# SUBSPACE HYPERCYCLICITY

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Abstract. A bounded linear operator $T$ on Hilbert space is subspace-hypercyclic for a subspace $\mathcal{M}$ if there exists a vector whose orbit under $T$ intersects the subspace in a relatively dense set. We construct examples to show that subspace-hypercyclicity is interesting, including a nontrivial subspace-hypercyclic operator that is not hypercyclic. There is a Kitai-like criterion that implies subspace-hypercyclicity and although the spectrum of a subspace-hypercyclic operator must intersect the unit circle, not every component of the spectrum will do so. We show that, like hypercyclicity, subspace-hypercyclicity is a strictly infinite-dimensional phenomenon. Additionally, compact or hyponormal operators can never be subspace-hypercyclic.

## 1. Introduction

An operator on a Banach space is called *hypercyclic* if there is a vector whose orbit under the operator is dense in the space; such a vector is called a hypercyclic vector for the operator. It is somewhat surprising that hypercyclic operators exist since they do not exist on finite-dimensional Banach spaces. The study of hypercyclicity goes back a long way, and has been studied in more general settings, for example in topological vector spaces. A good reference are the survey papers of K.-G. Grosse-Erdmann [9, 10], which contain a guide to what is known and not known about hypercyclicity (and universality, a more general notion). In this paper, we study the problem of when the orbit of a vector under an operator, intersected with a subspace, is dense in that subspace.

We start by giving some context to our research. The first (and still most famous) example of a hypercyclic operator on Banach spaces was given by Rolewicz [20] in 1969: the example is the backward shift on $\ell^p$ multiplied by a complex number of modulus bigger than 1. The first systematic study of hypercyclicity on Banach spaces occurred in Kitai's doctoral dissertation [14], where the famous Hypercyclicity Criterion was introduced. This criterion was rediscovered later by Gethner and Shapiro in [6]. It was an open question for many years whether (a stronger version of) this criterion was in fact equivalent to hypercyclicity: it was recently shown by de la Rosa and Read [5] that it is not. Furthermore, Bayart and Matheron showed in [2] that the equivalence fails on classical Banach spaces, and even on Hilbert space.

One reason the concept of hypercyclicity is interesting is because it relates to the invariant subset problem: does every bounded operator on a Banach space have a nontrivial invariant closed subset? An operator has no nontrivial invariant closed

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subsets if and only if every nonzero vector is hypercyclic. It is known that such operators exist on Banach spaces [19] but its existence is still an open problem in Hilbert space.

The question of hypercyclicity is really a dynamical one. As such, one is interested in the possible behaviour of the orbit of a vector under the operator. For example, if such an orbit is not dense, what other "forms" might it have? One possible direction for studying these questions was undertaken in [8] (as cited in [9, Section 1a]): namely, an operator is *hypercyclic for a nonempty closed set $A$* if there exists a vector $x$ such that the closure of the orbit of $x$ under the operator contains $A$.

Compare this concept with the remarkable result of Bourdon and Feldman [4]: if the orbit of a vector under an operator is somewhere dense, then it is everywhere dense; that is, if the closure of the orbit has nonempty interior, then the operator must be hypercyclic. Thus, if we want to study hypercyclicity for closed sets $A$ we must restrict ourselves to cases where $A$ has empty interior; for example, when $A$ is a (nontrivial) subspace.

In the present paper, we undertake the study of a special case. We ask ourselves whether it is possible for a bounded operator on Hilbert space to have the property that the orbit of some vector under the operator "touches a subspace enough times to fill it". In more technical jargon, if $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, we ask whether the orbit

$$\mathrm{Orb}(T, x) := \{T^n x : n \in \mathbb{N}_0\}$$

has the property that $\mathrm{Orb}(T, x) \cap \mathcal{M}$ is dense in $\mathcal{M}$ for a nonzero subspace $\mathcal{M}$. We call this concept subspace-hypercyclicity.

We should note that, although many of the results we present in this paper are undoubtedly true for Banach spaces, we prefer to deal with the Hilbert space case exclusively, for the sake of simplicity. Also, in Hilbert space one might follow other avenues of research. For example, given a subspace $\mathcal{M}$ and the orthogonal projection $P$ on it, one may ask if it is possible for $P(\mathrm{Orb}(T, x))$ to be dense in the subspace $\mathcal{M}$. We may investigate this question in further papers.

The paper is organized as follows. In the second section of this paper, we will introduce formally the concept of subspace-hypercyclicity and show some "trivial" examples. They will be trivial in the sense that the subspace $\mathcal{M}$ is invariant under the operator.

Next, in the third section, we show the existence of nontrivial examples. For this, we introduce the concept of subspace-transitivity and we show that a "Subspace-Hypercyclicity Criterion" holds. We also show, by giving a further example, that said criterion is not a necessary condition. The examples we show in this section are all based on the backward shift on $\ell^2$.

We prove in the fourth section of this paper that subspace-hypercyclicity, like hypercyclicity, is a purely infinite-dimensional concept. Namely, if an operator is subspace-hypercyclic for some subspace $\mathcal{M}$, then $\mathcal{M}$ is not finite-dimensional. Furthermore, there are no compact subspace-hypercyclic operators nor hyponormal subspace-hypercyclic operators.

In the last section of this paper, we conclude with some open questions for future research.

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## 2. Definition and Some Trivial Examples

In this note $\mathcal{H}$ always denotes a separable Hilbert space over $\mathbb{C}$, the field of complex numbers. Usually, it will be the case that $\mathcal{H}$ is infinite-dimensional and we will explicitly indicate when a result or definition only holds for finite or infinite dimensions. According to usual practice, whenever we talk about a subspace $\mathcal{M}$ of $\mathcal{H}$ we will assume that $\mathcal{M}$ is topologically closed.

We will denote by $\mathbf{B}(\mathcal{H})$ the set of all bounded linear operators on $\mathcal{H}$. We usually will refer to elements of $\mathbf{B}(\mathcal{H})$ as just "operators". The use of the symbol ":=" indicates a definition.

We start with our main definition.

**Definition 2.1.** *Let $T \in \mathbf{B}(\mathcal{H})$ and let $\mathcal{M}$ be a nonzero subspace of $\mathcal{H}$. We say that $T$ is* subspace-hypercyclic *for $\mathcal{M}$ if there exists $x \in \mathcal{H}$ such that* $\operatorname{Orb}(T,x) \cap \mathcal{M}$ *is dense in $\mathcal{M}$. We call $x$ a* subspace-hypercyclic vector*.*

The definition above reduces to the classical definition of hypercyclicity if $\mathcal{M} = \mathcal{H}$. Observe also that the subspace-hypercyclic vector $x$ is necessarily nonzero and we may assume, if needed, that $x$ belongs to $\mathcal{M}$.

