

Schrodinger's Equation in Riemann Spaces

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Abstract

We present some properties of the first and second order Beltrami differential operators in Riemann metric spaces. We also solve the Schrodinger's equation for a wide class of potentials and describe when the Hamiltonian of a physical system is self adjoint.

keywords Schrodinger's Equation; Metric Space; Riemann Geometry; Differential Operators; Laplacian

1 Introduction

In a 3-dimensional space, in the case of orthogonal coordinate system (x, y, z) of elementary geometry we make use of the differential operator:

$$\Delta_2 \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2}. \quad (1)$$

We call this differential operator, Laplacian operator, or Laplace-Beltrami differential parameter of second kind. One of many uses of this operator, is that help us describe the state of a particle in \mathbf{E}_3 , (the Euclidian three dimensional space). If in \mathbf{E}_3 exist also a potential $V = V(x, y, z, t)$, depending on position and time, then the Schroedinger's equation of the particle read as

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \left(\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} + \frac{\partial^2 \Psi}{\partial z^2} \right) + V\Psi \quad (2)$$

and $\Psi = \Psi(x, y, z, t)$ describes the state of the system.

In the present article we consider the case of a N dimensional metric space S of curvilinear coordinates x_l

$$\bar{\mathbf{S}}(x_1, x_2, \dots, x_N) = \{S_1(x_1, x_2, \dots, x_N), S_2(x_1, x_2, \dots, x_N), \dots, S_N(x_1, x_2, \dots, x_N)\} \quad (3)$$

with metric tensor

$$g_{ik} = g_{ik}(x^a) = \left\langle \frac{\partial \bar{\mathbf{S}}}{\partial x_i}, \frac{\partial \bar{\mathbf{S}}}{\partial x_k} \right\rangle_{reg}. \quad (4)$$

Note that the above product is the regular dot product of the vectors

$$\frac{\partial \bar{\mathbf{S}}}{\partial x_i} = \left\{ \frac{\partial S_1(x^a)}{\partial x_i}, \frac{\partial S_2(x^a)}{\partial x_i}, \dots, \frac{\partial S_N(x^a)}{\partial x_i} \right\}$$

and

$$\frac{\partial \bar{\mathbf{S}}}{\partial x_k} = \left\{ \frac{\partial S_1(x^a)}{\partial x_k}, \frac{\partial S_2(x^a)}{\partial x_k}, \dots, \frac{\partial S_N(x^a)}{\partial x_k} \right\},$$

i.e. for $\bar{A} = \{A_1, A_2, \dots, A_N\}$ and $\bar{B} = \{B_1, B_2, \dots, B_N\}$ we have

$$\langle \bar{A}, \bar{B} \rangle_{reg} = \sum_{i=1}^N A_i B_i.$$

Also $g^{ij} = (g_{ij})^t = (g_{ij})^{(-1)}$ and obviously $g_{ij} = g_{ji}$ and x^a denotes the vector $\{x_1, x_2, \dots, x_N\}$.

The linear element will be

$$ds^2 = \sum_{i,j=1}^N g_{ij} dx_i dx_j, \quad (5)$$

where $g^{ij} = (g_{ij})^t = (g_{ij})^{(-1)}$ and $g_{ij} = g_{ji}$. With the help of the above metric we define

Definition 1.

The 1-Beltrami differential operator is

$$\Delta_1(\Phi, \Psi) := \sum_{i,j=1}^N (g_{ij})^t (\partial_i \Phi \partial_j \Psi + \partial_j \Phi \partial_i \Psi) = 2 \sum_{i,j=1}^N (g_{ij})^t \partial_i \Phi \partial_j \Psi. \quad (6)$$

Definition 2.

The 2-Beltrami operator is

$$\Delta_2 \Phi = \sum_{i,j=1}^N (g_{ij})^t \left(\frac{\partial^2 \Phi}{\partial u_i \partial u_j} - \sum_{k=1}^N \Gamma_{ij}^k \partial_k \Phi \right), \quad (7)$$

where the $\Gamma_{ki}^j = \Gamma_{ik}^j$ are called Christoffel symbols and related with g_{ij} from the relations

$$\partial_l g_{ik} = \sum_{n=1}^N g_{nl} \Gamma_{ki}^n + \sum_{n=1}^N g_{kn} \Gamma_{il}^n \quad (8)$$

and

$$\Gamma_{kl}^i = \frac{1}{2} \sum_{n=1}^N (g_{in})^t (\partial_k g_{nl} + \partial_l g_{kn} - \partial_n g_{kl}). \quad (9)$$

Note.

The Beltrami operator of the first kind for one function is by notation

$$\Delta_1 \Phi := \frac{1}{2} \Delta_1(\Phi, \Phi) = \sum_{i,j=1}^N (g_{ij})^t \frac{\partial \Phi}{\partial u_i} \frac{\partial \Phi}{\partial u_j} = \sum_{i,j=1}^N (g_{ij})^t \partial_i \Phi \partial_j \Phi. \quad (10)$$

Hence in S the Schrodinger's equation with potential $V = V(x_1, x_2, \dots, x_N, t)$, (relaxed from the physical symbols which we may take them equal to 1, without loss of the generality of the problem) read as

$$i \frac{\partial \Psi(x_1, x_2, \dots, x_N, t)}{\partial t} = -\Delta_{2,x} \Psi(x_1, x_2, \dots, x_N, t) + V \Psi(x_1, x_2, \dots, x_N, t). \quad (11)$$

The x index in the Δ derivative in (11) means that only the x 's are differentiated and the t 's is parameter (the Beltrami derivative is for the space S).

Definition 3.

The space S in which all the $S_k(x_1, x_2, \dots, x_N)$, $k = 1, 2, \dots, N$ are homogeneous of degree μ , we call it S_μ^N (μ -Homogeneous Space).

Definition 4.

We call \mathbf{F}_μ^N , the space of all real smooth functions (f, g) such that

$$\lim_{r \rightarrow \{0, +\infty\}} r^{(N-1)\mu} (g(r)f'(r) - f(r)g'(r)) = 0 \quad (12)$$

and

$$(\mu - 1)(N - 1) \int_0^\infty r^{(N-1)\mu-1} (F'(r)G_1(r) - F_1(r)G_1'(r)) dr = 0, \quad (12.1)$$

where N is the dimension of S .

Note.

Observe that in F_1^N space we have only condition (12), since then (12.1) is always true.

Definition 5.

We will call the function $A = A(x^a) = A(x_1, x_2, \dots, x_N)$ harmonic if

$$\Delta_2 A(x^a) = 0. \quad (13)$$

We note here that with x^a we mean $x^a = \{x^a\} = \{x_1, x_2, \dots, x_N\}$, for the N first values. If A has more than N coordinates, say $x = \{x_1, x_2, \dots, x_N, x_{N+1}, \dots, x_M\}$, then we will write $x = \{x^a, x_{N+1}, \dots, x_{M+1}\}$. Hence the symbols a, x^a are protected.

2 Main tasks of the article

We will solve the Schroedinger's equation (11) for the potentials

$$V(x^a, t) = \frac{\sum_{\lambda=1}^{\infty} c_{\lambda}^* \lambda e^{-it\lambda} A_{\lambda}(x^a)}{\sum_{\lambda=1}^{\infty} c_{\lambda}^* e^{-it\lambda} A_{\lambda}(x^a) + \sum_{\lambda=1}^{\infty} c_{\lambda} \phi_{\nu, \lambda}(\Phi_0) e^{-it\lambda}}, \quad (14)$$

where $\phi_{\nu, \lambda}(x)$ are specific (null) functions and $A_{\lambda}(x^a) = A_{\lambda}(x_1, x_2, \dots, x_N)$ are arbitrary harmonic function of the space S . Also we have set

$$\Phi_0 = \Phi_0(x_1, x_2, \dots, x_N) = \sqrt{\sum_{k=1}^N S_k(x_1, x_2, \dots, x_N)^2}. \quad (15)$$

The function Φ_0 is what we call radial distance and its use is very important, since in many cases reduces considerably the complexity of the problem. Also in an Euclidian space, Φ_0 have the very well known distance meaning used in central potentials.

We also show that, in every space S exists an infinite class of harmonic functions $A(x^a)$, which will construct them (however we don't find all of them). In Euclidian spaces such functions have been found in full generality (see Note 3 in page 17 below), but in the cases of arbitrary spaces are hard to find.

Also in spaces S_{μ}^N over \mathbf{F}_{μ}^N , the potentials $V(x^a, t)$ are not only exactly solvable but produce together with the Beltrami derivative Δ_2 hamiltonians:

$$H(\cdot) = -\Delta_{2,x}(\cdot) + V(x^a, t)(\cdot),$$

which are self-adjoint i.e. Hermitian. The concept of self-adjointnes of the hamiltonian H is fundamental because gives a physical meaning to the quantum system. Without this one can not proceed further.

We give also several applications and solutions of certain equations in the most general Riemann spaces S . Note also the very interesting appearance of Bessel functions of the first and second kind J and Y , which are related with the Δ_2 Beltrami operators. Thsesse functions (J, Y), are generated in the middle of nowhere, when we consider the eigenvectors and eigenvalues of the Beltrami operators in curved spaces. Examples of solutions of equation (11) in spaces S_{μ}^N over \mathbf{F}_{μ}^N , with the Hamiltonian self adjoint and radial potentials of the form

$$\begin{aligned} V = V(x^a, t) &= \left(-\frac{C}{x} \frac{\partial}{\partial x} \log \left(\sum_{\lambda=1}^{\infty} c_{\lambda} e^{-it\lambda} \phi_{\nu, \lambda}(x) \right) \right)_{x=\Phi_0(x^a)} = \\ &= \left(-\frac{C}{x} \frac{\partial}{\partial x} \log(g(x, t)) \right)_{x=\Phi_0(x^a)}, \end{aligned} \quad (16)$$

where $\phi_{\nu, \lambda}(x) = J_{\nu}(\sqrt{\lambda}x) / x^{\nu}$, C -constant and ν real are given.

We also show that for the PDE

$$i\partial_t g(x^a, t) = -\Delta_2 g(x^a, t), \quad (17)$$

a class of solutions is

$$g(x^a, t) = \sum_{\lambda=1}^{\infty} c_{\lambda} e^{-it\lambda} \phi_{\nu, \lambda}(\Phi_0(x^a)), \quad (18)$$

where

$$\phi_{\nu, \lambda}(x) = x^{-\nu} \left(c_1 J_{\nu}(\sqrt{\lambda}x) + c_2 Y_{\nu}(\sqrt{\lambda}x) \right). \quad (19)$$

It is worth to mention that in Euclidian spaces \mathbf{R} , \mathbf{R}^2 and \mathbf{R}^3 relation (16) takes the form:

$$V = V(x^a, t) = \left(\frac{C}{x} \frac{\partial}{\partial x} \log \left(\sum_{\lambda=1}^{\infty} c_{\lambda} e^{-it\lambda} \phi_{\nu, \lambda}(x) \right) \right)_{x=\Phi_{1,2,3}}, \quad (20)$$

with

$$\Phi_1 = \Phi_0 = x_1, \quad \Phi_2 = \Phi_0 = \sqrt{x_1^2 + x_2^2}, \quad \Phi_3 = \Phi_0 = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

and $\nu = -1/2, 0, 1/2$ respectively. In the case of $S = \mathbf{R}$, we have $\Phi_0 = x_1 = x$ and hence we solve

$$i\partial_t \Psi(x, t) = -\partial_x^2 \Psi(x, t) + V(x, t) \Psi(x, t), \quad (21)$$

for the potential (20) in full generality. This equation admits exact solution

$$\Psi(x, t) = \sum_{p=1}^{\infty} \left(\frac{2\sqrt{x}}{J_{1/2}(\lambda_p)^2} \int_0^1 f(y) y^{1/2} J_{-1/2}(\lambda_p y) dy \right) e^{-i\lambda_p^2 t} J_{-1/2}(\lambda_p x), \quad (22)$$

where the numbers λ_p are the zeros of $J_{-1/2}(x) = 0$ in $x > 0$ and

$$V(x, t) = \frac{C}{x} \partial_x (\log(g(x, t))), \quad g(x, t) = \sum_{p=1}^{\infty} c_p e^{-it\lambda_p^2} \phi_{-1/2, \lambda_p^2}(x), \quad (23)$$

with $u(1, t) = 0$, $t > 0$ and $u(x, 0) = f(x)$. Note that when $c_2 = 0$ and $r_{\nu, p}$ are the solutions of $J_{\nu}(x) = 0$, then $x^{\nu+1/2} \phi_{\nu, r_{\nu, p}^2}(x)$ is base of $L[0, 1]$.

3 Properties of Beltrami differential operators and Schrodinger's equation

Proposition 1.

$$\Delta_1(\Phi\Psi) = \Phi^2(\Delta_1\Psi) + (\Delta_1\Phi)\Psi^2 + \Phi\Psi\Delta_1(\Phi, \Psi). \quad (24)$$

Proof.

