

NEW GENERAL INTEGRAL INEQUALITY FOR CONVEX FUNCTIONS AND APPLICATIONS

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ABSTRACT. In this paper, we establish new general inequality for convex functions. Then, we apply this inequality to obtain the midpoint, trapezoid and averaged midpoint-trapezoid integral inequality. Also, some applications for special means of real numbers are provided.

1. INTRODUCTION

Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a convex mapping defined on the interval I of real numbers and $a, b \in I$, with $a < b$. the following double inequality is well known in the literature as the Hermite-Hadamard inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2}.$$

Let $f : [a, b] \rightarrow \mathbb{R}$ be a differentiable mapping on (a, b) whose derivative $f' : (a, b) \rightarrow \mathbb{R}$ is bounded on (a, b) , i.e., $\|f'\|_\infty = \sup_{t \in (a, b)} |f'(t)| < \infty$. Then the following inequality

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[\frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty$$

holds. This result is known in the literature as the Ostrowski inequality[6].

In the realm of real functions of real variable, convex functions constitute a conspicuous body both because they are frequently encountered in practical applications, and because they satisfy a number of useful inequalities and theorems [see, [1]-[3], [5]]. The most important of the inequalities is of course the defining one which states that a real function $f(x)$ defined on a real-numbers interval $I = [a, b]$ is convex if, for any three elements x_1, x, x_2 of I and $x_1 < x_2$ such that $x_1 \leq x \leq x_2$,

$$f(x) \leq f(x_1)[(x_2 - x)/(x_2 - x_1)] + f(x_2)[(x - x_1)/(x_2 - x_1)].$$

Graphically, this means that the point $\{x, f(x)\}$ never falls above the straight line segment connecting the points $\{x_1, f(x_1)\}$ and $\{x_2, f(x_2)\}$.

Definition 1 ([7]). *Let $f : [a, b] \rightarrow \mathbb{R}$ be a given function. We say that f is an even function with respect to the point $t_0 = \frac{a+b}{2}$ if $f(a+b-t) = f(t)$ for $t \in [a, b]$. We say*

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that f is an odd function with respect to the point $t_0 = \frac{a+b}{2}$ if $f(a+b-t) = -f(t)$ for $t \in [a, b]$.

Here, we use the term even(odd) function for a given $f : [a, b] \rightarrow \mathbb{R}$ if f is even(odd) with respect to the point $t_0 = \frac{a+b}{2}$. We know that each function $f : [a, b] \rightarrow \mathbb{R}$ can be represented as a sum of one even and one odd function,

$$f(t) = f_1(t) + f_2(t)$$

where

$$f_1(t) = \frac{f(t) + f(a+b-t)}{2}$$

is an even function and

$$f_2(t) = \frac{f(t) - f(a+b-t)}{2}$$

is an odd function.

It is not difficult to verify the following facts:

- i) If f is an odd function, then $|f|$ is an even function.
- ii) If f, g are even or odd functions, then fg is an even function.
- iii) If f is an even function and g is an odd function, then fg is an odd function.

- iv) If f is an integrable and odd function, then $\int_a^b f(x)dx = 0$. Indeed, we have

$$\int_a^b f(x)dx = -\int_a^b f(a+b-x)dx = -\int_a^b f(u)du = -\int_a^b f(x)dx.$$

Thus, $2 \int_a^b f(x)dx = 0$ and we proved the above assertion.

- v) If f is an integrable and even function, then

$$\int_a^b f(x)dx = 2 \int_{\frac{a+b}{2}}^b f(x)dx = 2 \int_a^{\frac{a+b}{2}} f(x)dx.$$

Indeed, we have

$$\int_a^b f(x)dx = \int_a^{\frac{a+b}{2}} f(x)dx + \int_{\frac{a+b}{2}}^b f(x)dx,$$

and since f is an even function,

$$\int_a^{\frac{a+b}{2}} f(x)dx = \int_a^{\frac{a+b}{2}} f(a+b-x)dx = -\int_b^{\frac{a+b}{2}} f(u)du = \int_{\frac{a+b}{2}}^b f(x)dx.$$

Thus, the above assertion holds.