We start by showing the simplest example of a subspace-hypercyclic operator that is not hypercyclic.

**Example 2.2.** *Let $T$ be a hypercyclic operator on $\mathcal{H}$ with hypercyclic vector $x$ and let $I$ be the identity operator on $\mathcal{H}$. Then, the operator $T \oplus I : \mathcal{H} \oplus \mathcal{H} \to \mathcal{H} \oplus \mathcal{H}$ is subspace-hypercyclic for the subspace $\mathcal{M} := \mathcal{H} \oplus \{0\}$ with subspace-hypercyclic vector $x \oplus 0$. Clearly, $T \oplus I$ is not hypercyclic.*

The above example is trivial in the sense that $\mathcal{M}$ is an invariant subspace for $T \oplus I$. In fact, it is obvious that $T \oplus I \big|_{\mathcal{M}}$ is in fact a hypercyclic operator.

The following example is trivial in the same sense.

**Example 2.3.** *Let $T$ be a hypercyclic operator on $\mathcal{H}$ with hypercyclic vector $x$ and assume that $C \in \mathbf{B}(\mathcal{H})$ is nonzero and has closed range $\mathcal{M}$. If $A \in \mathbf{B}(\mathcal{H})$ satisfies the equation $AC = CT$, then it can easily be checked that $A$ is subspace-hypercyclic for $\mathcal{M}$ with subspace-hypercyclic vector $Cx$.*

The subspace-hypercyclicity of the above example is obviously due to the fact that $A \big|_{\mathcal{M}}$ is a hypercyclic operator, which can be proven easily. (Alternatively, this follows from well-known facts for transitive and hypercyclic operators: two references are the paper [18, Lemma 2.1] and the notes [23, Proposition 1.13]).

From the above two examples, one can obtain subspace-hypercyclic operators from known hypercyclic operators, the simplest examples of which are multiples of the backward shift.

Recall that on $\ell^2$, the Hilbert space of all square summable complex sequences, the backward shift $B$ is defined as

$$B(x_0, x_1, x_2, x_3, \ldots) := (x_1, x_2, x_3, x_4, \ldots).$$

As we mentioned in the introduction, it was shown in [20] that $\lambda B$ is a hypercyclic operator if $|\lambda| > 1$. Thus setting $T = \lambda B$ in Example 2.2 provides a concrete example of a subspace-hypercyclic operator.

One can find another concrete example by setting $T = \lambda B$ and finding operators $A$ and $C$ that satisfy the conditions of Example 2.3, but one should note that if $T$ is a multiple of the backward shift and $A$ is injective, $C$ would have to be zero if it is to satisfy the equation $AC = CT$. (The proof is easy, see [16] if needed.)

Nevertheless, one can obtain an interesting example. For the following, the reader should be familiar with some basic facts about the Hardy-Hilbert space $\mathbf{H}^2$. Let $\mathbb{D}$ be the open unit disk in $\mathbb{C}$ and $\phi : \mathbb{D} \to \mathbb{D}$ be an analytic funtion. Let $T_\phi$ denote the analytic Toeplitz operator on $\mathbf{H}^2$ defined by $T_\phi(f) = \phi f$, let $B$ be the backward shift on $\mathbf{H}^2$ and let $C_\phi$ denote the composition operator on $\mathbf{H}^2$ defined by $C_\phi(f) = f \circ \phi$ (see, for example, [17] for the definition of $\mathbf{H}^2$ and the basic properties of the above operators).

**Example 2.4.** *Let $\phi \in \mathbf{H}^2$ be an inner function with $\phi(0) = 0$ and $\phi$ not the identity function. Then $T_\phi^* C_\phi = C_\phi B$ and hence $\lambda T_\phi^*$ is subspace-hypercyclic for the subspace $\operatorname{ran} C_\phi$ if $|\lambda| > 1$.*

On the other hand, observe that $\lambda T_\phi^*$ is hypercyclic, by the characterization given by Godefroy and Shapiro [7]. Note that a subspace-hypercyclic vector for $\lambda T_\phi^*$ with respect to $\operatorname{ran} C_\phi$ is necesarily not a hypercyclic vector for $\lambda T_\phi^*$.

As in the case of hypercyclicity, analytic Toeplitz operators can never be subspace-hypercyclic. Indeed, suppose $T_\phi$ is an analytic Toeplitz operator which is subspace-hypercyclic for a subspace $\mathcal{M}$. Let $k_\lambda \in \mathbf{H}^2$ be the reproducing kernel for $\lambda \in \mathbb{D}$. Then, as is well-known, $k_\lambda \in \ker(T_\phi^* - \overline{\phi(\lambda)})$. By Proposition 4.6 below, we have that $k_\lambda \subseteq \mathcal{M}^\perp$ for all $\lambda \in \mathbb{D}$. The fact that the reproducing kernels have a dense span in $\mathbf{H}^2$ implies that $\mathbf{H}^2 \subseteq \mathcal{M}^\perp$, and hence that $\mathcal{M} = \{0\}$.

## 3. Some nontrivial examples and a subspace-hypercyclicity criterion

We would like to find examples of subspace-hypercyclic operators for a subspace $\mathcal{M}$ such that $\mathcal{M}$ is not invariant under the operator. The results presented in this section, including a subspace-hypercyclicity criterion, will help us achieve that goal.

Let us denote the set of subspace-hypercyclic vectors for $\mathcal{M}$ by

$$\mathrm{HC}(T, \mathcal{M}) := \{x \in \mathcal{H} : \mathrm{Orb}(T, x) \cap \mathcal{M} \text{ is dense in } \mathcal{M}\}.$$

The following lemma holds true in the classical setting and the proof in our setting is the same. We include here its short proof for the sake of completeness.

**Lemma 3.1.** *Let $T \in \mathbf{B}(\mathcal{H})$ and let $\mathcal{M}$ be a nonzero subspace of $\mathcal{H}$. Then*

$$\mathrm{HC}(T, \mathcal{M}) = \bigcap_{j=1}^{\infty} \bigcup_{n=0}^{\infty} T^{-n}(B_j)$$

*where $\{B_j\}$ is a countable open basis for the relative topology of $\mathcal{M}$ as a subspace of $\mathcal{H}$.*

*Proof.* We have that $x \in \bigcap_{j=1}^{\infty} \bigcup_{n=0}^{\infty} T^{-n}(B_j)$ if and only if, for any $j \in \mathbb{N}$, there exists a number $n \in \mathbb{N}_0$ such that $T^n x \in B_j$. But since $\{B_j\}$ is a basis for the relative topology of $\mathcal{M}$, this occurs if and only if $\mathrm{Orb}(T, x) \cap \mathcal{M}$ is dense in $\mathcal{M}$ or, equivalently, if $x \in HC(T, \mathcal{M})$. □

Hence, if the set in the display above is nonempty, $T$ is subspace-hypercyclic for $\mathcal{M}$. Our following lemma will obtain much more than what is needed to imply the nonemptiness of said set. The following definition will be convenient.

