A LINEARIZED KURAMOTO-SIVASHINSKY PDE VIA AN IMAGINARY-BROWNIAN-TIME-BROWNIAN-ANGLE PROCESS

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ABSTRACT. We introduce a new imaginary-Brownian-time-Brownian-angle process, which we also call the linear-Kuramoto-Sivashinsky process (LKSP). Building on our techniques in two recent articles involving the connection of Brownian-time processes to fourth order PDEs, we give an explicit solution to a linearized Kuramoto-Sivashinsky PDE in *d*-dimensional space: $u_t = -\frac{1}{8}\Delta^2 u - \frac{1}{2}\Delta u - \frac{1}{2}u$. The solution is given in terms of a functional of our LKSP.

1. Statements and discussions of results.

One of the prominent equations in modern applied mathematics is the celebrated Kuramoto-Sivashinsky (KS) PDE. This nonlinear equation has generated a lot of interest in the PDE literature (see e.g., [9, 10, 11, 12, 21] and many other papers). In the field of stochastic processes, a great deal of interest is directed at the study of processes in which time is replaced in one way or another by a Brownian motion, and this interest has picked up considerably (see e.g., [1, 2, 6, 7, 8, 19, 20, 16, 17, 13, 14]) after the fundamental work of Burdzy on iterated Brownian motion ([7, 8]). In [1, 2], we provided a unified framework for such iterated processes (including the IBM of Burdzy) and introduced several interesting new ones, through a large class of processes that we called Brownian-time processes. We then related them to different fourth order PDEs. In this article, and as announced in [2], we modify our process in Theorem 1.2 [2] and build on our methods in [2] to give an explicit solution to a linear version of the KS PDE. One modification needed is the introduction of $i = \sqrt{-1}$ in both the Brownian-time and the Brownian-exponential, and that leads to a new process we call imaginary-Brownian-time-Brownian-angle process IBTBAP, starting at $f : \mathbb{R}^d \to \mathbb{R}$:

(1.1)
$$\mathbb{A}_{B}^{f,X}(t,x) \stackrel{\scriptscriptstyle \triangle}{=} \begin{cases} f(X^{x}(iB(t))) \exp(iB(t)), & B(t) \ge 0; \\ f(iX^{-ix}(-iB(t))) \exp(iB(t)), & B(t) < 0; \end{cases}$$

where X^x is an \mathbb{R}^d -valued Brownian motion starting from $x \in \mathbb{R}^d$, X^{-ix} is an independent $i\mathbb{R}^d$ -valued BM starting at -ix (so that iX^{-ix} starts at x), and both are independent of the inner standard \mathbb{R} -valued Brownian motion B starting from 0. The time of the outer Brownian motions X^x and X^{-ix} is replaced by an imaginary

Date: 5/20/2002.

¹⁹⁹¹ Mathematics Subject Classification. Primary 35C15, 35G31, 35G46, 60H30, 60G60, 60J45, 60J35; Secondary 60J60, 60J65.

Key words and phrases. Brownian-time Brownian sheet, nonlinear fourth order coupled PDEs, linear systems of fourth order coupled PDEs, Brownian-time processes, initially perturbed fourth order PDEs, Brownian-time Feynman-Kac formula, iterated Brownian sheet, iterated Brownian sheet, random fields.

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positive Brownian time; and, when f is real-valued as we will assume here, the angle of $\mathbb{A}_B^{f,X}(t,x)$ is the Brownian motion B. We think of the imaginary-time processes $\{X^x(is), s \geq 0\}$ and $\{iX^{-ix}(-is), s \leq 0\}$ as having the same complex Gaussian distribution on \mathbb{R}^d with the corresponding complex distributional density

$$p_{is}^{(d)}(x,y) = \frac{1}{(2\pi i s)^{d/2}} e^{-|x-y|^2/2is}.$$

We will also call the process given by (1.1) the *d*-dimensional Linear-Kuramoto-Sivashinsky process (LKSP) starting at f (clearly $\mathbb{A}_B^{f,X}(0,x) = f(x)$). The dimension in *d*-dimensional IBTBAP (or *d*-dimensional LKSP) refers to the dimension of the BMs X^x and X^{-ix} , which is also the dimension of the spatial variable in the associated linearized KS PDE as we will see shortly.

Now, motivated by the definitions of v_{ϵ} and u_{ϵ} in the proof of Theorem 1.2 in [2], we let

(1.2)
$$v(s,x) \stackrel{\triangle}{=} \exp(is) \int_{\mathbb{R}^d} f(y) \frac{1}{(2\pi i s)^{d/2}} e^{-|x-y|^2/2is} dy$$
$$u(t,x) \stackrel{\triangle}{=} \int_{-\infty}^0 v(s,x) p_t(0,s) ds + \int_0^\infty v(s,x) p_t(0,s) ds$$

where $p_t(0,s)$ is the transition density of the inner (one-dimensional) Brownian motion B:

$$p_t(0,s) = \frac{1}{\sqrt{2\pi t}} e^{-s^2/2t}$$

We may think of v and u in terms of complex expectation by defining $v(s, x) \stackrel{\triangle}{=} \mathbb{E}^{\mathbb{C}} \left[f(X^x(is)) \exp(is) \right]$ and $u(t, x) \stackrel{\triangle}{=} \mathbb{E}^{\mathbb{C}} \left[\mathbb{A}_B^{f, X}(t, x) \right]$. A more detailed study of the rich connection between our process and its complex distribution to the KS PDE and its implications is the subject of an upcoming article [3]. We are now ready to state our main result.