In this article, our work is motivated by the works of N. Ujevic [7] and Z. Liu [4]. We obtain new general integral inequality for convex functions. Finally, new error bounds for the midpoint, trapezoid and other are obtained. Some applications for special means of real numbers are also provided.

2. MAIN RESULTS

In order to prove our main results, we need the following identity:

Lemma 1. *Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be twice differentiable function on I° with $f'' \in L_1[a, b]$, then*

$$\begin{aligned}
& 2 \int_a^b f(t) dt - (\beta - \alpha) [f(x) + f(a + b - x)] \\
& + (b - \beta)^2 f'(b) - (a - \alpha)^2 f'(a) + 2(a - \alpha) f(a) - 2(b - \beta) f(b) \\
(2.1) \quad & + (\beta - \alpha) \left[\left(x - \frac{3\alpha + \beta}{4} \right) f'(x) + \left(a + b - x - \frac{\alpha + 3\beta}{4} \right) f'(a + b - x) \right] \\
& = \int_a^b k(a, b, t) f''(t) dt
\end{aligned}$$

where

$$(2.2) \quad k(a, b, t) := \begin{cases} (t - \alpha)^2 & , a \leq t < x \\ \left(t - \frac{\alpha + \beta}{2} \right)^2 & x \leq t < a + b - x \text{ with } a \leq \alpha < \beta \leq b \\ (t - \beta)^2 & a + b - x \leq t \leq b \end{cases}$$

for any $x \in [a, \frac{a+b}{2}]$.

Proof. It suffices to note that

$$\begin{aligned}
I &= \int_a^b k(a, b, t) f''(t) dt \\
&= \int_a^x (t - \alpha)^2 f''(t) dt + \int_x^{a+b-x} \left(t - \frac{\alpha + \beta}{2} \right)^2 f''(t) dt + \int_{a+b-x}^b (t - \beta)^2 f''(t) dt \\
&= I_1 + I_2 + I_3.
\end{aligned}$$

By inegration by parts, we have the following identity

$$\begin{aligned}
I_1 &= \int_a^x (t - \alpha)^2 f''(t) dt \\
&= (x - \alpha)^2 f'(x) - (a - \alpha)^2 f'(a) + 2(a - \alpha) f(a) - 2(x - \alpha) f(x) + 2 \int_a^x f(t) dt.
\end{aligned}$$

Similarly, we observe that

$$\begin{aligned}
I_2 &= \int_x^{a+b-x} \left(t - \frac{\alpha + \beta}{2}\right)^2 f''(t) dt \\
&= \left(a + b - x - \frac{\alpha + \beta}{2}\right)^2 f'(a + b - x) - \left(x - \frac{\alpha + \beta}{2}\right)^2 f'(x) \\
&\quad + 2 \left(x - \frac{\alpha + \beta}{2}\right) f(x) - 2 \left(a + b - x - \frac{\alpha + \beta}{2}\right) f(a + b - x) + 2 \int_x^{a+b-x} f(t) dt
\end{aligned}$$

and

$$\begin{aligned}
I_3 &= \int_{a+b-x}^b (t - \beta)^2 f''(t) dt \\
&= (b - \beta)^2 f'(b) - (a + b - x - \beta)^2 f'(a + b - x) \\
&\quad + 2(a + b - x - \beta) f(a + b - x) - 2(b - \beta) f(b) + 2 \int_{a+b-x}^b f(t) dt.
\end{aligned}$$