The result follows from the differentiation of the product of two functions and the fact that g_{ij} is symmetric:

$$\begin{aligned}
\Delta_1(\Phi\Psi) &= \sum_{i,j=1}^N (g_{ij})^t \partial_i(\Phi\Psi) \partial_j(\Phi\Psi) = \\
&= \sum_{i,j=1}^N (g_{ij})^t [(\Psi(\partial_i\Phi) + \Phi(\partial_i\Psi))((\partial_j\Phi)\Psi + \Phi(\partial_j\Psi))] = \\
&= \sum_{i,j=1}^N (g_{ij})^t [\Psi^2(\partial_i\Phi)(\partial_j\Phi) + \Phi^2(\partial_i\Phi)(\partial_j\Psi) + \Phi\Psi(\partial_i\Phi)(\partial_j\Psi) + \Phi\Psi(\partial_j\Phi)(\partial_i\Psi)] = \\
&= \Phi^2(\Delta_1\Psi) + \Psi^2(\Delta_1\Phi) + \Phi\Psi\Delta_1(\Phi, \Psi).
\end{aligned}$$

Proposition 2.

$$\Delta_1(\Phi\Psi, Z) = \Phi\Delta_1(\Psi, Z) + \Psi\Delta_1(\Phi, Z). \quad (25)$$

Proof.

$$\begin{aligned}
\Delta_1(\Phi\Psi, Z) &= 2 \sum_{i,j=1}^N (g_{ij})^t \partial_i(\Phi\Psi) \partial_j Z = 2 \sum_{i,j=1}^N (g_{ij})^t (\Psi\partial_i\Phi + \Phi\partial_i\Psi) \partial_j Z = \\
&= 2\Phi \sum_{i,j=1}^N (g_{ij})^t (\partial_i\Psi)(\partial_j Z) + 2\Psi \sum_{i,j=1}^N (g_{ij})^t (\partial_i\Phi)(\partial_j Z)
\end{aligned}$$

and the result follows from the definition of $\Delta_1(\cdot, \cdot)$.

One can observe from Proposition 2 that Beltrami differential operator for products, obey the same rule as the classical differential operator $\frac{d}{dx}$ of functions of one variable. For example set $\Phi = \Psi$ in (25), we get:

$$\Delta_1(\Phi^2, Z) = 2\Phi\Delta_1(\Phi, Z). \quad (26)$$

From Propositions 1 and 2 we get by induction.

Proposition 3.

If $n = 1, 2, 3, \dots$, then

$$\Delta_1(\Phi^n, \Psi) = n\Phi^{n-1}\Delta_1(\Phi, \Psi). \quad (27)$$

The semi-linear property which is easy someone to see is

$$\Delta_1(\Phi + \Psi, Z) = \Delta_1(\Phi, Z) + \Delta_1(\Psi, Z). \quad (28)$$

Also if f is a function such that

$$f(z) = \sum_{n=1}^{\infty} c_n z^n, \quad (29)$$

then

Theorem 1.

$$\Delta_1(f(\Phi), Z) = f'(\Phi)\Delta_1(\Phi, Z). \quad (30)$$

Proof.

It follows from Proposition 3 and the semi-linear property.

Theorem 1 will help us to evaluate the second Beltrami operator $\Delta_2(f(\Phi))$ of the one variable function $f(x)$ on a general scalar Φ .

Proposition 4.

$$\Delta_2(\Phi\Psi) = \Psi\Delta_2(\Phi) + \Phi\Delta_2(\Psi) + \Delta_1(\Phi, \Psi). \quad (31)$$

Proof.

From the relations

$$\begin{aligned} \frac{\partial^2(\Phi\Psi)}{\partial x\partial y} &= \frac{\partial^2\Phi}{\partial x\partial y} + \frac{\partial^2\Psi}{\partial x\partial y} + \frac{\partial\Phi}{\partial x}\frac{\partial\Psi}{\partial y} + \frac{\partial\Phi}{\partial y}\frac{\partial\Psi}{\partial x}, \\ \frac{\partial(\Phi\Psi)}{\partial x} &= \frac{\partial\Phi}{\partial x}\Psi + \frac{\partial\Psi}{\partial x}\Phi \end{aligned}$$

and the definitions of 1 and 2-Beltrami derivatives, the proof easily follows.

Proposition 5.

For $n = 2, 3, \dots$

$$\Delta_2(\Phi^n) = n\Phi^{n-1}\Delta_2(\Phi) + n(n-1)\Phi^{n-2}\Delta_1(\Phi). \quad (32)$$

Proof.

Set $\Phi = \Psi$ in (31) then

$$\Delta_2(\Phi^2) = 2\Phi\Delta_2(\Phi) + \Delta_1(\Phi, \Phi). \quad (33)$$

Also

$$\Delta_2(\Phi^3) = \Delta_2(\Phi\Phi^2) = \Phi^2\Delta_2(\Phi) + \Phi\Delta_2(\Phi^2) + \Delta_1(\Phi^2, \Phi).$$

But from Theorem 1. and (19) we have

$$\begin{aligned}\Delta_2(\Phi^3) &= \Phi^2\Delta_2(\Phi) + \Phi(2\Phi\Delta_2(\Phi) + \Delta_1(\Phi, \Phi)) + 2\Phi\Delta_1(\Phi, \Phi) = \\ &= 3\Phi^2\Delta_2(\Phi) + 3\Phi\Delta_1(\Phi, \Phi).\end{aligned}$$

The result as someone can see follows easy from the above propositions and theorems by induction.

From the linearity of the 2-Beltrami operator and (32) we get

Theorem 2.

If f is a single variable function and analytic around 0 we have

$$\Delta_2(f(\Phi)) = f'(\Phi)\Delta_2(\Phi) + f''(\Phi)\Delta_1(\Phi). \quad (34)$$

Notes.

1) From Theorem 2 we get that if for some Φ in some space holds

$$\Delta_1(\Phi) = \Delta_2(\Phi) = 0, \quad (35)$$

then for every single variable function f analytic in a open set, containing the origin, we have

$$\Delta_2(f(\Phi)) = 0. \quad (36)$$

2) Set

$$|u| := \sqrt{\sum_{k=1}^N u_k^2}$$

and

$$\|u\|^2 := \sum_{i,j=1}^N (g_{ij})^t u_i u_j,$$

then

$$\Delta_1(|u|) = \frac{\|u\|^2}{|u|^2}. \quad (37)$$

Because

$$\begin{aligned}\Delta_1(|u|) &= \sum_{i,j=1}^N (g_{ij})^t \frac{\partial |u|}{\partial u_i} \frac{\partial |u|}{\partial u_j} = \sum_{i,j=1}^N (g_{ij})^t \frac{\partial \sqrt{\sum_{k=1}^N u_k^2}}{\partial u_i} \frac{\partial \sqrt{\sum_{k=1}^N u_k^2}}{\partial u_j} = \\ &= \sum_{i,j=1}^N (g_{ij})^t \frac{2u_i 2u_j}{4|u|^2} = \frac{\|u\|^2}{|u|^2}.\end{aligned}$$

Hence one can also see that

$$\Delta_1[f(|u|), g(|u|)] = 2f'(|u|)g'(|u|) \frac{\|u\|^2}{|u|^2}. \quad (38)$$

$$\Delta_2(f(|u|)) = f'(|u|)\Delta_2(|u|) + f''(|u|) \frac{\|u\|^2}{|u|^2}. \quad (39)$$

3) Theorems 1,2 can be used for calculation of the Beltrami derivatives. One can easily see that also hold the following relations

$$\Delta_1(u^\lambda) = (g_{\lambda\lambda})^t.$$

$$\Delta_2(u_\lambda) = - \sum_{i,j=1}^N (g_{ij})^t \Gamma_{ij}^\lambda = T_\lambda.$$

If $M \leq N$, then for all smooth functions f and Φ_k we have

$$\begin{aligned} & \Delta_1 \left(\sum_{k=1}^M f(\Phi_k) \right) = \\ & = \sum_{k=1}^M (f'(\Phi_k))^2 \Delta_1(\Phi_k) + 2 \sum_{(k<\mu), k,\mu=1}^M f'(\Phi_k) f'(\Phi_\mu) \Delta_1(\Phi_k, \Phi_\mu). \quad : (a) \end{aligned}$$

Hence for example if we have to evaluate 'say' $\Delta_2(\log(u_2 + u_3) + 1/u_1)$ in a arbitrary 3-dimensional space, then

$$\begin{aligned} & \Delta_2(\log(u_2 + u_3) + u_1^{-1}) = \\ & = \frac{1}{u_2 + u_3} \Delta_2(u_2 + u_3) + \frac{-1}{(u_2 + u_3)^2} \Delta_1(u_2 + u_3) + \frac{-1}{u_1^2} \Delta_2(u_1) + \frac{2}{u_1^3} \Delta_1(u_1) = \\ & = \frac{T_2 + T_3}{u_2 + u_3} - \frac{\Delta_1(u_2) + \Delta_1(u_3) + 2\Delta_1(u_2, u_3)}{(u_2 + u_3)^2} - \frac{T_1}{u_1^2} + \frac{2(g_{11})^t}{u_1^3} = \\ & = \frac{T_2 + T_3}{u_2 + u_3} - \frac{(g_{22})^t + (g_{33})^t + 2(g_{23})^t}{(u_2 + u_3)^2} - \frac{T_1}{u_1^2} + \frac{2(g_{11})^t}{u_1^3}. \end{aligned}$$

Now we considering the following parametrization

$$A = \{x_l : x_l = x_l(p), \text{ and } \Phi(x_1(p), x_2(p), \dots, x_N(p)) = p, p \in \mathbf{R}\}$$

and f such that $\Delta_2(f(\Phi))_A = 0$, from Theorem 2 we have

$$\Delta_2(f(\Phi))_A = f'(\phi)(\Delta_2(\Phi))_A + f''(\phi)(\Delta_1(\Phi))_A = 0,$$

which is an ordinary differential equation in a single variable (here the variable is ϕ), which the solution is

$$f(\phi) = \int \exp \left[- \int \frac{\Delta_2(\Phi)_A}{\Delta_1(\Phi)_A} d\phi \right] d\phi. \quad (40)$$

Hence we have the next

Theorem 4.

Let $\Phi = \Phi(x_1, x_2, \dots, x_N)$ be a function of a certain differentiable class. If the parametrization A is such that

$$\Phi(\{x_l(p)\}) = p.$$

Then

$$\Delta_2 \left(\int^{\Phi(x_1, x_2, \dots, x_N)} \exp \left[- \int \frac{\Delta_2(\Phi)_A}{\Delta_1(\Phi)_A} d\phi \right] d\phi \right)_A = 0. \quad (41)$$

Examples.

Consider the 4-dimensional space

$$\bar{S}(u_1, u_2, u_3, u_4) = \{u_1, u_1 + u_2, u_1 u_3, u_4\}.$$

Then

$$\bar{e}_i = \frac{\partial \bar{S}}{\partial u_i}$$

and

$$g_{ij} = \langle \bar{e}_i, \bar{e}_j \rangle_{reg}.$$

i) Set

$$\Phi(u_1, u_2, u_3, u_4) = \sqrt{u_1^2 + u_2^2 + u_3^2 + u_4^2}.$$

Then one parametrization is $u_2 = u_3 = u_4 = p$ and

$$u_1 = h(p) = \sqrt{-2}p.$$

The function f is

$$\begin{aligned} f(\phi) &= \int \exp \left[- \int \frac{\Delta_2(\Phi)_A}{\Delta_1(\Phi)_A} d\phi \right] d\phi = \\ &= \int \exp \left[\frac{3\sqrt{2} - 4i}{12\sqrt{2} - 6i} \left(2i \arctan \left(\frac{6\sqrt{2}\phi^2}{1 + 3\phi^2} \right) + \log(1 + 6\phi^2 + 81\phi^4) \right) \right] d\phi \end{aligned}$$

and

$$A_1 = \{ \sqrt{-2}p, p, p, p \}, p \in \mathbf{R}.$$

If one set the above values in the second order Beltrami derivative then

$$\Delta_2(f(\Phi))_{A_1} = 0.$$

ii) Set

$$\Phi(u_1, u_2, u_3, u_4) = \frac{u_1^2}{\sqrt{u_2 + u_3 + u_4}}.$$

Then one parametrization is $u_2 = u_3 = u_4 = p$, $u_1 = \sqrt{3}p$. The function f is

$$f(\phi) = -\frac{{}_2F_1\left[-\frac{1}{2}, -\frac{18}{29+7\sqrt{3}} - \frac{3\sqrt{3}}{29+7\sqrt{3}}; \frac{1}{2}; -(58 + 14\sqrt{3})\phi^2\right]}{\phi}$$

and

$$A_2 = \left\{ \sqrt{3}p, p, p, p \right\}, p \in \mathbf{R}.$$

Then indeed we get

$$\Delta_2(f(\Phi))_{A_2} = 0.$$

Theorem 5.

