A LINEARIZED KURAMOTO-SIVASHINSKY PDE VIA AN IMAGINARY-BROWNIAN-TIME-BROWNIAN-ANGLE PROCESS

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Abstract. We introduce a new imaginary-Brownian-time-Brownian-angle process, which we also call the linear-Kuramoto-Sivashinsky process (LKSP). Building on our techniques in two recent articles involving the connection of Brownian-time processes to fourth order PDEs, we give an explicit solution to a linearized Kuramoto-Sivashinsky PDE in d-dimensional space: $u_t = -\frac{1}{8} \Delta^2 u - \frac{1}{2} \Delta u - \frac{1}{2} u$. The solution is given in terms of a functional of our LKSP.

1. Statements and discussions of results.

One of the prominent equations in modern applied mathematics is the celebrated Kuramoto-Sivashinsky (KS) PDE. This nonlinear equation has generated a lot of interest in the PDE literature (see e.g., [\[9,](#page-4-0) [10,](#page-4-1) [11,](#page-4-2) [12,](#page-4-3) [21\]](#page-4-4) and many other papers). In the field of stochastic processes, a great deal of interest is directed at the study of processes in which time is replaced in one way or another by a Brownian motion, and this interest has picked up considerably (see e.g., $[1, 2, 6, 7, 8, 19, 20, 16, 17, 13, 14]$ $[1, 2, 6, 7, 8, 19, 20, 16, 17, 13, 14]$ $[1, 2, 6, 7, 8, 19, 20, 16, 17, 13, 14]$ $[1, 2, 6, 7, 8, 19, 20, 16, 17, 13, 14]$ $[1, 2, 6, 7, 8, 19, 20, 16, 17, 13, 14]$ $[1, 2, 6, 7, 8, 19, 20, 16, 17, 13, 14]$ $[1, 2, 6, 7, 8, 19, 20, 16, 17, 13, 14]$ $[1, 2, 6, 7, 8, 19, 20, 16, 17, 13, 14]$ $[1, 2, 6, 7, 8, 19, 20, 16, 17, 13, 14]$ $[1, 2, 6, 7, 8, 19, 20, 16, 17, 13, 14]$ $[1, 2, 6, 7, 8, 19, 20, 16, 17, 13, 14]$ $[1, 2, 6, 7, 8, 19, 20, 16, 17, 13, 14]$) after the fundamental work of Burdzy on iterated Brownian motion ([\[7,](#page-3-3) [8\]](#page-4-5)). In [\[1,](#page-3-0) [2\]](#page-3-1), we provided a unified framework for such iterated processes (including the IBM of Burdzy) and introduced several interesting new ones, through a large class of processes that we called Brownian-time processes. We then related them to different fourth order PDEs. In this article, and as announced in [\[2\]](#page-3-1), we modify our process in Theorem 1.2 [\[2\]](#page-3-1) and build on our methods in [\[2\]](#page-3-1) to give an explicit solution to a linear version of the KS PDE. One modification needed is the introduction of $i = \sqrt{-1}$ in both the Brownian-time and the Brownian-exponential, and that leads to a new process we call imaginary-Brownian-time-Brownian-angle process IBTBAP, starting at $f : \mathbb{R}^d \to \mathbb{R}$:

(1.1)
$$
\mathbb{A}_{B}^{f,X}(t,x) \stackrel{\triangle}{=} \begin{cases} f(X^x(iB(t))) \exp(iB(t)), & B(t) \ge 0; \\ f(iX^{-ix}(-iB(t))) \exp(iB(t)), & B(t) < 0; \end{cases}
$$

where X^x is an \mathbb{R}^d -valued Brownian motion starting from $x \in \mathbb{R}^d$, X^{-ix} is an independent $i\mathbb{R}^d$ -valued BM starting at $-i\tilde{x}$ (so that iX^{-ix} starts at x), and both are independent of the inner standard $\mathbb{R}\text{-}$ valued Brownian motion B starting from 0. The time of the outer Brownian motions X^x and X^{-ix} is replaced by an imaginary

