

A NEW INVARIANT OF QUADRATIC LIE ALGEBRAS

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ABSTRACT. We define a new invariant of quadratic Lie algebras and give a complete study and classification of singular quadratic Lie algebras, i.e. those for which the invariant does not vanish. The classification is related to $O(n)$ -adjoint orbits in $\mathfrak{o}(n)$.

0. INTRODUCTION

Let \mathfrak{g} be a non-Abelian quadratic Lie algebra equipped with a bilinear form B . We can associate to (\mathfrak{g}, B) a canonical non-zero 3-form $I \in \wedge^3(\mathfrak{g})^{\mathfrak{g}}$ defined by

$$I(X, Y, Z) := B([X, Y], Z), \quad \forall X, Y, Z \in \mathfrak{g}.$$

Let $\{\cdot, \cdot\}$ be the super-Poisson bracket on $\wedge(\mathfrak{g})$. The 3-form I satisfies (see [PU07]):

$$\{I, I\} = 0.$$

Conversely, given a quadratic vector space (\mathfrak{g}, B) and a non-zero 3-form $I \in \wedge^3(\mathfrak{g})$ such that $\{I, I\} = 0$, there is a non-Abelian quadratic Lie algebra structure on \mathfrak{g} such that I is the canonical 3-form associated to \mathfrak{g} ([PU07]).

Let $\mathcal{Q}(n)$ be the set of non-Abelian quadratic Lie algebra structures on the quadratic vector space \mathbb{C}^n . We identify

$$\mathcal{Q}(n) \leftrightarrow \left\{ I \in \wedge^3(\mathbb{C}^n) \mid \{I, I\} = 0 \right\}$$

and $\mathcal{Q}(n)$ is an affine variety in $\wedge^3(\mathbb{C}^n)$ (Proposition 2.8).

The *dup-number* of a non-Abelian quadratic Lie algebra \mathfrak{g} is defined by

$$\text{dup}(\mathfrak{g}) := \dim(\{\alpha \in \mathfrak{g}^* \mid \alpha \wedge I = 0\}),$$

where I is the 3-form associated to \mathfrak{g} . It measures the decomposability of the 3-form I and its range is $\{0, 1, 3\}$ (Proposition 1.1). For instance, I is decomposable if, and only if, $\text{dup}(\mathfrak{g}) = 3$ and the corresponding quadratic Lie algebras are classified in [PU07], up to i -isomorphism (i.e. isometric isomorphism). It is easy to check that the dup-number of \mathfrak{g} is invariant by i -isomorphism, that is, two i -isomorphic quadratic Lie algebras have the same dup-number (Lemma 2.1). We shall prove in this paper, a much stronger result:

the dup-number of \mathfrak{g} is invariant by isomorphism.

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To prove this result, we need to fully understand the structure of some particular Lie algebras. This study is interesting by itself and we shall describe it in the sequel.

We say that a non-Abelian quadratic Lie algebra \mathfrak{g} is *ordinary* if $\text{dup}(\mathfrak{g}) = 0$. Otherwise, \mathfrak{g} is called *singular*. Singular quadratic Lie algebras are of *type* S_1 if their dup-number is 1 and of *type* S_3 if their dup-number is 3.

For $n \geq 1$, let $\mathcal{O}(n)$ be the set of *ordinary* and $\mathcal{S}(n)$ be the set of *singular* quadratic Lie algebra structures on \mathbb{C}^n . We prove the following Theorem (Propositions 2.8, 2.10 and Appendix 2):

THEOREM 1:

- (1) $\mathcal{O}(n)$ is a Zariski-open subset of $\mathcal{Q}(n)$.
- (2) $\mathcal{S}(n)$ is a Zariski-closed subset of $\mathcal{Q}(n)$.
- (3) $\mathcal{Q}(n) \neq \emptyset$ if, and only if, $n \geq 3$.
- (4) $\mathcal{O}(n) \neq \emptyset$ if, and only if, $n \geq 6$.

As a consequence, non-Abelian quadratic Lie algebras with dimension higher than 6 are generically ordinary. In this work, we shall give a complete classification of singular quadratic Lie algebras, up to i -isomorphism and up to isomorphism.

Let us give some details of the main results of the paper. Section 3 contains a preparatory study of quadratic Lie algebras of type S_1 . It allow us to describe solvable singular Lie algebras in terms of double extensions, a useful method introduced by V. Kac and developed in [MR85] and [FS87]. First, we obtain (Propositions 4.3 and 4.4):

THEOREM 2:

- (1) Any quadratic Lie algebra of type S_1 is solvable and it is a double extension.
- (2) A quadratic Lie algebra is singular and solvable if, and only if, it is a double extension.

What about **non-solvable** singular Lie algebras? Such a Lie algebra \mathfrak{g} can be written as

$$\mathfrak{g} = \mathfrak{s} \overset{\perp}{\oplus} \mathfrak{z}$$

where \mathfrak{z} is a central ideal of \mathfrak{g} and $\mathfrak{s} \overset{i}{\simeq} \mathfrak{o}(3)$ equipped with a bilinear form $\lambda \kappa$ for some non-zero $\lambda \in \mathbb{C}$, where κ is the Killing form of $\mathfrak{o}(3)$ (Proposition 4.4).

In the remainder of the paper, we focus on the study of **solvable singular Lie algebras**. We denote by $\mathcal{S}_s(n+2)$ the set of these structures on \mathbb{C}^{n+2} , by $\widehat{\mathcal{S}}_s(n+2)$ the set of isomorphism classes of elements in $\mathcal{S}_s(n+2)$ and by $\widehat{\mathcal{S}}_s^i(n+2)$ the set of i -isomorphism classes. Also, we denote by $\mathbb{P}^1(\mathfrak{o}(n))$ the projective space of $\mathfrak{o}(n)$ and by $\widetilde{\mathbb{P}^1(\mathfrak{o}(n))}$ the set of orbits of elements in $\mathbb{P}^1(\mathfrak{o}(n))$ under the action induced by the $O(n)$ -adjoint action on $\mathfrak{o}(n)$. Given $\overline{C} \in \widetilde{\mathbb{P}^1(\mathfrak{o}(n))}$, there is an associated double extension $\mathfrak{g}_{\overline{C}} \in \mathcal{S}_s(n+2)$.

In Proposition 4.5 and Corollary 4.6, we characterize i-isomorphisms and isomorphisms. As a consequence, we prove the following result, conjectured and partially proved in [FS87] (Proposition 4.10):

THEOREM 3:

The map $\overline{\mathcal{C}} \rightarrow \mathfrak{g}_{\overline{\mathcal{C}}}$ induces a bijection from $\mathbb{P}^1(\widetilde{\mathfrak{o}(n)})$ onto $\widehat{\mathcal{S}}_s^i(n+2)$.

Theorem 3 gives a remarkable relation between solvable singular quadratic Lie algebra structures on \mathbb{C}^{n+2} and $O(n)$ -adjoint orbits in $\mathfrak{o}(n)$. A strong improvement to Theorem 3 will be given in Theorem 6.

Next, we detail some particular cases. Let $\mathcal{D}(n+2)$ be the set of *diagonalizable singular structures on \mathbb{C}^{n+2}* (i.e. $\overline{\mathcal{C}}$ is a semi-simple element of $\mathfrak{o}(n)$) and $\widehat{\mathcal{D}}^i(n+2)$ be the set of i-isomorphism classes in $\mathcal{D}(n+2)$. It is clear by Theorem 3 that $\widehat{\mathcal{D}}^i(n+2)$ is in bijection with the well-known set of semi-simple $O(n)$ -orbits in $\mathbb{P}^1(\mathfrak{o}(n))$ (see [CM93] for more details on this set). A description of the corresponding Lie algebra structures is given in Proposition 5.7, Corollary 5.8, Lemma 5.9 and Proposition 5.11.

Let $\mathcal{N}(n+2)$ be the set of *nilpotent singular structures on \mathbb{C}^{n+2}* , $\widehat{\mathcal{N}}^i(n+2)$ be the set of i-isomorphism classes and $\widehat{\mathcal{N}}(n+2)$ be the set of isomorphism classes of elements in $\mathcal{N}(n+2)$.

In the nilpotent case, we prove (Proposition 5.2):

THEOREM 4:

(1) *Let \mathfrak{g} and $\mathfrak{g}' \in \mathcal{N}(n+2)$. Then*

$$\mathfrak{g} \stackrel{i}{\simeq} \mathfrak{g}' \text{ if, and only if, } \mathfrak{g} \simeq \mathfrak{g}'.$$

Thus $\widehat{\mathcal{N}}^i(n+2) = \widehat{\mathcal{N}}(n+2)$.

(2) *Let $\widetilde{\mathcal{N}}(n)$ be the set of nilpotent $O(n)$ -orbits in $\mathfrak{o}(n)$. Then the map $\overline{\mathcal{C}} \mapsto \mathfrak{g}_{\overline{\mathcal{C}}}$ induces a bijection from $\widetilde{\mathcal{N}}(n)$ onto $\widehat{\mathcal{N}}^i(n+2) = \widehat{\mathcal{N}}(n+2)$.*

(3) *The set $\widehat{\mathcal{N}}(n+2)$ is finite.*

The classification of nilpotent $O(n)$ -orbits in $\mathfrak{o}(n)$ is known [CM93]. It uses deep results by Jacobson-Morosov and Kostant on $\mathfrak{sl}(2)$ -triples in semi-simple Lie algebras. Using this classification, we obtain a classification of $\widehat{\mathcal{N}}^i(n+2) = \widehat{\mathcal{N}}(n+2)$ in terms of *special partitions of n* and a characterization of the corresponding Lie algebras by means of amalgamated products of nilpotent Jordan-type Lie algebras (Proposition 5.5).

Before working on the general case, we define the notion of an *invertible singular Lie algebra* (i.e. $\overline{\mathcal{C}}$ is invertible). Let $\mathcal{S}_{\text{inv}}(2p+2)$ be the set of such structures on \mathbb{C}^{2p+2} and $\widehat{\mathcal{S}}_{\text{inv}}(2p+2)$ be the set of isomorphism classes of elements in $\mathcal{S}_{\text{inv}}(2p+2)$. The notions of i-isomorphism and isomorphism coincide in the invertible case as we show in Lemma 5.9.

Given a solvable singular Lie algebra \mathfrak{g} , realized as a double extension of \mathbb{C}^n by $\overline{C} \in \mathfrak{o}(n)$, we consider the Fitting components \overline{C}_I and \overline{C}_N of \overline{C} and the corresponding double extensions $\mathfrak{g}_I = \mathfrak{g}_{\overline{C}_I}$ and $\mathfrak{g}_N = \mathfrak{g}_{\overline{C}_N}$ that we call the *Fitting components* of \mathfrak{g} . We have \mathfrak{g}_I invertible, \mathfrak{g}_N nilpotent and we prove (Proposition 6.4):

THEOREM 5:

Let \mathfrak{g} and \mathfrak{g}' be solvable singular Lie algebras and let $\mathfrak{g}_N, \mathfrak{g}_I, \mathfrak{g}'_N, \mathfrak{g}'_I$ be their Fitting components. Then

$$\mathfrak{g} \stackrel{i}{\simeq} \mathfrak{g}' \text{ if, and only if } \begin{cases} \mathfrak{g}_N \stackrel{i}{\simeq} \mathfrak{g}'_N \\ \mathfrak{g}_I \stackrel{i}{\simeq} \mathfrak{g}'_I \end{cases}$$

The result remains valid if we replace $\stackrel{i}{\simeq}$ by \simeq .

Since i -isomorphism and isomorphism are equivalent notions in the case of nilpotent or invertible singular Lie algebras, we deduce as an immediate Corollary:

THEOREM 6:

Let \mathfrak{g} and \mathfrak{g}' be solvable singular Lie algebras. Then

$$\mathfrak{g} \simeq \mathfrak{g}' \text{ if, and only if } \mathfrak{g} \stackrel{i}{\simeq} \mathfrak{g}'.$$

Therefore $\widehat{\mathcal{S}}_s(n+2) = \widehat{\mathcal{S}}_s^i(n+2)$.

Theorem 6 is a really interesting and unexpected property of solvable singular quadratic Lie algebras.

Using Theorem 5, since the study of the nilpotent case is complete, we are left with the invertible case. First, we achieve the description of these structures in terms of amalgamated products of Jordan-type Lie algebras in Proposition 6.7. Then, we give a classification of invertible $O(n)$ -orbits in $\mathfrak{o}(n)$ (i.e. $O(n)$ -orbits of invertible elements). Let $\mathcal{S}(n)$ be the set of invertible elements in $\mathfrak{o}(n)$ and $\widetilde{\mathcal{S}}(n)$ be the set of $O(n)$ -adjoint orbits of elements in $\mathcal{S}(n)$. Notice that $\mathcal{S}(2p+1) = \emptyset$ (Appendix 1). Next, we consider

$$\mathcal{D} = \bigcup_{r \in \mathbb{N}^*} \{(d_1, \dots, d_r) \in \mathbb{N}^r \mid d_1 \geq d_2 \geq \dots \geq d_r \geq 1\}$$

and the map $\Phi : \mathcal{D} \rightarrow \mathbb{N}$ defined by $\Phi(d_1, \dots, d_r) = \sum_{i=1}^r d_i$. We introduce the set \mathcal{J}_p of all triples (Λ, m, d) such that:

- (1) Λ is a subset of $\mathbb{C} \setminus \{0\}$ with $\#\Lambda \leq 2p$ and $\lambda \in \Lambda$ if, and only if, $-\lambda \in \Lambda$.
- (2) $m : \Lambda \rightarrow \mathbb{N}^*$ satisfies $m(\lambda) = m(-\lambda)$, for all $\lambda \in \Lambda$ and $\sum_{\lambda \in \Lambda} m(\lambda) = 2p$.
- (3) $d : \Lambda \rightarrow \mathcal{D}$ satisfies $d(\lambda) = d(-\lambda)$, for all $\lambda \in \Lambda$ and $\Phi \circ d = m$.

To every $\overline{C} \in \mathcal{S}(2p)$, we can associate an element (Λ, m, d) of \mathcal{J}_p as follows: write $\overline{C} = S + N$ as a sum of its semi-simple and nilpotent parts. Then Λ is the spectrum of S , m is the multiplicity map on Λ and d gives the size of the Jordan blocks of N . Therefore, we obtain a map $i : \mathcal{S}(2p) \rightarrow \mathcal{J}_p$ and we prove (Proposition 6.10):

THEOREM 7:

The map $i : \mathcal{I}(2p) \rightarrow \mathcal{I}_p$ induces a bijection from $\tilde{\mathcal{I}}(2p)$ onto \mathcal{I}_p .

As a Corollary, we deduce a bijection from $\widehat{\mathcal{S}}_{\text{inv}}(2p+2)$ onto $\mathcal{I}_p/\mathbb{C}^*$ (Proposition 6.11) where the action of $\mu \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ on \mathcal{I}_p is defined by

$$\mu \cdot (\Lambda, m, d) := (\mu\Lambda, m', d'), \text{ with } m'(\mu\lambda) = m(\lambda) \text{ and } d'(\mu\lambda) = d(\lambda), \forall \lambda \in \Lambda$$

Combine Theorems 5, 4 and 7 to obtain a complete classification of $\widehat{\mathcal{S}}_s^i(n) = \widehat{\mathcal{S}}_s(n)$. As a by-product, we also obtain a complete classification of $\mathcal{O}(n)$ -orbits in $\mathfrak{o}(n)$, a result which is certainly known, but for which we have no available reference.

Finally, as a consequence of the preceding results, we prove in Section 7 (Proposition 7.3):

THEOREM 8:

The dup-number is invariant under isomorphism, i.e. if

$$\mathfrak{g} \simeq \mathfrak{g}' \text{ then } \text{dup}(\mathfrak{g}) = \text{dup}(\mathfrak{g}').$$

This result is rather unexpected. It is obtained through a computation of centro-morphisms in the reduced singular case (Proposition 7.2).

We also obtain the quadratic dimension of \mathfrak{g} [BB07] in this case:

$$\dim_{\mathbb{q}}(\mathfrak{g}) = 1 + \frac{\dim(\mathcal{Z}(\mathfrak{g}))(1 + \dim(\mathcal{Z}(\mathfrak{g})))}{2},$$

where $\mathcal{Z}(\mathfrak{g})$ is the center of \mathfrak{g} .

There are two Appendix. In the first one, we collect some well-known useful properties of elements of $\mathfrak{o}(n)$, shorts proofs are given for the sake of completeness. In Appendix 2, we show that $\mathcal{O}(5) = \emptyset$ and describe $\mathcal{Q}(5)$ up to i-isomorphism.

1. PRELIMINARIES

1.1. All vector spaces considered in the paper are finite-dimensional complex vector spaces.

Given a vector space V , we denote by V^* its dual space. Given a subset X of V , $X^{\perp*}$ denotes the *orthogonal subspace* of X in V^* .

We denote by $\mathcal{L}(V)$ the *algebra of linear operators* of V , by $\text{GL}(V)$ the *group of invertible operators* in $\mathcal{L}(V)$, by tA the *transpose* of an operator $A \in \mathcal{L}(V)$ and by $\wedge(V)$ the (\mathbb{Z} -graded) *Grassmann algebra* of skew-symmetric multilinear forms on V , i.e. $\wedge(V)$ is the exterior algebra of V^* . Recall that given an isomorphism A between two vector spaces V and V' , there is an algebra isomorphism from $\wedge(V')$ onto $\wedge(V)$ that extends the transpose ${}^tA : V'^* \rightarrow V^*$ and that we also denote by tA .