**Definition 3.2.** *Let $T \in \mathbf{B}(\mathcal{H})$ and let $\mathcal{M}$ be a nonzero subspace of $\mathcal{H}$. We say that $T$ is* subspace-transitive with respect to $\mathcal{M}$ *if for all nonempty sets $U \subseteq \mathcal{M}$ and $V \subseteq \mathcal{M}$, both relatively open, there exists $n \in \mathbb{N}_0$ such that $T^{-n}(U) \cap V$ contains a relatively open nonempty subset of $\mathcal{M}$.*

**Note added:** The referee has kindly pointed out the following equivalences for our definition above, which greatly simplify Lemma 3.4 and Theorem 3.6 below.

**Theorem 3.3.** *Let $T \in \mathbf{B}(\mathcal{H})$ and let $\mathcal{M}$ be a nonzero subspace of $\mathcal{H}$. Then the following conditions are equivalent:*

(i) *The operator $T$ is subspace-transitive with respect to $\mathcal{M}$.*
(ii) *For any nonempty sets $U \subseteq \mathcal{M}$ and $V \subseteq \mathcal{M}$, both relatively open, there exists $n \in \mathbb{N}_0$ such that $T^{-n}(U) \cap V$ is a relatively open nonempty subset of $\mathcal{M}$.*
(iii) *For any nonempty sets $U \subseteq \mathcal{M}$ and $V \subseteq \mathcal{M}$, both relatively open, there exists $n \in \mathbb{N}_0$ such that $T^{-n}(U) \cap V$ is nonempty and $T^n(\mathcal{M}) \subseteq \mathcal{M}$.*

*Proof.* The implication (ii) $\implies$ (i) is obvious. The impication (iii) $\implies$ (ii) is obvious once one observes that the operator $T^n \big|_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}$ is continuous and hence $T^{-n}(U)$ is relatively open in $\mathcal{M}$ if $U$ is relatively open in $\mathcal{M}$.

We will show that (i) $\implies$ (iii). Let $T$ be subspace-transitive with respect to $\mathcal{M}$ and let $U$ and $V$ be nonempty relatively open subsets of $\mathcal{M}$. By Definition 3.2 above, it follows that there exists $n \in \mathbb{N}_0$ such that $T^{-n}(U) \cap V$ contains a relatively open nonempty set, say $W$. Thus, in particular, $T^{-n}(U) \cap V$ is nonempty. Now, let $x \in \mathcal{M}$. Since $W \subseteq T^{-n}(U)$, it follows that $T^n(W) \subseteq \mathcal{M}$. Take $x_0 \in W$. Since $W$ is relatively open and $x \in \mathcal{M}$, for $r > 0$ small enough, we have $x_0 + rx \in W$, and hence $T^n(x_0 + rx) = T^n(x_0) + rT^n(x) \in \mathcal{M}$. Since $T^n(x_0) \in \mathcal{M}$, subtracting it and dividing by $r$ leads to $T^n(x) \in \mathcal{M}$, showing $T^n(\mathcal{M}) \subseteq \mathcal{M}$ and finishing the proof. □

The following lemma will achieve "half" of the classical equivalence of topological transitivity and hypercyclicity.

**Lemma 3.4.** *Let $T \in \mathbf{B}(\mathcal{H})$ and let $\mathcal{M}$ be a nonzero subspace of $\mathcal{H}$. Assume that $T$ is subspace-transitive with respect to $\mathcal{M}$. Then,*

$$\bigcap_{j=1}^{\infty} \bigcup_{n=0}^{\infty} T^{-n}(B_j) \cap \mathcal{M},$$

*is a dense subset of $\mathcal{M}$. Here $\{B_j\}$ is a countable open basis for the (relative) topology of $\mathcal{M}$.*

*Proof.* By Theorem 3.3 above, for each $j$ and $k$, there exists $n_{j,k} \in \mathbb{N}_0$ such that the set $T^{-n_{j,k}}(B_j) \cap B_k$ is nonempty and relatively open. Hence, the set

$$A_j := \bigcup_{k=1}^{\infty} T^{-n_{j,k}}(B_j) \cap B_k$$

is relatively open. Furthermore, each $A_j$ is dense, since it intersects each $B_k$. By the Baire Category Theorem, this implies that

$$\bigcap_{j=1}^{\infty} A_j = \bigcap_{j=1}^{\infty} \bigcup_{k=1}^{\infty} T^{-n_{j,k}}(B_j) \cap B_k$$

is a dense set. But clearly,

$$\bigcap_{j=1}^{\infty} \bigcup_{k=1}^{\infty} T^{-n_{j,k}}(B_j) \cap B_k \subseteq \bigcap_{j=1}^{\infty} \bigcup_{n=1}^{\infty} T^{-n}(B_j) \cap \mathcal{M},$$

and the result follows. □

The above lemmas clearly combine to imply the following theorem.

**Theorem 3.5.** *Let $T \in \mathbf{B}(\mathcal{H})$ and let $\mathcal{M}$ be a nonzero subspace of $\mathcal{H}$. If $T$ is subspace-transitive for $\mathcal{M}$ then $T$ is subspace-hypercyclic for $\mathcal{M}$.*

We will show (see the comment after Example 3.8 below) that the converse of the above theorem is not true.

The following theorem is a subspace-hypercyclicity criterion, stated in the style of [10]

**Theorem 3.6.** *Let $T \in \mathbf{B}(\mathcal{H})$ and let $\mathcal{M}$ be a nonzero subspace of $\mathcal{H}$. Assume there exist $X$ and $Y$, dense subsets of $\mathcal{M}$, and an increasing sequence of positive integers $\{n_k\}$ such that*

(i) *$T^{n_k} x \to 0$ for all $x \in X$,*

(ii) *for each $y \in Y$, there exist a sequence $\{x_k\}$ in $\mathcal{M}$ such that*

$$x_k \to 0 \quad \text{and} \quad T^{n_k} x_k \to y,$$

(iii) *$\mathcal{M}$ is an invariant subspace for $T^{n_k}$ for all $k \in \mathbb{N}$.*

*Then $T$ is subspace-transitive with respect to $\mathcal{M}$ and hence $T$ is subspace-hypercyclic for $\mathcal{M}$.*

*Proof.* Let $U$ and $V$ be nonempty relatively open subsets of $\mathcal{M}$. We will show that there exists $k \in \mathbb{N}_0$ such that $T^{-n_k}(U) \cap V$ is nonempty. By Theorem 3.3, since $T^{n_k}(\mathcal{M}) \subseteq \mathcal{M}$, it will follow that $T$ is subspace-transitive with respect to $\mathcal{M}$.