Thus, we can write

$$\begin{aligned}
I &= I_1 + I_2 + I_3 \\
&= (\beta - \alpha) \left[\left(x - \frac{3\alpha + \beta}{4}\right) f'(x) + \left(a + b - x - \frac{\alpha + 3\beta}{4}\right) f'(a + b - x) \right] \\
&\quad - (\beta - \alpha) [f(x) + f(a + b - x)] + (b - \beta)^2 f'(b) - (a - \alpha)^2 f'(a) \\
&\quad + 2(a - \alpha) f(a) - 2(b - \beta) f(b) + 2 \int_a^b f(t) dt
\end{aligned}$$

which gives the required identity (2.1). \square

Corollary 1. *Under the assumptions Lemma 1 with $\alpha = a$, $\beta = b$, we have the following identity:*

$$\begin{aligned}
&2 \int_a^b f(t) dt - (b - a) [f(x) + f(a + b - x)] + (b - a) \left(x - \frac{3a + b}{4}\right) [f'(x) - f'(a + b - x)] \\
&= \int_a^b k_1(a, b, t) f''(t) dt
\end{aligned}$$

where

$$k_1(a, b, t) = \begin{cases} (t-a)^2, & a \leq t < x \\ (t - \frac{a+b}{2})^2, & x \leq t < a+b-x \\ (t-b)^2, & a+b-x \leq t \leq b \end{cases}$$

for any $x \in [a, \frac{a+b}{2}]$.

The proof of the Corollary 1 is proved by Liu in [4]. Hence, our results in Lemma 1 are generalizations of the corresponding results of Liu [4].

Corollary 2. *Under the assumptions Lemma 1 with $\alpha = \beta = \frac{a+b}{2}$, we have the following identity:*

$$2 \int_a^b f(t)dt + \frac{(b-a)^2}{4} [f'(b) - f'(a)] - (b-a) [f(a) + f(b)] = \int_a^b \left(t - \frac{a+b}{2}\right)^2 f''(t)dt.$$

Let us show that the kernel $k(a, b, t)$ defined by (2.2) is an even function if $\alpha + \beta = a + b$. Indeed, for $t \in [a, x]$ we have

$$k(a, b, a+b-t) = (a+b-t-\beta)^2 = (t-\alpha)^2 = k(a, b, t).$$

For $t \in [x, a+b-x]$ we have

$$k(a, b, a+b-t) = \left(a+b-t - \frac{\alpha+\beta}{2}\right)^2 = \left(t - \frac{\alpha+\beta}{2}\right)^2 = k(a, b, t).$$

For $t \in [a+b-x, b]$ we have

$$k(a, b, a+b-t) = (a+b-t-\alpha)^2 = (t-\beta)^2 = k(a, b, t).$$

Hence, $k(a, b, t)$ is an even function.

Now, by using the above lemma, we prove our main theorems:

Theorem 1. *Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be twice differentiable function on I° such that $f'' \in L_1[a, b]$ where $a, b \in I$, $a < b$. If f' is a convex on $[a, b]$ and $f''(t) \geq 0$, $t \in [a, b]$, then the following inequality holds:*

$$\begin{aligned} & \left| 2 \int_a^b f(t)dt - (\beta - \alpha) [f(x) + f(a+b-x)] \right. \\ & + (b-\beta)^2 f'(b) - (a-\alpha)^2 f'(a) + 2(a-\alpha)f(a) - 2(b-\beta)f(b) \\ (2.3) \quad & \left. + (\beta - \alpha) \left[\left(x - \frac{3\alpha + \beta}{4}\right) f'(x) + \left(a+b-x - \frac{\alpha + 3\beta}{4}\right) f'(a+b-x) \right] \right| \\ & \leq \|k\|_\infty [f'(b) - f'(a)], \text{ for any } x \in [a, \frac{a+b}{2}] \end{aligned}$$

where $\|k\|_\infty = \max_{t \in [a, b]} |k(a, b, t)|$.

