{\mu}^{c}\left[0,T\right]\right)\\
&\quad=\mathbf{E}\exp\left(\lambda\sup_{0\leq s\leq T_{D}(c)\wedge T}\left(DU_{s}-c\right)_{+}+\lambda UTV_{\mu}^{c}\left[T_{D}\left(c\right)\wedge T,T\right]\right)\\
&=\int_{0}^{\infty}\mathbf{E}\exp\left(\lambda\sup_{0\leq s\leq t\wedge T}\left(DU_{s}-c\right)_{+}\right)\mathbf{E}\exp\left(\lambda UTV_{\mu}^{c}\left[0,T-t\wedge T\right]\right)\mathbf{P}\left(T_{D}\left(c\right)\in dt\right)\\
&=\int_{0}^{T}\int_{c}^{\infty}e^{\lambda(y-c)}\mathbf{E}\exp\left(\lambda UTV_{\mu}^{c}\left[0,T-t\right]\right)\mathbf{P}\left(T_{D}\left(c\right)\in dt,\sup_{0\leq s\leq T_{D}(c)}DU_{s}\in dy\right)\\
&\quad+\int_{0}^{T}\mathbf{E}\exp\left(\lambda UTV_{\mu}^{c}\left[0,T-t\right]\right)\mathbf{P}\left(T_{D}\left(c\right)\in dt,\sup_{0\leq s\leq T_{D}(c)}DU_{s}<c\right)\\
&\quad\quad+\int_{c}^{\infty}e^{\lambda(y-c)}\mathbf{P}\left(T_{D}\left(c\right)>T,\sup_{0\leq s\leq T}DU_{s}\in dy\right)\\
&\quad\quad\quad+\mathbf{P}\left(T_{D}\left(c\right)>T,\sup_{0\leq s\leq T}DU_{s}<c\right).
\end{aligned}$$

In the third line of the calculations above we have used iterated expectation, strong Markov property and the shift invariance of upward truncated variation for stopping time $T_D(c)$.

□

Hadjiliadis and Zhang in their recent paper (Hadjiliadis and Zhang [2009]) calculated for $a, b > 0$ the densities

$$p\left(t;a,b\right)dt=\mathbf{P}\left(T_{D}\left(a\right)\in dt,T_{U}\left(b\right)>t\right)$$

and

$$q\left(t;a,b\right)dt=\mathbf{P}\left(T_{U}\left(a\right)\in dt,T_{D}\left(b\right)>t\right).$$

Using these densities we are able to write equation (9) in more elegant form. Indeed, we have

Lemma 5. *The family $L(\lambda,T)$ satisfies the following integral equation*

$$L\left(\lambda,T\right)=\int_0^T L\left(\lambda,T-t\right)\left\{p\left(t;c,c\right)+\int_c^\infty e^{\lambda(y-c)}\frac{\partial p\left(t;c,y\right)}{\partial y}dy\right\}dt$$
$$-\int_0^T \mathbf{P}\left(T_D\left(c\right)>T-t\right)\left\{q\left(t;c,c\right)+\int_c^\infty e^{\lambda(y-c)}\frac{\partial q\left(t;y,c\right)}{\partial y}dy\right\}dt$$
$$+\mathbf{P}\left(T_D\left(c\right)>T\right). \quad (10)$$

Proof. We have

$$\mathbf{P}\left(T_D\left(c\right)\in dt,\sup_{0\le s\le T_D(c)}DU_s\in dy\right)$$
$$=\mathbf{P}\left(T_D\left(c\right)\in dt,T_U\left(y+dy\right)>t\right)-\mathbf{P}\left(T_D\left(c\right)\in dt,T_U\left(y\right)>t\right)$$
$$=\frac{\partial p\left(t;c,y\right)}{\partial y}dydt \quad (11)$$

and

$$\begin{aligned}\mathbf{P}\left(T_D\left(c\right)\in dt,\sup_{0\le s\le T_D(c)}DU_s<c\right) &= \mathbf{P}\left(T_D\left(c\right)\in dt,T_U\left(c\right)>t\right)\\ &= p\left(t;c,c\right)dt. \end{aligned} \quad (12)$$