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positive Brownian time; and, when f is real-valued as we will assume here, the angle of $\mathbb{A}_{B}^{f,X}(t,x)$ is the Brownian motion B. We think of the imaginary-time processes $\{X^x(is), s \geq 0\}$ and $\{iX^{-ix}(-is), s \leq 0\}$ as having the same complex Gaussian distribution on \mathbb{R}^d with the corresponding complex distributional density

$$
p_{is}^{(d)}(x,y) = \frac{1}{(2\pi is)^{d/2}} e^{-|x-y|^2/2is}.
$$

We will also call the process given by (1.1) the d-dimensional Linear-Kuramoto-Sivashinsky process (LKSP) starting at f (clearly $\mathbb{A}_{B}^{f,X}(0,x) = f(x)$). The dimension in d-dimensional IBTBAP (or d-dimensional LKSP) refers to the dimension of the BMs X^x and X^{-ix} , which is also the dimension of the spatial variable in the associated linearized KS PDE as we will see shortly.

Now, motivated by the definitions of v_{ϵ} and u_{ϵ} in the proof of Theorem 1.2 in [\[2\]](#page-3-1), we let

(1.2)

$$
v(s,x) \stackrel{\triangle}{=} \exp(is) \int_{\mathbb{R}^d} f(y) \frac{1}{(2\pi is)^{d/2}} e^{-|x-y|^2/2is} dy
$$

$$
u(t,x) \stackrel{\triangle}{=} \int_{-\infty}^0 v(s,x) p_t(0,s) ds + \int_0^\infty v(s,x) p_t(0,s) ds
$$

where $p_t(0, s)$ is the transition density of the inner (one-dimensional) Brownian motion B:

$$
p_t(0, s) = \frac{1}{\sqrt{2\pi t}} e^{-s^2/2t}
$$

.

We may think of v and u in terms of complex expectation by defining $v(s, x) \triangleq$ $\mathbb{E}^{\mathbb{C}}[f(X^x(is))\exp(is)]$ and $u(t,x)\overset{\triangle}{=}\mathbb{E}^{\mathbb{C}}[\mathbb{A}_B^{f,X}(t,x)]$. A more detailed study of the rich connection between our process and its complex distribution to the KS PDE and its implications is the subject of an upcoming article [\[3\]](#page-3-4). We are now ready to state our main result.

Theorem 1.1. Let $f \in C_c^2(\mathbb{R}^d; \mathbb{R})$ with $D_{ij}f$ Hölder continuous with exponent $0 < \alpha \leq 1$, for all $1 \leq i, j \leq d$. If $u(t, x)$ is given by [\(1.2\)](#page-1-0) then $u(t, x)$ solves the linearized Kuramoto-Sivashinsky PDE

(1.3)
$$
\begin{cases} \frac{\partial}{\partial t}u(t,x) = -\frac{1}{8}\Delta^2 u(t,x) - \frac{1}{2}\Delta u(t,x) - \frac{1}{2}u(t,x), & t > 0, x \in \mathbb{R}^d; \\ u(0,x) = f(x), & x \in \mathbb{R}^d. \end{cases}
$$

2. Proof of the main result

Proof of Theorem [1.1.](#page-1-1) Let u and v be as given in [\(1.2\)](#page-1-0). Differentiating $u(t, x)$ with respect to t and putting the derivative under the integral, which is easily justified by the dominated convergence theorem, then using the fact that $p_t(0, s)$ satisfies the heat equation

$$
\frac{\partial}{\partial t}p_t(0,s) = \frac{1}{2}\frac{\partial^2}{\partial s^2}p_t(0,s)
$$

and integrating by parts twice using the fact that the boundary terms vanish at $\pm\infty$ and that $(\partial/\partial s)p_t(0, s) = 0$ at $s = 0$, we obtain