Since $X$ and $Y$ are dense in $\mathcal{M}$, there exists $v \in X \cap V$ and $u \in Y \cap U$. Furthermore, since $U$ and $V$ are relatively open, there exists $\varepsilon > 0$ such that the $\mathcal{M}$-ball centered at $v$ of radius $\varepsilon$ is contained in $V$ and the $\mathcal{M}$-ball centered at $u$ of radius $\varepsilon$ is contained in $U$.

By hypothesis, given these $v \in X$ and $u \in Y$, one can choose $k$ large enough such that there exists $x_k \in \mathcal{M}$ with

$$\|T^{n_k} v\| < \frac{\varepsilon}{2}, \quad \|x_k\| < \varepsilon, \quad \text{and} \quad \|T^{n_k} x_k - u\| < \frac{\varepsilon}{2}.$$

We have:

- $v + x_k \in V$. Indeed, since $v \in \mathcal{M}$ and $x_k \in \mathcal{M}$, it follows that $v + x_k \in \mathcal{M}$. Also, since
$$\|(v + x_k) - v\| = \|x_k\| < \varepsilon,$$
it follows that $v + x_k$ is in the $\mathcal{M}$-ball centered at $v$ of radius $\varepsilon$ and hence $v + x_k \in V$.
- $T^{n_k}(v + x_k) \in U$. Indeed, since $v$ and $x_k$ are in $\mathcal{M}$ and $\mathcal{M}$ is invariant under $T^{n_k}$, it follows that $T^{n_k}(v + x_k) \in \mathcal{M}$. Also,
$$\|T^{n_k}(v + x_k) - u\| \le \|T^{n_k} v\| + \|T^{n_k} x_k - u\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$
and hence $T^{n_k}(v + x_k)$ is in the $\mathcal{M}$-ball centered at $u$ of radius $\varepsilon$ and thus $T^{n_k}(v + x_k) \in U$.

The two facts above imply that $v + x_k \in T^{-n_k}(U) \cap V$ and hence this set is nonempty. □

A natural question to ask is whether condition (iii) in the theorem above is really necessary. After a preliminary version of our paper was distributed, Le [15] proved an alternative Subspace-Hypercyclicity Criterion with a condition weaker than condition (iii) above. Evenmore, Le shows that Theorem 3.6 is false if only conditions (i) and (ii) hold.

The referee has pointed out to us that, with the conditions of Theorem 3.6, the sequence of operators $T^{n_k}\big|_{\mathcal{M}}\colon \mathcal{M} \to \mathcal{M}$ is universal, as follows from (for example) [9, Theorem 2], and thus $T$ is subspace-hypercyclic for $\mathcal{M}$. Note that Theorem 3.6 is stronger because it shows that $T$ is, in fact, subspace-transitive for $\mathcal{M}$.

In general, let $T : \mathcal{H} \to \mathcal{H}$ be an operator for which there exists an increasing sequence $\{n_k\}$ of natural numbers such that $T^{n_k}(\mathcal{M}) \subseteq \mathcal{M}$. If the sequence $T^{n_k}\big|_{\mathcal{M}}\colon \mathcal{M} \to \mathcal{M}$ is universal, it follows that $T$ is subspace-hypercyclic for $\mathcal{M}$. Contrast this with Example 3.8 which will show that an operator $T$ can be subspace-hypercyclic for a subspace $\mathcal{M}$ not invariant for any power of the operator.

The following is our first example of a subspace-hypercyclic operator for a subspace $\mathcal{M}$ such that $\mathcal{M}$ is not invariant for the operator. Recall that the *forward shift* $S$ on $\ell^2$ is the operator defined by

$$S(x_0, x_1, x_2, x_3, \ldots) := (0, x_0, x_1, x_2, \ldots).$$

Clearly $BS$ equals the identity on $\ell^2$. Observe also that $S$ is an isometry.

**Example 3.7.** *Let $\lambda \in \mathbb{C}$ be of modulus greater than 1 and consider $T := \lambda B$ where $B$ is the backward shift on $\ell^2$. Let $\mathcal{M}$ be the subspace of $\ell^2$ consisting of all sequences with zeroes on the even entries; that is,*

$$\mathcal{M} := \left\{ \{a_n\}_{n=0}^{\infty} \in \ell^2 \,:\, a_{2k} = 0 \textit{ for all } k \right\}.$$

*Then $T$ is subspace-hypercylic for $\mathcal{M}$.*

*Proof.* One can see that $T^2$ on $\mathcal{M}$ behaves like the hypercyclic operator $\lambda^2 B$ on $\ell^2$ and hence $T$ is subspace-hypercyclic for $\mathcal{M}$.

We will apply Theorem 3.6 to give an alternative proof. Let $X = Y$ be the subset of $\mathcal{M}$ consisting of all finite sequences; i.e., those sequences that only have a finite number of nonzero entries: this clearly is a dense subset of $\mathcal{M}$. Let $n_k := 2k$. Let us check that conditions (i), (ii) and (iii) hold.

Let $x \in X$. Since $x$ only has finitely many nonzero entries, $T^{2k}x$ will be zero eventually for $k$ large enough. Thus (i) holds.

Let $y \in Y$ and define $x_k := \frac{1}{\lambda^{2k}} S^{2k} y$, where $S$ is the forward shift on $\ell^2$. Each $x_k$ is in $\mathcal{M}$ since the even entries of $y$ are shifted by $S^{2k}$ into the even entries of $x_k$. We have

$$\|x_k\| = \frac{1}{|\lambda|^{2k}} \|y\|,$$

and thus it follows that $x_k \to 0$, since $|\lambda| > 1$. Also, because

$$T^{2k} x_k = (\lambda B)^{2k} x_k = (\lambda B)^{2k} \frac{1}{\lambda^{2k}} S^{2k} y = y,$$

we have that condition (ii) holds.

That condition (iii) holds follows from the fact that if a vector has a zero entry on all even positions then it will also have a zero entry on all even positions after the application of the backward shift any even number of times.