In order to express $\mathbf{P}\left(T_D\left(c\right)>T,\sup_{0\le s\le T}DU_s\in dy\right)$ with $p(t;a,b)$ and $q(t;a,b)$ let us notice that for $y>0$

$$\begin{aligned}&\mathbf{P}\left(T_D\left(c\right)>T,\sup_{0\le s\le T}DU_s\ge y\right)\\ &=\int_0^T \mathbf{P}\left(T_U\left(y\right)\in dt,T_D\left(c\right)>T\right)\\ &=\int_0^T \mathbf{P}\left(T_U\left(y\right)\in dt,T_D\left(c\right)>t\right)\mathbf{P}\left(T_D\left(c\right)>T-t\right)\\ &=\int_0^T q\left(t;y,c\right)\mathbf{P}\left(T_D\left(c\right)>T-t\right)dt \end{aligned} \quad (13)$$

The equality

$$\mathbf{P}\left(T_U\left(y\right)\in dt,T_D\left(c\right)>T\right)$$
$$=\mathbf{P}\left(T_U\left(y\right)\in dt,T_D\left(c\right)>t\right)\mathbf{P}\left(T_D\left(c\right)>T-t\right)$$

holds since the event $\{T_U(y) \in dt\}$ also means that the process W_t reaches a new maximum at the moment t. Now for $y > 0$ we calculate

$$\begin{aligned}
&\mathbf{P}\left(T_D(c) > T, \sup_{0\le s\le T} DU_s \in dy\right) \\
&= \mathbf{P}\left(T_D(c) > T, \sup_{0\le s\le T} DU_s \ge y\right) - \mathbf{P}\left(T_D(c) > T, \sup_{0\le s\le T} DU_s \ge y + dy\right) \\
&\qquad = \int_0^T \{q(t; y, c) - q(t; y + dy, c)\}\, \mathbf{P}(T_D(c) > T - t)\, dt \\
&\qquad\qquad = -\int_0^T \frac{\partial q(t; y, c)}{\partial y} \mathbf{P}(T_D(c) > T - t)\, dt dy. \quad (14)
\end{aligned}$$

Using similar reasoning, by (13) we also have

$$\begin{aligned}
\mathbf{P}\left(T_D(c) > T, \sup_{0\le s\le T} DU_s < c\right) &= \mathbf{P}(T_D(c) > T, T_U(c) > T) \\
&= \mathbf{P}(T_D(c) > T) - \int_0^T q(t; c, c)\, \mathbf{P}(T_D(c) > T - t)\, dt. \quad (15)
\end{aligned}$$

Thus, from (9), (11), (12), (14) and (15) we obtain the integral equation (10) satisfied by the family of moment-generating functions of upward truncated variation:

$$\begin{aligned}
L(\lambda, T) &= \int_0^T \int_c^\infty e^{\lambda(y-c)} L(\lambda, T-t) \frac{\partial p(t; c, y)}{\partial y} dy dt \\
&\quad + \int_0^T L(\lambda, T-t)\, p(t; c, c)\, dt \\
&\quad - \int_c^\infty e^{\lambda(y-c)} \int_0^T \frac{\partial q(t; y, c)}{\partial y} \mathbf{P}(T_D(c) > T - t)\, dt dy \\
&\quad + \mathbf{P}(T_D(c) > T) - \int_0^T q(t; c, c)\, \mathbf{P}(T_D(c) > T - t)\, dt \\
&= \int_0^T L(\lambda, T-t) \left\{ p(t; c, c) + \int_c^\infty e^{\lambda(y-c)} \frac{\partial p(t; c, y)}{\partial y} dy \right\} dt \\
&\quad - \int_0^T \mathbf{P}(T_D(c) > T - t) \left\{ q(t; c, c) + \int_c^\infty e^{\lambda(y-c)} \frac{\partial q(t; y, c)}{\partial y} dy \right\} dt \\
&\quad + \mathbf{P}(T_D(c) > T).
\end{aligned}$$