$$
\frac{\partial}{\partial t}u(t,x) = \int_{-\infty}^{0} v(s,x)\frac{\partial}{\partial t}p_t(0,s)ds + \int_{0}^{\infty} v(s,x)\frac{\partial}{\partial t}p_t(0,s)ds \n= \frac{1}{2}\left[\int_{-\infty}^{0} v(s,x)\frac{\partial^2}{\partial s^2}p_t(0,s)ds + \int_{0}^{\infty} v(s,x)\frac{\partial^2}{\partial s^2}p_t(0,s)ds\right] \n= \frac{1}{2}p_t(0,0)\left[\left(\frac{\partial}{\partial s}v(s,x)\right)\Big|_{s=0^{-}} + \left(\frac{\partial}{\partial s}v(s,x)\right)\Big|_{s=0^{+}}\right] \n+ \frac{1}{2}\int_{-\infty}^{0} p_t(0,s)\frac{\partial^2}{\partial s^2}v(s,x)ds + \frac{1}{2}\int_{0}^{\infty} p_t(0,s)\frac{\partial^2}{\partial s^2}v(s,x)ds \n= \frac{1}{2}\int_{-\infty}^{0} p_t(0,s)\left[-\frac{1}{4}\Delta^2v(s,x) - \Delta v(s,x) - v(s,x)\right]ds \n+ \frac{1}{2}\int_{0}^{\infty} p_t(0,s)\left[-\frac{1}{4}\Delta^2v(s,x) - \Delta v(s,x) - v(s,x)\right]ds \n= -\frac{1}{8}\Delta^2u(t,x) - \frac{1}{2}\Delta u(t,x) - \frac{1}{2}u(t,x)
$$

where for the last two equalities in (2.1) we have used the fact that

(2.2)
$$
\begin{aligned}\n\frac{\partial v}{\partial s} &= \frac{i}{2} \Delta v(s, x) + iv(s, x) \\
\frac{\partial^2 v}{\partial s^2} &= -\frac{1}{4} \Delta^2 v(s, x) - \Delta v(s, x) - v(s, x),\n\end{aligned}
$$

and the conditions on f to take the applications of the derivatives outside the integrals in [\(2.1\)](#page-2-0) and [\(2.2\)](#page-2-1) (the steps of Lemma 2.1 in [\[2\]](#page-3-1) easily translates to our setting here, see the discussion below). Clearly $u(0, x) = f(x)$, and the proof is complete. \Box

As we indicated above, only minor changes to Lemma 2.1 in [\[2\]](#page-3-1) are needed to justify pulling the derivatives outside the integrals in (2.1) under the conditions on f of Theorem [1.1.](#page-1-1) We now adapt Lemma 2.1 [\[2\]](#page-3-1) to our setting here, and we point out the necessary changes in its proof:

Lemma 2.1. Let $v(s, x)$ be given by (1.2) and let f be as in Theorem [1.1](#page-1-1). Let

(2.3)
$$
u_1(t,x) \stackrel{\triangle}{=} \int_{-\infty}^0 v(s,x)p_t(0,s)ds
$$
 and $u_2(t,x) \stackrel{\triangle}{=} \int_0^\infty v(s,x)p_t(0,s)ds$,

then $\Delta^2 u_1(t, x)$ and $\Delta^2 u_2(t, x)$ are finite and

$$
(2.4)
$$

$$
\Delta^2 u_1(t,x) = \int_{-\infty}^0 \Delta^2 v(s,x) p_t(0,s) ds \quad and \quad \Delta^2 u_2(t,x) = \int_0^\infty \Delta^2 v(s,x) p_t(0,s) ds.
$$

Proof. As in the proof of Lemma 2.1 [\[2\]](#page-3-1), letting $\mathbb{R}_{+} = (0, \infty)$ and $\mathbb{R}_{-} =$ $(-\infty, 0)$, it suffices to show

$$
(2.5) \qquad \frac{\partial^4}{\partial x_j^4} \int_{\mathbb{R}_+} v(s,x) p_t(0,s) ds = \int_{\mathbb{R}_+} \frac{\partial^4}{\partial x_j^4} v(s,x) p_t(0,s) ds, \qquad j=1,\ldots,d.
$$