Proof. From Lemma 1, we get,

$$\begin{aligned}
& \left| 2 \int_a^b f(t) dt - (\beta - \alpha) [f(x) + f(a + b - x)] \right. \\
& \quad \left. + (b - \beta)^2 f'(b) - (a - \alpha)^2 f'(a) + 2(a - \alpha) f(a) - 2(b - \beta) f(b) \right. \\
(2.4) \quad & \left. + (\beta - \alpha) \left[\left(x - \frac{3\alpha + \beta}{4} \right) f'(x) + \left(a + b - x - \frac{\alpha + 3\beta}{4} \right) f'(a + b - x) \right] \right| \\
& \leq \int_a^b |k(a, b, t)| |f''(t)| dt.
\end{aligned}$$

Let us consider the following notations

$$f_1''(t) = \frac{f''(t) + f''(a + b - t)}{2}, \quad f_2''(t) = \frac{f''(t) - f''(a + b - t)}{2},$$

then we have $f''(t) = f_1''(t) + f_2''(t)$ and $k(a, b, t) f_2''(t)$ is an odd function while $|f_1''(t)|$ and $k(a, b, t) f_1''(t)$ are even functions. Thus, by using properties (iv) we obtain

$$\begin{aligned}
\int_a^b k(a, b, t) f''(t) dt &= \int_a^b k(a, b, t) [f_1''(t) + f_2''(t)] dt \\
&= \int_a^b k(a, b, t) f_1''(t) dt.
\end{aligned}$$

Thus by using properties (v) we get

$$\begin{aligned}
(2.5) \quad \left| \int_a^b k(a, b, t) f''(t) dt \right| &= \left| \int_a^b k(a, b, t) f_1''(t) dt \right| \\
&\leq \int_a^b |k(a, b, t)| |f_1''(t)| dt \\
&\leq \|k\|_\infty \int_a^b |f_1''(t)| dt \\
&= 2 \|k\|_\infty \int_{\frac{a+b}{2}}^b |f_1''(t)| dt \\
&= \|k\|_\infty \int_{\frac{a+b}{2}}^b |f''(t) + f''(a + b - t)| dt.
\end{aligned}$$

Therefore, since $f''(t) \geq 0$, $t \in [a, b]$, we get

$$(2.6) \quad \int_{\frac{a+b}{2}}^b |f''(t) + f''(a+b-t)| dt = \int_{\frac{a+b}{2}}^b [f''(t) + f''(a+b-t)] dt \\ = f'(b) - f'(a).$$

Using (2.5) and (2.6) in (2.4), we obtain (2.3) which completes the proof. \square

Theorem 2. Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be twice differentiable function on I° such that $f'' \in L_1[a, b]$ where $a, b \in I$, $a < b$. If $|f''|$ is a convex on $[a, b]$, then the following inequality holds:

$$(2.7) \quad \left| 2 \int_a^b f(t) dt - (\beta - \alpha) [f(x) + f(a+b-x)] \right. \\ \left. + (b - \beta)^2 f'(b) - (a - \alpha)^2 f'(a) + 2(a - \alpha) f(a) - 2(b - \beta) f(b) \right. \\ \left. + (\beta - \alpha) \left[\left(x - \frac{3\alpha + \beta}{4} \right) f'(x) + \left(a + b - x - \frac{\alpha + 3\beta}{4} \right) f'(a + b - x) \right] \right| \\ \leq \frac{b-a}{4} \|k\|_\infty \left[|f''(a)| + |f''(b)| + 2 \left| f''\left(\frac{a+b}{2}\right) \right| \right], \text{ for any } x \in \left[a, \frac{a+b}{2} \right]$$

where $\|k\|_\infty = \max_{t \in [a, b]} |k(a, b, t)|$.