□

In order to shorten notation let introduce new functions of parameters t and λ

$$p(\lambda,t) := p(t;c,c) + \int_0^\infty e^{\lambda y} \frac{\partial p(t;c,y+c)}{\partial y} dy,$$
$$q(\lambda,t) := q(t;c,c) + \int_0^\infty e^{\lambda y} \frac{\partial q(t;y+c,c)}{\partial y} dy$$

and for such pairs of complex numbers (λ,ν) that the integral $\int_0^\infty e^{\nu t} L(\lambda,t)\,dt$ exists, let us define

$$M(\lambda,\nu) := \int_0^\infty e^{\nu t} L(\lambda,t)\,dt,$$
$$T(\nu) := \int_0^\infty e^{\nu t} \mathbf{P}(T_D(c) > t)\,dt.$$

By (10) we have

$$\begin{aligned} M(\lambda,\nu) &= \int_0^\infty e^{\nu\tau} L(\lambda,\tau)\,d\tau = \int_0^\infty e^{\nu\tau} \int_0^\tau L(\lambda,\tau-t)\,p(\lambda,t)\,dt d\tau \\ &\quad - \int_0^\infty e^{\nu\tau} \int_0^\tau \mathbf{P}(T_D(c) > \tau - t)\,q(\lambda,t)\,dt d\tau + T(\nu) \\ &= \int_0^\infty e^{\nu t} p(\lambda,t) \int_t^\infty e^{\nu(\tau-t)} L(\lambda,\tau-t)\,d\tau dt \\ &\quad - \int_0^\infty e^{\nu t} q(\lambda,t) \int_t^\infty e^{\nu(\tau-t)} \mathbf{P}(T_D(c) > \tau-t)\,d\tau dt + T(\nu) \\ &= M(\lambda,\nu) \int_0^\infty e^{\nu t} p(\lambda,t)\,dt - T(\nu) \int_0^\infty e^{\nu t} q(\lambda,t)\,dt + T(\nu). \end{aligned}$$

Thus we obtained a formula for the Laplace transform with respect to T of the moment-generating function of $UTV_\mu^c[0,T]$:

$$M(\lambda,\nu) = T(\nu) \frac{1 - \int_0^\infty e^{\nu t} q(\lambda,t)\,dt}{1 - \int_0^\infty e^{\nu t} p(\lambda,t)\,dt}. \tag{16}$$

Using results of Hadjiliadis and Zhang [2009] and Taylor [1975] we are able to compute $M(\lambda,\nu)$ more directly. We have

Theorem 6. *For ν with negative real part and any complex λ the following formula holds*

$$M(\lambda,\nu) = -\frac{1}{\nu} - \frac{\lambda e^{\mu c}}{\nu^2}\frac{\mu\sinh(cU_\mu(\nu)) - U_\mu(\nu)\cosh(cU_\mu(\nu))}{\frac{\lambda U_\mu(\nu)}{\nu} + \sinh(2cU_\mu(\nu)) - 2\frac{\lambda+\mu}{U_\mu(\nu)}\sinh^2(cU_\mu(\nu))}, \quad (17)$$

where $U_\mu(\nu) = \sqrt{\mu^2 - 2\nu}$.

Proof. Integrating by parts, we obtain

$$\begin{aligned} T(\nu) &= \int_0^\infty e^{\nu t}\mathbf{P}(T_D(c) > t)\,dt \\ &= \frac{e^{\nu t}}{\nu}\mathbf{P}(T_D(c) > t)\,|_{t=0}^{t=\infty} - \int_0^\infty \frac{e^{\nu t}}{\nu}\frac{d}{dt}\mathbf{P}(T_D(c) > t)\,dt \\ &= -\frac{1}{\nu} + \frac{1}{\nu}\mathbf{E}e^{\nu T_D(c)}. \end{aligned} \quad (18)$$