The subspace-hypercyclicity of $T$ now follows. □

As we commented before, the above is our first example of a subspace-hypercyclic operator $T$ for which $\mathcal{M}$ is not invariant under $T$. Observe that, nevertheless, the subspace $\mathcal{M}$ in the above example is invariant for $T^2$.

The example above can be generalized. Let $a \in \mathbb{N}_0$ and $b \in \mathbb{N}$ be some fixed numbers with $a < b$ and consider the subspace $\mathcal{M}$ of $\ell^2$ consisting of all sequences with zeroes on the entries indexed by the set $\{a + kb \,:\, k \in \mathbb{N}_0\}$. Then, the argument above holds for $X$ and $Y$ both equal to the set of all sequences in $\mathcal{M}$ with finitely many nonzero entries and $n_k := bk$. Hence, for $|\lambda| > 1$, the operator $\lambda B$ is subspace-hypercyclic for $\mathcal{M}$. In this case, the space $\mathcal{M}$ is $T^b$ invariant but not $T$ invariant.

As the referee so clearly expressed, all examples given so far are in some sense trivial, "because they contain some more or less hidden element of hypercyclicity". In Examples 2.2 and 2.3 the subspace is invariant under the operator, and hence the operator restricted to the subspace is hypercyclic. In Example 3.7 the subspace is invariant under a power of the operator. In Theorem 3.6, the subspace is invariant for the operators $T^{n_k}$.

The next example, although based on a hypercyclic operator, does not fit into any of the categories above, since the subspace is not invariant for any power of the operator.

**Example 3.8.** *Let $\lambda \in \mathbb{C}$ be of modulus greater than 1 and let $B$ be the backward shift on $\ell^2$. Let $m \in \mathbb{N}$ and $\mathcal{M}$ be the subspace of $\ell^2$ consisting of all sequences with zeroes on the first $m$ entries; that is,*

$$\mathcal{M} := \left\{ \{a_n\}_{n=0}^{\infty} \in \ell^2 \,:\, a_n = 0 \text{ for } n < m \right\}.$$

*Then $\lambda B$ is subspace-hypercylic for $\mathcal{M}$.*

*Proof.* The argument used here is really the same as the one Rolewicz [20] originally used to show hypercyclicity of $\lambda B$. We base our proof on the expositions of Halmos [11, p. 286] and of Jiménez-Munguía [13].

First of all, for a complex sequence $h = \{h_j\}_{j=0}^{\infty}$ with finitely-many nonzero entries, we define its *length* as

$$|h| := \min\{s \in \mathbb{N}_0 \,:\, h_k = 0 \text{ for all } k \geq s\}.$$

We can choose a countable dense subset of $\mathcal{M}$, called $\{f_j\}$, consisting of sequences which have at most a finite number of nonzero entries.

Define a (necessarily increasing) sequence of integers $k_j$ inductively as follows. Let $k_0 = 0$ and for each $j \in \mathbb{N}$ choose $k_j$ in such a way that

$$\frac{\|f_j\|}{|\lambda|^{k_j - k_{j-1}}} \leq \frac{1}{|\lambda|^j}, \tag{1}$$

and such that $k_j > k_{j-1} + |f_{j-1}|$.

Define the vector $f$ by

$$f := \sum_{j=0}^{\infty} \frac{S^{k_j} f_j}{\lambda^{k_j}},$$

where $S$ is the forward shift.

We must first show that $f \in \ell^2$. It follows from inequality (1) that

$$\left\| \frac{S^{k_j} f_j}{\lambda^{k_j}} \right\| = \frac{\|f_j\|}{|\lambda|^{k_j}} \leq \frac{1}{|\lambda|^{j+k_{j-1}}} \leq \frac{1}{|\lambda|^j},$$

and hence the infinite sum converges in norm to an $\ell^2$ vector.

Let $n \in \mathbb{N}$. Since for all $j$ the condition $k_j > k_{j-1} + |f_{j-1}|$ holds, it follows that $k_n > k_j + |f_j|$ for all $j < n$ and hence

$$(\lambda B)^{k_n} \frac{S^{k_j} f_j}{\lambda^{k_j}} = 0.$$

This implies that $(\lambda B)^{k_n} f$ "starts" with the vector $f_n$ and hence that $(\lambda B)^{k_n} f$ is in $\mathcal{M}$.

The condition $k_j > k_{j-1} + |f_{j-1}|$ also implies that the norm of the difference $(\lambda B)^{k_n} f - f_n$ is given by

$$\begin{aligned} \|(\lambda B)^{k_n} f - f_n\|^2 &= \sum_{j=n+1}^{\infty} \left\| \frac{S^{k_j - k_n} f_j}{\lambda^{k_j - k_n}} \right\|^2 \\ &= \sum_{j=n+1}^{\infty} \left( \frac{\|f_j\|}{|\lambda|^{k_j - k_n}} \right)^2 \\ &\leq \sum_{j=n+1}^{\infty} \left( \frac{\|f_j\|}{|\lambda|^{k_j - k_{j-1}}} \right)^2 \\ &\leq \sum_{j=n+1}^{\infty} \frac{1}{|\lambda|^{2j}}. \end{aligned}$$

Let $h \in \mathcal{M}$. Given $\epsilon > 0$, choose $N$ such that $\|h - f_N\| < \frac{\epsilon}{2}$ and such that

$$\left( \sum_{j=N+1}^{\infty} \frac{1}{|\lambda|^{2j}} \right)^{1/2} < \frac{\epsilon}{2}.$$

It then follows that

$$\|(\lambda B)^{k_N} f - h\| < \epsilon,$$

and hence that $\text{Orb}(\lambda B, f) \cap \mathcal{M}$ is dense in $\mathcal{M}$. □

In the example above, it is clear that it is impossible to find an increasing sequence of integers $n_k$ such that $\mathcal{M}$ is invariant for $T^{n_k}$ (since clearly, $\mathcal{M}$ is not invariant for $T^n$ for any $n$). Thus, condition (iii) in Theorem 3.6 is not necessary.

Observe that the operator above does not satisfy Theorem 3.3 and hence this implies that subspace-hypercyclicity for a subspace $\mathcal{M}$ does not imply subspace-transitivity for $\mathcal{M}$. After a preliminary version of this paper was distributed, another example of a subspace-hypercyclic operator for a subspace $\mathcal{M}$ which is not subspace-transitive for $\mathcal{M}$ was given in [15].

It should be noted that the procedure used in Example 3.8 above, could have been used to find a subspace-hypercyclic vector for the operator in Example 3.7.