Theorem 1.1. Let $f \in C_c^2(\mathbb{R}^d;\mathbb{R})$ with $D_{ij}f$ Hölder continuous with exponent $0 < \alpha \leq 1$, for all $1 \leq i, j \leq d$. If u(t,x) is given by (1.2) then u(t,x) solves the linearized Kuramoto-Sivashinsky PDE

(1.3)
$$\begin{cases} \frac{\partial}{\partial t}u(t,x) = -\frac{1}{8}\Delta^2 u(t,x) - \frac{1}{2}\Delta u(t,x) - \frac{1}{2}u(t,x), & t > 0, x \in \mathbb{R}^d; \\ u(0,x) = f(x), & x \in \mathbb{R}^d. \end{cases}$$

2. Proof of the main result

Proof of Theorem 1.1. Let u and v be as given in (1.2). Differentiating u(t, x) with respect to t and putting the derivative under the integral, which is easily justified by the dominated convergence theorem, then using the fact that $p_t(0, s)$ satisfies the heat equation

$$\frac{\partial}{\partial t}p_t(0,s) = \frac{1}{2}\frac{\partial^2}{\partial s^2}p_t(0,s)$$

and integrating by parts twice using the fact that the boundary terms vanish at $\pm \infty$ and that $(\partial/\partial s)p_t(0,s) = 0$ at s = 0, we obtain

$$\begin{aligned} \frac{\partial}{\partial t}u(t,x) &= \int_{-\infty}^{0} v(s,x)\frac{\partial}{\partial t}p_{t}(0,s)ds + \int_{0}^{\infty} v(s,x)\frac{\partial}{\partial t}p_{t}(0,s)ds \\ &= \frac{1}{2} \left[\int_{-\infty}^{0} v(s,x)\frac{\partial^{2}}{\partial s^{2}}p_{t}(0,s)ds + \int_{0}^{\infty} v(s,x)\frac{\partial^{2}}{\partial s^{2}}p_{t}(0,s)ds \right] \\ &= \frac{1}{2}p_{t}(0,0) \left[\left(\frac{\partial}{\partial s}v(s,x)\right) \right|_{s=0^{-}} + \left(\frac{\partial}{\partial s}v(s,x)\right) \right|_{s=0^{+}} \right] \\ &+ \frac{1}{2} \int_{-\infty}^{0} p_{t}(0,s)\frac{\partial^{2}}{\partial s^{2}}v(s,x)ds + \frac{1}{2} \int_{0}^{\infty} p_{t}(0,s)\frac{\partial^{2}}{\partial s^{2}}v(s,x)ds \\ &= \frac{1}{2} \int_{-\infty}^{0} p_{t}(0,s) \left[-\frac{1}{4}\Delta^{2}v(s,x) - \Delta v(s,x) - v(s,x) \right] ds \\ &+ \frac{1}{2} \int_{0}^{\infty} p_{t}(0,s) \left[-\frac{1}{4}\Delta^{2}v(s,x) - \Delta v(s,x) - v(s,x) \right] ds \\ &= -\frac{1}{8}\Delta^{2}u(t,x) - \frac{1}{2}\Delta u(t,x) - \frac{1}{2}u(t,x) \end{aligned}$$

where for the last two equalities in (2.1) we have used the fact that

(2.2)
$$\frac{\partial v}{\partial s} = \frac{i}{2} \Delta v (s, x) + iv (s, x)$$
$$\frac{\partial^2 v}{\partial s^2} = -\frac{1}{4} \Delta^2 v (s, x) - \Delta v (s, x) - v (s, x)$$

and the conditions on f to take the applications of the derivatives outside the integrals in (2.1) and (2.2) (the steps of Lemma 2.1 in [2] easily translates to our setting here, see the discussion below). Clearly u(0, x) = f(x), and the proof is complete. \Box

As we indicated above, only minor changes to Lemma 2.1 in [2] are needed to justify pulling the derivatives outside the integrals in (2.1) under the conditions on f of Theorem 1.1. We now adapt Lemma 2.1 [2] to our setting here, and we point out the necessary changes in its proof:

Lemma 2.1. Let v(s, x) be given by (1.2) and let f be as in Theorem 1.1. Let

(2.3)
$$u_1(t,x) \stackrel{\triangle}{=} \int_{-\infty}^0 v(s,x) p_t(0,s) ds \quad and \quad u_2(t,x) \stackrel{\triangle}{=} \int_0^\infty v(s,x) p_t(0,s) ds,$$

then $\Delta^2 u_1(t,x)$ and $\Delta^2 u_2(t,x)$ are finite and

$$\Delta^{2} u_{1}(t,x) = \int_{-\infty}^{0} \Delta^{2} v(s,x) p_{t}(0,s) ds \quad and \quad \Delta^{2} u_{2}(t,x) = \int_{0}^{\infty} \Delta^{2} v(s,x) p_{t}(0,s) ds$$