Similarly, we have

$$\begin{aligned} p(\lambda,t) &= p(t;c,c) + \int_0^\infty e^{\lambda y}\frac{\partial p(t;c,y+c)}{\partial y}dy \\ &= -\lambda\int_0^\infty e^{\lambda y}p(t;c,y+c)\,dy, \end{aligned}$$

hence

$$\begin{aligned} \int_0^\infty e^{\nu t}p(\lambda,t)\,dt &= -\lambda\int_0^\infty\int_0^\infty e^{\lambda y}p(t;c,y+c)\,dydt \\ &= -\lambda\int_0^\infty e^{\lambda y}\int_0^\infty e^{\nu t}\mathbf{P}(T_D(c)\in dt, T_U(y+c) > t)\,dy \\ &= -\lambda\int_0^\infty e^{\lambda y}\mathbf{E}e^{\nu T_D(c)}I_{\{T_U(y+c)>T_D(c)\}}dy. \end{aligned} \quad (19)$$

Using notation from Hadjiliadis and Zhang [2009], page 11, we have

$$\mathbf{E}e^{\nu T_D(c)}I_{\{T_U(y+c)>T_D(c)\}} = \left(1 - L_0^{-W}(-\nu;c)\,e^{T_{-\mu,1}(-\nu,c)y}\right)\mathbf{E}e^{\nu T_D(c)}$$

thus

$$\begin{aligned} &\int_0^\infty e^{\lambda y}\mathbf{E}e^{\nu T_D(c)}I_{\{T_U(y+c)>T_D(c)\}}dy \\ &= \left(\int_0^\infty e^{\lambda y}\left[1 - L_0^{-W}(-\nu,c)\exp\left(T_{-\mu,1}(-\nu,a)\,y\right)\right]dy\right)\mathbf{E}e^{\nu T_D(c)} \\ &= \left(\frac{L_0^{-W}(-\nu,c)}{T_{-\mu,1}(-\nu,c)+\lambda} - \frac{1}{\lambda}\right)\mathbf{E}e^{\nu T_D(c)} \end{aligned}$$

and finally from (19) we obtain

$$\int_0^\infty e^{\nu t} p(\lambda, t)\, dt = \left(1 - \lambda \frac{L_0^{-W}(-\nu, c)}{T_{-\mu,1}(-\nu, c) + \lambda}\right) \mathbf{E} e^{\nu T_D(c)}. \tag{20}$$

Similarly

$$\begin{aligned} q(\lambda, t) &= q(t; c, c) + \int_0^\infty e^{\lambda y} \frac{\partial q(t; y + c, c)}{\partial y} dy \\ &= -\lambda \int_0^\infty e^{\lambda y} q(t; y + c, c)\, dy, \end{aligned}$$

hence

$$\begin{aligned} \int_0^\infty e^{\nu t} q(\lambda, t)\, dt &= -\lambda \int_0^\infty e^{\nu t} \int_0^\infty e^{\lambda y} q(t; y + c, c)\, dy dt \\ &= -\lambda \int_0^\infty e^{\lambda y} \mathbf{E} e^{\nu T_U(y+c)} I_{\{T_U(y+c) < T_D(c)\}} dy. \end{aligned}$$

Again, by results of Hadjiliadis and Zhang [2009] and using symmetry of standard Brownian motion, we have

$$\mathbf{E} e^{\nu T_U(y+c)} I_{\{T_U(y+c) < T_D(c)\}} = L_0^{-W}(-\nu; c)\, e^{T_{-\mu,1}(-\nu; c) y},$$

and finally we get

$$\begin{aligned} \int_0^\infty e^{\nu t} q(\lambda, t)\, dt &= -\lambda \int_0^\infty e^{\lambda y} \mathbf{E} e^{\nu T_U(y+c)} I_{\{T_U(y+c) < T_D(c)\}} dy \\ &= -\lambda \int_0^\infty e^{\lambda y} L_0^{-W}(-\nu, c)\, e^{T_{-\mu,1}(-\nu, c) y} dy \\ &= \lambda \frac{L_0^{-W}(-\nu, c)}{T_{-\mu,1}(-\nu, c) + \lambda}. \end{aligned} \tag{21}$$