Let us contrast the behaviour of the subspace-hypercyclic vector in the above two examples. In Example 3.7, the orbit of any subspace-hypercyclic vector $x$ under $T$ goes in and out of the space $\mathcal{M}$ at regular intervals. In Example 3.8, by judiciously choosing the sequence of finite vectors $\{f_j\}$ and the sequence of natural numbers $\{k_j\}$, the orbit of $f$ under $T$ goes in and out of the space $\mathcal{M}$ at irregular intervals; namely, one can find arbitrarily long consecutive elements of the orbit that stay inside the space and arbitrarily long consecutive elements of the orbit that stay outside the space.

We do not know if the vector $f$ in Example 3.8 is hypercyclic. An easy way to obtain a subspace-hypercyclic operator which is not hypercyclic and has the properties of Example 3.8 above, follows.

**Example 3.9.** *Let $|\lambda| > 1$, and consider the operator $T := (\lambda B) \oplus I$ on $\mathcal{H} := \ell^2 \oplus \ell^2$. Let $\mathcal{M}$ be as in Example 3.8 and let $f$ be a subspace-hypercyclic vector for $\mathcal{M}$. Define $\mathcal{N} := \mathcal{M} \oplus \{0\}$. Then $f \oplus 0$ is a subspace-hypercyclic vector for $\mathcal{N}$, but $f \oplus 0$ is not hypercyclic for $\mathcal{H}$. Also, $\mathcal{N}$ is not an invariant subspace for $T^k$ for any $k$.*

## 4. Finite Dimensions

The following easy observation will be useful.

**Proposition 4.1.** *Let $T \in \mathbf{B}(\mathcal{H})$ be subspace-hypercyclic for $\mathcal{M}$. If $\mathcal{N}$ is an invariant subspace for $T$ and $\mathcal{M} \subseteq \mathcal{N}$, then $T\,\big|_{\mathcal{N}} \colon \mathcal{N} \to \mathcal{N}$ is subspace-hypercyclic for $\mathcal{M}$.*

We remind the reader of the following well known definitions.

**Definition 4.2.** *Let $\mathcal{M}$ and $\mathcal{N}$ be subspaces of $\mathcal{H}$. If $\mathcal{M} \cap \mathcal{N} = \{0\}$ and $\mathcal{M} + \mathcal{N} = \mathcal{H}$ we say that $\mathcal{M}$ and $\mathcal{N}$ are* complementary.

**Definition 4.3.** *Let $\mathcal{M}$ and $\mathcal{N}$ be complementary subspaces of $\mathcal{H}$. The* projection onto $\mathcal{M}$ along $\mathcal{N}$ *is the function $P : \mathcal{H} \to \mathcal{H}$ defined as*

$$P(x + y) = x,$$

*where $x \in \mathcal{M}$ and $y \in \mathcal{N}$.*

It can easily be shown that $P$ is a bounded operator. This allows us to obtain the following theorem. An analogous result for hypercyclic operators is due to Herrero [12].

**Theorem 4.4.** *Let $\mathcal{M}$ and $\mathcal{N}$ be complementary subspaces of $\mathcal{H}$ and let $P$ be the projection onto $\mathcal{M}$ along $\mathcal{N}$. Let $T \in \mathbf{B}(\mathcal{H})$ and suppose that $\mathcal{N}$ is invariant under $T$. If $T$ is subspace-hypercyclic for some $\mathcal{L} \subseteq \mathcal{M}$, then $PT\,\big|_{\mathcal{M}}$ is subspace-hypercyclic for $\mathcal{L}$.*

*Proof.* Assume $T$ is subspace-hypercyclic for $\mathcal{L}$ with subspace-hypercyclic vector $x \in \mathcal{L}$. Since $\mathrm{Orb}(T, x) \cap \mathcal{L}$ is dense in $\mathcal{L}$, and $\mathcal{L} \subseteq \mathcal{M}$ it follows that $P(\mathrm{Orb}(T, x)) \cap \mathcal{L}$ is dense in $\mathcal{L}$.

Also, since $\mathcal{N}$ is an invariant subspace for $T$, it follows that $PTP = PT$ and hence that $(PT)^k = PT^k$ for all $k \in \mathbb{N}$. Thus

$$P(\mathrm{Orb}(T, x)) = \mathrm{Orb}(PT\,\big|_{\mathcal{M}}, x).$$

It follows that $\mathrm{Orb}(PT\,\big|_{\mathcal{M}}, x) \cap \mathcal{L}$ is dense in $\mathcal{L}$, as desired. □

The following proposition was shown to us by A. Peris.

**Theorem 4.5.** *Let $T \in \mathbf{B}(\mathcal{H})$. If $T$ is subspace-hypercyclic for some subspace then $\sigma(T) \cap S^1 \neq \varnothing$.*

*Proof.* Assume the intersection is empty. Then, there exist (possibly empty) sets $K_1$ and $K_2$ such that $\sigma(T) = K_1 \cup K_2$ with $K_1 \subseteq \mathbb{D}$ and $K_2 \subseteq \overline{\mathbb{D}}^c$. By the Riesz Decomposition Theorem [21, Theorem 2.10], there exist complementary invariant subspaces $\mathcal{M}_1$ and $\mathcal{M}_2$ such that

$$\sigma\left(T\mid_{\mathcal{M}_1}\right) \subseteq K_1 \quad \text{and} \quad \sigma\left(T\mid_{\mathcal{M}_2}\right) \subseteq K_2.$$

Let $x \in \mathcal{H}$. Then, there exist $x_1 \in \mathcal{M}_1$ and $x_2 \in \mathcal{M}_2$ such that $x = x_1 + x_2$. If $x_2$ was equal to zero, then

$$T^n x = T^n x_1 = \left(T\mid_{\mathcal{M}_1}\right)^n x_1$$

which converges to zero because $\sigma\left(T\mid_{\mathcal{M}_1}\right) \subseteq \mathbb{D}$. Thus, $\mathrm{Orb}(T, x)$ is bounded and hence its intersection with any subspace cannot be dense in that subspace.