Let P be any point of a metric space S . Let also that S is described by the vector

$$\overline{OP} = \overline{S} = \{S_1(x_1, x_2, \dots, x_N), S_2(x_1, x_2, \dots, x_N), \dots, S_N(x_1, x_2, \dots, x_N)\}. \quad (41.1)$$

Then it holds

$$\Delta_2 f(\Phi_0(x_1, x_2, \dots, x_N)) = \left[x^{-(N-1)} \frac{d}{dx} (x^{N-1} f'(x)) \right]_{x=\Phi_0(x_1, x_2, \dots, x_N)} \quad (42)$$

and

$$\Delta_1(f(\Phi_0)) = f'(\Phi_0)^2, \quad (43)$$

where

$$\Phi_0(x_1, x_2, \dots, x_N) = \sqrt{\sum_{k=1}^N S_k(x_1, x_2, \dots, x_N)^2}. \quad (44)$$

We give the idea of how we arrived to this result. The calculations are not proper but describe the idea:

From (34) it holds

$$\frac{\Delta_2(f(\Phi))}{\Delta_1(\Phi)} = f'(\Phi) \frac{\Delta_2(\Phi)}{\Delta_1(\Phi)} + f''(\Phi).$$

Or if we use the parametrization of Theorem 4

$$e^{\int \frac{\Delta_2(\Phi)}{\Delta_1(\Phi)} d\Phi} \frac{\Delta_2(f(\Phi))}{\Delta_1(\Phi)} = \frac{d}{d\Phi} \left(e^{\int \frac{\Delta_2(\Phi)}{\Delta_1(\Phi)} d\Phi} f'(\Phi) \right)$$

Or

$$\int e^{-\int \frac{\Delta_2(\Phi)}{\Delta_1(\Phi)} d\Phi} \int e^{\int \frac{\Delta_2(\Phi)}{\Delta_1(\Phi)} d\Phi} \frac{\Delta_2(f(\Phi))}{\Delta_1(\Phi)} d\Phi = f(\Phi). \quad (45)$$

Now, if Φ is that of (44), then $\Delta_2(f(\Phi))$, $\Delta_1(\Phi)$, $\Delta_2(\Phi)$ are functions of Φ and we have

$$\Delta_1(\Phi) = 1 \quad (46)$$

and

$$\Delta_2(\Phi) = \frac{N-1}{\sqrt{\sum_{k=1}^N S_k(x_1, x_2, \dots, x_N)^2}} = \frac{N-1}{\Phi}. \quad (47)$$

From (44),(45),(46),(47) and setting

$$H(x) = \frac{\Delta_2(f(\Phi))}{\Delta_1(f(\Phi))},$$

we arrive to

$$\int x^{-(N-1)} \left(\int x^{(N-1)} H(x) dx \right) dx = f(x).$$

Hence solving with respect to $H(x)$ this last equation, we find the value of $\Delta_2 f(\Phi)$, where

$$\Phi = \Phi_0 = \sqrt{\sum_{n=1}^N S_n(x_1, x_2, \dots, x_N)^2}.$$

Theorem 6.

Consider the N -dimensional metric space S (as in Theorem 5) and $\nu = (N-2)/2$. Then the PDE

$$\partial_t U(x^a, t) = \Delta_{2,x} U(x^a, t), \quad (48)$$

admits solution

$$\begin{aligned} U(x_1, x_2, \dots, x_N, t) = U(x^a, t) &= \frac{C_1}{\Phi_0(x^a)^\nu} \sum_{\lambda=1}^{\infty} c_\lambda e^{-\lambda t} J_\nu \left(\sqrt{\lambda} \Phi_0(x^a) \right) + \\ &+ \frac{C_2}{\Phi_0(x^a)^\nu} \sum_{\lambda=1}^{\infty} c_\lambda e^{-\lambda t} Y_\nu \left(\sqrt{\lambda} \Phi_0(x^a) \right), \end{aligned} \quad (49)$$

where J_ν and Y_ν are the usual Bessel functions of the first and second kind of order ν and

$$\Phi_0(x^a) = \sqrt{\sum_{k=1}^N S_k(x^a)^2}.$$

Proof.

Set $X_\lambda(x^a) = X_\lambda(\Phi_0)$ and observe from (42) that the DE

$$\Delta_2 X_\lambda(x^a) = -\lambda X_\lambda(x^a)$$

have solution

$$X_\lambda(x^a) = y_\lambda(\Phi_0),$$

where

$$y_\lambda(x) = C_1 x^{-\nu} J_\nu(\sqrt{\lambda}x) + C_2 x^{-\nu} Y_\nu(\sqrt{\lambda}x).$$

This lead us to the desired result.

Notes.

- 1) The main idea of Theorem 6 remains the same if we take instead of λ an arbitrary sequence λ_p . Then the summation will be with respect to p .
- 2) The function

$$U_1(x_1, x_2, \dots, x_N, t) = U(x^a, t) = \frac{C_1}{\Phi_0(x^a)^\nu} \sum_{\lambda=1}^{\infty} c_\lambda e^{-i\lambda t} J_\nu(\sqrt{\lambda}\Phi_0(x^a)) + \frac{C_2}{\Phi_0(x^a)^\nu} \sum_{\lambda=1}^{\infty} c_\lambda e^{-i\lambda t} Y_\nu(\sqrt{\lambda}\Phi_0(x^a)), \quad (50)$$

satisfies the equation

$$i\partial_t U_1(x^a, t) = -\Delta_{2,x} U_1(x^a, t). \quad (51)$$

- 3) For any function $f(x)$ of one variable and $\Phi_0 = \sqrt{\sum_{k=1}^N S_k(x^a)^2}$, holds

$$\Delta_{2,x} f(\Phi_0) = f''(\Phi_0) + \frac{N-1}{\Phi_0} f'(\Phi_0). \quad (52)$$

Theorem 7.

Consider the N -dimensional metric space S (as in Theorem 5). The PDE

$$\Delta_2 U(x_1, x_2, \dots, x_N) = -\lambda U(x_1, x_2, \dots, x_N), \quad \lambda > 0, \quad (53)$$

admits solution

$$U(x_1, x_2, \dots, x_N) = \frac{C_1}{\Phi_0(x_1, x_2, \dots, x_N)^{N/2-1}} J_{N/2-1}(\sqrt{\lambda}\Phi_0(x_1, x_2, \dots, x_N)) + \frac{C_2}{\Phi_0(x_1, x_2, \dots, x_N)^{N/2-1}} Y_{N/2-1}(\sqrt{\lambda}\Phi_0(x_1, x_2, \dots, x_N)). \quad (54)$$

Proof.

Easy.

Theorem 8.

In the space S , we set the Beltrami-D'Alembert wave operator to be

$$\Pi_2 \equiv \Delta_2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}, \quad (55)$$

where the Δ_2 is defined in the space S as in (7). Then the equation

$$\Pi_2 U(x^a, t) = 0,$$

where $x^a = \{x_1, x_2, \dots, x_N\}$ admits solution

$$U(x^a, t) = \sum_{\lambda=1}^{\infty} B_{\lambda} e^{itc\sqrt{\lambda}} U_{\lambda}(x^a). \quad (56)$$

The function $U_{\lambda}(x^a) = U(x^a)$, $U(x^a)$ is that of (54).

Proof.

Easy

Theorem 9.

Let S^+ be the space produced as

$$\begin{aligned} \overline{S}^+(x^a, x_{N+1}) &= \\ &= \{S_1(x^a, x_{N+1}), S_2(x^a, x_{N+1}), \dots, S_N(x^a, x_{N+1}), S_{N+1}(x^a, x_{N+1})\}. \end{aligned}$$

Then if

$$\begin{aligned} U_{\lambda}(x_1, x_2, \dots, x_N, x_{N+1}) &= \\ = U_{\lambda}(x^a, x_{N+1}) &= \frac{C_1}{\Phi_0(x^a, x_{N+1})^{N/2-1}} J_{N/2-1} \left(\sqrt{\lambda} \Phi_0(x^a, x_{N+1}) \right) + \\ &+ \frac{C_2}{\Phi_0(x^a, x_{N+1})^{N/2-1}} Y_{N/2-1} \left(\sqrt{\lambda} \Phi_0(x^a, x_{N+1}) \right), \end{aligned}$$

is that of relation (54) (with $N + 1$ coordinates) and

$$\Phi_0(x^a, x_{N+1}) = \sqrt{\sum_{k=1}^N S_k^+(x^a, x_{N+1})^2},$$

then the function

$$\begin{aligned} U^+ &= U^+(x_1, x_2, \dots, x_N, x_{N+1}) = \\ &= U^+(x^a, x_{N+1}) = \sum_{\lambda=1}^{\infty} c_{\lambda} e^{\sqrt{\lambda} S_{N+1}(x^a, x_{N+1})} U_{\lambda}(x^a, x_{N+1}), \end{aligned} \quad (57)$$

have 2-Beltrami derivative

$$\Delta_2 (U^+(x_1, x_2, \dots, x_N, x_{N+1})) = 0, \quad (58)$$

where $\nu = (N + 1 - 3)/2$. Here for no confusion the second Beltrami derivative refers to all coordinates $\{x^a, x_{N+1}\} = \{x_1, x_2, \dots, x_N, x_{N+1}\}$.

Proof.

Let x^a, x_{N+1} as defined in the state of the theorem. We will use the following acceptable change of variables:

$$x_j = f_j((x^a)', x'_{N+1}), j = 1, 2, \dots, N + 1,$$

such that:

$$S_k(x^a, x_{N+1}) = S_k((x^a)', 0), \text{ for } 1 \leq k \leq N \text{ and } 1 \leq a \leq N \quad : (\nu_1)$$

and

$$S_{N+1}(x^a, x_{N+1}) = icx'_{N+1}. \quad : (\nu_2)$$

Also let $\Phi_0 = \sqrt{\sum_{k=1}^N S_k(x^a)^2}$ considers the values of S_k for $k = 1, \dots, N$ only. Then it holds

$$\Pi_2 \left(\sum_{\lambda=1}^{\infty} c_\lambda e^{ic\sqrt{\lambda}x'_{N+1}} U_\lambda(x'_1, x'_2, \dots, x'_N) \right) = 0$$

and also

$$\Delta_2 U^+(x^a, x_{N+1}) = \Delta_2 \left(\sum_{\lambda=1}^{\infty} c_\lambda e^{\sqrt{\lambda}S_{N+1}(x^a, x_{N+1})} U_\lambda(x_1, x_2, \dots, x_N, x_{N+1}) \right).$$

But from the change of variables (ν_1) and (ν_2) we have

$$\begin{aligned} \Delta_2 U^+(x^a, x_{N+1}) &= \Delta_2 \left(\sum_{\lambda=1}^{\infty} c_\lambda e^{i\sqrt{\lambda}cx'_{N+1}} U_\lambda(x'_1, x'_2, \dots, x'_N) \right) = \\ &= \Pi_2 \left(\sum_{\lambda=1}^{\infty} c_\lambda e^{ic\sqrt{\lambda}x'_{N+1}} U_\lambda(x'_1, x'_2, \dots, x'_N) \right) = 0, \end{aligned}$$

according to Theorem 8. Hence we complete the proof.

Note.

The above theorem show us that in every metric space are attached the harmonic functions (57), which by Theorems 6,7 are related with the Bessel functions $J_\nu(x)$ and $Y_\nu(x)$.

Theorem 10.

Let S^{++} be a metric space with

$$\begin{aligned} \overline{S}^{++}(x_1, x_2, \dots, x_N, x_{N+1}, x_{N+2}) &= \overline{S}^{++}(x^a, x_{N+1}, x_{N+2}) = \\ &= \{S_1(x^a), S_2(x^a), \dots, S_N(x^a), x_{N+1}, x_{N+2}\} \end{aligned}$$

and

$$\Phi_0 = \Phi_0(x^a) = \sqrt{\sum_{k=1}^N S_k^2(x^a)}.$$

Then the equation

$$\Delta_2 U^{++} - N \sum_{k,m=N+1}^{N+2} \epsilon_{k,m} \frac{\partial^2 U^{++}}{\partial x_k \partial x_m} = 0,$$

where $\epsilon_{N+1,N+1} = \epsilon_{N+2,N+2} = 1$ and $\epsilon_{N+1,N+2} = \epsilon_{N+2,N+1} = -1/2$, admits solution

$$U^{++} = \sum_{k=1}^{\infty} A_k e^{ix_{N+1}\lambda_k} e^{ix_{N+2}\lambda_k} \frac{J_{N/2-1}(\Phi_0\lambda_k)}{\Phi_0^{N/2-1}}$$

and Δ_2 is referring to all coordinates $\{x_1, x_2, \dots, x_N, x_{N+1}, x_{N+2}\}$.

Theorem 11.

Let S be the 3-dimensional flatten metric space. Set $x^a = \{x_1, x_2, x_3\}$ and let $A_\lambda = A_\lambda(x^a)$ such that

$$\Delta_2(A_\lambda(x^a)) = 0. \quad (59)$$

Moreover let

$$\phi_\lambda(x) = \frac{\sqrt{2}}{\sqrt{\pi}\lambda^{1/4}x} \left(C_1 \sin(\sqrt{\lambda}x) + C_2 \cos(\sqrt{\lambda}x) \right)$$

and $\phi_\lambda = \phi_\lambda(\Phi_0)$, with $\Phi_0 = \sqrt{x_1^2 + x_2^2 + x_3^2}$.