Proof. By similar computation the proof of Theorem 1, we get

$$(2.8) \quad \left| 2 \int_a^b f(t) dt - (\beta - \alpha) [f(x) + f(a+b-x)] \right. \\ \left. + (b - \beta)^2 f'(b) - (a - \alpha)^2 f'(a) + 2(a - \alpha) f(a) - 2(b - \beta) f(b) \right. \\ \left. + (\beta - \alpha) \left[\left(x - \frac{3\alpha + \beta}{4} \right) f'(x) + \left(a + b - x - \frac{\alpha + 3\beta}{4} \right) f'(a + b - x) \right] \right| \\ \leq \|k\|_\infty \int_{\frac{a+b}{2}}^b [|f''(t)| + |f''(a+b-t)|] dt.$$

Since $|f''|$ is a convex on $[a, b]$, by Hermite-Hadamard's integral inequality we have

$$\begin{aligned}
 & \int_{\frac{a+b}{2}}^b [|f''(t)| + |f''(a+b-t)|] dt \\
 (2.9) \quad &= \int_{\frac{a+b}{2}}^b |f''(t)| dt + \int_a^{\frac{a+b}{2}} |f''(t)| dt \\
 &\leq \frac{b-a}{4} \left[|f''(a)| + |f''(b)| + 2 \left| f''\left(\frac{a+b}{2}\right) \right| \right].
 \end{aligned}$$

Therefore, using (2.9) in (2.8), we obtain (2.7) which completes the proof. \square

3. APPLICATIONS TO QUADRATURE FORMULAS

In this section we point out some particular inequalities which generalize some classical results such as : trapezoid inequality, Ostrowski's inequality, midpoint inequality and others.

Proposition 1. *Under the assumptions Theorem 1, we have*

$$\begin{aligned}
 & \left| \int_a^b f(t) dt - \frac{b-a}{2} [f(x) + f(a+b-x)] + \frac{b-a}{2} \left(x - \frac{3a+b}{4}\right) [f'(x) - f'(a+b-x)] \right| \\
 (3.1) \quad & \leq \frac{f'(b) - f'(a)}{3} \left[(x-a)^3 + \left(\frac{a+b}{2} - x\right)^3 \right], \text{ for any } x \in \left[a, \frac{a+b}{2}\right].
 \end{aligned}$$

Proof. If we choose $\alpha = a$, $\beta = b$ in (2.2), then we obtain $\|k\|_\infty = \frac{2}{3} [(x-a)^3 + (\frac{a+b}{2} - x)^3]$. Thus, from the inequality (2.3) it follows that (3.1) holds. \square

Remark 1. *If we put $x = \frac{a+b}{2}$ in (3.1), we get the "midpoint inequality":*

$$(3.2) \quad \left| \frac{1}{b-a} \int_a^b f(t) dt - f\left(\frac{a+b}{2}\right) \right| \leq \frac{(b-a)^2}{24} [f'(b) - f'(a)].$$

Proposition 2. *Under the assumptions Theorem 1, we have*

$$\begin{aligned}
 & \left| \frac{1}{b-a} \int_a^b f(t) dt - \frac{f(a) + f(b)}{2} + \frac{(b-a)}{8} [f'(b) - f'(a)] \right| \\
 (3.3) \quad & \leq \frac{(b-a)^2}{48} [f'(b) - f'(a)].
 \end{aligned}$$

Proof. If we choose $\alpha = \beta = \frac{a+b}{2}$ in (2.2), then we obtain $\|k\|_\infty = \frac{(b-a)^3}{24}$. Thus, from the inequality (2.3) it follows that (3.3) holds. \square

Another particular integral inequality with many applications is the following one:

Proposition 3. *Under the assumptions Theorem 2, we have*

$$(3.4) \quad \left| \frac{1}{b-a} \int_a^b f(t) dt - \frac{f(a)+f(b)}{2} + \frac{(b-a)}{8} [f'(b) - f'(a)] \right| \\ \leq \frac{(b-a)^3}{192} \left[|f''(a)| + |f''(b)| + 2 \left| f'' \left(\frac{a+b}{2} \right) \right| \right].$$

Proof. If we choose $\alpha = \beta = \frac{a+b}{2}$ in (2.2), then we obtain $\|k\|_\infty = \frac{(b-a)^3}{24}$. Thus, from the inequality (2.7) it follows that (3.4) holds. \square