Finally from (16), (18), (20) and (21) we obtain

$$\begin{aligned} M(\lambda, \nu) &= \left(-\frac{1}{\nu} + \frac{1}{\nu} \mathbf{E} e^{\nu T_D(c)}\right) \frac{1 - \lambda \frac{L_0^{-W}(-\nu, c)}{T_{-\mu,1}(-\nu, c) + \lambda}}{1 - \left(1 - \lambda \frac{L_0^{-W}(-\nu, c)}{T_{-\mu,1}(-\nu, c) + \lambda}\right) \mathbf{E} e^{\nu T_D(c)}} \\ &= -\frac{1}{\nu} \frac{\left(1 - \lambda \frac{L_0^{-W}(-\nu, c)}{T_{-\mu,1}(-\nu, c) + \lambda}\right)\left(1 - \mathbf{E} e^{\nu T_D(c)}\right)}{1 - \left(1 - \lambda \frac{L_0^{-W}(-\nu, c)}{T_{-\mu,1}(-\nu, c) + \lambda}\right) \mathbf{E} e^{\nu T_D(c)}}. \end{aligned} \tag{22}$$

It is possible to express the obtained formula for $M(\lambda,\nu)$ with the elementary functions. We have (cf. Hadjiliadis and Zhang [2009] and Taylor [1975]):

$$L_0^{-W}(-\nu,c)=\frac{U_\mu(\nu)}{-2\nu}\left\{\frac{e^{\mu c}(U_\mu(\nu)\coth(cU_\mu(\nu))-\mu)}{\sinh(cU_\mu(\nu))}-\frac{U_\mu(\nu)}{\sinh^2(cU_\mu(\nu))}\right\},$$

$$\mathbf{E}e^{\nu T_D(c)}=\frac{U_\mu(\nu)\,e^{-\mu c}}{U_\mu\cosh(cU_\mu(\nu))-\mu\sinh(cU_\mu(\nu))}$$

and

$$T_{-\mu,1}(-\nu,c)=\mu-U_\mu(\nu)\coth(cU_\mu(\nu)),$$

where

$$U_\mu(\nu)=\sqrt{\mu^2-2\nu}.$$

Substituting the above formulas in (22) we obtain (17). □

4. Examples of applications

The direct application of the derived formula may be calculation of the moment-generating function $L(\lambda,T)$ (with the use of the inverse Laplace transform formula). However, we will start with simpler formulae.

4.1. Exact formula for the expected value of $UTV_\mu^c[0,T]$.

Using formula

$$\mathbf{E}UTV_\mu^c[0,T]=\lim_{\lambda\to0}\frac{1}{\lambda}(L(\lambda,T)-1)$$

we obtain

$$\begin{aligned}&\lim_{\lambda\to0}\frac{1}{\lambda}(M(\nu,\lambda)-M(\nu,0))\\&=\lim_{\lambda\to0}\frac{1}{\lambda}\left(\int_0^\infty e^{\nu t}L(\lambda,t)\,dt-\int_0^\infty e^{\nu t}L(0,t)\,dt\right)\\&=\int_0^\infty e^{\nu t}\lim_{\lambda\to0}\frac{1}{\lambda}[L(\lambda,t)-1]\,dt\\&=\int_0^\infty e^{\nu t}\mathbf{E}UTV_\mu^c[0,t]\,dt.\end{aligned}\tag{23}$$