Assume $x_2$ is not equal to zero. We have

$$\|T^n x\| = \|T^n x_1 + T^n x_2\| \geq \|T^n x_2\| - \|T^n x_1\|,$$

and as before, $\|T^n x_1\|$ goes to zero. Since $\sigma\left(T\mid_{\mathcal{M}_2}\right) \subseteq \overline{\mathbb{D}}^c$, it follows that $\|T^n x_2\|$ goes to infinity and hence $\|T^n x\|$ goes to infinity. Thus only finitely many elements of $\mathrm{Orb}(T, x)$ intersect any bounded set, and hence $\mathrm{Orb}(T, x)$ intersected with a subspace cannot be dense in that subspace. □

As we did in Example 2.2, it can easily be shown that the operator $(2B) \oplus (3I) : \ell^2 \oplus \ell^2 \to \ell^2 \oplus \ell^2$ is subspace hypercyclic for $\mathcal{M} := \ell^2 \oplus \{0\}$ with subspace hypercyclic vector $f \oplus 0$, where $f$ is a hypercyclic vector for $2B$. Observe that $\sigma((2B) \oplus (3I))$ is the closed disk of radius 2 union the singleton $\{3\}$. Thus, it is not true that every component of the spectrum must intersect the unit circle for a subspace-hypercyclic operator, as is the case for classical hypercyclicity.

Is there any spectral restriction besides the one given by Theorem 4.5? Namely, given an arbitrary nonempty compact subset $K$ of $\mathbb{C}$ which intersects $S^1$, is there a subspace-hypercyclic operator with that set as its spectrum?

Yes. Let $K_0$ be a component of $K$ which intersects $S^1$. By a theorem of Shkarin [24], one can construct a hypercyclic operator $T$ with spectrum $K_0$. Take the direct sum of $T$ with an operator whose spectrum is the closure of $K \setminus K_0$ to obtain the desired operator. This proof was communicated to us independently by A. Peris and by the referee.

**Proposition 4.6.** *Let $T \in \mathbf{B}(\mathcal{H})$ be subspace-hypercyclic for $\mathcal{M}$. Then* $\ker(T^* - \lambda) \subseteq \mathcal{M}^\perp$ *for all* $\lambda \in \mathbb{C}$.

*Proof.* Assume that $\mathrm{Orb}(T, x) \cap \mathcal{M}$ is dense in $\mathcal{M}$. Fix $\lambda \in \mathbb{C}$ and let $y$ be in $\ker(T^* - \lambda)$. Let $\varphi : \mathcal{M} \to \mathbb{C}$ be the functional defined by $\varphi(x) := \langle x, y\rangle$. Clearly $\varphi$ is surjective if and only if $y \notin \mathcal{M}^\perp$.

Observe that

$$\langle T^n x, y\rangle = \langle x, T^{*n} y\rangle = \langle x, \lambda^n y\rangle = \overline{\lambda}^n \langle x, y\rangle,$$

and hence

$$\varphi(\mathrm{Orb}(T, x) \cap \mathcal{M}) = \left\{\overline{\lambda}^n \langle x, y\rangle \ : \ \text{there exists } n \text{ such that } T^n x \in \mathcal{M}\right\}. \tag{2}$$

But, if $\mathrm{Orb}(T,x)\cap\mathcal{M}$ is dense in $\mathcal{M}$, then $\varphi(\mathrm{Orb}(T,x)\cap\mathcal{M})$ must be dense in $\mathbb{C}$, unless $\varphi$ is not surjective. Since the set in equation (2) is clearly not dense in $\mathbb{C}$, it follows that $\varphi$ is not surjective and hence $y\in\mathcal{M}^\perp$. □

The result above can be generalized. We will need the following fact.

**Lemma 4.7.** *Let $T\in\mathbf{B}(\mathcal{H})$ and $\lambda\in\mathbb{C}$. If $(T-\lambda)^p y=0$ for some $p\in\mathbb{N}$, then for $n\geq p$*

$$T^n y=\sum_{k=1}^{p}\binom{p}{k}\binom{n}{p}\frac{k}{n-p+k}(-1)^{k-1}\lambda^{n-p+k}\,T^{p-k}y.$$

*Proof.* This follows by a straightforward (but tedious) induction argument. □

The case $p=1$ of the following proposition was shown in Proposition 4.6 above.

**Proposition 4.8.** *Let $T\in\mathbf{B}(\mathcal{H})$ be subspace-hypercyclic for $\mathcal{M}$. Then* $\ker(T^*-\lambda)^p\subseteq\mathcal{M}^\perp$ *for all $\lambda\in\mathbb{C}$ and all $p\in\mathbb{N}$.*

*Proof.* Assume that $\mathrm{Orb}(T,x)\cap\mathcal{M}$ is dense in $\mathcal{M}$. Set $\lambda\in\mathbb{C}$, $p\in\mathbb{N}$ and let $y$ be in $\ker(T^*-\lambda)^p$. As before, let $\varphi:\mathcal{M}\to\mathbb{C}$ be the functional defined by $\varphi(x):=\langle x,y\rangle$. Again, $\varphi$ is surjective if and only if $y\notin\mathcal{M}^\perp$.

Observe that, for $n\geq p$ we have, by Lemma 4.7,

$$\begin{aligned}\langle T^n x,y\rangle &= \langle x,T^{*n}y\rangle\\ &= \sum_{k=1}^{p}\binom{p}{k}\binom{n}{p}\frac{k}{n-p+k}(-1)^{k-1}\overline{\lambda}^{n-p+k}\left\langle x,T^{*(p-k)}y\right\rangle\\ &= \overline{\lambda}^{n-p}\sum_{k=1}^{p}\binom{p}{k}\binom{n}{p}\frac{k}{n-p+k}(-1)^{k-1}\overline{\lambda}^{k}\left\langle T^{p-k}x,y\right\rangle.\end{aligned}$$

Since

$$\binom{n}{p}=\frac{n(n-1)\cdots(n-p+k)\cdots(n-p+1)}{p!},$$

for each $k$ less than or equal to $p$, it follows that $\binom{n}{p}\frac{1}{n-p+k}$ is a polynomial in the variable $n$ of degree $p-1$. Hence

$$\langle T^n x,y\rangle=\overline{\lambda}^{n-p}Q(n)$$

where $Q(n)$ is a polynomial in the variable $n$ of degree at most $p-1$ with complex coefficients (the coefficients depend, of course, on $\lambda$, $p$, $T$, $x$ and $y$).

Hence we have

$$\begin{aligned}&\varphi(\mathrm{Orb}(T,x)\cap\mathcal{M})=\\ &\qquad\left\{\left\langle T^j x,y\right\rangle:\ \text{there exists } j=0,1,\ldots,p-1 \text{ such that } T^j x\in\mathcal{M}\right\}\\ &\qquad\qquad\cup\left\{\overline{\lambda}^{n-p}Q(n):\ \text{there exists } n\geq p \text{ such that } T^n x\in\mathcal{M}\right\}.\end{aligned}$$

It can be checked that the above set is never dense in $\mathbb{C}$.