1.2. Let $I \in \wedge^k(V)$, for $k \geq 1$. We introduce two subspaces of V^* :

$$\begin{aligned}\mathcal{V}_I &:= \{\alpha \in V^* \mid \alpha \wedge I = 0\} \\ \mathcal{W}_I &:= \{v \in V \mid \iota_v(I) = 0\}^{\perp*} = \{\iota_{v \wedge v'}(I) \mid v, v' \in V\}\end{aligned}$$

where ι_v is the derivation of $\wedge(V)$ defined by:

$$\iota_v(\Omega)(v_1, \dots, v_{r-1}) = \Omega(v, v_1, \dots, v_{r-1}), \forall \Omega \in \wedge^r(V), v_1, \dots, v_{r-1} \in V.$$

The following result is well known, see for instance [Bou58].

Proposition 1.1. *Let $I \in \wedge^k(V)$, $I \neq 0$. Then:*

- (1) $\mathcal{V}_I \subset \mathcal{W}_I$, $\dim(\mathcal{V}_I) \leq k$ and $\dim(\mathcal{W}_I) \geq k$.
- (2) If $\{\alpha_1, \dots, \alpha_r\}$ is a basis of \mathcal{V}_I , then $\alpha_1 \wedge \dots \wedge \alpha_r$ divides I . Moreover, I belongs to the k -th exterior power of \mathcal{W}_I , also denoted by $\wedge^k(\mathcal{W}_I)$.
- (3) I is decomposable if, and only if, $\dim(\mathcal{V}_I) = k$ or $\dim(\mathcal{W}_I) = k$. In this case, $\mathcal{V}_I = \mathcal{W}_I$ and if $\{\alpha_1, \dots, \alpha_k\}$ is a basis of \mathcal{V}_I , one has for some non-zero $\lambda \in \mathbb{C}$,

$$I = \lambda \alpha_1 \wedge \dots \wedge \alpha_k.$$

1.3. A vector space V equipped with a non-degenerate symmetric bilinear form B is called a *quadratic vector space*. In this case, there is an isomorphism ϕ from V onto V^* defined by

$$\phi(v)(v') := B(v, v'), \forall v, v' \in V.$$

Given a subspace W of V , we denote by W^\perp the *orthogonal subspace* of W in V with respect to the bilinear form B . One has $V = W \oplus W^\perp$ if, and only if, the restriction $B|_{W \times W}$ is non degenerate and in this case, we use the notation

$$V = W \oplus^\perp W^\perp.$$

1.4. Let (V, B) and (V', B') be two quadratic vector spaces. An *isometry* is a bijective map $A : V \rightarrow V'$ that satisfies

$$B'(A(v), A(w)) = B(v, w), \forall v, w \in V.$$

We denote by $A^* \in \mathcal{L}(V)$ the *adjoint map* of an element $A \in \mathcal{L}(V)$. Remark that A is an isometry of V if, and only if, $A^{-1} = A^*$.

The *group of isometries* of V is denoted by $O(V, B)$ (or simply $O(V)$) and its Lie algebra is denoted by $\mathfrak{o}(V, B)$ (or simply $\mathfrak{o}(V)$). An element A of $\mathfrak{o}(V) \subset \mathcal{L}(V)$ satisfies $A^* = -A$ (that means A is skew-symmetric with respect to B). Notice that $\text{Tr}(A) = 0$, for all $A \in \mathfrak{o}(V)$. The *adjoint action* Ad of $O(V)$ on $\mathfrak{o}(V)$ is given by

$$\text{Ad}_U(C) := UCU^{-1}, \forall U \in O(V), C \in \mathfrak{o}(V).$$

We denote by \mathcal{O}_C , the *orbit* of an element $C \in \mathfrak{o}(V)$.

Let $V = \mathbb{C}^n$. Consider the canonical basis $\mathcal{B} = \{E_1, \dots, E_n\}$ of V . If n even, $n = 2p$, write $\mathcal{B} = \{E_1, \dots, E_p, F_1, \dots, F_p\}$ and if n is odd, $n = 2p + 1$, write $\mathcal{B} = \{E_1, \dots, E_p, G, F_1, \dots, F_p\}$. The *canonical bilinear form* B on V is defined by:

- if $n = 2p$:

$$B(E_i, F_j) = \delta_{ij}, B(E_i, E_j) = B(F_i, F_j) = 0, \forall 1 \leq i, j \leq p$$

- if $n = 2p + 1$:

$$\begin{cases} B(E_i, F_j) = \delta_{ij}, B(E_i, E_j) = B(F_i, F_j) = 0, \forall 1 \leq i, j \leq p \\ B(E_i, G) = B(F_j, G) = 0, \\ B(G, G) = 1 \end{cases}$$

In that case, $O(n)$ stands for $O(\mathbb{C}^n, B)$ and $\mathfrak{o}(n)$ stands for $\mathfrak{o}(\mathbb{C}^n, B)$.

Finally, if V is an n -dimensional quadratic vector space, then V is isometrically isomorphic (i-isomorphic) to the quadratic space \mathbb{C}^n [Bou59].

1.5. Let (V, B) be a quadratic vector space. We define the super-Poisson bracket on $\bigwedge(V)$ as follows (see [PU07] for details): fix an orthonormal basis $\{v_1, \dots, v_n\}$ of V . Then

$$\{\Omega, \Omega'\} := (-1)^{k+1} \sum_{j=1}^n \iota_{v_j}(\Omega) \wedge \iota_{v_j}(\Omega'), \forall \Omega \in \bigwedge^k(V), \Omega' \in \bigwedge(V).$$

For instance, if $\alpha \in V^*$, one has

$$\{\alpha, \Omega\} = \iota_{\phi^{-1}(\alpha)}(\Omega), \forall \Omega \in \bigwedge(V),$$

and if $\alpha' \in V^*$, $\{\alpha, \alpha'\} = B(\phi^{-1}(\alpha), \phi^{-1}(\alpha'))$. This definition does not depend on the choice of the basis.

For any $\Omega \in \bigwedge^k(V)$, define $\text{ad}_p(\Omega)$ by

$$\text{ad}_p(\Omega)(\Omega') := \{\Omega, \Omega'\}, \forall \Omega' \in \bigwedge(V).$$

Then $\text{ad}_p(\Omega)$ is a super-derivation of degree $k - 2$ of the exterior algebra $\bigwedge(V)$. One has:

$$\text{ad}_p(\Omega)(\{\Omega', \Omega''\}) = \{\text{ad}_p(\Omega)(\Omega'), \Omega''\} + (-1)^{kk'} \{\Omega', \text{ad}_p(\Omega)(\Omega'')\},$$

for all $\Omega' \in \bigwedge^{k'}(V)$, $\Omega'' \in \bigwedge(V)$. That implies that $\bigwedge(V)$ is a graded Lie algebra for the super-Poisson bracket.

1.6. A *quadratic Lie algebra* (\mathfrak{g}, B) is a quadratic vector space \mathfrak{g} equipped with a bilinear form B and a Lie algebra structure on \mathfrak{g} such that B is invariant (that means, $B([X, Y], Z) = B(X, [Y, Z])$, for all $X, Y, Z \in \mathfrak{g}$).

If (\mathfrak{g}, B) is a quadratic Lie algebra, recall that

$$[\mathfrak{g}, \mathfrak{g}] = \mathcal{Z}(\mathfrak{g})^\perp$$

where $\mathcal{Z}(\mathfrak{g})$ is the center of \mathfrak{g} . There is a canonical invariant $I \in \bigwedge^3(\mathfrak{g})$ defined by

$$I(X, Y, Z) := B([X, Y], Z), \forall X, Y, Z \in \mathfrak{g}.$$

This invariant satisfies $\{I, I\} = 0$ (see [PU07]) and it is easy to check that

$$\mathcal{W}_I = \phi([\mathfrak{g}, \mathfrak{g}]).$$

We say that I is the *3-form associated to* \mathfrak{g} .

On the other hand, given a quadratic vector space (\mathfrak{g}, B) and $I \in \wedge^3(\mathfrak{g})$, define

$$[X, Y] := \phi^{-1}(\iota_{X \wedge Y}(I)), \quad \forall X, Y \in \mathfrak{g}.$$

This bracket satisfies the Jacobi identity if, and only if, $\{I, I\} = 0$ [PU07]. In this case, \mathfrak{g} becomes a quadratic Lie algebra with invariant bilinear form B .

Definition 1.2. Let (\mathfrak{g}, B) and (\mathfrak{g}', B') be two quadratic Lie algebras. We say that (\mathfrak{g}, B) and (\mathfrak{g}', B') are *isometrically isomorphic* (or *i-isomorphic*) if there exists a Lie algebra isomorphism A from \mathfrak{g} onto \mathfrak{g}' satisfying

$$B'(A(X), A(Y)) = B(X, Y), \quad \forall X, Y \in \mathfrak{g}.$$

In other words, A is an i-isomorphism if it is a Lie algebra isomorphism and an isometry. We write $\mathfrak{g} \stackrel{i}{\simeq} \mathfrak{g}'$.

Consider two quadratic Lie algebras (\mathfrak{g}, B) and (\mathfrak{g}, B') (same Lie algebra) with $B' = \lambda B$, $\lambda \in \mathbb{C}$, $\lambda \neq 0$. They are not necessarily i-isomorphic, as shown by the example below:

Example 1.3. Let $\mathfrak{g} = \mathfrak{o}(3)$ and B its Killing form. Then A is a Lie algebra automorphism of \mathfrak{g} if, and only if, $A \in O(\mathfrak{g})$. So (\mathfrak{g}, B) and $(\mathfrak{g}, \lambda B)$ cannot be i-isomorphic if $\lambda \neq 1$.

2. THE DUP NUMBER OF A QUADRATIC LIE ALGEBRA

2.1. Let \mathfrak{g} and \mathfrak{g}' be quadratic Lie algebras with associated invariants I and I' (see (1.6)). The following Lemma is straightforward:

Lemma 2.1. *Let A be an i-isomorphism from \mathfrak{g} onto \mathfrak{g}' . Then $I = {}^t A(I')$, $\mathcal{V}_I = {}^t A(\mathcal{V}_{I'})$ and $\mathcal{W}_I = {}^t A(\mathcal{W}_{I'})$.*

It results from the previous Lemma that $\dim(\mathcal{V}_I)$ and $\dim(\mathcal{W}_I)$ are invariant under i-isomorphisms. This is not new for $\dim(\mathcal{W}_I)$, since $\dim(\mathcal{W}_I) = \dim([\mathfrak{g}, \mathfrak{g}])$.

For $\dim(\mathcal{V}_I)$, to our knowledge this fact was not remarked up to now, so we introduce the following definition:

Definition 2.2. Let \mathfrak{g} be a quadratic Lie algebra. The *dup number* $\text{dup}(\mathfrak{g})$ is defined by

$$\text{dup}(\mathfrak{g}) := \dim(\mathcal{V}_I).$$

Remark 2.3. By Proposition 1.1, when \mathfrak{g} is non-Abelian, one has $\text{dup}(\mathfrak{g}) \leq 3$. Actually $\text{dup}(\mathfrak{g}) \in \{0, 1, 3\}$. Notice that $\dim(\mathcal{W}_I) \geq 3$, so $\dim([\mathfrak{g}, \mathfrak{g}]) \geq 3$ (see [PU07]), a simple but rather interesting remark.

2.2. We shall use the decomposition result below:

Proposition 2.4. [PU07]

Let (\mathfrak{g}, B) be a non-Abelian quadratic Lie algebra. Then there exists a central ideal \mathfrak{z} and an ideal $\mathfrak{l} \neq \{0\}$ such that:

$$(1) \quad \mathfrak{g} = \mathfrak{z} \oplus^{\perp} \mathfrak{l}$$

- (2) $(\mathfrak{z}, B|_{\mathfrak{z} \times \mathfrak{z}})$ and $(\mathfrak{l}, B|_{\mathfrak{l} \times \mathfrak{l}})$ are quadratic Lie algebras. Moreover, \mathfrak{l} is non-Abelian.
- (3) The center $\mathcal{Z}(\mathfrak{l})$ is totally isotropic, i.e. $\mathcal{Z}(\mathfrak{l}) \subset [\mathfrak{l}, \mathfrak{l}]$.
- (4) Let \mathfrak{g}' be a quadratic Lie algebra and $A : \mathfrak{g} \rightarrow \mathfrak{g}'$ be a Lie algebra isomorphism. Then

$$\mathfrak{g}' = \mathfrak{z}' \oplus \mathfrak{l}'$$

where $\mathfrak{z}' = A(\mathfrak{z})$ is central, $\mathfrak{l}' = A(\mathfrak{l})^\perp$, $\mathcal{Z}(\mathfrak{l}')$ is totally isotropic and \mathfrak{l} and \mathfrak{l}' are isomorphic. Moreover if A is an i -isomorphism, then \mathfrak{l} and \mathfrak{l}' are i -isomorphic.

Proof. We prove (4) : recall that \mathfrak{z} is any complementary subspace of $\mathcal{Z}(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}]$ in $\mathcal{Z}(\mathfrak{g})$ (see [PU07]) and that \mathfrak{l} is defined as the orthogonal subspace of \mathfrak{z} , $\mathfrak{l} = \mathfrak{z}^\perp$.

One has $A(\mathcal{Z}(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}]) = \mathcal{Z}(\mathfrak{g}') \cap [\mathfrak{g}', \mathfrak{g}']$ and $\mathcal{Z}(\mathfrak{g}') = \mathfrak{z}' \oplus (\mathcal{Z}(\mathfrak{g}') \cap [\mathfrak{g}', \mathfrak{g}'])$.

Therefore \mathfrak{l}' satisfies $\mathfrak{g}' = \mathfrak{z}' \oplus \mathfrak{l}'$ and $\mathcal{Z}(\mathfrak{l}')$ is totally isotropic. Since A is an isomorphism from \mathfrak{z} onto \mathfrak{z}' , A induces an isomorphism from $\mathfrak{g}/\mathfrak{z}$ onto $\mathfrak{g}'/\mathfrak{z}'$, and it results that \mathfrak{l} and \mathfrak{l}' are isomorphic Lie algebras. Same reasoning works for A i -isomorphism. \square

It is clear that $\mathfrak{z} = \{0\}$ if, and only if, $\mathcal{Z}(\mathfrak{g})$ is totally isotropic and that

$$\text{dup}(\mathfrak{g}) = \text{dup}(\mathfrak{l}).$$

Definition 2.5. A quadratic Lie algebra \mathfrak{g} is *reduced* if:

- (1) $\mathfrak{g} \neq \{0\}$
- (2) $\mathcal{Z}(\mathfrak{g})$ is totally isotropic.

Notice that a reduced quadratic Lie algebra is necessarily non-Abelian.

2.3. We separate non-Abelian quadratic Lie algebras as follows:

Definition 2.6.

Let \mathfrak{g} be a non-Abelian quadratic Lie algebra.

- (1) \mathfrak{g} is an *ordinary* quadratic Lie algebra if $\text{dup}(\mathfrak{g}) = 0$.
- (2) \mathfrak{g} is a *singular* quadratic Lie algebra if $\text{dup}(\mathfrak{g}) \geq 1$.
 - (i) \mathfrak{g} is a *singular* quadratic Lie algebra of *type* S_1 if $\text{dup}(\mathfrak{g}) = 1$.
 - (i) \mathfrak{g} is a *singular* quadratic Lie algebra of *type* S_3 if $\text{dup}(\mathfrak{g}) = 3$.

Now, given a non-Abelian n -dimensional quadratic Lie algebra \mathfrak{g} , we can assume, up to i -isomorphism, that $\mathfrak{g} = \mathbb{C}^n$ equipped with its canonical bilinear form B (as a quadratic space) (1.4). So we introduce the following sets:

Definition 2.7. For $n \geq 1$:

- (1) $\mathcal{Q}(n)$ is the set of non-Abelian quadratic Lie algebra structures on \mathbb{C}^n .
- (2) $\mathcal{O}(n)$ is the set of *ordinary* quadratic Lie algebra structures on \mathbb{C}^n .
- (3) $\mathcal{S}(n)$ is the set of *singular* quadratic Lie algebra structures on \mathbb{C}^n .

By (1.6), there is a one to one map from $\mathcal{Q}(n)$ onto the subset

$$\left\{ I \in \bigwedge^3(\mathbb{C}^n) \mid I \neq 0, \{I, I\} = 0 \right\} \subset \bigwedge^3(\mathbb{C}^n).$$

In the sequel, we identify these two sets, so that $\mathcal{Q}(n) \subset \bigwedge^3(\mathbb{C}^n)$.

Proposition 2.8. *One has:*

- (1) $\mathcal{Q}(n)$ is an affine variety in $\bigwedge^3(\mathbb{C}^n)$.
- (2) $\mathcal{O}(n)$ is a Zariski-open subset of $\mathcal{Q}(n)$.
- (3) $\mathcal{S}(n)$ is a Zariski-closed subset of $\mathcal{Q}(n)$.

Proof. The map $I \mapsto \{I, I\}$ is a polynomial map from $\bigwedge^3(\mathbb{C}^n)$ into $\bigwedge^4(\mathbb{C}^n)$, so the first claim follows.