On the other hand, from (22) we have

$$\lim_{\lambda\to 0}\frac{1}{\lambda}\left(M\left(\nu,\lambda\right)-M\left(\nu,0\right)\right)$$
$$=\lim_{\lambda\to 0}\frac{1}{\lambda}\left(-\frac{1}{\nu}\frac{\left(1-\lambda\frac{L_0^{-W}(-\nu,c)}{T_{-\mu,1}(-\nu,c)+\lambda}\right)\left(1-\mathbf{E}e^{\nu T_D(c)}\right)}{1-\left(1-\lambda\frac{L_0^{-W}(-\nu,c)}{T_{-\mu,1}(-\nu,c)+\lambda}\right)\mathbf{E}e^{\nu T_D(c)}}+\frac{1}{\nu}\right)$$
$$=\frac{L_0^{-W}\left(-\nu,c\right)}{\nu T_{-\mu,1}\left(-\nu,c\right)\left(1-\mathbf{E}e^{\nu T_D(c)}\right)}, \tag{24}$$

which, by (23) and after substituting in (24) the formulas for $L_0^{-W}\left(-\nu,c\right)$, $\mathbf{E}e^{\nu T_D(c)}$ and $T_{-\mu,1}\left(-\nu,c\right)$ yields

$$\int_0^\infty e^{\nu t}\mathbf{E}UTV_\mu^c\left[0,t\right]dt=\frac{e^{\mu c}\sqrt{\mu^2-2\nu}}{2\nu^2\sinh\left(c\sqrt{\mu^2-2\nu}\right)}. \tag{25}$$

Inverting the formula (25) we are able to obtain exact formula for the expected value of $\mathbf{E}UTV_\mu^c\left[0,T\right]$. Let $\mathcal{L}_s^{-1}(g)$ denote inverse of the Laplace transform of the function $g(s)=\int_0^\infty e^{-s\cdot t}f\left(t\right)dt$, i.e. the function $f\left(t\right)$. We have

$$\mathcal{L}_s^{-1}\left(s^{-2}\right)=t \tag{26}$$

and, by the last formula on page 641 of Borodin, Salminen [2002]

$$\mathcal{L}_s^{-1}\left(\frac{\sqrt{2s}}{\sinh\left(c\sqrt{2s}\right)}\right)=\frac{\sqrt{2}}{\sqrt{\pi}t^{5/2}}\sum_{k=0}^{\infty}\left(\left(2k+1\right)^2c^2-t\right)e^{-(2k+1)^2c^2/(2t)}.$$

Hence, by properties of Laplace transform

$$\mathcal{L}_s^{-1}\left(\frac{\sqrt{2s+\mu^2}}{\sinh\left(c\sqrt{2s+\mu^2}\right)}\right)$$
$$=\frac{\sqrt{2}}{\sqrt{\pi}t^{5/2}}e^{-\mu^2 t}\sum_{k=0}^{\infty}\left(\left(2k+1\right)^2c^2-t\right)e^{-(2k+1)^2c^2/(2t)}. \tag{27}$$

Finally, by (26), (27) and Borel convolution theorem for the Laplace transform, we obtain

$$\begin{aligned}
&\mathbf{E}UTV_{\mu}^{c}[0,T] \\
&= \mathcal{L}_{s}^{-1}\left(\frac{e^{\mu c}\sqrt{2s+\mu^{2}}}{2s^{2}\sinh\left(c\sqrt{2s+\mu^{2}}\right)}\right) \\
&= \frac{e^{\mu c}}{\sqrt{2\pi}}\int_{0}^{T}(T-t)\frac{e^{-\mu^{2}t}}{t^{5/2}}\sum_{k=0}^{\infty}\left((2k+1)^{2}c^{2}-t\right)e^{-(2k+1)^{2}c^{2}/(2t)}dt \\
&= \frac{e^{\mu c}}{\sqrt{2\pi}}\sum_{k=0}^{\infty}\int_{0}^{T}(T-t)\frac{(2k+1)^{2}c^{2}-t}{t^{5/2}}e^{-\mu^{2}t-(2k+1)^{2}c^{2}/(2t)}dt. \qquad (28)
\end{aligned}$$

4.2. Estimation of the expected value of $UTV_{\mu}^{c}[0,T]$.

In Łochowski [2008] we obtained a formula for function $F(\mu,c,T)$, such that

$$\mathbf{E}TV_{\mu}^{c}[0,T] \sim F(|\mu|,c,T), \qquad (29)$$

where relation "$\sim$" means that the ratio $\mathbf{E}TV_{\mu}^{c}[0,T]/F(|\mu|,c,T)$ is separated from 0 and infinity by universal constants, which do not depend on μ,c,T.