But, if $\mathrm{Orb}(T,x)\cap\mathcal{M}$ were dense in $\mathcal{M}$, then $\varphi(\mathrm{Orb}(T,x)\cap\mathcal{M})$ would be dense in $\mathbb{C}$, which we just agreed was impossible. Thus $\varphi$ is not surjective and hence $y\in\mathcal{M}^\perp$. This finishes the proof. □

**Theorem 4.9.** *Let $\mathcal{H}$ be finite-dimensional. If $T \in \mathbf{B}(\mathcal{H})$ then $T$ is not subspace hypercyclic for any $\mathcal{M}$.*

*Proof.* Since $\mathcal{H}$ is finite-dimensional and $T^* \in \mathbf{B}(\mathcal{H})$, it is well known (e.g., [1, p. 174]) that there exist complex numbers $\lambda_1, \lambda_2, \dots, \lambda_s$ and natural numbers $p_1, p_2, \dots, p_s$ such that $\mathcal{H}$ is the direct sum of the subspaces

$$\ker(T^* - \lambda_1)^{p_1}, \ker(T^* - \lambda_2)^{p_2}, \dots, \ker(T^* - \lambda_s)^{p_s}.$$

If $T$ was subspace-hypercyclic for some $\mathcal{M} \neq \{0\}$, then by Proposition 4.8, for each $j$, we have

$$\ker(T^* - \lambda_j)^{p_j} \subseteq \mathcal{M}^\perp.$$

Hence $\mathcal{H} \subseteq \mathcal{M}^\perp$ and thus $\mathcal{M} = \{0\}$, a contradiction. □

**Theorem 4.10.** *Let $T \in \mathbf{B}(\mathcal{H})$. If $T$ is subspace-hypercyclic for $\mathcal{M}$, then $\mathcal{M}$ is not finite-dimensional.*

*Proof.* Assume $T$ was subspace-hypercyclic for a finite-dimensional subspace $\mathcal{M}$ and let $x \in \mathcal{M}$ such that $\mathrm{Orb}(T, x) \cap \mathcal{M}$ is dense in $\mathcal{M}$. It then follows that the infinite set $\mathrm{Orb}(T, x) \cap \mathcal{M}$ in the finite-dimensional space $\mathcal{M}$ has a finite subset of (nonzero) linearly dependent vectors, say $\{T^{n_1}x, T^{n_2}x, \dots, T^{n_k}x\}$. Let $m = \max\{n_1, n_2, \dots, n_k\}$. An easy induction argument shows that $\mathrm{Orb}(T, x) \subseteq \mathrm{span}\{x, Tx, T^2x, \dots, T^{m-1}x\}$. Hence the closed linear span $\mathcal{N}$ of $\mathrm{Orb}(T, x)$ is finite-dimensional. Observe that the density in $\mathcal{M}$ of $\mathrm{Orb}(T, x) \cap \mathcal{M}$ implies that $\mathcal{M} \subseteq \mathcal{N}$. Since clearly $\mathcal{N}$ is an invariant subspace for $T$, Proposition 4.1 shows that $T\big|_{\mathcal{N}}$ is subspace-hypercyclic for $\mathcal{M}$. But Theorem 4.9 contradicts this, and so the proof is finished. □

For the following theorem, we assume the reader is familiar with some basic facts about compact operators on Hilbert space, which can be found in many standard references. A particularly nice exposition can be found in [22, p. 140–142].

**Theorem 4.11.** *Let $T \in \mathbf{B}(\mathcal{H})$. If $T$ is compact, then $T$ is not subspace-hypercyclic for any subspace.*

*Proof.* Assume that $T$ is compact and subspace-hypercyclic for some $\mathcal{L}$. Since $T$ is compact, Theorem 4.5 implies that there exists $\lambda$, an eigenvalue of $T$, such that $\lambda \in S^1$.

Since $T$ is compact there exists $N \in \mathbb{N}$ such that $\mathrm{ran}(T - \lambda)^N = \mathrm{ran}(T - \lambda)^{N+k}$ for all $k \in \mathbb{N}$. Also, Proposition 4.8 gives that $\ker((T - \lambda)^*)^N \subseteq \mathcal{L}^\perp$ and hence $\mathcal{L} \subseteq (\ker((T - \lambda)^*)^N)^\perp = \mathrm{ran}(T - \lambda)^N$, since $\mathrm{ran}(T - \lambda)^N$ is closed. Since $\mathcal{N} := \ker(T - \lambda)^N$ and $\mathcal{M} := \mathrm{ran}(T - \lambda)^N$ are complementary and invariant under $T$, and since we showed that $\mathcal{L} \subseteq \mathcal{M}$, we can apply Theorem 4.4 to obtain that $T\big|_{\mathcal{M}}$ is subspace-hypercyclic for $\mathcal{L}$. Also, $\sigma(T\big|_{\mathcal{M}}) = \sigma(T) \setminus \{\lambda\}$ and $T\big|_{\mathcal{M}}$ is a compact operator.

The process above can be repeated starting with the compact operator $T\big|_{\mathcal{M}}$. Since the original operator $T$ is compact, it can have only a finite number of eigenvalues of modulus equal to 1. Hence, eventually, repeating this process leaves us with a compact operator which is subspace-hypercyclic for $\mathcal{L}$ but whose spectrum does not intersect $S^1$, contradicting Theorem 4.5. □

We should mention that hyponormal operators (and hence normal operators) cannot be subspace-hypercyclic since the norms of the elements of the orbit are either decreasing or eventually increasing, as proved by P. Bourdon [3].

## 5. Some questions

(i) Let $T$ be an invertible operator. If $T$ is subspace-hypercyclic for some $\mathcal{M}$, is $T^{-1}$ subspace-hypercyclic for some space? If so, for which space?

(ii) If $T$ is subspace-hypercyclic for some $\mathcal{M}$ and $\lambda$ is of modulus 1, is $\lambda T$ subspace-hypercyclic for $\mathcal{M}$?

(iii) If $T$ is hypercyclic, must there be a proper subspace $\mathcal{M}$ such that $T$ is subspace-hypercylic for $T$?

(iv) Coanalytic Toeplitz operators can be subspace-hypercyclic (see Example 2.4). Is there a classification of when this occurs (for example, *á la* Godefroy-Shapiro [7])?

(v) Can Proposition 4.8 be generalized? That is, if $T$ is subspace-hypercyclic for some $\mathcal{M}$ and $q$ is a complex polynomial, is it true that $\ker q(T^*) \subseteq \mathcal{M}^{\perp}$? This question was suggested to us by the referee.

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