If

$$V(x^a, t) = \frac{\sum_{\lambda=1}^{\infty} c_\lambda^* \lambda e^{-it\lambda} A_\lambda(x^a)}{\sum_{\lambda=1}^{\infty} c_\lambda^* e^{-it\lambda} A_\lambda(x^a) + \sum_{\lambda=1}^{\infty} c_\lambda \phi_\lambda(\Phi_0) e^{-it\lambda}}, \quad (60)$$

then a solution of the Schrodinger's equation

$$i\partial_t \Psi(x^a, t) = -\Delta_{2,x} \Psi(x^a, t) + V(x^a, t) \Psi(x^a, t), \quad (61)$$

is

$$\Psi(x^a, t) = \sum_{\lambda=1}^{\infty} c_\lambda \phi_\lambda(\Phi_0) e^{-it\lambda} + \sum_{\lambda=1}^{\infty} c_\lambda^* e^{-it\lambda} A_\lambda(x^a). \quad (62)$$

Proof.

Let $\Psi = \Psi(x^a, t) = U(x^a, t) + G(x^a)$, where $U(x^a, t) = \sum_{\lambda=1}^{\infty} c_\lambda \phi_\lambda(\Phi_0) e^{-it\lambda}$ (as in Theorem 6), but now satisfying $i\partial_t U = -\Delta_2 U$ and potential $V(x_1, x_2, x_3, t) = V(x^a, t)$, depended from x^a, t . Let also $G = G(x^a, t)$ be harmonic with respect to x^a , i.e. $\Delta_2 G = 0$. Then

$$i\partial_t \Psi = -\Delta_2 \Psi + V\Psi,$$

or equivalently

$$i\partial_t U + i\partial_t G = -\Delta_2 U - \Delta_2 G + VU + VG.$$

Hence if also U is such that $i\partial_t U = -\Delta_2 U$, we have

$$i\partial_t G = (U + G)V,$$

or equivalently

$$V = \frac{i\partial_t G}{U + G}.$$

QED

Notes.

1) All the potentials of the form

$$V(x, y, z, t) = V(x, y, z) = \frac{c_1 f \left(e^{-i \cot^{-1} \left(\frac{z}{y} \right)} \sqrt{x^2 + y^2} \right)}{c_2 f \left(e^{-i \cot^{-1} \left(\frac{z}{y} \right)} \sqrt{x^2 + y^2} \right) - P \left(\sqrt{x^2 + y^2 + z^2} \right)}, \quad (63)$$

where f is arbitrary smooth function of \mathbf{R} and $P(x) = \sum_{\lambda=1}^{\infty} c_{\lambda} \phi_{\lambda}(x)$, are solvable (c_{λ} are arbitrary numbers such the series $P(x)$ converges).

2) If F is a smooth function and

$$A(x_1, x_2, x_3) = \int \int \int_{\mathbf{R}^3} \frac{F(x'_1, x'_2, x'_3)}{\left[1 + \sum_{k=1}^3 (x_k - x'_k)^2 \right]^2} dx'_1 dx'_2 dx'_3, \quad (64)$$

then $A(x_1, x_2, x_3)$ is Harmonic function in \mathbf{R}^3 . The opposite is also true: For every harmonic function A exists F such that (64) holds.

Example.

In the case of \mathbf{E}_3 , for $\partial V / \partial t = 0$ and $0 = c_2 = c_3 = \dots$, we have that all the potentials

$$V(x_1, x_2, x_3) = \frac{A(x_1, x_2, x_3)}{2(x_1^2 + x_2^2 + x_3^2)^{-1/2} \cos \left(\sqrt{x_1^2 + x_2^2 + x_3^2} \right) - A(x_1, x_2, x_3)},$$

are solvable.

Theorem 11 can expanded to every metric non Euclidian space as follows

Theorem 12.

Let

$$\begin{aligned} \bar{S}(x_1, x_2, \dots, x_N) = \\ = \{S_1(x_1, x_2, \dots, x_N), S_2(x_1, x_2, \dots, x_N), \dots, S_N(x_1, x_2, \dots, x_N)\} \end{aligned}$$

be an arbitrary metric space. In this case we have $\nu = (N - 2)/2$ and

$$\Delta_2(\phi_{\nu, \lambda}(\Phi_0)) = -\lambda \phi_{\nu, \lambda}(\Phi_0), \quad (64.1)$$

where

$$\phi_{\nu, \lambda}(x) = c_1 x^{-\nu} J_{\nu}(\sqrt{\lambda} x) + c_2 x^{-\nu} Y_{\nu}(\sqrt{\lambda} x). \quad (64.2)$$

Also if $\Phi_0 = \sqrt{\sum_{k=1}^N S_k(x_1, x_2, \dots, x_N)^2}$, $x^a = \{x_1, x_2, \dots, x_N\}$, then for a given potential

$$V(x^a, t) = \frac{\sum_{\lambda=1}^{\infty} c_{\lambda}^* \lambda e^{-it\lambda} A_{\lambda}(x^a)}{\sum_{\lambda=1}^{\infty} c_{\lambda}^* e^{-it\lambda} A_{\lambda}(x^a) + \sum_{\lambda=1}^{\infty} c_{\lambda} \phi_{\nu, \lambda}(\Phi_0) e^{-it\lambda}}, \quad (65)$$

with $\Delta_2(A_\lambda(x^a)) = 0$, a solution of

$$i\partial_t \Psi(x^a, t) = -\Delta_{2,x} \Psi(x^a, t) + V(x^a, t) \Psi(x^a, t), \quad (66)$$

is

$$\Psi(x^a, t) = \sum_{\lambda=1}^{\infty} c_\lambda e^{-it\lambda} \phi_{\nu,\lambda}(\Phi_0) + \sum_{\lambda=1}^{\infty} c_\lambda^* e^{-it\lambda} A_\lambda(x^a). \quad (67)$$

Theorem 13.

If the potentials $V(x^a, t)$ are of the form

$$V(x^a, t) = \frac{\sum_{k=1}^{\infty} c_k^* k e^{-itk} A_k(x^a)}{\sum_{k=1}^{\infty} c_k^* e^{-itk} A_k(x^a) + \sum_{k=1}^{\infty} c_k \phi_{\nu,k}(\Phi_0) e^{-itk}}, \quad (68)$$

where

$$A_k(x^a) = \sum_{\lambda=1}^{\infty} a_{k,\lambda} e^{\sqrt{\lambda} S_N(x^a)} U_\lambda(\Phi_0^-),$$

with $\Phi_0^- = \sqrt{\sum_{\lambda=1}^{N-1} S_k(x^a)^2}$, then a solution of (66) is

$$\Psi = \Psi(x^a, t) = \sum_{k=1}^{\infty} c_k e^{-itk} \phi_{\nu,k}(\Phi_0) + \sum_{k=1}^{\infty} c_k^* e^{-itk} A_k(x^a). \quad (69)$$

Proof.

Observe that in any space S holds

$$\Delta_2(A_k(x^a)) = 0 \quad (70)$$

Theorem 14.

The potentials of the form

$$\begin{aligned} V = V(x^a, t) &= -\frac{\sum_{\lambda=1}^{\infty} c_\lambda \sqrt{\lambda} e^{-it\lambda} \phi_{\nu+1,\lambda}(\Phi_0)}{\sum_{\lambda=1}^{\infty} c_\lambda e^{-it\lambda} \phi_{\nu,\lambda}(\Phi_0)} = \\ &= \left(\frac{1}{x} \frac{\partial}{\partial x} \log \left(\sum_{\lambda=1}^{\infty} c_\lambda e^{-it\lambda} \phi_{\nu,\lambda}(x) \right) \right)_{x=\Phi_0}, \end{aligned} \quad (71)$$

are solvable in view of Theorem 16 below.

Theorem 15.

Consider the case of $S = \mathbf{R}$. Then if

$$V(x, t) = -\frac{2\nu+1}{x} \partial_x (\log(u(x, t))) \quad (72)$$

and $u(x, t)$ satisfies the PDE

$$i\partial_t u(x, t) = -\partial_x^2 u(x, t) - \frac{2\nu+1}{x} \partial_x u(x, t), \quad (73)$$

the Schrodinger's equation

$$i\partial_t \Psi(x, t) = -\partial_x^2 \Psi(x, t) + V(x, t) \Psi(x, t) \quad (74)$$

is solvable and the solution is $\Psi(x, t) = u(x, t)$.

Theorem 16.

The functions

$$u_0(x, t) = \sum_{\lambda=1}^{\infty} c_{\lambda} e^{-it\lambda} \phi_{\nu, \lambda}(x), \quad (75)$$

are solutions of PDE

$$i\partial_t u(x, t) = -\partial_x^2 u(x, t) - \frac{2\nu+1}{x} \partial_x u(x, t). \quad (76)$$

The oposite is also true. If $u(x, t)$ satisfy (76), then $u(x, t)$ are of the form (75). Also if the dimension of S is N and the potential's are of the form

$$V(x^a, t) = -\frac{2\nu+2-N}{\Phi_0} (\partial_x \log u(x, t))_{x=\Phi_0}, \quad (77)$$

with $u(x, t)$ solution of (76), then equation

$$i\partial_t \Psi(x^a, t) = -\Delta_{2,x} \Psi(x^a, t) + V(x^a, t) \Psi(x^a, t), \quad (78)$$

have solution $\Psi(x^a, t) = u(\Phi_0, t)$.

Assume now $r_{\nu, \lambda}$ is the λ -th root of $J_{\nu}(x) = 0$ and $\rho_{\nu, \lambda} = r_{\nu, \lambda}^2$. When $f(x, t) \in D = L[0, 1] \times \mathbf{R}$, then the functions $x^{\nu+1/2} \phi_{\nu, \rho_{\nu, \lambda}}(x)$, (for $c_2 = 0$ in (64.2)) are bases in D and writing

$$f(x, t) = \sum_{p=1}^{\infty} C_p e^{-it\rho_{\nu, p}} x^{\nu+1/2} \phi_{\nu, \rho_{\nu, p}}(x), \quad (79)$$

we have

$$\begin{aligned} \int_0^1 x^{2\nu+1} \phi_{\nu, \rho_{\nu, n}}(x) \phi_{\nu, \rho_{\nu, m}}(x) dx &= \int_0^1 x J_{\nu}(xr_{\nu, n}) J_{\nu}(xr_{\nu, m}) dx = \\ &= \frac{1}{2} J_{\nu+1}(r_{\nu, n}) J_{\nu+1}(r_{\nu, m}) \delta_{n, m}. \end{aligned}$$

The $\delta_{n, m}$ is 1 if $n = m$ and 0 otherwise. Hence the coefficients C_p in (79) are

$$C_p = \frac{2e^{it\rho_{\nu, p}}}{J_{\nu+1}(r_{\nu, p})^2} \int_0^1 f(x, t) x^{\nu+1/2} \phi_{\nu, \rho_{\nu, p}}(x) dx. \quad (80)$$

In case

$$\frac{2e^{it\rho_{\nu, p}}}{J_{\nu+1}(r_{\nu, p})^2} \int_0^1 f(x, t) x^{1/2} J_{\nu}(r_{\nu, p}x) dx$$

is independent of t we have that indeed $f(x, t)$ is solution of (76). Hence

$$\int_0^1 f(x, t) x^{\nu+1/2} \phi_{\nu, \rho_{\nu, p}}(x) dx = \frac{1}{2} C_p J_{\nu+1}(r_{\nu, p})^2 e^{-itr_{\nu, p}^2}, \quad (81)$$

where C_p is constant.