On the other hand, we see that the exact formula (28) for $\mathbf{E}UTV_{\mu}^{c}[0,T]$ may be stated in the form

$$\mathbf{E}UTV_{\mu}^{c}[0,T] = e^{\mu c}G(|\mu|,c,T), \qquad (30)$$

where

$$G(|\mu|,c,T) = \frac{1}{\sqrt{2\pi}}\sum_{k=0}^{\infty}\int_{0}^{T}(T-t)\frac{(2k+1)^{2}c^{2}-t}{t^{5/2}}e^{-\mu^{2}t-(2k+1)^{2}c^{2}/(2t)}dt$$

does not depend on the sign of μ. Using (2), (30) and the fact that $DTV_{\mu}^{c}[0,T]$ has the same distribution as $UTV_{-\mu}^{c}[0,T]$ we see that

$$\begin{aligned}
\mathbf{E}TV_{\mu}^{c}[0,T] &\sim \mathbf{E}UTV_{\mu}^{c}[0,T]+\mathbf{E}DTV_{\mu}^{c}[0,T] \\
&= \mathbf{E}UTV_{\mu}^{c}[0,T]+\mathbf{E}UTV_{-\mu}^{c}[0,T] \\
&\sim e^{|\mu|c}G(|\mu|,c,T). \qquad (31)
\end{aligned}$$

Comparing (29) and (31) we see that

$$G(|\mu|,c,T) \sim e^{-|\mu|c}F(|\mu|,c,T)$$

and finally we get estimates up to universal constants for $\mathbf{E}UTV_{\mu}^{c}[0,T]$:

$$\mathbf{E}UTV_{\mu}^{c}[0,T] \sim e^{\mu c-|\mu|c}F(|\mu|,c,T)$$
$$= e^{\mu c-|\mu|c}\begin{cases} T/c+|\mu|T & \text{if } \sqrt{T}\geq\chi(c,\mu); \\ 2\sqrt{T}+|\mu|T-c & \text{if } \sqrt{T}\in(c-|\mu|T,\chi(c,\mu)); \\ T^{3/2}\frac{\exp(-(c-|\mu|T)^2/(2T))}{(c-|\mu|T)^2} & \text{if } \sqrt{T}\leq c-|\mu|T, \end{cases}$$

where $\chi(c,\mu)=\sqrt{\frac{e^{2\mu|c|}-2\mu|c|-1}{2\mu^2}}=c\sqrt{1+\frac{2}{3}|\mu|c+\ldots}\geq c.$

4.3. Laplace transform of the second moment of $UTV_{\mu}^{c}[0,T]$

Similarly as (28) we may obtain a formula for the Laplace transform of the second moment of $UTV_{\mu}^{c}[0,T]$:

$$\int_0^\infty e^{\nu t}\mathbf{E}\left(UTV_{\mu}^{c}[0,t]\right)^2 dt = \left[\frac{\partial^2}{\partial\lambda^2}M(\nu,\lambda)\right]_{\lambda=0}$$
$$= -\frac{2L_0^{-W}(-\nu,c)\left(1-\mathbf{E}e^{\nu T_D(c)}+L_0^{-W}(-\nu,c)\,\mathbf{E}e^{\nu T_D(c)}\right)}{\nu\left(T_{-\mu,1}(-\nu,c)\left(1-\mathbf{E}e^{\nu T_D(c)}\right)\right)^2}. \tag{32}$$