Letting $p_{is}^{(d)}(x,y) = (2\pi i s)^{-d/2} e^{-|x-y|^2/2is}$ and using the conditions on f, we easily get

(2.6)
$$
\frac{\partial^4}{\partial x_j^4} v(s, x) p_t(0, s) = \exp(is) \left(\int_{\mathbb{R}^d} f(y) \frac{\partial^4}{\partial y_j^4} p_{is}^{(d)}(x, y) dy \right) p_t(0, s)
$$

$$
= \exp(is) \left(\int_{\mathbb{R}^d} \frac{\partial^2}{\partial y_j^2} f(y) \frac{\partial^2}{\partial y_j^2} p_{is}^{(d)}(x, y) dy \right) p_t(0, s).
$$

Rewriting the last term in [\(2.6\)](#page-3-5), and letting $h_j(y) \stackrel{\triangle}{=} \partial^2 f(y) / \partial y_j^2$, we have

$$
(2.7)
$$
\n
$$
\begin{aligned}\n&\left|\exp\left(is\right)\left(\int_{\mathbb{R}^d} (2\pi i s)^{-d/2} \left(\frac{-(x_j - y_j)^2 + i s}{s^2}\right) e^{-|x-y|^2/2i s} h_j(y) dy\right) \frac{e^{-s^2/2t}}{\sqrt{2\pi t}}\right| \\
&= \frac{e^{-s^2/2t}}{\sqrt{2\pi t}} \left| \left(\int_{\mathbb{R}^d} (2\pi i s)^{-d/2} \left(\frac{-(x_j - y_j)^2 + i s}{s^2}\right) e^{-|x-y|^2/2i s} (h_j(y) - h_j(x)) dy\right) \right| \\
&\leq \frac{e^{-s^2/2t}}{\sqrt{2\pi t}} \int_{\mathbb{R}^d} (2\pi |s|)^{-d/2} \left| \frac{-(\tilde{x}_j - y_j)^2 + |s|}{s^2} \right| e^{-|\tilde{x}-y|^2/2|s|} |h_j(y) - h_j(\tilde{x})| dy \\
&= \frac{e^{-s^2/2t}}{\sqrt{2\pi t}} \mathbb{E}_{\mathbb{P}} \left| \left(\frac{(\tilde{x}_j - W_j^{\tilde{x}}(|s|))^2 - |s|}{s^2}\right) \left(h_j(W^{\tilde{x}}(|s|)) - h_j(\tilde{x})\right) \right|,\n\end{aligned}
$$

for some $\tilde{x} \in \mathbb{R}^d$ where $\tilde{x}_j = \pm x_j$ for $j = 1 \ldots, d$; and where $W^{\tilde{x}} : \Omega \times \mathbb{R}_+ \to \mathbb{R}^d$ is a standard Brownian motion starting at $\tilde{x} \in \mathbb{R}^d$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and $W_j^{\tilde{x}}$ is its j-th component. The inequality in [\(2.7\)](#page-3-6) follows easily if h_j is a polynomial, and standard approximation yields the inequality for $h_j \in C_c(\mathbb{R}^d; \mathbb{R})$. Now, exactly as in [\[2\]](#page-3-1) (2.6) and (2.7); we use the Brownian motion scaling for $W^{\tilde{x}}$, the Cauchy-Shwarz inequality on the last term in (2.7) , and the Hölder condition on h_i to deduce that the last term in [\(2.7\)](#page-3-6) is bounded above by $K \exp(-s^2/2t)/(\sqrt{2\pi t}|s|^{1-\alpha/2}) \in$ $L^1((-\infty,0),ds) \cap L^1((0,\infty),ds)$; hence $\left|\frac{\partial^4}{\partial x_j^4}v(s,x)p_t(0,s)\right| \in L^1((-\infty,0),ds) \cap$ $L^1((0,\infty), ds)$, which completes the proof by standard analysis.

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