Proof. As in the proof of Lemma 2.1 [2], letting $\overset{\circ}{\mathbb{R}}_+ = (0,\infty)$ and $\overset{\circ}{\mathbb{R}}_- = (-\infty,0)$, it suffices to show

(2.5)
$$\frac{\partial^4}{\partial x_j^4} \int_{\mathbb{R}_{\pm}}^{\circ} v(s,x) p_t(0,s) ds = \int_{\mathbb{R}_{\pm}}^{\circ} \frac{\partial^4}{\partial x_j^4} v(s,x) p_t(0,s) ds, \qquad j = 1, \dots, d.$$

Letting $p_{is}^{(d)}(x,y) = (2\pi i s)^{-d/2} e^{-|x-y|^2/2is}$ and using the conditions on f, we easily

(2.6)
$$\frac{\partial^4}{\partial x_j^4} v(s,x) p_t(0,s) = \exp\left(is\right) \left(\int_{\mathbb{R}^d} f(y) \frac{\partial^4}{\partial y_j^4} p_{is}^{(d)}(x,y) dy \right) p_t(0,s)$$
$$= \exp\left(is\right) \left(\int_{\mathbb{R}^d} \frac{\partial^2}{\partial y_j^2} f(y) \frac{\partial^2}{\partial y_j^2} p_{is}^{(d)}(x,y) dy \right) p_t(0,s).$$

Rewriting the last term in (2.6), and letting $h_j(y) \stackrel{\triangle}{=} \partial^2 f(y) / \partial y_i^2$, we have $(2 \ 7)$

$$\begin{aligned} \left| \exp\left(is\right) \left(\int_{\mathbb{R}^d} (2\pi is)^{-d/2} \left(\frac{-(x_j - y_j)^2 + is}{s^2} \right) e^{-|x - y|^2/2is} h_j(y) dy \right) \frac{e^{-s^2/2t}}{\sqrt{2\pi t}} \right| \\ &= \frac{e^{-s^2/2t}}{\sqrt{2\pi t}} \left| \left(\int_{\mathbb{R}^d} (2\pi is)^{-d/2} \left(\frac{-(x_j - y_j)^2 + is}{s^2} \right) e^{-|x - y|^2/2is} (h_j(y) - h_j(x)) dy \right) \right| \\ &\leq \frac{e^{-s^2/2t}}{\sqrt{2\pi t}} \int_{\mathbb{R}^d} (2\pi |s|)^{-d/2} \left| \frac{-(\tilde{x}_j - y_j)^2 + |s|}{s^2} \right| e^{-|\tilde{x} - y|^2/2|s|} |h_j(y) - h_j(\tilde{x})| dy \\ &= \frac{e^{-s^2/2t}}{\sqrt{2\pi t}} \mathbb{E}_{\mathbb{P}} \left| \left(\frac{(\tilde{x}_j - W_j^{\tilde{x}}(|s|))^2 - |s|}{s^2} \right) \left(h_j(W^{\tilde{x}}(|s|)) - h_j(\tilde{x}) \right) \right|, \end{aligned}$$

for some $\tilde{x} \in \mathbb{R}^d$ where $\tilde{x}_j = \pm x_j$ for $j = 1, \ldots, d$; and where $W^{\tilde{x}} : \Omega \times \mathbb{R}_+ \to \mathbb{R}^d$ is a standard Brownian motion starting at $\tilde{x} \in \mathbb{R}^d$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and $W_j^{\tilde{x}}$ is its *j*-th component. The inequality in (2.7) follows easily if h_j is a polynomial, and standard approximation yields the inequality for $h_j \in C_c(\mathbb{R}^d; \mathbb{R})$. Now, exactly as in [2] (2.6) and (2.7); we use the Brownian motion scaling for $W^{\tilde{x}}$, the Cauchy-Shwarz inequality on the last term in (2.7), and the Hölder condition on h_i to deduce that the last term in (2.7) is bounded above by $K \exp(-s^2/2t)/(\sqrt{2\pi t}|s|^{1-\alpha/2}) \in$ $L^{1}((-\infty,0), ds) \cap L^{1}((0,\infty), ds);$ hence $\left| \frac{\partial^{4}}{\partial x_{i}^{4} v(s,x) p_{t}(0,s)} \right| \in L^{1}((-\infty,0), ds) \cap$ $L^1((0,\infty), ds)$, which completes the proof by standard analysis.

Acknowledgements. I'd like to thank Ciprian Foias for his encouragement to pursue this project and for his support. I also enjoyed several one on one fruitful discussions with him. This research is supported in part by NSA grant MDA904-02 - 1 - 0083.

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