Fix $I \in \bigwedge^3(\mathbb{C}^n)$ such that $\{I, I\} = 0$. Consider the map $m : (\mathbb{C}^n)^* \rightarrow \bigwedge^4(\mathbb{C}^n)$ defined by $m(\alpha) = \alpha \wedge I$, for all $\alpha \in (\mathbb{C}^n)^*$. Then, if \mathfrak{g} is the quadratic Lie algebra associated to I , one has $\text{dup}(\mathfrak{g}) = 0$ if, and only if, $\text{rank}(m) = n$. This can never happen for $n \leq 4$. Assume $n \geq 5$. Let M be a matrix of m and Δ_i be the minors of order n , for $1 \leq i \leq \binom{n}{4}$. Then $\mathfrak{g} \in \mathcal{O}(n)$ if, and only if, there exists i such that $\Delta_i \neq 0$. But Δ_i is a polynomial function and from that the second and the third claims follow. \square

Lemma 2.9. *Let \mathfrak{g}_1 and \mathfrak{g}_2 be non-Abelian quadratic Lie algebras. Then $\mathfrak{g}_1 \perp \mathfrak{g}_2$ is an ordinary quadratic Lie algebra.*

Proof. Set $\mathfrak{g} = \mathfrak{g}_1 \perp \mathfrak{g}_2$. Denote by I, I_1 and I_2 the non-trivial 3-forms associated to $\mathfrak{g}, \mathfrak{g}_1$ and \mathfrak{g}_2 respectively.

One has $\bigwedge(\mathfrak{g}) = \bigwedge(\mathfrak{g}_1) \otimes \bigwedge(\mathfrak{g}_2)$, $\bigwedge^k(\mathfrak{g}) = \bigoplus_{r+s=k} \bigwedge^r(\mathfrak{g}_1) \otimes \bigwedge^s(\mathfrak{g}_2)$ and $I = I_1 + I_2$, with $I_1 \in \bigwedge^3(\mathfrak{g}_1)$ and $I_2 \in \bigwedge^3(\mathfrak{g}_2)$. It immediately results that for $\alpha = \alpha_1 + \alpha_2 \in \mathfrak{g}_1^* \oplus \mathfrak{g}_2^*$, one has $\alpha \wedge I = 0$ if, and only if, $\alpha_1 = \alpha_2 = 0$. \square

Proposition 2.10. *One has:*

- (1) $\mathcal{Q}(n) \neq \emptyset$ if, and only if, $n \geq 3$.
- (2) $\mathcal{O}_3 = \mathcal{O}_4 = \emptyset$ and $\mathcal{O}(n) \neq \emptyset$ if $n \geq 6$.

Proof. If \mathfrak{g} is a non-Abelian quadratic Lie algebra, using Remark 2.3, one has $\dim([\mathfrak{g}, \mathfrak{g}]) \geq 3$, so $\mathcal{Q}(n) = \emptyset$ if $n < 3$.

We shall now use some elementary quadratic Lie algebras given in Section 6 of [PU07]. We denote these algebras by \mathfrak{g}_i , according to their dimension, so that $\dim(\mathfrak{g}_i) = i$, for $3 \leq i \leq 6$. Note that $\mathfrak{g}_3 = \mathfrak{o}(3)$, $\mathfrak{g}_4, \mathfrak{g}_5$ and \mathfrak{g}_6 are examples of elements of $\mathcal{Q}(3), \mathcal{Q}(4), \mathcal{Q}(5)$ and $\mathcal{Q}(6)$, respectively.

Consider

$$\mathfrak{g} := \bigoplus_{3 \leq i \leq 6} \left(\mathfrak{g}_i \perp \overbrace{\dots \perp \mathfrak{g}_i}^{k_i \text{ times}} \right).$$

Then $\dim(\mathfrak{g}) = \sum_{i=3}^6 ik_i$ and by Lemma 2.9, $\text{dup}(\mathfrak{g}) = 0$, so we obtain $\mathcal{O}(n) \neq \emptyset$ if $n \geq 6$.

Finally, let \mathfrak{g} be a non-Abelian quadratic Lie algebra of dimension 3 or 4 with associated 3-form I . Then I is decomposable, so \mathfrak{g} is singular. Therefore $\mathcal{O}_3 = \mathcal{O}_4 = \emptyset$. □

Remark 2.11. We shall prove in Appendix 2 that $\mathcal{O}_5 = \emptyset$. So, generically a non-Abelian quadratic Lie algebra is ordinary if $n \geq 6$.

Definition 2.12. A quadratic Lie algebra \mathfrak{g} is *indecomposable* if $\mathfrak{g} = \mathfrak{g}_1 \oplus^\perp \mathfrak{g}_2$, with \mathfrak{g}_1 and \mathfrak{g}_2 ideals of \mathfrak{g} , imply \mathfrak{g}_1 or $\mathfrak{g}_2 = \{0\}$.

The Proposition below gives another characterization of reduced singular quadratic Lie algebras.

Proposition 2.13. *Let \mathfrak{g} be a singular quadratic Lie algebra. Then \mathfrak{g} is reduced if, and only if, \mathfrak{g} is indecomposable.*

Proof. If \mathfrak{g} is indecomposable, by Proposition 2.4, \mathfrak{g} is reduced. If \mathfrak{g} is reduced and $\mathfrak{g} = \mathfrak{g}_1 \oplus^\perp \mathfrak{g}_2$, with \mathfrak{g}_1 and \mathfrak{g}_2 ideals of \mathfrak{g} , then $\mathcal{Z}(\mathfrak{g}_i) \subset [\mathfrak{g}_i, \mathfrak{g}_i]$ for $i = 1, 2$. So \mathfrak{g}_i is reduced or $\mathfrak{g}_i = \{0\}$. But if \mathfrak{g}_1 and \mathfrak{g}_2 are both reduced, by Lemma 2.9, one has $\text{dup}(\mathfrak{g}) = 0$. Hence \mathfrak{g}_1 or $\mathfrak{g}_2 = \{0\}$. □

3. QUADRATIC LIE ALGEBRAS OF TYPE S_1

3.1. Let (\mathfrak{g}, B) be a quadratic vector space and I be a non-zero 3-form in $\wedge^3(\mathfrak{g})$. As in (1.6), we define a Lie bracket on \mathfrak{g} by:

$$[X, Y] := \phi^{-1}(t_{X \wedge Y}(I)), \quad \forall X, Y \in \mathfrak{g}.$$

Then \mathfrak{g} becomes a quadratic Lie algebra with an invariant bilinear form B if, and only if, $\{I, I\} = 0$ [PU07].

In the sequel, we assume that $\dim(\mathcal{V}_I) = 1$. Fix $\alpha \in \mathcal{V}_I$ and choose $\Omega \in \wedge^2(\mathfrak{g})$ such that $I = \alpha \wedge \Omega$ as follows: let $\{\alpha, \alpha_1, \dots, \alpha_r\}$ be a basis of \mathcal{W}_I . Then, $I \in \wedge^3(\mathcal{W}_I)$ by Proposition 1.1. We set:

$$X_0 := \phi^{-1}(\alpha) \quad \text{and} \quad X_i := \phi^{-1}(\alpha_i), \quad 1 \leq i \leq r.$$

So, we can choose $\Omega \in \wedge^2(V)$ where $V = \text{span}\{X_1, \dots, X_r\}$. Note that Ω is an indecomposable bilinear form, so $\dim(V) > 3$.

We define $C : \mathfrak{g} \rightarrow \mathfrak{g}$ by

$$B(C(X), Y) := \Omega(X, Y).$$

Therefore C is skew-symmetric with respect to B .

Lemma 3.1. *The following are equivalent:*

- (1) $\{I, I\} = 0$
- (2) $\{\alpha, \alpha\} = 0$ and $\{\alpha, \Omega\} = 0$
- (3) $B(X_0, X_0) = 0$ and $C(X_0) = 0$

In this case, one has $\dim([\mathfrak{g}, \mathfrak{g}]) > 4$, $\mathcal{Z}(\mathfrak{g}) \subset \ker(C)$, $\text{Im}(C) \subset [\mathfrak{g}, \mathfrak{g}]$ and $X_0 \in \mathcal{Z}(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}]$.

Proof. It is easy to see that:

$$\{I, I\} = 0 \Leftrightarrow \{\alpha, \alpha\} \wedge \Omega \wedge \Omega = 2I \wedge \{\alpha, \Omega\}.$$

If $\Omega \wedge \Omega = 0$, then Ω is decomposable and that is a contradiction since $\dim(\mathcal{V}_I) = 1$. So $\Omega \wedge \Omega \neq 0$.

If $\{\alpha, \alpha\} \neq 0$, then α divides $\Omega \wedge \Omega \in \wedge^4(V)$, another contradiction. That implies $\{\alpha, \alpha\} = 0 = B(X_0, X_0)$. It results that $\{\alpha, \Omega\} \in \mathcal{V}_I = \mathbb{C}\alpha$, hence $\{\alpha, \Omega\} = \lambda\alpha$ for some $\lambda \in \mathbb{C}$. But $\{\alpha, \Omega\}$ is an element of $\wedge^1(V)$, so λ must be zero and by (1.5), $\iota_{X_0}(\Omega) = 0$, therefore $C(X_0) = 0$. Moreover, since $\{\alpha, \alpha\} = \{\alpha, \Omega\} = 0$, using $I = \alpha \wedge \Omega$, we deduce that $\{\alpha, I\} = 0$. Again by (1.5), it results that $B(X_0, [X, Y]) = \{\alpha, I\}(X \wedge Y) = 0$, for all $X, Y \in \mathfrak{g}$. So $X_0 \in [\mathfrak{g}, \mathfrak{g}]^\perp = \mathcal{Z}(\mathfrak{g})$. Also, $\mathcal{V}_I \subset \mathcal{W}_I$, so $X_0 = \phi^{-1}(\alpha) \in \phi^{-1}(\mathcal{W}_I) = [\mathfrak{g}, \mathfrak{g}]$.

Write $\Omega = \sum_{i < j} a_{ij} \alpha_i \wedge \alpha_j$, with $a_{ij} \in \mathbb{C}$. Since $\mathcal{W}_I = \phi([\mathfrak{g}, \mathfrak{g}])$ and $X_1, \dots, X_r \in [\mathfrak{g}, \mathfrak{g}]$, we deduce that

$$C = \sum_{i < j} a_{ij} (\alpha_i \otimes X_j - \alpha_j \otimes X_i)$$

Hence $\text{Im}(C) \subset [\mathfrak{g}, \mathfrak{g}]$. Since C is skew-symmetric, one has $\ker(C) = \text{Im}(C)^\perp$ and it follows $\mathcal{Z}(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{g}]^\perp \subset \ker(C)$.

Finally, $[\mathfrak{g}, \mathfrak{g}] = \mathbb{C}X_0 \oplus V$ and since $\dim(V) > 3$, we conclude that $\dim([\mathfrak{g}, \mathfrak{g}]) > 4$. \square

Remark 3.2. It is important to notice that our choice of Ω such that $I = \alpha \wedge \Omega$ is not unique, it depends on the choice of V , so C is not uniquely defined. Assume we consider another vector space V' and $I = \alpha \wedge \Omega'$. Then $\Omega' = \Omega + \alpha \wedge \beta$ for some $\beta \in \mathfrak{g}^*$. Let $X_1 = \phi^{-1}(\beta)$ and let C' be the map associated to Ω' . By a straightforward computation, $C' = C + \alpha \otimes X_1 - \beta \otimes X_0$. Since $C'(X_0) = 0$, we must have $B(X_0, X_1) = 0$.

3.2. We keep the notation as in the previous subsection. Assume that $\{I, I\} = 0$. Hence \mathfrak{g} is a quadratic Lie algebra of type S_1 .

Lemma 3.3. *There exists $Y_0 \in V^\perp$ such that*

$$V^\perp = \mathcal{Z}(\mathfrak{g}) \oplus \mathbb{C}Y_0, \quad B(Y_0, Y_0) = 0 \quad \text{and} \quad B(X_0, Y_0) = 1.$$

Moreover

$$C(Y_0) = 0.$$

Proof. One has $\phi^{-1}(\mathcal{W}_I) = [\mathfrak{g}, \mathfrak{g}] = \mathbb{C}X_0 \oplus V$, therefore $\mathcal{Z}(\mathfrak{g}) \subset V^\perp$ and $\dim(\mathcal{Z}(\mathfrak{g})) = \dim(\mathfrak{g}) - \dim([\mathfrak{g}, \mathfrak{g}]) = \dim(V^\perp) - 1$. So there exists $Y \in V^\perp$ such that $V^\perp = \mathcal{Z}(\mathfrak{g}) \oplus \mathbb{C}Y$. Now, Y cannot be orthogonal to X_0 , since it would be orthogonal to $[\mathfrak{g}, \mathfrak{g}]$ and therefore an element of $\mathcal{Z}(\mathfrak{g})$. So we can assume that $B(X_0, Y) = 1$.

Replace Y by $Y_0 = Y - \frac{1}{2}B(Y, Y)X_0$ to obtain $B(Y_0, Y_0) = 0$ (recall $B(X_0, X_0) = 0$).

By Lemma 3.1, $\text{Im}(C) \subset V$ and that implies $B(Y_0, C(X)) = -B(C(Y_0), X) = 0$, for all $X \in \mathfrak{g}$. Then $C(Y_0) = 0$. \square

Proposition 3.4. *We keep the previous notation and assumptions. Then:*

$$(1) \quad [X, Y] = B(X_0, X)C(Y) - B(X_0, Y)C(X) + B(C(X), Y)X_0, \quad \text{for all } X, Y \in \mathfrak{g}.$$

- (2) $C = \text{ad}(Y_0)$ and $\text{rank}(C)$ is even.
- (3) $\ker(C) = \mathcal{Z}(\mathfrak{g}) \oplus \mathbb{C}Y_0$, $\text{Im}(C) = V$ and $[\mathfrak{g}, \mathfrak{g}] = \mathbb{C}X_0 \oplus \text{Im}(C)$.
- (4) the Lie algebra \mathfrak{g} is solvable. Moreover, \mathfrak{g} is nilpotent if, and only if, C is nilpotent.
- (5) the dimension of $[\mathfrak{g}, \mathfrak{g}]$ is greater or equal to 5 and it is odd.

Proof.

- (1) This is a straightforward computation, use $B([X, Y], Z) = (\alpha \wedge \Omega)(X, Y, Z)$, $\alpha(X) = B(X_0, X)$ and $\Omega(X, Y) = B(C(X), Y)$, for all $X, Y, Z \in \mathfrak{g}$.
- (2) Set $X = Y_0$ in (1) and use Lemma 3.3 to show $C = \text{ad}(Y_0)$. Since $C(\mathfrak{g}) = \text{ad}(Y_0)(\mathfrak{g}) = \phi^{-1}(\text{ad}(\mathfrak{g})(\phi(Y_0)))$, the rank of C is the dimension of the coadjoint orbit through $\phi(Y_0)$, so it is even (see also Appendix 1).
- (3) We may assume that \mathfrak{g} is reduced. Then $\mathcal{Z}(\mathfrak{g})$ is totally isotropic and $\mathcal{Z}(\mathfrak{g}) \subset X_0^\perp$. Write $X_0^\perp = \mathcal{Z}(\mathfrak{g}) \oplus \mathfrak{h}$ with \mathfrak{h} a complementary subspace of $\mathcal{Z}(\mathfrak{g})$. Therefore $\mathfrak{g} = \mathcal{Z}(\mathfrak{g}) \oplus \mathfrak{h} \oplus \mathbb{C}Y_0$ and for an element $X = Z + H + \lambda Y_0 \in \ker(C)$, we deduce $H \in \ker(C)$ by Lemmas 3.1 and 3.3.

But $B(X_0, H) = 0$, so using (1), $H \in \mathcal{Z}(\mathfrak{g})$. It results that $H = 0$. Then $\ker(C) = \mathcal{Z}(\mathfrak{g}) \oplus \mathbb{C}Y_0$. In addition,

$$\dim(\text{Im}(C)) = \dim(\mathfrak{h}) = \dim(X_0^\perp) - \dim(\mathcal{Z}(\mathfrak{g})) = \dim([\mathfrak{g}, \mathfrak{g}]) - 1.$$

Our choice of V implies that $[\mathfrak{g}, \mathfrak{g}] = \phi^{-1}(\mathcal{W}_I) = \mathbb{C}X_0 \oplus V$ and $\text{Im}(C) \subset V$ (see the proof of Lemma 3.1). Therefore $\text{Im}(C) = V$ and $[\mathfrak{g}, \mathfrak{g}] = \mathbb{C}X_0 \oplus \text{Im}(C)$.

- (4) Since $B(X_0, \text{Im}(C)) = 0$, then $[[\mathfrak{g}, \mathfrak{g}], [\mathfrak{g}, \mathfrak{g}]] = [\text{Im}(C), \text{Im}(C)] \subset \mathbb{C}X_0$. We conclude that \mathfrak{g} is solvable. If \mathfrak{g} is nilpotent, then $C = \text{ad}(Y_0)$ is nilpotent. If C is nilpotent, using $\text{Im}(C) \subset X_0^\perp$, we obtain by induction that $(\text{ad}(X))^k(\mathfrak{g}) \subset \mathbb{C}X_0 \oplus \text{Im}(C^k)$ for any $k \in \mathbb{N}$. So $\text{ad}(X)$ is nilpotent, for all $X \in \mathfrak{g}$ and that implies \mathfrak{g} nilpotent.
- (5) Notice that $[\mathfrak{g}, \mathfrak{g}] = \mathbb{C}X_0 \oplus \text{Im}(C)$ and $\text{rank}(C)$ is even, so $\dim([\mathfrak{g}, \mathfrak{g}])$ is odd. By (3.1), $\dim([\mathfrak{g}, \mathfrak{g}]) \geq 5$.

□

3.3. Recall that C is not unique (see Remark 3.2) and it depends on the choice of V . Let

$$\mathfrak{a} := X_0^\perp / \mathbb{C}X_0.$$

We denote by \widehat{X} the class of an element $X \in \mathfrak{g}$.

Proposition 3.5.

Keep the notation above. One has:

- (1) the Lie algebra \mathfrak{a} is Abelian.
- (2) Define

$$\widehat{B}(\widehat{X}, \widehat{Y}) := B(X, Y), \quad \forall X, Y \in \mathfrak{g}.$$

Then \widehat{B} is a non degenerate symmetric bilinear form on \mathfrak{a} .