Also assuming the initial condition $f(x, 0) = f_0(x)$, we get

$$f(x, t) = \sum_{p=1}^{\infty} \left(\frac{2x^{-\nu}}{J_{\nu+1}(r_{\nu, p})^2} \int_0^1 f_0(y) y^{1/2} J_{\nu}(r_{\nu, p} y) dy \right) e^{-ir_{\nu, p}^2 t} J_{\nu}(r_{\nu, p} x). \quad (82)$$

Now if we assume that the function $f(x, t)$ have an expansion of the form

$$f(x, t) = \sum_{q=1}^{\infty} c_q A_q(t) B_q(x), \quad (83)$$

then

$$\begin{aligned} \int_0^1 f(x, t) x^{\nu+1/2} \phi_{\nu, \rho_{\nu, p}}(x) dx &= \sum_{q=1}^{\infty} c_q A_q(t) \int_0^1 B_q(x) x^{1/2} J_{\nu}(r_{\nu, p} x) dx = \\ &= \frac{1}{2} C_p J_{\nu+1}(r_{\nu, p})^2 e^{-itr_{\nu, p}^2}. \end{aligned}$$

Hence there exists constants $C_{\nu, q, p}$, $E_{\nu, p}$, such that

$$\sum_{q=1}^{\infty} C_{\nu, q, p} A_q(t) = E_{\nu, p} e^{-itr_{\nu, p}^2}, \quad \forall t \in \mathbf{R}. \quad (84)$$

Write

$$A_q(t) = \sum_{l=1}^{\infty} \eta_{q, l} e^{-itr_{\nu, l}^2}. \quad (85)$$

Then

$$\begin{aligned} \sum_{q=1}^{\infty} C_{\nu, q, p} \sum_{l=1}^{\infty} \eta_{q, l} e^{-itr_{\nu, l}^2} &= E_{\nu, p} e^{-itr_{\nu, p}^2} \Rightarrow \\ \sum_{l=1}^{\infty} \left(\sum_{q=1}^{\infty} \eta_{q, l} C_{\nu, q, p} \right) e^{-itr_{\nu, l}^2} &= E_{\nu, p} e^{-itr_{\nu, p}^2} \Rightarrow \\ \sum_{q=1}^{\infty} \eta_{q, l} C_{\nu, q, p} &= \delta_{p, l} E_{\nu, l}. \end{aligned} \quad (86)$$

Hence also

$$\sum_{q=1}^{\infty} \eta_{q, p} C_{\nu, q, p} = E_{\nu, p}. \quad (87)$$

Assume that $\eta_{l,a}^*$ is such that $\sum_{l=1}^{\infty} \eta_{q,l} \eta_{l,a}^* = \delta_{q,a}$, then

$$\begin{aligned} \sum_{q=1}^{\infty} \sum_{l=1}^{\infty} \eta_{q,l} \eta_{l,a}^* C_{\nu,q,p} &= \sum_{l=1}^{\infty} \delta_{p,l} \eta_{l,a}^* E_{\nu,l} \Rightarrow \sum_{q=1}^{\infty} \delta_{q,a} C_{\nu,q,p} = \eta_{p,a}^* E_{\nu,p} \Rightarrow \\ C_{\nu,a,p} &= \eta_{p,a}^* E_{\nu,p}. \end{aligned} \quad (88)$$

Hence $A_q(t)$ have the form (85) and staisfy (86),(87). From (86) with $p \neq l$, we have

$$\sum_{q=1}^{\infty} \eta_{q,l} C_{\nu,q,p} = 0, \quad l \neq p.$$

Hence

$$\begin{aligned} \sum_{q=1}^{\infty} \sum_{l \neq p} \eta_{q,l} \eta_{l,a}^* C_{\nu,q,p} &= 0 \Leftrightarrow \\ \sum_{q=1}^{\infty} \left[\left(\sum_{l=1}^{\infty} \eta_{q,l} \eta_{l,a}^* \right) - \eta_{q,p} \eta_{p,a}^* \right] C_{\nu,q,p} &= 0 \Leftrightarrow \\ \sum_{q=1}^{\infty} (\delta_{q,a} - \eta_{q,p} \eta_{p,a}^*) C_{\nu,q,p} &= 0 \Leftrightarrow \\ C_{\nu,a,p} - \eta_{p,a}^* \sum_{q=1}^{\infty} \eta_{q,p} C_{\nu,q,p} &= 0 \Leftrightarrow \\ \eta_{p,a}^* E_{\nu,p} - \eta_{p,a}^* \sum_{q=1}^{\infty} \eta_{q,p} C_{\nu,q,p} &= 0. \end{aligned}$$

Hence $\eta_{p,a}^* = 0$ or if $\eta_{p,a}^* \neq 0$, we have

$$E_{\nu,p} = \sum_{q=1}^{\infty} \eta_{q,p} C_{\nu,q,p}.$$

Now since $x^{\nu+1/2} \phi_{\nu,\rho_{\nu,l}} = x^{1/2} J_{\nu}(r_{\nu,l}x)$ is a base of $L[0, 1]$ every function $B_q(x)$ have expansion of the form

$$B_q(x) = B_{\nu,q}(x) = \sum_{k=1}^{\infty} c_{\nu,q,k}^* x^{\nu+1/2} \phi_{\nu,\rho_{\nu,k}}(x) dx = x^{1/2} \sum_{k=1}^{\infty} c_{\nu,q,k}^* J_{\nu}(r_{\nu,k}x). \quad (89)$$

Hence

$$\begin{aligned} c_q \int_0^1 B_q(x) x^{\nu+1/2} \phi_{\nu,\rho_{\nu,q}}(x) dx &= C_{\nu,q,p} \Leftrightarrow \\ c_q c_{\nu,q,p}^* \frac{1}{2} J_{\nu+1}(r_{\nu,p})^2 &= C_{\nu,q,p}. \end{aligned}$$

Using (89) and (85) in (83) we get

$$\begin{aligned}
f(x, t) &= \sum_{q=1}^{\infty} c_q \left(\sum_{l=1}^{\infty} \eta_{q,l} e^{-itr_{\nu,l}^2} \right) \left(x^{1/2} \sum_{k=1}^{\infty} c_{\nu,q,k}^* J_{\nu}(r_{\nu,k}x) \right) = \\
&= x^{1/2} \sum_{q,l,k=1}^{\infty} c_q \eta_{q,l} c_{\nu,q,k}^* e^{-itr_{\nu,l}^2} J_{\nu}(r_{\nu,k}x) = \\
&= \frac{2x^{1/2}}{J_{\nu+1}(r_{\nu,p})^2} \sum_{q,k,l=1}^{\infty} \eta_{q,l} C_{\nu,q,k} e^{-itr_{\nu,l}^2} J_{\nu}(r_{\nu,k}x) = \\
&= \frac{2x^{1/2}}{J_{\nu+1}(r_{\nu,p})^2} \sum_{k,l=1}^{\infty} \delta_{k,l} E_{\nu,l} e^{-itr_{\nu,l}^2} J_{\nu}(r_{\nu,k}x) = \\
&= \frac{2x^{1/2}}{J_{\nu+1}(r_{\nu,p})^2} \sum_{k=1}^{\infty} E_{\nu,k} e^{-itr_{\nu,k}^2} J_{\nu}(r_{\nu,k}x) = \\
&= \frac{2x^{1/2}}{J_{\nu+1}(r_{\nu,p})^2} \sum_{k=1}^{\infty} \frac{1}{2} C_k J_{\nu+1}(r_{\nu,k})^2 e^{-itr_{\nu,k}^2} J_{\nu}(r_{\nu,k}x) = \\
&= x^{1/2} \sum_{k=1}^{\infty} C_k e^{-itr_{\nu,k}^2} J_{\nu}(r_{\nu,k}x).
\end{aligned}$$

Hence we have the next

Theorem 17.

Assume that given $f(x, t)$, this is of the form (for example from a PDE problem):

$$f(x, t) = \sum_{p=1}^{\infty} c_p A_p(t) B_p(x). \quad (90)$$

Assume also that

$$e^{itr_{\nu,p}^2} \int_0^1 f(x, t) x^{1/2} J_{\nu}(r_{\nu,p}x) dx = \frac{1}{2} C_p J_{\nu+1}(r_{\nu,p})^2, \quad (91)$$

for some constants C_p . Then $f(x, t)$ satisfies the PDE

$$i\partial_t u(x, t) = -\partial_x^2 u(x, t) - \frac{2\nu+1}{x} \partial_x u(x, t) \quad (92)$$

and the general solution $u(x, t)$ of (92) as also the function $f(x, t)$ are given by

$$u(x, t) = f(x, t) = x^{1/2} \sum_{p=1}^{\infty} C_p e^{-itr_{\nu,p}^2} J_{\nu}(r_{\nu,p}x). \quad (93)$$

4 The self-adjoint property of the Hamiltonian

The Hamiltonian of equation (11) in the space

$$S : \overline{\mathbf{S}}(x^a) = \{S_1(x^a), S_2(x^a), \dots, S_N(x^a)\}, \quad (94)$$

is

$$H\Psi = -\Delta_{2,x}\Psi + V(x^a, t)\Psi. \quad (95)$$

We define the inner product of the functions $f = f(x^a)$ and $g = g(x^a)$ with

$$\langle f, g \rangle := \int_{\mathbf{R}^N} f(x_1, x_2, \dots, x_N) \overline{g(x_1, x_2, \dots, x_N)} dx_1 dx_2 \dots dx_N = \int_{\mathbf{R}^N} f \overline{g} dx^a. \quad (96)$$

We will find the conditions under in what spaces and what functions F_1, G_1 holds.

$$\int_{\mathbf{R}^N} \left(HF_1(\Phi_0) \overline{G_1(\Phi_0)} - \overline{HG_1(\Phi_0)} F_1(\Phi_0) \right) dx^a = 0. \quad (97)$$

We assume first that F_1, G_1 are single valued. We have

$$\int_{\mathbf{R}^N} \left[\overline{G_1(\Phi_0)} \Delta_2 F_1(\Phi_0) - F_1(\Phi_0) \overline{\Delta_2 G_1(\Phi_0)} \right] dx^a = 0. \quad (98)$$

For this we make change of variables of the form

$$x'_i = f_i(x^a), \quad i = 1, 2, \dots, N.$$

More precisely we set

$$x'_k = S_k(x^a), \quad k = 1, 2, \dots, N. \quad (T)$$

Then equation (98) is transformed into

$$\begin{aligned} & \int_{\mathbf{R}^N} \left[\overline{G_1 \left(\sqrt{x_1'^2 + x_2'^2 + \dots + x_N'^2} \right)} \Delta_2 F_1 \left(\sqrt{x_1'^2 + x_2'^2 + \dots + x_N'^2} \right) - \right. \\ & \left. - F_1 \left(\sqrt{x_1'^2 + x_2'^2 + \dots + x_N'^2} \right) \overline{\Delta_2 G_1 \left(\sqrt{x_1'^2 + x_2'^2 + \dots + x_N'^2} \right)} \right] \times \\ & \quad \times \frac{D(x_1, x_2, \dots, x_N)}{D(x'_1, x'_2, \dots, x'_N)} dx'_1 dx'_2 \dots dx'_N = 0. \end{aligned}$$

Applying again the change of variables into polar-type

$$\begin{aligned} x'_1 &= r \cos(\theta_1) \cos(\theta_2) \dots \cos(\theta_{N-2}) \cos(\theta_{N-1}), \\ x'_2 &= r \cos(\theta_1) \cos(\theta_2) \dots \cos(\theta_{N-2}) \sin(\theta_{N-1}), \\ x'_3 &= r \cos(\theta_1) \cos(\theta_2) \dots \cos(\theta_{N-3}) \sin(\theta_{N-2}), \\ x'_4 &= r \cos(\theta_1) \cos(\theta_2) \dots \cos(\theta_{N-4}) \sin(\theta_{N-3}), \end{aligned}$$

$$\begin{aligned}
& \vdots \\
x'_{N-1} &= r \cos(\theta_1) \sin(\theta_2), \\
x'_N &= r \sin(\theta_1), \tag{99}
\end{aligned}$$

and $A_{pol} = \{0 \leq r < +\infty, \theta_1, \theta_2, \dots, \theta_{N-2} \in (-\pi/2, \pi/2), \theta_{N-1} \in (0, 2\pi)\}$, we have to show:

$$\begin{aligned}
& \int_{A_{pol}} [G_1(r) \left(\frac{N-1}{r} F_1^{(1)}(r) + F_1^{(2)}(r) \right) - F_1(r) \left(\frac{N-1}{r} G_1^{(1)}(r) + G_1^{(2)}(r) \right)] \times \\
& \quad \times \frac{D(x_1, x_2, \dots, x_N)}{D(x'_1, x'_2, \dots, x'_N)} \frac{D(x'_1, x'_2, \dots, x'_N)}{D(r, \theta_2, \dots, \theta_{N-1})} dr d\theta_1 \dots d\theta_{N-1} = 0.
\end{aligned}$$

Or if we assume that

$$\frac{D(x_1, x_2, \dots, x_N)}{D(x'_1, x'_2, \dots, x'_N)} \frac{D(x'_1, x'_2, \dots, x'_N)}{D(r, \theta_2, \dots, \theta_{N-1})} = r^{(N-1)\mu} T(\theta_1, \theta_2, \dots, \theta_{N-1}). \tag{100}$$

Then equivalently we have

$$\begin{aligned}
& \int_0^\infty r^{(N-1)\mu} [G_1(r) \left(\frac{N-1}{r} F_1'(r) + F_1''(r) \right) - F_1(r) \left(\frac{N-1}{r} G_1'(r) + G_1''(r) \right)] dr \times \\
& \quad \times \int T(\theta_1, \theta_2, \dots, \theta_{N-1}) \cos(\theta_1)^{N-2} \cos(\theta_2)^{N-1} \dots \cos(\theta_{N-1}) d\theta_1 \dots d\theta_{N-1} = 0,
\end{aligned}$$

which is true for functions in \mathbf{F}_μ^N , since the following integral expansion is valid

$$\begin{aligned}
& \int_0^\infty r^{(N-1)\mu} \left(G_1(r) \left(\frac{N-1}{r} F_1'(r) + F_1''(r) \right) - F_1(r) \left(\frac{N-1}{r} G_1'(r) + G_1''(r) \right) \right) dr = \\
& \quad = \left[r^{(N-1)\mu} (G_1(r) F_1'(r) - F_1(r) G_1'(r)) \right]_0^\infty - \\
& \quad - (\mu - 1)(N - 1) \int_0^\infty r^{(N-1)\mu-1} (F_1'(r) G_1(r) - F_1(r) G_1'(r)) dr = 0.
\end{aligned}$$

Hence we conclude that

Theorem 18.