Remark 2. *It is clear that the best estimation we can have in (3.4) for $f'(b) = f'(a)$ is getting the "trapezoid inequality":*

$$(3.5) \quad \left| \frac{1}{b-a} \int_a^b f(t) dt - \frac{f(a)+f(b)}{2} \right| \leq \frac{(b-a)^3}{192} \left[|f''(a)| + |f''(b)| + 2 \left| f'' \left(\frac{a+b}{2} \right) \right| \right].$$

4. APPLICATIONS FOR SPECIAL MEANS

Recall the following means:

(a) The arithmetic mean

$$A = A(a, b) := \frac{a+b}{2}, \quad a, b \geq 0;$$

(b) The geometric mean

$$G = G(a, b) := \sqrt{ab}, \quad a, b \geq 0;$$

(c) The harmonic mean

$$H = H(a, b) := \frac{2ab}{a+b}, \quad a, b > 0;$$

(d) The logarithmic mean

$$L = L(a, b) := \begin{cases} a & \text{if } a = b \\ \frac{b-a}{\ln b - \ln a} & \text{if } a \neq b \end{cases}, \quad a, b > 0;$$

(e) The identric mean

$$I = I(a, b) := \begin{cases} a & \text{if } a = b \\ \frac{1}{e} \left(\frac{b^b}{a^a} \right)^{\frac{1}{b-a}} & \text{if } a \neq b \end{cases}, \quad a, b > 0;$$

(f) The p -logarithmic mean:

$$L_p = L_p(a, b) := \begin{cases} \left[\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} \right]^{\frac{1}{p}} & \text{if } a \neq b \\ a & \text{if } a = b \end{cases}, \quad p \in \mathbb{R} \setminus \{-1, 0\}; \quad a, b > 0.$$

It is also known that L_p is monotonically nondecreasing in $p \in \mathbb{R}$ with $L_{-1} := L$ and $L_0 := I$. The following simple relationships are known in the literature

$$H \leq G \leq L \leq I \leq A.$$

Now, using the results of Section 3, some new inequalities is derived for the above means.

Proposition 4. *Let $p > 1$ and $0 < a < b$. Then we have the inequality:*

$$\left| L_p^p(a, b) - A^p(a, b) \right| \leq p \frac{(b-a)^2}{12} (b^{p-1} - a^{p-1}).$$

Proof. The assertion follows from (3.2) applied for $f(t) = t^p$, $t \in [a, b]$. We omitted the details. \square

Proposition 5. *Let $p > 1$ and $0 < a < b$. Then we have the inequality:*

$$\left| L_p^p(a, b) - A^p(a, b) + p(p-1) \frac{(b-a)^2}{8} L_p^{p-2}(a, b) \right| \leq p \frac{(b-a)^2}{24} (b^{p-1} - a^{p-1}).$$

Proof. The assertion follows from (3.3) applied for $f(x) = x^p$, $x \in [a, b]$. \square

Proposition 6. *Let $0 < a < b$. Then we have the inequality:*

$$\left| \ln [I(a, b)G(a, b)] + \frac{(b-a)^2}{8} G^{-2}(a, b) \right| \leq \frac{(b-a)^3}{96} \left[H^{-1}(a^2, b^2) + \frac{1}{2} A^{-2}(a, b) \right].$$

Proof. The assertion follows from (3.4) applied for $f(x) = -\ln t$, $t \in [a, b]$. \square

Proposition 7. *Let $0 < a < b$. Then we have the inequality:*

$$\left| L^{-1}(a, b) - H^{-1}(a, b) \right| \leq \frac{(b-a)^3}{48} \left[H^{-1}(a^3, b^3) + \frac{1}{2} A^{-3}(a, b) \right].$$

Proof. The assertion follows from (3.5) applied for $f(x) = \frac{1}{x}$, $t \in [a, b]$. \square

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