(3) Define

$$\widehat{C}(\widehat{X}) := C(X), \forall X \in \mathfrak{g}.$$

Then $\widehat{C} \in \mathcal{L}(\mathfrak{a})$ is a skew-symmetric map with $\text{rank}(\widehat{C}) = \text{rank}(C)$ even and $\text{rank}(\widehat{C}) \geq 4$.

- (4) \widehat{C} does not depend on the choice of V . More precisely, if $\mathcal{W}_I = \mathbb{C}\alpha \oplus \phi(V')$ and C' is the associated map to V' (see Remark 3.2), then $\widehat{C}' = \widehat{C}$.
- (5) the Lie algebra \mathfrak{g} is reduced if, and only if, $\ker(\widehat{C}) \subset \text{Im}(\widehat{C})$.

Proof.

- (1) It follows from Proposition 3.4 (1).
- (2) It is clear that \widehat{B} is well-defined. Now, since $B(X_0, Y_0) = 1$, $B(X_0, X_0) = B(Y_0, Y_0) = 0$, the restriction of B to $\text{span}\{X_0, Y_0\}$ is non degenerate. So

$$\mathfrak{g} = \text{span}\{X_0, Y_0\} \oplus \text{span}\{X_0, Y_0\}^\perp,$$

$X_0^\perp = \mathbb{C}X_0 \oplus \text{span}\{X_0, Y_0\}^\perp$ and $X_0^{\perp\perp} = X_0^\perp \cap \text{span}\{X_0, Y_0\} = \mathbb{C}X_0$. We conclude that \widehat{B} is non degenerate.

- (3) We have $C(X_0^\perp) = \text{ad}(Y_0)(X_0^\perp) \subset X_0^\perp$ since X_0^\perp is an ideal of \mathfrak{g} . Moreover, $C(X_0) = 0$, so \widehat{C} is well-defined. The image of C is contained in X_0^\perp and $\text{Im}(C) \cap \mathbb{C}X_0 = \{0\}$, therefore $\dim(\text{Im}(C)/\mathbb{C}X_0) = \dim(\text{Im}(\widehat{C})) = \dim(\text{Im}(C))$. Now it is enough to apply Proposition 3.4.
- (4) By Remark 3.2, we have $C' = C + \alpha \otimes X_1 - \beta \otimes X_0$. But $\alpha(X_0) = 0$, so $\widehat{C}' = \widehat{C}$.
- (5) By Proposition 3.4, we have $\ker(C) = \mathcal{Z}(\mathfrak{g}) \oplus \mathbb{C}Y_0$ and by Lemma 3.1, we have $\mathcal{Z}(\mathfrak{g}) \subset X_0^\perp$. Again by Proposition 3.4, we conclude that $\ker(\widehat{C}) = \mathcal{Z}(\mathfrak{g})/\mathbb{C}X_0$. Applying Proposition 3.4 once more, we have $[\mathfrak{g}, \mathfrak{g}] = \mathbb{C}X_0 \oplus \text{Im}(C)$, so $\text{Im}(\widehat{C}) = [\mathfrak{g}, \mathfrak{g}]/\mathbb{C}X_0$. Then $\ker(\widehat{C}) \subset \text{Im}(\widehat{C})$ if, and only if, $\mathcal{Z}(\mathfrak{g}) \subset [\mathfrak{g}, \mathfrak{g}] + \mathbb{C}X_0$. But $X_0 \in [\mathfrak{g}, \mathfrak{g}]$ (see Lemma 3.1), so the result follows. \square

We should notice that \widehat{C} still depends on the choice of α (see Remark 3.2): if we replace α by $\lambda\alpha$, for a non-zero $\lambda \in \mathbb{C}$, that will change \widehat{C} into $\frac{1}{\lambda}\widehat{C}$. So there is not a *unique* map \widehat{C} associated to \mathfrak{g} but rather a *family* $\{\lambda\widehat{C} \mid \lambda \in \mathbb{C} \setminus \{0\}\}$ of associated maps. In other words, there is a line

$$[\widehat{C}] := \{\lambda\widehat{C} \mid \lambda \in \mathbb{C}\} \in \mathbb{P}^1(\mathfrak{o}(\mathfrak{a}))$$

where $\mathbb{P}^1(\mathfrak{o}(\mathfrak{a}))$ is the projective space associated to the space $\mathfrak{o}(\mathfrak{a})$.

Definition 3.6. We call $[\widehat{C}]$ the *line of skew-symmetric maps* associated to the quadratic Lie algebra \mathfrak{g} of type S_1 .

Remark 3.7. The unicity of $[\widehat{C}]$ is valuable, but the fact that \widehat{C} acts on a quotient space and not on a subspace of \mathfrak{g} could be a problem. Hence it is convenient to use the following decomposition of \mathfrak{g} : the restriction of B to $\mathbb{C}X_0 \oplus \mathbb{C}Y_0$ is non

degenerate, so we can write $\mathfrak{g} = (\mathbb{C}X_0 \oplus \mathbb{C}Y_0) \overset{\perp}{\oplus} \mathfrak{q}$ where $\mathfrak{q} = (\mathbb{C}X_0 \oplus \mathbb{C}Y_0)^\perp$. Since $C(X_0) = C(Y_0) = 0$ and $C \in \mathfrak{o}(\mathfrak{g})$, C maps \mathfrak{q} into \mathfrak{q} . Let $\pi : X_0^\perp \rightarrow X_0^\perp / \mathbb{C}X_0$ be the canonical surjection and $\overline{C} = C|_{\mathfrak{q}}$. Then the restriction $\pi_{\mathfrak{q}} : \mathfrak{q} \rightarrow X_0^\perp / \mathbb{C}X_0$ is an isometry and $\widehat{C} = \pi_{\mathfrak{q}} \overline{C} \pi_{\mathfrak{q}}^{-1}$.

Remark that Y_0 is not unique, but if Y'_0 satisfies Lemma 3.3, consider $C' = \text{ad}(Y'_0)$ and \mathfrak{q}' such that $\mathfrak{g} = (\mathbb{C}X_0 \oplus \mathbb{C}Y'_0) \overset{\perp}{\oplus} \mathfrak{q}'$, therefore $\widehat{C} = \pi'_{\mathfrak{q}} \overline{C}' \pi'^{-1}_{\mathfrak{q}}$ with the obvious notation. It results that $\pi'^{-1}_{\mathfrak{q}} \pi_{\mathfrak{q}}$ is an isometry from \mathfrak{q} to \mathfrak{q}' and that

$$\overline{C}' = (\pi'^{-1}_{\mathfrak{q}} \pi_{\mathfrak{q}}) \overline{C} (\pi'^{-1}_{\mathfrak{q}} \pi_{\mathfrak{q}})^{-1}.$$

We shall develop this aspect in the next Section.

4. SOLVABLE SINGULAR QUADRATIC LIE ALGEBRAS AND DOUBLE EXTENSIONS

4.1. Double extensions are a very effective method initiated by V. Kac to construct quadratic Lie algebras (see [Kac85, FS87, MR85]). Here, we only need a particular case that we shall recall:

Definition 4.1.

- (1) Let $(\mathfrak{q}, B_{\mathfrak{q}})$ be a quadratic vector space and $\overline{C} : \mathfrak{q} \rightarrow \mathfrak{q}$ be a skew-symmetric map. Let $(\mathfrak{t} = \text{span}\{X_1, Y_1\}, B_{\mathfrak{t}})$ be a 2-dimensional quadratic vector space with $B_{\mathfrak{t}}$ defined by

$$B_{\mathfrak{t}}(X_1, X_1) = B_{\mathfrak{t}}(Y_1, Y_1) = 0, \quad B_{\mathfrak{t}}(X_1, Y_1) = 1.$$

Consider

$$\mathfrak{g} = \mathfrak{q} \overset{\perp}{\oplus} \mathfrak{t}$$

equipped with a bilinear form $B := B_{\mathfrak{q}} + B_{\mathfrak{t}}$ and define a bracket on \mathfrak{g} by

$$[X + \lambda X_1 + \mu Y_1, Y + \lambda' X_1 + \mu' Y_1] := \mu \overline{C}(Y) - \mu' \overline{C}(X) + B(\overline{C}(X), Y) X_1,$$

for all $X, Y \in \mathfrak{q}, \lambda, \mu, \lambda', \mu' \in \mathbb{C}$. Then (\mathfrak{g}, B) is a quadratic solvable Lie algebra. We say that \mathfrak{g} is the *double extension* of \mathfrak{q} by \overline{C} .

- (2) Let \mathfrak{g}_i be double extensions of quadratic vector spaces (\mathfrak{q}_i, B_i) by skew-symmetric maps $\overline{C}_i \in \mathcal{L}(\mathfrak{q}_i)$, for $1 \leq i \leq k$. The *amalgamated product*

$$\mathfrak{g} = \mathfrak{g}_1 \underset{\mathfrak{a}}{\times} \mathfrak{g}_2 \underset{\mathfrak{a}}{\times} \dots \underset{\mathfrak{a}}{\times} \mathfrak{g}_k$$

is defined as follows:

- consider (\mathfrak{q}, B) be the quadratic vector space with $\mathfrak{q} = \mathfrak{q}_1 \oplus \mathfrak{q}_2 \oplus \dots \oplus \mathfrak{q}_k$ and the bilinear form B such that $B(\sum_{i=1}^k X_i, \sum_{i=1}^k Y_i) = \sum_{i=1}^k B_i(X_i, Y_i)$, for $X_i, Y_i \in \mathfrak{q}_i, 1 \leq i \leq k$.
- the skew-symmetric map $\overline{C} \in \mathcal{L}(\mathfrak{q})$ is defined by $\overline{C}(\sum_{i=1}^k X_i) = \sum_{i=1}^k \overline{C}_i(X_i)$, for $X_i \in \mathfrak{q}_i, 1 \leq i \leq k$.

Then \mathfrak{g} is the double extension of \mathfrak{q} by \overline{C} .

In this Section, we will show that double extensions are highly related to singular quadratic Lie algebras. Amalgamated products will be used in Sections 5 and 6 to **decompose** double extensions.

We notice that if $\mathfrak{g}_1 \stackrel{i}{\simeq} \mathfrak{g}'_1$ and $\mathfrak{g}_2 \stackrel{i}{\simeq} \mathfrak{g}'_2$, it may happen that $\mathfrak{g}_1 \times_a \mathfrak{g}_2$ and $\mathfrak{g}'_1 \times_a \mathfrak{g}'_2$ are not even isomorphic. So, amalgamated products have a bad behavior with respect to i -isomorphisms. An example will be given in Section 5, Remark 5.12.

Lemma 4.2. *We keep the notation above.*

(1) *Let \mathfrak{g} be the double extension of \mathfrak{q} by \overline{C} . Then*

$$[X, Y] = B(X_1, X)C(Y) - B(X_1, Y)C(X) + B(C(X), Y)X_1, \forall X, Y \in \mathfrak{g},$$

where $C = \text{ad}(Y_1)$. Moreover, $X_1 \in \mathcal{Z}(\mathfrak{g})$ and $C|_{\mathfrak{q}} = \overline{C}$.

(2) *Let \mathfrak{g}' be the double extension of \mathfrak{q} by $\overline{C}' = \lambda \overline{C}$, $\lambda \in \mathbb{C}$, $\lambda \neq 0$. Then \mathfrak{g} and \mathfrak{g}' are i -isomorphic.*

Proof.

(1) This is a straightforward computation.

(2) Write $\mathfrak{g} = \mathfrak{q} \oplus^{\perp} \mathfrak{t} = \mathfrak{g}'$. Denote by $[\cdot, \cdot]'$ the Lie bracket on \mathfrak{g}' . Define $A : \mathfrak{g} \rightarrow \mathfrak{g}'$ by $A(X_1) = \lambda X_1$, $A(Y_1) = \frac{1}{\lambda} Y_1$ and $A|_{\mathfrak{q}} = \text{Id}_{\mathfrak{q}}$. Then $A([Y_1, X]) = C(X) = [A(Y_1), A(X)]'$ and $A([X, Y]) = [A(X), A(Y)]'$, for all $X, Y \in \mathfrak{q}$. So A is an i -isomorphism. □

4.2. A natural consequence of formulas in Lemma 4.2 and Proposition 3.4 (1) is given by the Proposition below:

Proposition 4.3.

(1) *Consider the notation in Section 3, Remark 3.7. Let \mathfrak{g} be quadratic Lie algebra of type S_1 (that is, $\text{dup}(\mathfrak{g}) = 1$). Then \mathfrak{g} is the double extension of $\mathfrak{q} = (\mathbb{C}X_0 \oplus \mathbb{C}Y_0)^{\perp}$ by $\overline{C} = \text{ad}(Y_0)|_{\mathfrak{q}}$.*

(2) *Let (\mathfrak{g}, B) be a quadratic Lie algebra. Let \mathfrak{g}' be a double extension of a quadratic vector space (\mathfrak{q}', B') by a map \overline{C}' . Let A be an i -isomorphism of \mathfrak{g}' onto \mathfrak{g} and write $\mathfrak{q} = A(\mathfrak{q}')$. Then \mathfrak{g} is a double extension of $(\mathfrak{q}, B|_{\mathfrak{q} \times \mathfrak{q}})$ by the map $\overline{C} = \overline{A} \overline{C}' \overline{A}^{-1}$ where $\overline{A} = A|_{\mathfrak{q}'}$.*

(3) *Let \mathfrak{g} be the double extension of a quadratic vector space \mathfrak{q} by a map $\overline{C} \neq 0$. Then \mathfrak{g} is a singular solvable quadratic Lie algebra. Moreover:*

- (a) \mathfrak{g} is of type S_3 if, and only if, $\text{rank}(\overline{C}) = 2$.
- (b) \mathfrak{g} is of type S_1 if, and only if, $\text{rank}(\overline{C}) \geq 4$.
- (c) \mathfrak{g} is reduced if, and only if, $\ker(\overline{C}) \subset \text{Im}(\overline{C})$.
- (d) \mathfrak{g} is nilpotent if, and only if, \overline{C} is nilpotent.

Proof.

- (1) Let $\mathfrak{b} = \mathbb{C}X_0 \oplus \mathbb{C}Y_0$. Then $B|_{\mathfrak{b} \times \mathfrak{b}}$ is non degenerate and $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{q}$. Since $\text{ad}(Y_0)(\mathfrak{b}) \subset \mathfrak{b}$ and $\text{ad}(Y_0)$ is skew-symmetric, we have $\text{ad}(Y_0)(\mathfrak{q}) \subset \mathfrak{q}$. By Proposition 3.4 (1), we have

$$[X, X'] = B(\overline{C}(X), X')X_0, \forall X, X' \in \mathfrak{q}.$$

Set $X_1 := X_0$ and $Y_1 := Y_0$ to obtain the result.

- (2) Write $\mathfrak{g}' = (\mathbb{C}X'_1 \oplus \mathbb{C}Y'_1) \oplus \mathfrak{q}'$. Let $X_1 = A(X'_1)$ and $Y_1 = A(Y'_1)$. Then $\mathfrak{g} = (\mathbb{C}X_1 \oplus \mathbb{C}Y_1) \oplus \mathfrak{q}$ and

$$[Y_1, X] = (A\overline{C}'A^{-1})(X), \forall X \in \mathfrak{q}, \text{ and}$$

$$[X, Y] = B((A\overline{C}'A^{-1})(X), Y)X'_1, \forall X, Y \in \mathfrak{q}.$$

and this proves the result.

- (3) Let $\mathfrak{g} = (\mathbb{C}X_1 \oplus \mathbb{C}Y_1) \oplus \mathfrak{q}$, $C = \text{ad}(Y_1)$, $\alpha = \phi(X_1)$, $\Omega(X, Y) = B(C(X), Y)$, for all $X, Y \in \mathfrak{g}$ and I be the 3-form associated to \mathfrak{g} . Then the formula for the Lie bracket in Lemma 4.2(1) can be translated as $I = \alpha \wedge \Omega$, hence $\text{dup}(\mathfrak{g}) \geq 1$ and \mathfrak{g} is singular.

Let W_Ω be the set $W_\Omega = \{\iota_X(\Omega), X \in \mathfrak{g}\}$. Then $W_\Omega = \phi(\text{Im}(\overline{C}))$. Therefore $\text{rank}(\overline{C}) \geq 2$ by Proposition 1.1 and Ω is decomposable if, and only if, $\text{rank}(\overline{C}) = 2$.

If $\text{rank}(\overline{C}) > 2$, then \mathfrak{g} is of type S_1 and by Proposition 3.5, we have $\text{rank}(\overline{C}) \geq 4$.

Finally, $\mathfrak{Z}(\mathfrak{g}) = \mathbb{C}X_1 \oplus \ker(\overline{C})$ and $[\mathfrak{g}, \mathfrak{g}] = \mathbb{C}X_1 \oplus \text{Im}(\overline{C})$, so \mathfrak{g} is reduced if, and only if, $\ker(\overline{C}) \subset \text{Im}(\overline{C})$.