If in the metric space S the transformation (T) have the property

$$\frac{D(x_1, x_2, \dots, x_N)}{D(x'_1, x'_2, \dots, x'_N)} \frac{D(x'_1, x'_2, \dots, x'_N)}{D(r, \theta_2, \dots, \theta_{(N-1)\mu})} = r^{(N-1)\mu} T(\theta_1, \theta_2, \dots, \theta_{N-1}), \tag{101}$$

then the Hamiltonian is a self adjoint operator over \mathbf{F}_μ^N (Definition 4, pg 3), provided that $T(\theta_1, \theta_2, \dots, \theta_{N-1})$ is continuous.

A consequence of this is the next

Theorem 19.

In a space S_μ^N the Hamiltonian H is self adjoint over \mathbf{F}_μ^N .

Proof.

If the transformation is $x'_k(x^a) = S_k(x^a)$, $k = 1, 2, \dots, N$ and is homogeneous of degree μ then the inverse is $x_k = S'_k(x'^a)$ and is also homogeneous of degree μ . Also $\Phi_0(x^a) = \sqrt{\sum_{k=1}^N S_k(x^a)^2}$ is homogeneous of degree μ and every first order partial derivative is of degree $\mu - 1$. Hence the determinant

$$\frac{D(x_1, x_2, \dots, x_N)}{D(x'_1, x'_2, \dots, x'_N)} = \det(H_{k,j}) = \det \left[\left(\frac{\partial S'_k(x'_1, x'_2, \dots, x'_N)}{\partial x'_j} \right)_{k,j} \right],$$

is homogeneous function of degree $(\mu - 1)N$.

$$\begin{aligned} dx_1 dx_2 \dots dx_N &= \frac{D(x_1, x_2, \dots, x_N)}{D(x'_1, x'_2, \dots, x'_N)} dx'_1 dx'_2 \dots dx'_N = \\ &= E(x'_1, x'_2, \dots, x'_N) dx'_1 dx'_2 \dots dx'_N = \\ &= \det \left[\left(\frac{\partial S'_k(x'_1, x'_2, \dots, x'_N)}{\partial x'_j} \right)_{k,j} \right] dx'_1 dx'_2 \dots dx'_N. \end{aligned}$$

Setting now $x'_j \rightarrow \xi y'_j$, $j = 1, 2, \dots, N$: (σ) , we have

$$dx'_1 dx'_2 \dots dx'_N = \xi^N dy'_1 dy'_2 \dots dy'_N.$$

Hence $E(x'_1, x'_2, \dots, x'_N)$ is homogeneous of degree μN . Lastly, taking the polar coordinates (99), from the homogeneity of E we have that r is a common factor (i.e r behaves like ξ) and for a certain T holds

$$E = r^{(N-1)\mu} T(\theta_1, \theta_2, \dots, \theta_{N-1}), \quad \partial_r T = 0.$$

This is condition (101) of Theorem 18.

5 The Euclidian Space and the Polar Coordinates

In the Euclidean space \mathbf{E}_3 the metric is $g_{ij} = \delta_{ij}$ and

$$\Delta_2(\Phi) = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2}.$$

We make the change of coordinates

$$x = r \sin(\theta) \cos(\phi), \quad y = r \sin(\theta) \sin(\phi), \quad z = r \cos(\theta).$$

Then

$$\Delta_2(\Phi) = \frac{\partial^2 \Phi}{\partial r^2} + \frac{2}{r} \frac{\partial \Phi}{\partial r} + \frac{1}{r^2} \left(\frac{1}{\sin^2(\theta)} \frac{\partial^2 \Phi}{\partial \phi^2} + \cot(\theta) \frac{\partial \Phi}{\partial \theta} + \frac{\partial^2 \Phi}{\partial \theta^2} \right)$$

and holds

$$\Delta_2 (f(e^{-i\phi} r \sin(\theta))) = \Delta_1 (f(e^{-i\phi} r \sin(\theta))) = 0. \quad (102)$$

Expanding this idea in \mathbf{E}_N -Euclidian space with dimension $N = 2n$, for any function f analytic at the origin we have

Theorem 20.

For f analytic in the origin set

$$\Phi_0(r, \theta_1, \dots, \theta_{N-1}) = r e^{i\theta_1} e^{\psi(\theta_2)} e^{i\theta_3} e^{\psi(\theta_4)} \dots e^{i\psi(\theta_{N-2})} e^{i\theta_{N-1}}, \quad (103)$$

where $N = 2n$, $n = 1, 2, 3, \dots$ and

$$\psi(x) = 2 \operatorname{arctanh} \left(\tan \left(\frac{x}{2} \right) \right). \quad (104)$$

Then

$$\Delta_{1S}(f(\Phi_0)) = 0. \quad (105)$$

For every f .

Proof.

If $x = (x_1, x_2, \dots, x_N)$ is an element of \mathbf{R}^N then we have

$$r = \sqrt{x_1^2 + x_2^2 + \dots + x_N^2}$$

and

$$\theta_1, \theta_2, \dots, \theta_{N-1} \in \left(-\frac{\pi}{2}, \frac{\pi}{2} \right),$$

$$\theta_{N-1} \in (0, 2\pi),$$

then

$$x_1 = r \cos(\theta_1) \cos(\theta_2) \dots \cos(\theta_{N-2}) \cos(\theta_{N-1}),$$

$$x_2 = r \cos(\theta_1) \cos(\theta_2) \dots \cos(\theta_{N-2}) \sin(\theta_{N-1}),$$

$$x_3 = r \cos(\theta_1) \cos(\theta_2) \dots \cos(\theta_{N-3}) \sin(\theta_{N-2}),$$

$$x_4 = r \cos(\theta_1) \cos(\theta_2) \dots \cos(\theta_{N-4}) \sin(\theta_{N-3}),$$

\vdots

$$x_{N-1} = r \cos(\theta_1) \sin(\theta_2),$$

$$x_N = r \sin(\theta_1). \quad (106)$$

Writing

$$\Phi_1(u_1, u_2, \dots, u_N) = a_1(u_1)a_2(u_2) \dots a_N(u_N),$$

we try to solve the spherical partial differential equation

$$\Delta_{1S}(\Phi_1) = 0$$

(by Δ_{1S} we note the 1–Beltrami operator in spherical coordinates).

One can manage to find g_{ij} at any rate N , using Mathematica Program. We first find the solutions for $N = 2, 3, 4, 5, 6, 7$ and then we derive numerically our results for arbitrary dimensions. Actually the operator $\Delta_{1S}(\Phi_1)$ is quite simple, since the g_{ij} form a diagonal matrix. We have:

$$\begin{aligned} & \left(u_1 \prod_{k=1}^N a_k(u_k) \right)^{-2} \Delta_{1S}(\Phi_1) = \\ & = \frac{a'_1(u_1)^2 u_1^2}{a_1(u_1)^2} + \frac{a'_2(u_2)^2}{a_2(u_2)^2} + \sum_{k=3}^N \frac{a'_k(u_k)^2}{a_k(u_k)^2} \prod_{j=2}^{k-1} \sec(u_j)^2 = 0. \end{aligned} \quad (107)$$

Hence the problem reduces to prove that g_{ij} is diagonal and the values in the diagonal are

$$\left\{ 1, u_1^2, u_1^2 \cos(u_2)^2, u_1^2 \cos(u_2)^2 \cos(u_3)^2, \dots, u_1^2 \prod_{j=2}^{N-1} \cos(u_j)^2 \right\}.$$

If we assure that, then we solve the three simple differential equations $xy'/y = c$, $y'/y = 1$ and $y'/y = c \sec(x)$ and the result follows.

Another interesting proposition for evaluations of spherical 1–Beltrami operators is

Theorem 21.

Let f be analytic in an open set containing the origin, then if $N = 2n$, $n = 2, 3, \dots$ and

$$\begin{aligned} & G(x_1, x_2, x_3, x_4, \dots, x_{N-3}, x_{N-2}, x_{N-1}, x_N) = \\ & = f_1(x_1 x_2) f_2(x_3 x_4) f(x_5 x_6) \dots f_{n-1}(x_{N-3} x_{N-2}) f_n(x_{N-1} x_N), \end{aligned}$$

where $x_1 = r$, $x_2 = e^{i\theta_1}$, $x_3 = e^{\psi(\theta_2)}$, $x_4 = e^{i\theta_3}$, $x_5 = e^{\psi(\theta_4)}$, \dots , $x_{N-1} = e^{\psi(\theta_{N-2})}$, $x_N = e^{i\theta_{N-1}}$, then also

$$\Delta_{1S}(G) = 0, \quad (108)$$

where f_j , $j = 1, 2, \dots, n$ are single valued.

Proof.

Equation (108) follows from (107) where we can set the terms of (107) respectively with $p_1, p_2, \dots, p_{N-1}, p_N$ and then take $p_k + p_{k+1} = 0$, for the equation to hold (Note that we use, as in Theorem 20, the method of separate variables). The p 's in the general solution appear as powers $e^{i(p_k \theta_k + p_{k+1} \psi(\theta_{k+1}))}$.

6 Appendix. (The general solution of space independent Schrodinger equation)

I have read many books in Quantum physics that treat with the subject of time independent Schrodinger equation (TISE). None of these books have treated with SISE (space independent Schrodinger equation). I thought that would be nice to present a general solution of SISE here.

The SISE in a potential $V(t)$ read as

$$i\hbar \frac{\partial}{\partial t} \Psi(x, t) = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \Psi(x, t) + V(t) \Psi(x, t). \quad (1)$$

For a smooth function $\Psi(x, t)$ in a certain L_p space, the Fourier transform is given from

$$\widehat{\Psi}(s, t) = \int_{\mathbf{R}} \Psi(x, t) e^{-ixs} dx \Leftrightarrow \Psi(x, t) = \frac{1}{2\pi} \int_{\mathbf{R}} \widehat{\Psi}(s, t) e^{isx} ds. \quad (2)$$

Assume also that Ψ satisfies the conditions

$$\lim_{x \rightarrow \pm\infty} \Psi(x, t) = \lim_{x \rightarrow \pm\infty} \partial_x \Psi(x, t) = 0. \quad (3)$$

Taking the Fourier transform in (1) with respect to x , we have

$$\begin{aligned} i\hbar \frac{\partial}{\partial t} \int_{\mathbf{R}} \Psi(x, t) e^{-ixs} dx &= -\frac{\hbar^2}{2m} \int_{\mathbf{R}} \frac{\partial^2}{\partial x^2} \Psi(x, t) e^{-ixs} dx + V(t) \widehat{\Psi}(s, t) \Leftrightarrow \\ i\hbar \partial_t \widehat{\Psi}(s, t) &= -\frac{\hbar^2}{2m} \left([\partial_x \Psi(x, t) e^{-ixs}]_{x=-\infty}^{x=+\infty} + is \int_{\mathbf{R}} \frac{\partial}{\partial x} \Psi(x, t) e^{-ixs} dx \right) + \\ &\quad + V(t) \widehat{\Psi}(s, t) \Leftrightarrow \\ i\hbar \partial_t \widehat{\Psi}(s, t) &= -\frac{\hbar^2}{2m} is \int_{\mathbf{R}} \frac{\partial}{\partial x} \Psi(x, t) e^{-ixs} dx + V(t) \widehat{\Psi}(s, t) \Leftrightarrow \\ i\hbar \partial_t \widehat{\Psi}(s, t) &= -\frac{\hbar^2}{2m} is \left([\Psi(x, t) e^{-ixs}]_{x=-\infty}^{x=+\infty} + is \int_{\mathbf{R}} \Psi(x, t) e^{-ixs} dx \right) + \\ &\quad + V(t) \widehat{\Psi}(s, t) \Leftrightarrow \\ i\hbar \partial_t \widehat{\Psi}(s, t) &= \left(\frac{\hbar^2}{2m} s^2 + V(t) \right) \widehat{\Psi}(s, t) \end{aligned} \quad (4)$$

This last equation can be solved easily. We get

$$i\hbar \log \left(\widehat{\Psi}(s, t) \right) = \frac{\hbar^2}{2m} s^2 t + \int V(t) dt + C(s) \Leftrightarrow$$

$$\widehat{\Psi}(s, t) = \exp\left(-\frac{i}{\hbar}C(s)\right) \exp\left(-\frac{i\hbar}{2m}s^2t - \frac{i}{\hbar} \int V(t)dt\right).$$

Lastly inverting the Fourier transform we get

$$\Psi(x, t) = \frac{1}{2\pi} \int_{\mathbf{R}} \exp\left(-\frac{i}{\hbar}C(s)\right) \exp\left(-\frac{i\hbar}{2m}s^2t - \frac{i}{\hbar} \int V(t)dt\right) e^{isx} ds$$

and the general solution of (1) is

$$\Psi(x, t) = \frac{1}{2\pi} \exp\left(-\frac{i}{\hbar} \int V(t)dt\right) \int_{\mathbf{R}} \exp\left(-\frac{i}{\hbar}C(s) - \frac{i\hbar}{2m}s^2t\right) e^{isx} ds, \quad (5)$$

where $C(s)$ is arbitrary smooth function defined from the needs of the problem. Actually then

$$\left|\widehat{\Psi}(s, t)\right| = \exp\left(\frac{Im(C(s))}{\hbar}\right). \quad (6)$$