After substituting in formula (32) the formulas for $L_0^{-W}(-\nu,c)$, $\mathbf{E}e^{\nu T_D(c)}$ and $T_{-\mu,1}(-\nu,c)$, it simplifies to

$$\int_0^\infty e^{\nu t}\mathbf{E}\left(UTV_{\mu}^{c}[0,t]\right)^2 dt$$
$$= -\frac{e^{\mu c}U_\mu(\nu)\left[U_\mu^2(\nu)+\nu\left(1-\cosh\left(2cU_\mu(\nu)\right)\right)\right]}{2\nu^3\left[U_\mu(\nu)\cosh\left(cU_\mu(\nu)\right)-\mu\sinh\left(cU_\mu(\nu)\right)\right]\sinh^2\left(cU_\mu(\nu)\right)}.$$

Remark 7. *Using formulas from Borodin, Salminen [2002] (page 642) it is possible to invert the above formula and obtain expression for* $\mathbf{E}\left(UTV_{\mu}^{c}[0,t]\right)^2$ *in terms of parabolic cylinder functions.*

5. Interpretation of upward truncated variation in financial mathematics

As it was mentioned earlier, upward truncated variation appears naturally in the expression for **the least upper bound** for the rate of return from any trading of a financial asset, dynamics of which follows geometric Brownian motion, in the presence of flat commission. Similar result was proved in Łochowski [2008] for truncated variation, however, truncated variation is not the least upper bound.

Indeed, similarly as in Łochowski [2008], let us assume that the dynamics of the prices P_t of some financial asset (e.g. stock) is the following $P_t = \exp(\mu t + \sigma B_t)$. We are interested in the maximal possible profit coming from trading this single instrument during time interval $[0, T]$. We buy the instrument at the moments $0 \leq t_1 < ... < t_n < T$ and sell it at the moments $s_1 < ... < s_n \leq T$, such that $t_1 < s_1 < t_2 < s_2 < ... < t_n < s_n$, in order to obtain the maximal possible profit. Furthermore we assume that for every transaction we have to pay a flat commission and γ is the ratio of the transaction value paid for the commission.

The maximal possible rate of return from our strategy reads as (cf. Łochowski [2008])

$$\sup_n \sup_{0 \leq t_1 < s_1 < ... < t_n < s_n \leq T} \frac{P_{s_1}}{P_{t_1}} \frac{1-\gamma}{1+\gamma} ... \frac{P_{s_n}}{P_{t_n}} \frac{1-\gamma}{1+\gamma} - 1.$$

Let M_n be the set of all partitions

$$\pi = \{0 \leq t_1 < s_1 < ... < t_n < s_n \leq T\}.$$

To see that $\exp\left(\sigma UTV_{\mu/\sigma}^{c/\sigma}[0, T]\right) - 1$ with $c = \ln \frac{1+\gamma}{1-\gamma}$ is the least upper bound for maximal possible rate of return let us substitute

$$\begin{aligned}
\sup_n \sup_{M_n} \prod_{i=1}^n \left\{ \frac{P_{s_i}}{P_{t_i}} \frac{1-\gamma}{1+\gamma} \right\} &= \sup_n \sup_{M_n} \prod_{i=1}^n \left\{ \frac{\exp(\mu s_i + \sigma B_{s_i})}{\exp(\mu t_i + \sigma B_{t_i})} e^{-c} \right\} \\
&= \sup_n \sup_{M_n} \exp\left(\sigma \sum_{i=1}^n \left\{ \left(\frac{\mu}{\sigma} s_i + B_{s_i} \right) - \left(\frac{\mu}{\sigma} t_i + B_{t_i} \right) - \frac{c}{\sigma} \right\} \right) \\
&= \exp\left(\sigma \sup_n \sup_{M_n} \sum_{i=1}^n \left\{ \left(\frac{\mu}{\sigma} s_i + B_{s_i} \right) - \left(\frac{\mu}{\sigma} t_i + B_{t_i} \right) - \frac{c}{\sigma} \right\} \right) \\
&= \exp\left(\sigma UTV_{\mu/\sigma}^{c/\sigma}[0, T] \right).
\end{aligned}$$

This gives the claimed bound.

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