The proof of the last claim is exactly the same as in Proposition 3.4 (4). \square

4.3. A complete classification (up to i -isomorphism) of quadratic Lie algebras of type S_3 is given in [PU07]. We shall recall the characterization of these algebras here:

Proposition 4.4. *Let \mathfrak{g} be a quadratic Lie algebra of type S_3 . Then \mathfrak{g} is i -isomorphic to an algebra $\mathfrak{l} \oplus \mathfrak{z}$ where \mathfrak{z} is a central ideal of \mathfrak{g} and \mathfrak{l} is one of the following algebras:*

- (1) $\mathfrak{g}_3(\lambda) = \mathfrak{o}(3)$ equipped with the bilinear form $B = \lambda \kappa$ where κ is the Killing form and $\lambda \in \mathbb{C}$, $\lambda \neq 0$.
- (2) \mathfrak{g}_4 , a 4-dimensional Lie algebra: consider $\mathfrak{q} = \mathbb{C}^2$, $\{E_1, E_2\}$ its canonical basis and the bilinear form B defined by $B(E_1, E_1) = B(E_2, E_2) = 0$ and $B(E_1, E_2) = 1$. Then \mathfrak{g}_4 is the double extension of \mathfrak{q} by the skew-symmetric map

$$\overline{C} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Moreover, \mathfrak{g}_4 is solvable, but it is not nilpotent.

- (3) \mathfrak{g}_5 , a 5-dimensional Lie algebra: consider $\mathfrak{q} = \mathbb{C}^3$, $\{E_1, E_2, E_3\}$ its canonical basis and the bilinear form B defined by $B(E_1, E_1) = B(E_2, E_2) = B(E_1, E_2) = B(E_2, E_3) = 0$ and $B(E_1, E_3) = B(E_2, E_2) = 1$. Then \mathfrak{g}_5 is the double extension of \mathfrak{q} by the skew-symmetric map

$$\bar{C} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Moreover, \mathfrak{g}_5 is nilpotent.

- (4) \mathfrak{g}_6 , a 6-dimensional Lie algebra: consider $\mathfrak{q} = \mathbb{C}^4$, $\{E_1, E_2, E_3, E_4\}$ its canonical basis and the bilinear form B defined by $B(E_1, E_3) = B(E_2, E_4) = 1$ and $B(E_i, E_j) = 0$ otherwise. Then \mathfrak{g}_6 is the double extension of \mathfrak{q} by the skew-symmetric map

$$\bar{C} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Moreover, \mathfrak{g}_6 is nilpotent.

All solvable quadratic Lie algebras of type S_3 are double extensions of a quadratic vector space by a skew-symmetric map.

4.4. Let (\mathfrak{q}, B) be a quadratic vector space. We recall that $O(\mathfrak{q})$ is the group of orthogonal maps and $\mathfrak{o}(\mathfrak{q})$ is its Lie algebra, i.e. the Lie algebra of skew-symmetric maps. Recall that the *adjoint action* is the action of $O(\mathfrak{q})$ on $\mathfrak{o}(\mathfrak{q})$ by conjugation.

Proposition 4.5. *Let (\mathfrak{q}, B) be a quadratic vector space. Let $\mathfrak{g} = (\mathbb{C}X_1 \oplus \mathbb{C}Y_1) \overset{\perp}{\oplus} \mathfrak{q}$ and $\mathfrak{g}' = (\mathbb{C}X'_1 \oplus \mathbb{C}Y'_1) \overset{\perp}{\oplus} \mathfrak{q}$ be double extensions of \mathfrak{q} , by skew-symmetric maps \bar{C} and \bar{C}' respectively. Then:*

- (1) *there exists a Lie algebra isomorphism between \mathfrak{g} and \mathfrak{g}' if, and only if, there exists an invertible map $P \in \mathcal{L}(\mathfrak{q})$ and a non-zero $\lambda \in \mathbb{C}$ such that $\bar{C}' = \lambda P \bar{C} P^{-1}$ and $P^* P \bar{C} = \bar{C}$, where P^* is the adjoint map of P with respect to B .*
- (2) *there exists an i -isomorphism between \mathfrak{g} and \mathfrak{g}' if, and only if, \bar{C}' is in the $O(\mathfrak{q})$ -adjoint orbit through $\lambda \bar{C}$ for some non-zero $\lambda \in \mathbb{C}$.*

Proof.

- (1) Let $A : \mathfrak{g} \rightarrow \mathfrak{g}'$ be a Lie algebra isomorphism. We know by Proposition 4.3 that \mathfrak{g} and \mathfrak{g}' are singular. Assume that \mathfrak{g} is of type S_3 . Then $3 = \dim([\mathfrak{g}, \mathfrak{g}]) = \dim([\mathfrak{g}', \mathfrak{g}'])$. So \mathfrak{g}' is also of type S_3 ([PU07]). Therefore, \mathfrak{g} and \mathfrak{g}' are either both of type S_1 or both of type S_3 . Let us study these two cases.

- (i) First, assume that \mathfrak{g} and \mathfrak{g}' are both of type S_1 . We start by proving that $A(\mathbb{C}X_1 \oplus \mathfrak{q}) = \mathbb{C}X'_1 \oplus \mathfrak{q}$. If this is not the case, there is $X \in \mathfrak{q}$ such

that $A(X) = \beta X'_1 + \gamma Y'_1 + Y$ with $Y \in \mathfrak{q}$ and $\gamma \neq 0$. Then

$$[A(X), \mathbb{C}X'_1 \oplus \mathfrak{q}]' = \gamma \overline{C'}(\mathfrak{q}) + [Y, \mathfrak{q}]'.$$

Since \mathfrak{g}' is of type S_1 , we have $\text{rank}(\overline{C'}) \geq 4$ (see Proposition 4.3) and it follows that $\dim([A(X), \mathbb{C}X'_1 \oplus \mathfrak{q}]') \geq 4$. On the other hand, $[A(X), \mathbb{C}X'_1 \oplus \mathfrak{q}]'$ is contained in $A([X, \mathfrak{g}])$ and $\dim([X, \mathfrak{g}]) \leq 2$, so we obtain a contradiction.

Next, we prove that $A(X_1) \in \mathbb{C}X'_1$. Since $X_1 \in [\mathfrak{g}, \mathfrak{g}]$, then there exists $X, Y \in \mathfrak{q}$ such that $X_1 = [X, Y]$. Then $A(X_1) = [A(X), A(Y)]' \in [\mathbb{C}X'_1 \oplus \mathfrak{q}, \mathbb{C}X'_1 \oplus \mathfrak{q}]' = \mathbb{C}X'_1$. Hence $A(X_1) = \mu X'_1$ for some non-zero $\mu \in \mathbb{C}$.

Now, write $A|_{\mathfrak{q}} = P + \beta \otimes X'_1$ with $P : \mathfrak{q} \rightarrow \mathfrak{q}$ and $\beta \in \mathfrak{q}^*$. If $X \in \ker(P)$, then $A\left(X - \frac{1}{\mu}\beta(X)X_1\right) = 0$, so $X = 0$ and therefore, P is invertible.

For all $X, Y \in \mathfrak{q}$, we have $A([X, Y]) = \mu B(\overline{C'}(X), Y)X'_1$. Also,

$$\begin{aligned} A([X, Y]) &= [P(X) + \beta(X)X'_1, P(Y) + \beta(Y)X'_1]' \\ &= B(\overline{C'}P(X), P(Y))X'_1. \end{aligned}$$

So it results that $P^* \overline{C'} P = \mu \overline{C}$.

Moreover, $A([Y_1, X]) = P(C(X) + \beta(C(X))X'_1)$, for all $X \in \mathfrak{q}$. Let $A(Y_1) = \gamma Y'_1 + Y + \delta X'_1$, with $Y \in \mathfrak{q}$. Therefore

$$A([Y_1, X]) = \gamma \overline{C'}P(X) + B(\overline{C'}(Y), P(X))X'_1$$

and we conclude that $P \overline{C} P^{-1} = \gamma \overline{C'}$ and since $P^* \overline{C'} P = \mu \overline{C}$, then $P^* P \overline{C} = \gamma \mu \overline{C}$.

Set $Q = \frac{1}{(\mu\gamma)^{\frac{1}{2}}}P$. It follows that $Q \overline{C} Q^{-1} = \gamma \overline{C'}$ and $Q^* Q \overline{C} = \overline{C}$. This

finishes the proof in the case \mathfrak{g} and \mathfrak{g}' of type S_1 .

- (ii) We proceed to the case when \mathfrak{g} and \mathfrak{g}' of type S_3 : the proof is a straightforward case-by-case verification. By Proposition 2.4, we can assume that \mathfrak{g} and \mathfrak{g}' are reduced. Then $\dim(\mathfrak{q}) = 2, 3$ or 4 by Proposition 4.4.

Recall that \mathfrak{g} is nilpotent if, and only if, \overline{C} is nilpotent (see Proposition 4.3 (3)). The same is valid for \mathfrak{g}' .

If $\dim(\mathfrak{q}) = 2$, then \mathfrak{g} is not nilpotent, so \overline{C} is not nilpotent, $\text{Tr}(\overline{C}) = 0$ and \overline{C} must be semi-simple. Therefore we can find a basis $\{e_1, e_2\}$ of \mathfrak{q} such that $B(e_1, e_2) = 1$, $B(e_1, e_1) = B(e_2, e_2) = 0$ and the matrix of \overline{C} is $\begin{pmatrix} \mu & 0 \\ 0 & -\mu \end{pmatrix}$. The same holds for $\overline{C'}$: there exists a basis $\{e'_1, e'_2\}$ of \mathfrak{q} such that $B(e'_1, e'_2) = 1$ and $B(e'_1, e_1)' = B(e'_2, e'_2) = 0$ such that the matrix of $\overline{C'}$ is $\begin{pmatrix} \mu' & 0 \\ 0 & -\mu' \end{pmatrix}$. It results that $\overline{C'}$ and $\frac{\mu'}{\mu} \overline{C}$ are $\text{O}(\mathfrak{q})$ -conjugate and we are done.

If $\dim(\mathfrak{q}) = 3$ or 4 , then \mathfrak{g} and \mathfrak{g}' are nilpotent. We use the classification of nilpotent orbits given for instance in [CM93]: there is only one orbit in dimension 3 or 4, so \overline{C} and $\overline{C'}$ are conjugate by $\text{O}(\mathfrak{q})$.

This finishes the proof of the necessary condition. To prove the sufficiency, we replace \overline{C}' by $\lambda P \overline{C} P^{-1}$ to obtain $P^* \overline{C}' P = \lambda \overline{C}$. Then we define $A : \mathfrak{g} \rightarrow \mathfrak{g}'$ by $A(X_1) = \lambda X'_1$, $A(Y_1) = \frac{1}{\lambda} Y'_1$ and $A(X) = P(X)$, for all $X \in \mathfrak{q}$. By a direct computation, we have for all X and $Y \in \mathfrak{q}$:

$$A([X, Y]) = [A(X), A(Y)]' \quad \text{and} \quad A([Y_1, X]) = [A(Y_1), A(X)]',$$

so A is a Lie algebra isomorphism between \mathfrak{g} and \mathfrak{g}' .

- (2) If \mathfrak{g} and \mathfrak{g}' are i-isomorphic, then the isomorphism A in the proof of (1) is an isometry. Hence $P \in O(\mathfrak{q})$ and $P^* \overline{C}' P = \mu \overline{C}$ gives the result.

Conversely, define A as above (sufficiency of (1)). Then A is an isometry and it is easy to check that A is an i-isomorphism. □

Corollary 4.6. *Let (\mathfrak{g}, B) and (\mathfrak{g}', B') be double extensions of $(\mathfrak{q}, \overline{B})$ and $(\mathfrak{q}', \overline{B}')$ respectively, where $\overline{B} = B|_{\mathfrak{q} \times \mathfrak{q}}$ and $\overline{B}' = B'|_{\mathfrak{q}' \times \mathfrak{q}'}$. Write $\mathfrak{g} = (\mathbb{C}X_1 \oplus \mathbb{C}Y_1) \overset{\perp}{\oplus} \mathfrak{q}$ and $\mathfrak{g}' = (\mathbb{C}X'_1 \oplus \mathbb{C}Y'_1) \overset{\perp}{\oplus} \mathfrak{q}'$. Then:*

- (1) *there exists an i-isomorphism between \mathfrak{g} and \mathfrak{g}' if, and only if, there exists an isometry $\overline{A} : \mathfrak{q} \rightarrow \mathfrak{q}'$ such that $\overline{C}' = \lambda \overline{A} \overline{C} \overline{A}^{-1}$, for some non-zero $\lambda \in \mathbb{C}$.*
- (2) *there exists a Lie algebra isomorphism between \mathfrak{g} and \mathfrak{g}' if, and only if, there exist invertible maps $\overline{Q} : \mathfrak{q} \rightarrow \mathfrak{q}'$ and $\overline{P} \in \mathcal{L}(\mathfrak{q})$ such that*
 - (i) $\overline{C}' = \lambda \overline{Q} \overline{C} \overline{Q}^{-1}$ for some non-zero $\lambda \in \mathbb{C}$,
 - (ii) $\overline{P}^* \overline{P} \overline{C} = \overline{C}$ and
 - (iii) $\overline{Q} \overline{P}^{-1}$ is an isometry from \mathfrak{q} onto \mathfrak{q}' .

Proof.

- (1) We can assume that $\dim(\mathfrak{g}) = \dim(\mathfrak{g}')$. Define a map $F : \mathfrak{g}' \rightarrow \mathfrak{g}$ by $F(X'_1) = X_1$, $F(Y'_1) = Y_1$ and $\overline{F} = F|_{\mathfrak{q}'}$ is an isometry from \mathfrak{q}' onto \mathfrak{q} . Then define a new Lie bracket on \mathfrak{g} by

$$[X, Y]'' = F([F^{-1}(X), F^{-1}(Y)]'), \quad \forall X, Y \in \mathfrak{g}.$$

Denote by $(\mathfrak{g}'', [\cdot, \cdot]'')$ this new Lie algebra. So F is an i-isomorphism from \mathfrak{g}' onto \mathfrak{g}'' .

Moreover $\mathfrak{g}'' = (\mathbb{C}X_1 \oplus \mathbb{C}Y_1) \overset{\perp}{\oplus} \mathfrak{q}$ is the double extension of \mathfrak{q} by \overline{C}'' with $\overline{C}'' = \overline{F} \overline{C}' \overline{F}^{-1}$. Then \mathfrak{g} and \mathfrak{g}' are i-isomorphic if, and only if, \mathfrak{g} and \mathfrak{g}'' are i-isomorphic. Applying Proposition 4.5, this is the case if, and only if, there exists $\overline{A} \in O(\mathfrak{q})$ such that $\overline{C}'' = \lambda \overline{A} \overline{C} \overline{A}^{-1}$ for some non-zero complex λ . That implies

$$\overline{C}' = \lambda (\overline{F}^{-1} \overline{A}) \overline{C} (\overline{F}^{-1} \overline{A})^{-1}$$

and proves (1).

(2) We keep the notation in (1). We have that \mathfrak{g} and \mathfrak{g}' are isomorphic if, and only if, \mathfrak{g} and \mathfrak{g}'' are isomorphic. Applying Proposition 4.5, \mathfrak{g} and \mathfrak{g}'' are isomorphic if, and only if, there exists an invertible map $\bar{P} \in \mathcal{L}(\mathfrak{q})$ and a non-zero $\lambda \in \mathbb{C}$ such that $\bar{C}'' = \lambda \bar{P} \bar{C} \bar{P}^{-1}$ and $\bar{P}^* \bar{P} \bar{C} = \bar{C}$ and we conclude that $\bar{C}' = \lambda \bar{Q} \bar{C} \bar{Q}^{-1}$ with $\bar{Q} = \bar{F}^{-1} \bar{P}$. Finally, $\bar{F}^{-1} = \bar{Q} \bar{P}^{-1}$ is an isometry from \mathfrak{q} to \mathfrak{q}' .

On the other hand, if $\bar{C}' = \lambda \bar{Q} \bar{C} \bar{Q}^{-1}$ and $\bar{P}^* \bar{P} \bar{C} = \bar{C}$ with $\bar{P} = \bar{F} \bar{Q}$ for some isometry $\bar{F} : \mathfrak{q}' \rightarrow \mathfrak{q}$, then construct \mathfrak{g}'' as in (1). We deduce $\bar{C}'' = \lambda \bar{P} \bar{C} \bar{P}^{-1}$ and $\bar{P}^* \bar{P} \bar{C} = \bar{C}$. So, by Proposition 4.5, \mathfrak{g} and \mathfrak{g}'' are isomorphic and therefore, \mathfrak{g} and \mathfrak{g}' are isomorphic. □

Remark 4.7. Let \mathfrak{g} be a solvable singular quadratic Lie algebra. Consider \mathfrak{g} as a double extension of two quadratic vectors spaces \mathfrak{q} and \mathfrak{q}' :

$$\mathfrak{g} = (\mathbb{C}X_1 \oplus \mathbb{C}Y_1) \overset{\perp}{\oplus} \mathfrak{q} \quad \text{and} \quad \mathfrak{g} = (\mathbb{C}X'_1 \oplus \mathbb{C}Y'_1) \overset{\perp}{\oplus} \mathfrak{q}'.$$

Let $\bar{C} = \text{ad}(Y_1)|_{\mathfrak{q}}$ and $\bar{C}' = \text{ad}(Y'_1)|_{\mathfrak{q}'}$. Since $\text{Id}_{\mathfrak{g}}$ is obviously an i-isomorphism, there exists an isometry $\bar{A} : \mathfrak{q} \rightarrow \mathfrak{q}'$ and a non-zero $\lambda \in \mathbb{C}$ such that

$$\bar{C}' = \lambda \bar{A} \bar{C} \bar{A}^{-1}.$$

Remark 4.8. A weak form of Corollary 4.6 (1) was stated in [FS87], in the case of i-isomorphisms satisfying some (dispensable) conditions. So (1) is an improvement. To our knowledge, (2) is completely new. Corollary 4.6 and Remark 4.7 can be applied directly to solvable singular Lie algebras: by Propositions 4.3 and 4.4, they are double extensions of quadratic vector spaces by skew-symmetric maps.