Hence from Parseval's Identity:

$$f, g \in L_2(\mathbf{R}) \Rightarrow \int_{\mathbf{R}} f(t)\overline{g(t)}dt = \frac{1}{2\pi} \int_{\mathbf{R}} \widehat{f}(\gamma)\overline{\widehat{g}(\gamma)}d\gamma,$$

we get

$$\begin{aligned} \int_{\mathbf{R}} |\Psi(x, t)|^2 dx &= \frac{1}{2\pi} \int_{\mathbf{R}} \left|\widehat{\Psi}(s, t)\right|^2 ds = \frac{1}{2\pi} \int_{\mathbf{R}} \exp\left(\frac{2Im(C(s))}{\hbar}\right) ds = \\ &= c_0 = \text{constant}. \end{aligned}$$

Hence the normalized solution is

$$\Psi(x, t) = \frac{1}{2\pi\sqrt{c_0}} \exp\left(-\frac{i}{\hbar} \int V(t)dt\right) \int_{\mathbf{R}} \exp\left(-\frac{i}{\hbar}C(s) - \frac{i\hbar}{2m}s^2t\right) e^{isx} ds, \quad (7)$$

where

$$c_0 = \frac{1}{2\pi} \int_{\mathbf{R}} \exp\left(\frac{2Im(C(s))}{\hbar}\right) ds. \quad (8)$$

Hence the probability density is

$$P(x, t) = |\Psi(x, t)|^2 = \frac{1}{4\pi^2 c_0} \left| \int_{\mathbf{R}} \exp\left(-\frac{i}{\hbar}C(s) - \frac{i\hbar}{2m}s^2t\right) e^{isx} ds \right|^2$$

and is independent from the potential. If we define as usual the inner product

$$\langle f, g \rangle = \int_{-\infty}^{+\infty} f(t)\overline{g(t)}dt, \quad \forall f, g \in L_2(\mathbf{R})$$

and the norm of $L_2(\mathbf{R})$:

$$\|f\|_2 := \sqrt{\langle f, f \rangle} = \sqrt{\int_{-\infty}^{+\infty} |f(t)|^2 dt}, \quad \forall f \in L_2(\mathbf{R}).$$

The expected value of a operator \mathbf{A} is

$$\langle \mathbf{A} \rangle = \langle \mathbf{A}\Psi(x, t), \Psi(x, t) \rangle.$$

One can easily see that the operator of energy is $\mathbf{E} = i\hbar\partial_t$ and the operator of momentum $\mathbf{p} = -i\hbar\partial_x$ and both are self-adjoint. The Hamiltonian of the system is

$$\mathbf{H} = \frac{\mathbf{p}^2}{2m} + V(t).$$

Also holds

$$\mathbf{E}\Psi = \mathbf{H}\Psi$$

and this is (1). We will evaluate the expected value of energy \mathbf{E} .

$$\begin{aligned} \langle \mathbf{E} \rangle &= \langle \mathbf{E}\Psi, \Psi \rangle = \frac{1}{2\pi} \langle i\hbar\partial_t \widehat{\Psi}, \widehat{\Psi} \rangle = \frac{1}{2\pi} \left\langle \left(\frac{\hbar^2}{2m} s^2 + V(t) \right) \widehat{\Psi}, \widehat{\Psi} \right\rangle \Rightarrow \\ \langle \mathbf{E} \rangle &= \frac{\hbar^2}{4\pi m} \int_{-\infty}^{+\infty} s^2 |\widehat{\Psi}|^2 ds + V(t). \end{aligned} \quad (8.1)$$

Hence exists constant c_1 such that

$$\langle \mathbf{E} \rangle = V(t) + c_1.$$

From this we get that energy is not conserved. However the expected value of the momentum is

$$\begin{aligned} \langle \mathbf{p} \rangle &= \langle -i\hbar\partial_x \Psi, \Psi \rangle = -i\hbar \langle \partial_x \Psi, \Psi \rangle = \frac{-i\hbar}{2\pi} \left\langle \int_{\mathbf{R}} \partial_x \Psi(x, t) e^{-ixs} dx, \widehat{\Psi}(s, t) \right\rangle = \\ &= \frac{-i\hbar}{2\pi} \left\langle \int_{\mathbf{R}} \partial_x \Psi(x, t) e^{-ixs} dx, \widehat{\Psi}(s, t) \right\rangle = \\ &= \frac{-i\hbar}{2\pi} \left\langle - \int_{\mathbf{R}} \Psi(x, t) (-is) e^{-ixs} dx, \widehat{\Psi}(s, t) \right\rangle = \\ &= \frac{\hbar}{2\pi} \langle s \widehat{\Psi}, \widehat{\Psi} \rangle = \frac{\hbar}{2\pi} \int_{\mathbf{R}} s |\widehat{\Psi}(s, t)|^2 ds \Rightarrow \\ \langle \mathbf{p} \rangle &= \frac{\hbar}{2\pi} \int_{\mathbf{R}} s \exp\left(\frac{2Im(C(s))}{\hbar}\right) ds = \frac{\hbar}{2\pi} \int_{\mathbf{R}} s |\widehat{\Psi}|^2 ds = constant. \end{aligned} \quad (8.2)$$

Since the expected value of momentum is constant, the momentum is conserved. We also can write

$$\begin{aligned} \langle \mathbf{p}^2 \rangle &= \langle -\hbar^2 \partial_x^2 \Psi, \Psi \rangle = \langle 2mi\hbar\partial_t \Psi - 2mV(t)\Psi, \Psi \rangle = \\ &= 2m \langle \mathbf{E} \rangle - 2mV(t) = 2mV(t) + 2mc_1 - 2mV(t) = 2mc_1 = \\ &= \frac{\hbar^2}{2\pi} \int_{-\infty}^{+\infty} s^2 |\widehat{\Psi}|^2 ds. \end{aligned} \quad (8.3)$$

Hence

$$(\Delta \mathbf{p})^2 = \langle \mathbf{p}^2 \rangle - \langle \mathbf{p} \rangle^2 = \frac{\hbar^2}{2\pi} \int_{-\infty}^{+\infty} s^2 |\widehat{\Psi}|^2 ds - \frac{\hbar^2}{4\pi^2} \left(\int_{\mathbf{R}} s |\widehat{\Psi}|^2 ds \right)^2 = \text{const.} \quad (8.4)$$

The mean value of position is

$$\begin{aligned} \langle \mathbf{x} \rangle &= \langle x \Psi, \Psi \rangle = \frac{1}{2\pi} \left\langle \left(x \widehat{\Psi} \right), \widehat{\Psi} \right\rangle = \frac{1}{2\pi} \left\langle \int_{-\infty}^{+\infty} x \Psi(x, t) e^{-ixs} dx, \widehat{\Psi} \right\rangle = \\ &= \frac{i}{2\pi} \left\langle \partial_s \widehat{\Psi}, \widehat{\Psi} \right\rangle. \end{aligned}$$

But

$$\widehat{\Psi}(s, t) = \left| \widehat{\Psi}(s, t) \right| \exp(i\theta(s, t)),$$

where

$$\theta(s, t) = -\frac{1}{\hbar} \text{Re}(C(s)) - \frac{\hbar}{2m} s^2 t - \frac{1}{\hbar} \int V(t) dt.$$

Hence

$$\begin{aligned} \partial_s \widehat{\Psi}(s, t) &= \frac{\text{Im}(C'(s))}{\hbar} e^{\text{Im}(C(s))/\hbar} e^{i\theta(s, t)} + e^{\text{Im}(C(s))/\hbar} e^{i\theta(s, t)} i \partial_s \theta(s, t) \Rightarrow \\ \partial_s \widehat{\Psi} &= e^{\text{Im}(C(s))/\hbar} e^{i\theta(s, t)} \left(\frac{\text{Im}(C'(s))}{\hbar} + i \partial_s \theta(s, t) \right) \end{aligned}$$

Hence

$$\left| \partial_s \widehat{\Psi} \right|^2 = e^{2\text{Im}(C(s))/\hbar} \left(\left(\frac{\text{Im}(C'(s))}{\hbar} \right)^2 + \left(\frac{\text{Re}(C'(s))}{\hbar} + \frac{\hbar s t}{m} \right)^2 \right). \quad (9)$$

Also

$$\begin{aligned} (\partial_s \widehat{\Psi}) \overline{\widehat{\Psi}} &= \exp\left(\frac{2\text{Im}(C(s))}{\hbar}\right) \left(\frac{\text{Im}(C'(s))}{\hbar} - i \left(\frac{\text{Re}(C'(s))}{\hbar} + \frac{\hbar s t}{m} \right) \right) = \\ &= -i |\widehat{\Psi}|^2 \frac{C'(s)}{\hbar} - \frac{i\hbar}{m} s t |\widehat{\Psi}|^2 = -i |\widehat{\Psi}|^2 \left(\frac{C'(s)}{\hbar} + \frac{\hbar s t}{m} \right). \quad (10) \end{aligned}$$

Hence we have

$$\begin{aligned} \langle \mathbf{x} \rangle &= \frac{1}{2\pi\hbar} \int_{-\infty}^{+\infty} |\widehat{\Psi}|^2 C'(s) ds + \frac{\hbar t}{2\pi m} \int_{-\infty}^{+\infty} s |\widehat{\Psi}|^2 ds = \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} |\widehat{\Psi}|^2 \left(\frac{C'(s)}{\hbar} + \frac{\hbar s t}{m} \right) ds = C_1 + \frac{\hbar t}{2\pi m} \int_{-\infty}^{+\infty} s |\widehat{\Psi}|^2 ds \end{aligned}$$

and

$$\frac{d\langle \mathbf{x} \rangle}{dt} = \frac{\hbar}{2\pi m} \int_{-\infty}^{+\infty} s |\widehat{\Psi}|^2 ds = \text{constant}.$$

Hence the mean value of the position of the particle is lineary depended with time. If happens

$$\int_{-\infty}^{+\infty} s|\widehat{\Psi}|^2 ds = \int_{-\infty}^{+\infty} s \cdot \exp\left(\frac{2Im(C(s))}{\hbar}\right) ds = 0,$$

then the position of particle is conserved. However the particle may move. This phenomenon is related with the parity operator. Since the Hamiltonian is independed on the change $x \rightarrow -x$, from (5), we have

$$\begin{aligned} \Psi(-x, t) &= \frac{1}{2\pi} \exp\left(-\frac{i}{\hbar} \int V(t)\right) \int_{\mathbf{R}} \exp\left(-\frac{i}{\hbar} C(s) - \frac{i\hbar}{2m} s^2 t\right) e^{-isx} ds = \\ &= \frac{1}{2\pi} \exp\left(-\frac{i}{\hbar} \int V(t)\right) \int_{\mathbf{R}} \exp\left(-\frac{i}{\hbar} C(-s) - \frac{i\hbar}{2m} s^2 t\right) e^{isx} ds. \end{aligned}$$

Hence $C(-s) = C(s)$ if and only if $\Psi(-x, t) = \Psi(x, t)$. But then

$$\int_{-\infty}^{+\infty} s|\widehat{\Psi}|^2 ds = \int_{-\infty}^{+\infty} s \exp(2Im(C(s))/\hbar) ds = 0$$

and thus the position in this case is conserved.
Also

$$\begin{aligned} \langle \mathbf{x}^2 \rangle &= \langle x^2 \Psi, \Psi \rangle = \frac{1}{2\pi} \langle (\widehat{x^2 \Psi}), \widehat{\Psi} \rangle = \frac{1}{2\pi} \left\langle \int_{-\infty}^{+\infty} x^2 \Psi(x, t) e^{-ixs} dx, \widehat{\Psi} \right\rangle = \\ &= -\frac{1}{2\pi} \langle \partial_{ss}^2 \widehat{\Psi}, \widehat{\Psi} \rangle \end{aligned}$$

and from (10) we have

$$\begin{aligned} \left(\partial_{ss}^2 \widehat{\Psi}\right) \overline{\widehat{\Psi}} + |\partial_s \widehat{\Psi}|^2 &= -\frac{2i}{\hbar^2} |\widehat{\Psi}|^2 Im(C'(s)) C'(s) - \frac{i}{\hbar} |\widehat{\Psi}|^2 C''(s) - \\ &-\frac{i\hbar t}{m} |\widehat{\Psi}|^2 - \frac{2ist}{m} |\widehat{\Psi}|^2 Im(C'(s)). \end{aligned}$$