4.5. We shall now classify solvable singular Lie algebra structures on \mathbb{C}^{n+2} up to i-isomorphism in terms of $O(n)$ -orbits in $\mathbb{P}^1(\mathfrak{o}(n))$. We need the Lemma below:

Lemma 4.9. *Let V be a quadratic vector space such that $V = (\mathbb{C}X_1 \oplus \mathbb{C}Y_1) \overset{\perp}{\oplus} \mathfrak{q}'$ with X_1, Y_1 isotropic elements and $B(X_1, Y_1) = 1$. Let \mathfrak{g} be a solvable singular quadratic Lie algebra with $\dim(\mathfrak{g}) = \dim(V)$. Then, there exists a skew-symmetric map $\bar{C}' : \mathfrak{q}' \rightarrow \mathfrak{q}'$ such that V considered as the the double extension of \mathfrak{q}' by \bar{C}' is i-isomorphic to \mathfrak{g} .*

Proof. By Propositions 4.3 and 4.4, \mathfrak{g} is a double extension. Let us write $\mathfrak{g} = (\mathbb{C}X_0 \oplus \mathbb{C}Y_0) \overset{\perp}{\oplus} \mathfrak{q}$ and $\bar{C} = \text{ad}(Y_0)|_{\mathfrak{q}}$. Define $A : \mathfrak{g} \rightarrow V$ by $A(X_0) = X_1, A(Y_0) = Y_1$ and $\bar{A} = A|_{\mathfrak{q}}$ any isometry from $\mathfrak{q} \rightarrow \mathfrak{q}'$. It is clear that A is an isometry from \mathfrak{g} to V . Now, define the Lie bracket on V by:

$$[X, Y] = A([A^{-1}(X), A^{-1}(Y)]), \quad \forall X, Y \in V.$$

Then V is a quadratic Lie algebra, that is i-isomorphic to \mathfrak{g} , by definition. Moreover, V is obviously a double extension of \mathfrak{q}' by $\bar{C}' = \bar{A} \bar{C} \bar{A}^{-1}$. □

We can now apply our results to the classification (up to i-isomorphism) of solvable elements of $\mathcal{S}(n+2)$ (the set of singular Lie algebras structures on \mathbb{C}^{n+2}), for $n \geq 2$. We denote by $\mathcal{S}_s(n+2)$ the set of solvable elements of $\mathcal{S}(n+2)$. Given $\mathfrak{g} \in \mathcal{S}(n+2)$, we denote by $[\mathfrak{g}]_i$ its i-isomorphism class and by $\widehat{\mathcal{S}}_s^i(n+2)$ the set of classes. For $[\overline{C}] \in \mathbb{P}^1(\mathfrak{o}(n))$, we denote by $\mathcal{O}_{[\overline{C}]}$ its $O(n)$ -adjoint orbit and by $\widetilde{\mathbb{P}^1(\mathfrak{o}(n))}$ the set of orbits.

Proposition 4.10. *There exists a bijection $\theta : \widetilde{\mathbb{P}^1(\mathfrak{o}(n))} \rightarrow \widehat{\mathcal{S}}_s^i(n+2)$.*

Proof. We consider $\mathcal{O}_{[\overline{C}]} \in \widetilde{\mathbb{P}^1(\mathfrak{o}(n))}$. There is a double extension \mathfrak{g} of $\mathfrak{q} = \text{span}\{E_2, \dots, E_{n+1}\}$ by \overline{C} realized on $\mathbb{C}^{n+2} = (\mathbb{C}E_1 \oplus \mathbb{C}E_{n+2}) \oplus^\perp \mathfrak{q}$. Then, by Corollary 4.6, $\mathfrak{g} \in \mathcal{S}_s(n+2)$ and $[\mathfrak{g}]_i$ does not depend on the choice of \overline{C} . We define $\theta(\mathcal{O}_{[\overline{C}]}) = [\mathfrak{g}]_i$. If $\mathfrak{g}' \in \mathcal{S}_s(n+2)$ then by Lemma 4.9, \mathfrak{g}' can be realized (up to i-isomorphism) as a double extension on $\mathbb{C}^{n+2} = (\mathbb{C}E_1 \oplus \mathbb{C}E_{n+2}) \oplus^\perp \mathfrak{q}$. So θ is onto. Finally, θ is one-to-one by Corollary 4.6. \square

5. NILPOTENT AND DIAGONALIZABLE CASES

5.1. Let us denote by $\mathcal{N}(n+2)$ the set of nilpotent elements of $\mathcal{S}(n+2)$, for $n \geq 1$. Given $\mathfrak{g} \in \mathcal{N}(n+2)$, we denote by $[\mathfrak{g}]$ its isomorphism class and by $[\mathfrak{g}]_i$ its i-isomorphism class. The set $\widehat{\mathcal{N}}(n+2)$ is the set of all isomorphism classes and $\widehat{\mathcal{N}}^i(n+2)$ is the set of all i-isomorphism classes of elements in $\mathcal{N}(n+2)$.

Let $\mathcal{N}(n)$ be the set of non-zero nilpotent elements of $\mathfrak{o}(n)$. Given $\overline{C} \in \mathcal{N}(n)$, we denote by $\mathcal{O}_{\overline{C}}$ its $O(n)$ -adjoint orbit. The set of nilpotent orbits is denoted by $\widetilde{\mathcal{N}}(n)$.

Lemma 5.1. *Let \overline{C} and $\overline{C}' \in \mathcal{N}(n)$. Then \overline{C} is conjugate to $\lambda\overline{C}'$ modulo $O(n)$ for some non-zero $\lambda \in \mathbb{C}$ if, and only if, \overline{C} is conjugate to \overline{C}' .*

Proof. It is enough to show that \overline{C} and $\lambda\overline{C}$ are conjugate, for any non-zero $\lambda \in \mathbb{C}$. By [CM93], there exists a $\mathfrak{sl}(2)$ -triple $\{X, H, \overline{C}\}$ in $\mathfrak{o}(n)$ such that $[H, \overline{C}] = 2\overline{C}$, so $e^{t \text{ad}(H)}(\overline{C}) = e^{2t}\overline{C}$, $\forall t \in \mathbb{C}$. We choose t such that $e^{2t} = \lambda$, then $e^{tH}\overline{C}e^{-tH} = \lambda\overline{C}$ and $e^{tH} \in O(n)$. \square

Proposition 5.2. *One has:*

- (1) *Let \mathfrak{g} and $\mathfrak{g}' \in \mathcal{N}(n+2)$. Then \mathfrak{g} and \mathfrak{g}' are isomorphic if, and only if, they are i-isomorphic, so $[\mathfrak{g}]_i = [\mathfrak{g}]$ and $\widehat{\mathcal{N}}^i(n+2) = \widehat{\mathcal{N}}(n+2)$.*
- (2) *There is a bijection $\tau : \widetilde{\mathcal{N}}(n) \rightarrow \widehat{\mathcal{N}}(n+2)$.*
- (3) *$\widehat{\mathcal{N}}(n+2)$ is finite.*

Proof.

- (1) Using Lemma 4.9, Proposition 4.3(3) and Corollary 4.6, it is enough to show that for \overline{C} and $\overline{C}' \in \mathcal{N}(n+2)$, if there exists $P \in GL(n)$ such that $\overline{C}' = \lambda P\overline{C}P^{-1}$, for some non-zero $\lambda \in \mathbb{C}$, then \overline{C} and \overline{C}' are conjugate under

$\mathcal{O}(n)$. By Lemma 5.1, we can assume that $\lambda = 1$, and then the result is well known (see e.g. [CM93]).

- (2) As in the proof of Proposition 4.10, for a given $\mathcal{O}_{\bar{C}} \in \widetilde{\mathcal{N}}(n)$, we construct the double extension \mathfrak{g} of $\mathfrak{q} = \text{span}\{E_2, \dots, E_{n+1}\}$ by \bar{C} realized on \mathbb{C}^{n+2} . Then, by Proposition 4.3 (3), $\mathfrak{g} \in \mathcal{N}(n+2)$ and $[\mathfrak{g}]$ does not depend on the choice of \bar{C} . We define $\tau(\mathcal{O}_{\bar{C}}) = [\mathfrak{g}]$. Then by (1) and Corollary 4.6, τ is one-to-one and onto.
- (3) $\widehat{\mathcal{N}}(n+2)$ is finite since the set of nilpotent orbits $\widetilde{\mathcal{N}}(n)$ is finite (see e.g. [CM93]).

□

Definition 5.3. Let $p \in \mathbb{N} \setminus \{0\}$. We denote the *Jordan block of size p* by $J_1 := (0)$ and for $p \geq 2$,

$$J_p := \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \dots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

Next, we define nilpotent Jordan-type Lie algebras. There are two types:

- for $p \geq 2$, we consider $\mathfrak{q} = \mathbb{C}^{2p}$ equipped with its canonical bilinear form \bar{B} and the map \bar{C}_{2p}^J with matrix

$$\begin{pmatrix} J_p & 0 \\ 0 & -{}^t J_p \end{pmatrix}$$

in the canonical basis. Then $\bar{C}_{2p}^J \in \mathfrak{o}(2p)$ and we denote by \mathfrak{j}_{2p} the double extension of \mathfrak{q} by \bar{C}_{2p}^J . So $\mathfrak{j}_{2p} \in \mathcal{N}(2p+2)$.

- for $p \geq 1$, we consider $\mathfrak{q} = \mathbb{C}^{2p+1}$ equipped with its canonical bilinear form \bar{B} and the map \bar{C}_{2p+1}^J with matrix

$$\begin{pmatrix} J_{p+1} & M \\ 0 & -{}^t J_p \end{pmatrix}$$

in the canonical basis, where $M = (m_{ij})$ denotes the $(p+1) \times p$ -matrix with $m_{p+1,p} = -1$ and $m_{ij} = 0$ otherwise. Then $\bar{C}_{2p+1}^J \in \mathfrak{o}(2p+1)$ and we denote by \mathfrak{j}_{2p+1} the double extension of \mathfrak{q} by \bar{C}_{2p+1}^J . So $\mathfrak{j}_{2p+1} \in \mathcal{N}(2p+3)$.

Lie algebras \mathfrak{j}_{2p} or \mathfrak{j}_{2p+1} will be called *nilpotent Jordan-type Lie algebras*.

Let $n \in \mathbb{N}$, $n \neq 0$. We consider partitions $[d] := (d_1, \dots, d_r)$ of n of a special type:

- each even d_i must occur with even multiplicity.
- $[d]$ can be written as $(p_1, p_1, p_2, p_2, \dots, p_k, p_k, 2q_1 + 1, \dots, 2q_\ell + 1)$ with all p_i even, $p_1 \geq p_2 \geq \dots \geq p_k$ and $q_1 \geq q_2 \geq \dots \geq q_\ell$.

We denote by $\mathcal{P}'(n)$ the set of partitions satisfying the above conditions. To every $[d] \in \mathcal{P}'(n)$, we associate a map $\overline{C}_{[d]} \in \mathfrak{o}(n)$: write $[d] = (p_1, p_1, p_2, p_2, \dots, p_k, p_k, 2q_1 + 1, \dots, 2q_\ell + 1)$. Then $\overline{C}_{[d]}$ is the map with matrix

$$\text{diag}_{\mathfrak{g}_{2k+\ell}}(\overline{C}_{2p_1}^J, \overline{C}_{2p_2}^J, \dots, \overline{C}_{2p_k}^J, \overline{C}_{2q_1+1}^J, \dots, \overline{C}_{2q_\ell+1}^J).$$

in the canonical basis of \mathbb{C}^n .

Moreover, we denote by $\mathfrak{g}_{[d]}$ the double extension of \mathbb{C}^n by $\overline{C}_{[d]}$. Then $\mathfrak{g}_{[d]} \in \mathcal{N}(n+2)$ and $\mathfrak{g}_{[d]}$ is an amalgamated product of nilpotent Jordan-type Lie algebras, more precisely,

$$\mathfrak{g}_{[d]} = \mathfrak{j}_{2p_1} \times_{\mathfrak{a}} \mathfrak{j}_{2p_2} \times_{\mathfrak{a}} \dots \times_{\mathfrak{a}} \mathfrak{j}_{2p_k} \times_{\mathfrak{a}} \mathfrak{j}_{2q_1+1} \times_{\mathfrak{a}} \dots \times_{\mathfrak{a}} \mathfrak{j}_{2q_\ell+1}.$$

The following fundamental result classifies all nilpotent $O(n)$ -orbits in $\mathfrak{o}(n)$ (see [CM93]).

Lemma 5.4. *The map $[d] \mapsto \overline{C}_{[d]}$ from $\mathcal{P}'(n)$ to $\mathfrak{o}(n)$ induces a bijection from $\mathcal{P}'(n)$ onto $\widehat{\mathcal{N}}(n)$.*

Using Propositions 5.2 and 5.4, we deduce:

Proposition 5.5.

- (1) *The map $[d] \mapsto \mathfrak{g}_{[d]}$ from $\mathcal{P}'(n)$ to $\mathcal{N}(n+2)$ induces a bijection from $\mathcal{P}'(n)$ onto $\widehat{\mathcal{N}}(n+2)$.*
- (2) *Each nilpotent singular $n+2$ -dimensional Lie algebra is i -isomorphic to a unique amalgamated product $\mathfrak{g}_{[d]}$, $[d] \in \mathcal{P}'(n)$ of nilpotent Jordan-type Lie algebras.*

5.2. We introduce some notation:

Definition 5.6. Let \mathfrak{g} be a solvable singular quadratic Lie algebra and write $\mathfrak{g} = (\mathbb{C}X_0 \oplus \mathbb{C}Y_0) \oplus_{\perp} \mathfrak{q}$ a decomposition of \mathfrak{g} as a double extension (Proposition 4.3 and Lemma 4.4). Let $\overline{C} = \text{ad}(Y_0)|_{\mathfrak{q}}$. We say that \mathfrak{g} is a *diagonalizable* if \overline{C} is diagonalizable.

We denote by $\mathcal{D}(n+2)$ the set of such structures on the quadratic space \mathbb{C}^{n+2} , by $\mathcal{D}_{\text{red}}(n+2)$ the reduced ones, by $\widehat{\mathcal{D}}(n+2)$, $\widehat{\mathcal{D}}^i(n+2)$, $\widehat{\mathcal{D}}_{\text{red}}(n+2)$, $\widehat{\mathcal{D}}_{\text{red}}^i(n+2)$ the corresponding sets of isomorphism and i -isomorphism classes of elements in $\mathcal{D}(n+2)$ and $\mathcal{D}_{\text{red}}(n+2)$.

Remark that the property of being diagonalizable does not depend on the chosen decomposition of \mathfrak{g} (see Remark 4.7). By Corollary 4.6 and a proof completely similar to Proposition 4.10 or Proposition 5.2, we conclude:

Proposition 5.7. *There is a bijection between $\widehat{\mathcal{D}}^i(n+2)$ and the set of semi-simple $O(n)$ -orbits in $\mathbb{P}^1(\mathfrak{o}(n))$. The same result holds for $\widehat{\mathcal{D}}_{\text{red}}^i(n+2)$ and semi-simple invertible orbits in $\mathbb{P}^1(\mathfrak{o}(n))$.*

Proof. Proceed exactly as in Proposition 4.10 or Proposition 5.2, but notice that a diagonalizable \overline{C} satisfies $\ker(\overline{C}) \subset \text{Im}(\overline{C})$ if, and only if, $\ker(\overline{C}) = \{0\}$. \square

5.3. The classification of semi-simple adjoint orbits of a semi-simple Lie algebra \mathfrak{g} is fully known (see e.g. [CM93]). Given a Cartan subalgebra \mathfrak{h} of \mathfrak{g} , there is a bijection between the set of semi-simple adjoint orbits and \mathfrak{h}/W , where W is the Weyl group.

Here, we deal with $O(n)$ -adjoint and not $SO(n)$ -adjoint orbits. Hence, slight changes must be done. Let us recall the result: write $n = 2p$ if n is even and $n = 2p + 1$ if n is odd. Let \mathfrak{h} be a Cartan subalgebra given by the vector space of diagonal matrices of type $\text{diag}_{2p}(\lambda_1, \dots, \lambda_p, \lambda_1, \dots, \lambda_p)$ if n is even and of type $\text{diag}_{2p+1}(\lambda_1, \dots, \lambda_p, 0, -\lambda_1, \dots, -\lambda_p)$ if n is odd. Any diagonalizable $\bar{C} \in \mathfrak{o}(n)$ is conjugate to an element of \mathfrak{h} (see Appendix 1 for a direct proof). If \bar{C} is invertible, then n is even (see Appendix 1).

If n is even, the Weyl group consists of all permutations and even sign changes of $(\lambda_1, \dots, \lambda_p)$. Thus, to describe $O(n)$ -orbits we must admit any number of sign changes. We denote by G_p the corresponding group. If n is odd, the Weyl group is G_p and there is nothing to add.

However, we are interested in $O(n)$ -orbits in $\mathbb{P}^1(\mathfrak{o}(n))$. So, we must add maps $(\lambda_1, \dots, \lambda_p) \mapsto \lambda(\lambda_1, \dots, \lambda_p)$, $\forall \lambda \in \mathbb{C}$, $\lambda \neq 0$ to the group G_p . We obtain a group denoted by H_p . Now, let $\Lambda_p = \{(\lambda_1, \dots, \lambda_p) \mid \lambda_1, \dots, \lambda_p \in \mathbb{C}, \lambda_i \neq 0 \text{ for some } i\}$ and $\Lambda_p^+ = \{(\lambda_1, \dots, \lambda_p) \mid \lambda_1, \dots, \lambda_p \in \mathbb{C}, \lambda_i \neq 0, \forall i\}$.

By Proposition 5.7, we obtain the Corollary:

Corollary 5.8. *There is a bijection between $\widehat{\mathcal{D}}^i(n+2)$ and Λ_p/H_p . Moreover, if $n = 2p + 1$, $\widehat{\mathcal{D}}_{\text{red}}(n+2) = \emptyset$ and if $n = 2p$, then $\widehat{\mathcal{D}}_{\text{red}}(2p+2)$ is in bijection with Λ_p^+/H_p .*

5.4. To go further in the study of diagonalizable reduced case, we need the following Lemma that will also be used in Section 6:

Lemma 5.9.

Let \mathfrak{g}' and \mathfrak{g}'' be solvable singular quadratic Lie algebras, $\mathfrak{g}' = (\mathbb{C}X'_1 \oplus \mathbb{C}Y'_1) \overset{\perp}{\oplus} \mathfrak{q}'$ a decomposition of \mathfrak{g}' as a double extension and $\bar{C}' = \text{ad}(Y'_1)|_{\mathfrak{q}'}$. We assume that \bar{C}' is invertible. Then \mathfrak{g}' and \mathfrak{g}'' are isomorphic if, and only if, they are i-isomorphic.

Proof. Write $\mathfrak{g}'' = (\mathbb{C}X''_1 \oplus \mathbb{C}Y''_1) \overset{\perp}{\oplus} \mathfrak{q}''$ a decomposition of \mathfrak{g}'' as a double extension and $\bar{C}'' = \text{ad}(Y''_1)|_{\mathfrak{q}''}$.

Assume that \mathfrak{g}' and \mathfrak{g}'' are isomorphic. By Corollary 4.6, there exist $\bar{Q}: \mathfrak{q}' \rightarrow \mathfrak{q}''$ and $\bar{P} \in \mathcal{L}(\mathfrak{q}')$ such that $\bar{Q}\bar{P}^{-1}$ is an isometry, $\bar{P}^* \bar{P} \bar{C}' = \bar{C}''$ and $\bar{C}'' = \lambda \bar{Q} \bar{C}' \bar{Q}^{-1}$ for some non-zero $\lambda \in \mathbb{C}$. But \bar{C}' is invertible, so $\bar{P}^* \bar{P} = \text{Id}_{\mathfrak{q}'}$. Therefore, \bar{P} is an isometry of \mathfrak{q}' and then \bar{Q} is an isometry from \mathfrak{q}' to \mathfrak{q}'' . The conditions of Corollary 4.6 (1) are satisfied, so \mathfrak{g}' and \mathfrak{g}'' are i-isomorphic. \square

Corollary 5.10. *One has:*

$$\widehat{\mathcal{D}}_{\text{red}}(2p+2) = \widehat{\mathcal{D}}_{\text{red}}^i(2p+2), \forall p \geq 1.$$

Next, we describe diagonalizable reduced singular Lie algebras using amalgamated products. First, let $\mathfrak{g}_4(\lambda)$ be the double extension of $\mathfrak{q} = \mathbb{C}^2$ by $\overline{C} = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}$. By Lemma 4.2, $\mathfrak{g}_4(\lambda)$ is i -isomorphic to $\mathfrak{g}_4(1)$, call it \mathfrak{g}_4 .

Proposition 5.11. *Let (\mathfrak{g}, B) be a diagonalizable reduced singular Lie algebra. Then \mathfrak{g} is an amalgamated product of singular Lie algebras all i -isomorphic to \mathfrak{g}_4 .*

Proof. We write $\mathfrak{g} = (\mathbb{C}X_0 \oplus \mathbb{C}Y_0) \overset{\perp}{\oplus} \mathfrak{q}$, $C = \text{ad}(Y_0)$, $\overline{C} = C|_{\mathfrak{q}}$ and $\overline{B} = B_{\mathfrak{q} \times \mathfrak{q}}$. Then \overline{C} is a diagonalizable invertible element of $\mathfrak{o}(\mathfrak{q}, \overline{B})$. Apply Appendix 1 to obtain a basis $\{e_1, \dots, e_p, f_1, \dots, f_p\}$ of \mathfrak{q} and $\lambda_1, \dots, \lambda_p \in \mathbb{C}$, all non-zero, such that $B(e_i, e_j) = B(f_i, f_j) = 0$, $B(e_i, f_j) = \delta_{ij}$ and $\overline{C}(e_i) = \lambda_i e_i$, $\overline{C}(f_i) = -\lambda_i f_i$, for all $1 \leq i, j \leq p$. Let $\mathfrak{q}_i = \text{span}\{e_i, f_i\}$, $1 \leq i \leq p$. Then

$$\mathfrak{q} = \overset{\perp}{\oplus}_{i=1}^p \mathfrak{q}_i.$$

Furthermore, $\mathfrak{h}_i = (\mathbb{C}X_0 \oplus \mathbb{C}Y_0) \overset{\perp}{\oplus} \mathfrak{q}_i$ is a Lie subalgebra of \mathfrak{g} for all $1 \leq i \leq p$ and

$$\mathfrak{g} = \mathfrak{h}_1 \underset{a}{\times} \mathfrak{h}_2 \underset{a}{\times} \dots \underset{a}{\times} \mathfrak{h}_p \text{ with } \mathfrak{h}_i \overset{i}{\simeq} \mathfrak{g}_4(\lambda_i) \overset{i}{\simeq} \mathfrak{g}_4.$$

□

Remark 5.12. For non-zero $\lambda, \mu \in \mathbb{C}$, consider the amalgamated product:

$$\mathfrak{g}(\lambda, \mu) = \mathfrak{g}_4(\lambda) \underset{a}{\times} \mathfrak{g}_4(\mu).$$

Then $\mathfrak{g}(\lambda, \mu)$ is the double extension of \mathbb{C}^4 by

$$\begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \mu & 0 & 0 \\ 0 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & -\mu \end{pmatrix}.$$

Therefore $\mathfrak{g}(\lambda, \mu)$ is isomorphic to $\mathfrak{g}(1, 1)$ if, and only if, $\mu = \pm\lambda$ (Lemma 5.9 and (5.3)). So, though $\mathfrak{g}_4(\lambda)$ and $\mathfrak{g}_4(\mu)$ are i -isomorphic to \mathfrak{g}_4 , the amalgamated product $\mathfrak{g}(\lambda, \mu)$ is not even isomorphic to $\mathfrak{g}(1, 1) = \mathfrak{g}_4 \underset{a}{\times} \mathfrak{g}_4$ if $\mu \neq \pm\lambda$. This illustrates that amalgamated products may have a rather bad behavior with respect to isomorphisms.

6. THE GENERAL CASE

6.1. Let \mathfrak{g} be a solvable singular quadratic Lie algebra. We fix a realization of \mathfrak{g} as a double extension, $\mathfrak{g} = (\mathbb{C}X_0 \oplus \mathbb{C}Y_0) \overset{\perp}{\oplus} \mathfrak{q}$ (Propositions 4.3 and 4.5). Let $C = \text{ad}(Y_0)$ and $\overline{C} = C|_{\mathfrak{q}}$. We consider the Fitting decomposition of \overline{C} :

$$\mathfrak{q} = \mathfrak{q}_N \oplus \mathfrak{q}_I,$$

where \mathfrak{q}_N and \mathfrak{q}_I are \overline{C} -stable, $\overline{C}_N = \overline{C}|_{\mathfrak{q}_N}$ is nilpotent and $\overline{C}_I = \overline{C}|_{\mathfrak{q}_I}$ is invertible.

Since \bar{C} is skew-symmetric, one has $\mathfrak{q}_I = \mathfrak{q}_N^\perp$. Therefore, the restrictions $\bar{B}_N = \bar{B}|_{\mathfrak{q}_N \times \mathfrak{q}_N}$ and $\bar{B}_I = \bar{B}|_{\mathfrak{q}_I \times \mathfrak{q}_I}$ of $\bar{B} = B|_{\mathfrak{q} \times \mathfrak{q}}$ are non degenerate, \bar{C}_N and \bar{C}_I are skew-symmetric and $[\mathfrak{q}_I, \mathfrak{q}_N] = 0$. Let $\mathfrak{g}_N = (\mathbb{C}X_0 \oplus \mathbb{C}Y_0) \oplus^\perp \mathfrak{q}_N$ and $\mathfrak{g}_I = (\mathbb{C}X_0 \oplus \mathbb{C}Y_0) \oplus^\perp \mathfrak{q}_I$. Then \mathfrak{g}_N and \mathfrak{g}_I are Lie subalgebras of \mathfrak{g} , \mathfrak{g}_N is the double extension of \mathfrak{q}_N by \bar{C}_N , \mathfrak{g}_I is the double extension of \mathfrak{q}_I by \bar{C}_I and \mathfrak{g} is a nilpotent singular quadratic Lie algebra. To study \mathfrak{g}_I , we introduce the following definition:

Definition 6.1. A double extension is called an *invertible quadratic Lie algebra* if the corresponding skew-symmetric map is invertible.

Remark 6.2.

- By Remark 4.7, the property of being an invertible quadratic Lie algebra does not depend on the chosen decomposition.
- By Appendix 1, the dimension of an invertible quadratic Lie algebra is even.
- By Lemma 5.9, two invertible quadratic Lie algebras are isomorphic if, and only if, they are i-isomorphic.

With the above definition, \mathfrak{g}_I is an invertible quadratic Lie algebra and we have

$$\mathfrak{g} = \mathfrak{g}_N \times_{\mathfrak{a}} \mathfrak{g}_I.$$

Definition 6.3. The Lie subalgebras \mathfrak{g}_N and \mathfrak{g}_I are respectively the *nilpotent* and *invertible Fitting components* of \mathfrak{g} .

This definition is justified by:

Proposition 6.4. Let \mathfrak{g} and \mathfrak{g}' be solvable singular quadratic Lie algebras and $\mathfrak{g}_N, \mathfrak{g}_I, \mathfrak{g}'_N, \mathfrak{g}'_I$ be their Fitting components. Then

- (1) $\mathfrak{g} \stackrel{i}{\simeq} \mathfrak{g}'$ if, and only if, $\mathfrak{g}_N \stackrel{i}{\simeq} \mathfrak{g}'_N$ and $\mathfrak{g}_I \stackrel{i}{\simeq} \mathfrak{g}'_I$. The result remains valid if we replace $\stackrel{i}{\simeq}$ by \simeq .
- (2) $\mathfrak{g} \simeq \mathfrak{g}'$ if, and only of, $\mathfrak{g} \stackrel{i}{\simeq} \mathfrak{g}'$.

Proof. We assume that $\mathfrak{g} \simeq \mathfrak{g}'$. Then by Corollary 4.6, there exists an invertible $\bar{P} : \mathfrak{q} \rightarrow \mathfrak{q}'$ and a non-zero $\lambda \in \mathbb{C}$ such that $\bar{C}' = \lambda \bar{P} \bar{C} \bar{P}^{-1}$, so $\mathfrak{q}'_N = \bar{P}(\mathfrak{q}_N)$ and $\mathfrak{q}'_I = \bar{P}(\mathfrak{q}_I)$, then $\dim(\mathfrak{q}'_N) = \dim(\mathfrak{q}_N)$ and $\dim(\mathfrak{q}'_I) = \dim(\mathfrak{q}_I)$. Thus, there exist isometries $F_N : \mathfrak{q}'_N \rightarrow \mathfrak{q}_N$ and $F_I : \mathfrak{q}'_I \rightarrow \mathfrak{q}_I$ and we can define an isometry $\bar{F} : \mathfrak{q}' \rightarrow \mathfrak{q}$ by $\bar{F}(X'_N + X'_I) = F_N(X'_N) + F_I(X'_I), \forall X'_N \in \mathfrak{q}'_N$ and $X'_I \in \mathfrak{q}'_I$. We now define $F : \mathfrak{g}' \rightarrow \mathfrak{g}$ by $F(X'_I) = X_I, F(Y'_I) = Y_I, F|_{\mathfrak{q}'} = \bar{F}$ and a new Lie bracket on \mathfrak{g} :

$$[X, Y]'' = F([F^{-1}(X), F^{-1}(Y)]'), \forall X, Y \in \mathfrak{g}.$$

Call \mathfrak{g}'' this new quadratic Lie algebra. We have $\mathfrak{g}'' = (\mathbb{C}X_1 \oplus \mathbb{C}Y_1) \oplus^\perp \mathfrak{q}$, i.e., $\mathfrak{q}'' = \mathfrak{q}$ and $\bar{C}'' = \bar{F} \bar{C}' \bar{F}^{-1}$. So $\mathfrak{q}''_N = F(\mathfrak{q}'_N) = \mathfrak{q}_N$ and $\mathfrak{q}''_I = F(\mathfrak{q}'_I) = \mathfrak{q}_I$. But $\mathfrak{g} \simeq \mathfrak{g}''$, so there exists an invertible $Q : \mathfrak{q} \rightarrow \mathfrak{q}$ such that $\bar{C}'' = \lambda \bar{Q} \bar{C} \bar{Q}^{-1}$ for some non-zero $\lambda \in \mathbb{C}$ (Corollary 4.6). It follows that $\mathfrak{q}''_N = Q(\mathfrak{q}_N)$ and $\mathfrak{q}_I = Q(\mathfrak{q}_I)$, so $Q(\mathfrak{q}_N) = \mathfrak{q}_N$ and $Q(\mathfrak{q}_I) = \mathfrak{q}_I$.

Moreover, we have $Q^* Q \bar{C} = \bar{C}$ (Corollary 4.6), so $Q^* Q \bar{C}^k = \bar{C}^k$ for all k . There exists k such that $\mathfrak{q}_I = \text{Im}(C^k)$ and $(Q^* Q \bar{C}^k)(X) = \bar{C}^k(X)$, for all $X \in \mathfrak{g}$. So $Q^* Q|_{\mathfrak{q}_I} = \text{Id}_{\mathfrak{q}_I}$ and $Q_I = Q|_{\mathfrak{q}_I}$ is an isometry. Since $\bar{C}'_I = \lambda \bar{Q}_I \bar{C}_I \bar{Q}_I^{-1}$, then $\mathfrak{g}_I \stackrel{i}{\simeq} \mathfrak{g}''_I$ (Corollary 4.6).

Let $Q_N = Q|_{\mathfrak{q}_N}$. Then $\bar{C}'_N = \lambda \bar{Q}_N \bar{C}_N \bar{Q}_N^{-1}$ and $Q_N^* Q_N \bar{C}_N = \bar{C}_N$, so by Corollary 4.6, $\mathfrak{g}_N \simeq \mathfrak{g}''_N$. Since \mathfrak{g}_N and \mathfrak{g}''_N are nilpotent, then $\mathfrak{g}''_N \stackrel{i}{\simeq} \mathfrak{g}_N$ by Proposition 5.2.

Conversely, assume that $\mathfrak{g}_N \simeq \mathfrak{g}'_N$ and $\mathfrak{g}_I \simeq \mathfrak{g}'_I$. Then $\mathfrak{g}_N \stackrel{i}{\simeq} \mathfrak{g}'_N$ and $\mathfrak{g}_I \stackrel{i}{\simeq} \mathfrak{g}'_I$ by Proposition 5.2 and Lemma 5.9.

So, there exist isometries $P_N : \mathfrak{g}_N \rightarrow \mathfrak{g}'_N$, $P_I : \mathfrak{g}_I \rightarrow \mathfrak{g}'_I$ and non-zero λ_N and $\lambda_I \in \mathbb{C}$ such that $\bar{C}'_N = \lambda_N \bar{P}_N \bar{C}_N \bar{P}_N^{-1}$ and $\bar{C}'_I = \lambda_I \bar{P}_I \bar{C}_I \bar{P}_I^{-1}$. By Lemma 5.1, since \mathfrak{g}_N and \mathfrak{g}'_N are nilpotent, we can assume that $\lambda_N = \lambda_I = \lambda$. Now we define $P : \mathfrak{q} \rightarrow \mathfrak{q}'$ by $P(X_N + X_I) = P_N(X_N) + P_I(X_I)$, $\forall X_N \in \mathfrak{q}_N, X_I \in \mathfrak{q}_I$, so P is an isometry. Moreover, since $\bar{C}(X_N + X_I) = \bar{C}_N(X_N) + \bar{C}_I(X_I)$, $\forall X_N \in \mathfrak{q}_N, X_I \in \mathfrak{q}_I$ and $\bar{C}'(X'_N + X'_I) = \bar{C}'_N(X'_N) + \bar{C}'_I(X'_I)$, for all $X'_N \in \mathfrak{q}_N, X'_I \in \mathfrak{q}_I$, we conclude $\bar{C}' = \lambda P \bar{C} P^{-1}$ and finally, $\mathfrak{g} \stackrel{i}{\simeq} \mathfrak{g}'$, by Corollary 4.6. \square

Remark 6.5. The class of solvable singular quadratic Lie algebras has the remarkable property that two Lie algebras in this class are isomorphic if, and only if, they are i -isomorphic. In addition, the Fitting components do not depend on the realizations of the Lie algebra as a double extension and they completely characterize the Lie algebra (up to isomorphism).

6.2. To classify all solvable singular Lie algebras (up to isomorphism), we have to classify nilpotent and invertible ones (see Proposition 6.4). The nilpotent case is completely achieved in Proposition 5.5, so we are left with the invertible case.

For $p \geq 1$ and $\lambda \in \mathbb{C}$, let $J_p(\lambda) = \text{diag}_p(\lambda, \dots, \lambda) + J_p$ and

$$\bar{C}'_{2p}(\lambda) = \begin{pmatrix} J_p(\lambda) & 0 \\ 0 & -{}^t J_p(\lambda) \end{pmatrix}.$$

Then $\bar{C}'_{2p}(\lambda) \in \mathfrak{o}(2p)$.