Hence using (9)

$$\begin{aligned} \left(\partial_{ss}^2 \widehat{\Psi}\right) \overline{\widehat{\Psi}} &= -\frac{2i}{\hbar^2} |\widehat{\Psi}|^2 Im(C'(s)) C'(s) - \frac{i}{\hbar} |\widehat{\Psi}|^2 C''(s) - \frac{i\hbar t}{m} |\widehat{\Psi}|^2 - \\ &-\frac{2ist}{m} |\widehat{\Psi}|^2 Im(C'(s)) - |\widehat{\Psi}|^2 \left(\frac{Im(C'(s))}{\hbar}\right)^2 - |\widehat{\Psi}|^2 \left(\frac{Re(C'(s))}{\hbar}\right)^2 - \\ &-\frac{|\widehat{\Psi}|^2 \hbar^2 s^2 t^2}{m^2} - |\widehat{\Psi}|^2 \frac{2Re(C'(s))st}{m} = \\ &= -\frac{2st}{m} |\widehat{\Psi}|^2 C'(s) + \frac{2}{\hbar^2} |\widehat{\Psi}|^2 Im(C'(s))^2 - \frac{2i}{\hbar^2} |\widehat{\Psi}|^2 Im(C'(s)) Re(C'(s)) - \end{aligned}$$

$$\begin{aligned}
& -\frac{i}{\hbar}|\widehat{\Psi}|^2 C''(s) - \frac{i\hbar t}{m}|\widehat{\Psi}|^2 - |\widehat{\Psi}|^2 \frac{|C'(s)|^2}{\hbar^2} - |\widehat{\Psi}|^2 \frac{\hbar^2 s^2 t^2}{m^2} = \\
& = -\frac{2st}{m}|\widehat{\Psi}|^2 C'(s) + \frac{1}{\hbar^2}|\widehat{\Psi}|^2 \text{Im}(C'(s))^2 + \frac{|\widehat{\Psi}|^2}{\hbar^2} \text{Re}(C'(s))^2 - \frac{|\widehat{\Psi}|^2}{\hbar^2} C'(x)^2 - \\
& \quad -\frac{i}{\hbar}|\widehat{\Psi}|^2 C''(s) - \frac{i\hbar t}{m}|\widehat{\Psi}|^2 - |\widehat{\Psi}|^2 \frac{|C'(s)|^2}{\hbar^2} - |\widehat{\Psi}|^2 \frac{\hbar^2 s^2 t^2}{m^2} = \\
& = -\frac{2st}{m}|\widehat{\Psi}|^2 C'(s) - \frac{|\widehat{\Psi}|^2}{\hbar^2} C'(s)^2 - \frac{i|\widehat{\Psi}|^2}{\hbar} C''(s) - \frac{i\hbar t}{m}|\widehat{\Psi}|^2 - |\widehat{\Psi}|^2 \frac{\hbar^2 s^2 t^2}{m^2} = \\
& = -|\widehat{\Psi}|^2 \left(\frac{C'(s)}{\hbar} + \frac{st\hbar}{m} \right)^2 - i|\widehat{\Psi}|^2 \left(\frac{C''(s)}{\hbar} + \frac{\hbar t}{m} \right). \tag{11}
\end{aligned}$$

Hence

$$\begin{aligned}
\langle \mathbf{x}^2 \rangle & = -\frac{1}{2\pi} \langle \partial_{ss}^2 \widehat{\Psi}, \widehat{\Psi} \rangle = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |\widehat{\Psi}|^2 \left(\frac{C'(s)}{\hbar} + \frac{st\hbar}{m} \right)^2 ds + \\
& \quad + \frac{i}{2\pi} \int_{-\infty}^{+\infty} |\widehat{\Psi}|^2 \left(\frac{C''(s)}{\hbar} + \frac{\hbar t}{m} \right) ds.
\end{aligned}$$

Also

$$i\hbar \frac{d}{dt} \langle \mathbf{x}^2 \rangle = \frac{i\hbar}{m\pi} \int_{-\infty}^{+\infty} C'(s)s|\widehat{\Psi}|^2 ds - \frac{\hbar^2}{2\pi m} \int_{-\infty}^{+\infty} |\widehat{\Psi}|^2 ds + \frac{it\hbar^3}{\pi m^2} \int_{-\infty}^{+\infty} |\widehat{\Psi}|^2 s^2 ds. \tag{11.1}$$

Finally

$$\begin{aligned}
(\Delta \mathbf{x})^2 & = \langle \mathbf{x}^2 \rangle - \langle \mathbf{x} \rangle^2 = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |\widehat{\Psi}|^2 \left(\frac{C'(s)}{\hbar} + \frac{st\hbar}{m} \right)^2 ds + \\
& \quad + \frac{i}{2\pi} \int_{-\infty}^{+\infty} |\widehat{\Psi}|^2 \left(\frac{C''(s)}{\hbar} + \frac{\hbar t}{m} \right) ds - \left(\frac{1}{2\pi} \int_{-\infty}^{+\infty} |\widehat{\Psi}|^2 \left(\frac{C'(s)}{\hbar} + \frac{\hbar t s}{m} \right) ds \right)^2. \tag{12}
\end{aligned}$$

We have

$$\begin{aligned}
[\mathbf{x}^2, \mathbf{H}]\Psi & = [\mathbf{x} \ \mathbf{x}, \mathbf{H}]\Psi = [\mathbf{x}, \mathbf{H}]\mathbf{x}\Psi + \mathbf{x}[\mathbf{x}, \mathbf{H}]\Psi = i\hbar \frac{\mathbf{p}}{m} \mathbf{x}\Psi + i\hbar \mathbf{x} \frac{\mathbf{p}}{m} \Psi = \\
& = \frac{i\hbar}{m} (-i\hbar \partial_x (x\Psi) - i\hbar x \partial_x \Psi) = \frac{\hbar^2}{m} (\Psi + x \partial_x \Psi + x \partial_x \Psi) = \\
& = \frac{\hbar^2}{m} (\Psi + 2x \partial_x \Psi).
\end{aligned}$$

Hence

$$\begin{aligned}
\langle [\mathbf{x}^2, \mathbf{H}] \rangle \Psi & = \frac{\hbar^2}{m} (\|\Psi\|_2^2 + 2 \langle x \partial_x \Psi, \Psi \rangle) = \\
& = \frac{\hbar^2}{2\pi m} \|\widehat{\Psi}\|_2^2 + \frac{2\hbar^2}{m} \langle x \partial_x \Psi, \Psi \rangle = \frac{\hbar^2}{2\pi m} \int_{-\infty}^{+\infty} |\widehat{\Psi}|^2 ds + \frac{\hbar^2}{\pi m} \langle \widehat{\partial_x \Psi}, \widehat{x\Psi} \rangle =
\end{aligned}$$

$$\begin{aligned}
&= c_1 + \frac{\hbar^2}{\pi m} \left\langle -(-is) \int_{-\infty}^{+\infty} \Psi(x, t) e^{-ixs} dx, i \int_{-\infty}^{+\infty} (-ix) \Psi(x, t) e^{-ixs} dx \right\rangle = \\
&= c_1 + \frac{\hbar^2}{\pi m} \langle s \widehat{\Psi}, \partial_s \widehat{\Psi} \rangle = \\
&= c_1 + \frac{\hbar^2}{\pi m} \left\langle s |\widehat{\Psi}| e^{i\theta(s,t)}, \partial_s (|\widehat{\Psi}| e^{i\theta(s,t)} + |\widehat{\Psi}| e^{i\theta(s,t)} i \partial_s \theta(s,t)) \right\rangle = \\
c_1 + \frac{\hbar^2}{\pi m} \int_{-\infty}^{+\infty} s |\widehat{\Psi}|^2 \frac{Im(C'(s))}{\hbar} ds - \frac{i\hbar^2}{\pi m} \int_{-\infty}^{+\infty} s |\widehat{\Psi}|^2 \left(-\frac{1}{\hbar} Re(C'(s)) - \frac{\hbar s t}{m} \right) ds = \\
&= c_1 + \frac{\hbar}{\pi m} \int_{-\infty}^{+\infty} s |\widehat{\Psi}|^2 Im(C'(s)) ds + \frac{i\hbar}{\pi m} \int_{-\infty}^{+\infty} s |\widehat{\Psi}|^2 Re(C'(s)) ds + \\
&\quad + \frac{i\hbar^3 t}{\pi m^2} \int_{-\infty}^{+\infty} s^2 |\widehat{\Psi}|^2 ds.
\end{aligned}$$

Hence

$$\langle [\mathbf{x}^2, \mathbf{H}] \rangle \Psi = \frac{\hbar^2}{2\pi m} \int_{-\infty}^{+\infty} |\widehat{\Psi}|^2 ds + \frac{i\hbar}{\pi m} \int_{-\infty}^{+\infty} s |\widehat{\Psi}|^2 \overline{C'(s)} ds + \frac{i\hbar^3 t}{\pi m^2} \int_{-\infty}^{+\infty} s^2 |\widehat{\Psi}|^2 ds. \quad (13)$$

But there holds the general law

$$i\hbar \frac{d}{dt} \langle \mathbf{A} \rangle = \langle [\mathbf{A}, \mathbf{H}] \rangle + i\hbar \left\langle \frac{\partial \mathbf{A}}{\partial t} \right\rangle. \quad (14)$$

Now we now that

$$\langle \mathbf{x}^n \rangle = \int_{-\infty}^{+\infty} x^n |\Psi(x, t)|^2 dx \in \mathbf{R}.$$

Also $\langle \frac{\partial \mathbf{x}^n}{\partial t} \rangle = 0$. Hence (14) is equivalent to

$$i\hbar \frac{d}{dt} \langle \mathbf{x}^2 \rangle + \overline{\langle [\mathbf{x}^2, \mathbf{H}] \rangle} = 0 \quad (15)$$

Using (11.1) and (13) we can easily verify (15) and hence (14). Note here that we did not use (14) in first place because the function $C(s)$ satisfies conditions imposed by the Schrodinger equation and the nature of the problem and every-time we solve (1) we must find these conditions. For example $\langle \mathbf{x}^n \rangle \in \mathbf{R} \dots$ etc gives some conditions on $C(s) \dots$ etc.

Examples of (14) are:

1) Setting $\mathbf{A} = \mathbf{x}$, then

$$[\mathbf{x}, \mathbf{H}] = i\hbar \frac{\partial \mathbf{H}}{\partial \mathbf{p}} = \frac{i\hbar}{m} \mathbf{p}$$

Hence

$$i\hbar \frac{d \langle \mathbf{x} \rangle}{dt} = \frac{i\hbar}{m} \langle \mathbf{p} \rangle.$$

Hence the position is not in general preserved.

2) Setting $\mathbf{A} = \mathbf{H}$, then (since $[\mathbf{H}, \mathbf{H}] = 0$):

$$\frac{d\langle \mathbf{E} \rangle}{dt} = \frac{d\langle \mathbf{H} \rangle}{dt} = \left\langle \frac{\partial \mathbf{H}}{\partial t} \right\rangle = \left\langle \frac{d\mathbf{V}}{dt} \right\rangle. \text{ Energy not preserved.}$$

3) If $\mathbf{A} = \mathbf{p}$, then from

$$[\mathbf{p}, \mathbf{H}] = -i\hbar \frac{\partial \mathbf{H}}{\partial x} = 0,$$

we have

$$i\hbar \frac{d\langle \mathbf{p} \rangle}{dt} = \langle [\mathbf{p}, \mathbf{H}] \rangle + i\hbar \left\langle \frac{\partial \mathbf{p}}{\partial t} \right\rangle = 0. \text{ Momentum is always preserved.}$$

4) If \mathbf{P} =parity operator, then

$$\begin{aligned} \langle [\mathbf{P}, \mathbf{H}] \rangle &= \mathbf{P}\mathbf{H}\Psi(x, t) - \mathbf{H}\mathbf{P}\Psi(x, t) = \mathbf{P}\mathbf{E}\Psi(x, t) - \mathbf{H}\Psi(-x, t) - \\ &= i\hbar\partial_t\Psi(-x, t) - \mathbf{H}\Psi(-x, t) = i\hbar\partial_t\Psi_e(-x, t) + i\hbar\partial_t\Psi_o(-x, t) + \\ &+ \frac{\hbar^2}{2m}\partial_{xx}^2\Psi_e(-x, t) + \frac{\hbar^2}{2m}\partial_{xx}^2\Psi_o(-x, t) - V(t)(\Psi_e(-x, t) + \Psi_o(-x, t)) \end{aligned}$$

The above quantity is zero in either case of $\Psi = \Psi_e$ (even) and $\Psi = \Psi_o$ (odd). Hence the parity is always preserved.

References

- [1]: Nirmala Prakash. "Differential Geometry An Integrated Approach". Tata McGraw-Hill Publishing Company Limited. New Delhi. 1981.
- [2]: Murray R. Spiegel. 'Schaum's Outline of Theory and Problems of Fourier Analysis with Applications to Boundary Value Problems'. McGraw-Hill, New York. 1974.
- [3]: Frigyes Riesz and Bela Sz.-Nagy. 'Functional Analysis'. Dover Publications, Inc. New York. 1990.
- [4]: Eduard Prugovecki. 'Quantum Mechanics in Hilbert Space'. Academic Press. New York and London. 1971.
- [5]: Jerome A. Goldstein. 'Semigroups of Linear Operators and Applications'. Oxford University Press-New York. Clarendon Press-Oxford. 1985.