Definition 6.6. For $\lambda \in \mathbb{C}$, let $\mathfrak{j}_{2p}(\lambda)$ be the double extension of \mathbb{C}^{2p} by $\bar{C}'_{2p}(\lambda)$. We say that $\mathfrak{j}_{2p}(\lambda)$ is a *Jordan-type quadratic Lie algebra*.

When $\lambda = 0$ and $p \geq 2$, we recover the nilpotent Jordan-type Lie algebras \mathfrak{j}_{2p} from Definition 5.3.

When $\lambda \neq 0$, $\mathfrak{j}_{2p}(\lambda)$ is an invertible singular quadratic lie algebra and

$$\mathfrak{j}_{2p}(-\lambda) \simeq \mathfrak{j}_{2p}(\lambda).$$

Proposition 6.7. *Let \mathfrak{g} be a solvable singular quadratic Lie algebra. Then \mathfrak{g} is an invertible Lie algebra if, and only if, \mathfrak{g} is an amalgamated product of Lie algebras all i -isomorphic to Jordan-type Lie algebras $\mathfrak{j}_{2p}(\lambda)$, with $\lambda \neq 0$.*

Proof. Let $\mathfrak{g} = (\mathbb{C}X_0 \oplus \mathbb{C}Y_0) \overset{\perp}{\oplus} \mathfrak{q}$, B be the bilinear form of \mathfrak{g} , $\overline{B} = B|_{\mathfrak{q} \times \mathfrak{q}}$, $C = \text{ad}(Y_0)$ and $\overline{C} = C|_{\mathfrak{q}} \in \mathfrak{o}(\mathfrak{q}, \overline{B})$. We decompose \overline{C} into its semi-simple and nilpotent parts, $\overline{C} = \overline{S} + \overline{N}$. It is well known that \overline{S} and $\overline{N} \in \mathfrak{o}(\mathfrak{q}, \overline{B})$.

Let $\Lambda \subset \mathbb{C} \setminus \{0\}$ be the spectrum of \overline{S} . We have that $\lambda \in \Lambda$ if, and only if, $-\lambda \in \Lambda$ (see Appendix 1). Let V_λ be the eigenspace corresponding to the eigenvalue λ . We have $\dim(V_\lambda) = \dim(V_{-\lambda})$. Denote by $\mathfrak{q}(\lambda)$ the direct sum $\mathfrak{q}(\lambda) = V_\lambda \oplus V_{-\lambda}$. If $\mu \in \Lambda$, $\mu \neq \pm\lambda$, then $\mathfrak{q}(\lambda)$ and $\mathfrak{q}(\mu)$ are orthogonal (Appendix 1). Choose Λ_+ such that $\Lambda = \Lambda_+ \cup (-\Lambda_+)$ and $\Lambda_+ \cap (-\Lambda_+) = \emptyset$. We have (see Appendix 1):

$$\mathfrak{q} = \overset{\perp}{\oplus}_{\lambda \in \Lambda_+} \mathfrak{q}(\lambda).$$

So the restriction $B_\lambda = B|_{\mathfrak{q}(\lambda) \times \mathfrak{q}(\lambda)}$ is non degenerate. Moreover, V_λ and $V_{-\lambda}$ are maximal isotropic subspaces in $\mathfrak{q}(\lambda)$.

Now, consider the map $\Psi : V_{-\lambda} \rightarrow V_\lambda^*$ defined by $\Psi(u)(v) = B_\lambda(u, v)$, $\forall u \in V_{-\lambda}$, $v \in V_\lambda$. Then Ψ is an isomorphism. Given any basis $\mathcal{B}(\lambda) = \{e_1(\lambda), \dots, e_{n_\lambda}(\lambda)\}$ of V_λ , there is a basis $\mathcal{B}(-\lambda) = \{e_1(-\lambda), \dots, e_{n_\lambda}(-\lambda)\}$ of $V_{-\lambda}$ such that $B_\lambda(e_i(\lambda), e_j(-\lambda)) = \delta_{ij}$, $\forall 1 \leq i, j \leq n_\lambda$: simply define $e_i(-\lambda) = \Psi^{-1}(e_i(\lambda)^*)$, for all $1 \leq i \leq n_\lambda$.

Remark that \overline{N} and \overline{S} commute, so $\overline{N}(V_\lambda) \subset V_\lambda$, $\forall \lambda \in \Lambda$. Define $\overline{N}_\lambda = \overline{N}|_{\mathfrak{q}(\lambda)}$, then $\overline{N}_\lambda \in \mathfrak{o}(\mathfrak{q}(\lambda), B_\lambda)$. Hence, if $\overline{N}_\lambda|_{V_\lambda}$ has a matrix M_λ with respect to $\mathcal{B}(\lambda)$, then $\overline{N}_\lambda|_{V_{-\lambda}}$ has a matrix $-{}^t M_\lambda$ with respect to $\mathcal{B}(-\lambda)$. We choose the basis $\mathcal{B}(\lambda)$ such that M_λ is of Jordan type, i.e.

$$\mathcal{B}(\lambda) = \mathcal{B}(\lambda, 1) \cup \dots \cup \mathcal{B}(\lambda, r_\lambda),$$

the multiplicity m_λ of λ is $m_\lambda = \sum_{i=1}^{r_\lambda} d_\lambda(i)$ where $d_\lambda(i) = \#\mathcal{B}(\lambda, i)$ and

$$M_\lambda = \text{diag}_{n_\lambda} (J_{d_\lambda(1)}, \dots, J_{d_\lambda(r_\lambda)}).$$

The matrix of $C|_{\mathfrak{q}(\lambda)}$ written on the basis $\mathcal{B}(\lambda) \cup \mathcal{B}(-\lambda)$ is:

$$\text{diag}_{n_\lambda} (J_{d_\lambda(1)}(\lambda), \dots, J_{d_\lambda(r_\lambda)}(\lambda), -{}^t J_{d_\lambda(1)}(\lambda), \dots, -{}^t J_{d_\lambda(r_\lambda)}(\lambda)).$$

Let $\mathfrak{q}(\lambda, i)$ be the subspace generated by $\mathcal{B}(\lambda, i) \cup \mathcal{B}(-\lambda, i)$, for all $1 \leq i \leq r_\lambda$ and let $C(\lambda, i) = C|_{\mathfrak{q}(\lambda, i)}$. We have

$$\mathfrak{q}(\lambda) = \overset{\perp}{\oplus}_{1 \leq i \leq r_\lambda} \mathfrak{q}(\lambda, i).$$

The matrix of $C(\lambda, i)$ written on the basis of $\mathfrak{q}(\lambda, i)$ is $C_{2d_\lambda(i)}^J(\lambda)$. Let $\mathfrak{g}(\lambda, i)$, $\lambda \in \Lambda_+$, $1 \leq i \leq r_\lambda$ be the double extension of $\mathfrak{q}(\lambda, i)$ by $C(\lambda, i)$. Then $\mathfrak{g}(\lambda, i)$ is i -isomorphic to $j_{2d_\lambda(i)}(\lambda)$. But

$$\mathfrak{g} = \overset{\perp}{\oplus}_{\substack{\lambda \in \Lambda_+ \\ 1 \leq i \leq r_\lambda}} \mathfrak{q}(\lambda, i) \text{ and } C|_{\mathfrak{q}(\lambda, i)} = C(\lambda, i).$$

Therefore, \mathfrak{g} is the amalgamated product

$$\mathfrak{g} = \times_a \mathfrak{g}(\lambda, i).$$

$$\lambda \in \Lambda_+ \\ 1 \leq i \leq r_\lambda$$

□

6.3. Denote by $\mathcal{S}_{\text{inv}}(2p+2)$ the set of invertible singular Lie algebra structures on \mathbb{C}^{2p+2} , by $\widehat{\mathcal{S}}_{\text{inv}}(2p+2)$ the set of isomorphism (or i-isomorphism) classes of $\mathcal{S}_{\text{inv}}(2p+2)$. Next, we will give a classification of $\mathcal{S}_{\text{inv}}(2p+2)$. Using Propositions 6.4 and 5.5, a classification of $\widehat{\mathcal{S}}_{\text{s}}(n+2)$ can finally be achieved.

We shall need the following Lemma;

Lemma 6.8. *Let (V, B) be a quadratic vector space. We assume that $V = V_+ \oplus V_-$ with V_{\pm} totally isotropic vector subspaces.*

- (1) *Let $N \in \mathcal{L}(V)$ such that $N(V_{\pm}) \subset V_{\pm}$. We define maps N_{\pm} by $N_+|_{V_+} = N|_{V_+}$, $N_+|_{V_-} = 0$, $N_-|_{V_-} = N|_{V_-}$ and $N_-|_{V_+} = 0$. Then $N \in \mathfrak{o}(V)$ if, and only if, $N_- = -N_+^*$ and, in this case, $N = N_+ - N_+^*$.*
- (2) *Let $U_+ \in \mathcal{L}(V)$ such that U_+ is invertible, $U_+(V_+) = V_+$ and $U_+|_{V_-} = \text{Id}_{V_-}$. We define $U \in \mathcal{L}(V)$ by $U|_{V_+} = U_+$ and $U|_{V_-} = (U_+^{-1})^*$. Then $U \in \mathfrak{O}(V)$.*
- (3) *Let $N' \in \mathfrak{o}(V)$ such that N' satisfies the assumptions of (1). Define N_{\pm} as in (1). Moreover, we assume that there exists $U_+ \in \mathcal{L}(V_+)$, U_+ invertible such that*

$$N'_+|_{V_+} = (U_+ N_+ U_+^{-1})|_{V_+}.$$

We extend U_+ to V by $U_+|_{V_-} = \text{Id}_{V_-}$ and define $U \in \mathfrak{O}(V)$ as in (2). Then

$$N' = U N U^{-1}.$$

Proof. The proof is a straightforward computation. □

Let us now consider $C \in \mathfrak{o}(n)$, C invertible. Then, n is even, $n = 2p$ (see Appendix 1). We decompose $C = S + N$ into semi-simple and nilpotent parts, $S, N \in \mathfrak{o}(2p)$. We have $\lambda \in \Lambda$ if, and only if, $-\lambda \in \Lambda$ (Appendix 1), where Λ is the spectrum of C . Also $m(\lambda) = m(-\lambda)$, for all $\lambda \in \Lambda$ with multiplicity $m(\lambda)$. Since N and S commute, we have $N(V(\pm\lambda)) \subset V(\pm\lambda)$ where V_{λ} is the eigenspace of S corresponding to $\lambda \in \Lambda$. Denote by $W(\lambda)$ the direct sum

$$W(\lambda) = V_{\lambda} \oplus V_{-\lambda}.$$

Define the equivalence relation \mathcal{R} on Λ by:

$$\lambda \mathcal{R} \mu \text{ if, and only if, } \lambda = \pm \mu.$$

Then

$$\mathbb{C}^{2p} = \bigoplus_{\lambda \in \Lambda/\mathcal{R}}^{\perp} W(\lambda),$$

and each $(W(\lambda), B_{\lambda})$ is a quadratic vector space with $B_{\lambda} = B|_{W(\lambda) \times W(\lambda)}$.

Fix $\lambda \in \Lambda$. We write $W(\lambda) = V_+ \oplus V_-$ with $V_{\pm} = V_{\pm\lambda}$. Then, with the notation in Lemma 6.8, define $N_{\pm\lambda} = N_{\pm}$. Since $N|_{V_-} = -N_{\lambda}^*$, it is easy to verify that the matrices of $N|_{V_+}$ and $N|_{V_-}$ have the same Jordan form. Let $(d_1(\lambda), \dots, d_{r_{\lambda}}(\lambda))$ be the size of the Jordan blocks in the Jordan decomposition of $N|_{V_+}$. This does not depend on a possible choice between $N|_{V_+}$ or $N|_{V_-}$ since both maps have the same Jordan type.

Next, we consider

$$\mathcal{D} = \bigcup_{r \in \mathbb{N}^*} \{(d_1, \dots, d_r) \in \mathbb{N}^r \mid d_1 \geq d_2 \geq \dots \geq d_r \geq 1\}$$

Define $d : \Lambda \rightarrow \mathcal{D}$ by $d(\lambda) = (d_1(\lambda), \dots, d_{r_\lambda}(\lambda))$. It is clear that $\Phi \circ d = m$, where $\Phi : \mathcal{D} \rightarrow \mathbb{N}$ is the map defined by $\Phi(d_1, \dots, d_r) = \sum_{i=1}^r d_i$.

Finally, we can associate to $C \in \mathfrak{o}(n)$ a triple (Λ, m, d) defined as above.

Definition 6.9. Let \mathcal{I}_p be the set of all triples (Λ, m, d) such that:

- (1) Λ is a subset of $\mathbb{C} \setminus \{0\}$ with $\#\Lambda \leq 2p$ and $\lambda \in \Lambda$ if, and only if, $-\lambda \in \Lambda$.
- (2) $m : \Lambda \rightarrow \mathbb{N}^*$ satisfies $m(\lambda) = m(-\lambda)$, for all $\lambda \in \Lambda$ and $\sum_{\alpha \in \Lambda} m(\alpha) = 2p$.
- (3) $d : \Lambda \rightarrow \mathcal{D}$ satisfies $d(\lambda) = d(-\lambda)$, for all $\lambda \in \Lambda$ and $\Phi \circ d = m$.

Let $\mathcal{S}(2p)$ be the set of invertible elements in $\mathfrak{o}(2p)$ and $\tilde{\mathcal{S}}(2p)$ be the set of $O(2p)$ -adjoint orbits of elements in $\mathcal{S}(2p)$. By the preceding remarks, there is a map $i : \mathcal{S}(2p) \rightarrow \mathcal{I}_p$. The following Proposition classifies $\tilde{\mathcal{S}}(2p)$:

Proposition 6.10.

The map $i : \mathcal{S}(2p) \rightarrow \mathcal{I}_p$ induces a bijection $\tilde{i} : \tilde{\mathcal{S}}(2p) \rightarrow \mathcal{I}_p$.

Proof. Let C and $C' \in \mathcal{S}(2p)$ such that $C' = U C U^{-1}$ with $U \in O(2p)$. Let S, S', N, N' be respectively the semi-simple and nilpotent parts of C and C' . Write $i(C) = (\Lambda, m, \lambda)$ and $i(C') = (\Lambda', m', \lambda')$.

Then $S' = U S U^{-1}$. So $\Lambda' = \Lambda$ and $m' = m$. Also, $U(V_\lambda) = V'_\lambda$, for all $\lambda \in \Lambda$. Since $N' = U N U^{-1}$, then $N'|_{V'_\lambda} = U|_{V_\lambda} N|_{V_\lambda} U^{-1}|_{V_\lambda}$. Hence, $N|_{V_\lambda}$ and $N'|_{V'_\lambda}$ have the same Jordan decomposition, so $d = d'$ and \tilde{i} is well defined.

To prove that \tilde{i} is onto, we start with $\Lambda = \{\lambda_1, -\lambda_1, \dots, \lambda_k, -\lambda_k\}$, m and d as in Definition 6.9. Define on the canonical basis:

$$S = \text{diag}_{2p} \left(\overbrace{\lambda_1, \dots, \lambda_1}^{m(\lambda_1)}, \dots, \overbrace{\lambda_k, \dots, \lambda_k}^{m(\lambda_k)}, \overbrace{-\lambda_1, \dots, -\lambda_1}^{m(\lambda_1)}, \dots, \overbrace{-\lambda_k, \dots, -\lambda_k}^{m(\lambda_k)} \right).$$

For all $1 \leq i \leq k$, let $d(\lambda_i) = (d_1(\lambda_i) \geq \dots \geq d_{r_{\lambda_i}}(\lambda_i) \geq 1)$ and define

$$N_+(\lambda) = \text{diag}_{d(\lambda_i)} \left(J_{d_1(\lambda_i)}, J_{d_2(\lambda_i)}, \dots, J_{d_{r_{\lambda_i}}(\lambda_i)} \right)$$

on the eigenspace V_{λ_i} and 0 on the eigenspace $V_{-\lambda_i}$ where J_d is the Jordan block of size d .

By Lemma 6.8, $N(\lambda_i) := N_+(\lambda_i) - N_+(\lambda_i)^*$ is skew-symmetric on $V_{\lambda_i} \oplus V_{-\lambda_i}$. Finally,

$$\mathbb{C}^{2p} = \bigoplus_{i=1}^k (V_{\lambda_i} \oplus V_{-\lambda_i}).$$

Define $N \in \mathfrak{o}(2p)$ by $N(\sum_{i=1}^k v_i) = \sum_{i=1}^k N(\lambda_i)(v_i)$, $v_i \in V_{\lambda_i} \oplus V_{-\lambda_i}$ and $C = S + N \in \mathfrak{o}(2p)$. By construction, $i(C) = (\Lambda, m, d)$, so \tilde{i} is onto.

To prove that \tilde{i} is one-to-one, assume that $C, C' \in \mathcal{S}(2p)$ and that $i(C) = i(C') = (\Lambda, m, d)$. Using the previous notation, since their respective semi-simple parts S and S' have the same spectrum and same multiplicities, there exist $U \in O(2p)$ such

that $S' = USU^{-1}$. For $\lambda \in \Lambda$, we have $U(V_\lambda) = V'_\lambda$ for eigenspaces V_λ and V'_λ of S and S' .

Also, for $\lambda \in \Lambda$, if N and N' are the nilpotent parts of C and C' , then $N''(V_\lambda) \subset V_\lambda$, with $N'' = U^{-1}N'U$. Since $i(C) = i(C')$, then $N|_{V_\lambda}$ and $N'|_{V'_\lambda}$ have the same Jordan type. Since $N'' = U^{-1}N'U$, then $N''|_{V_\lambda}$ and $N'|_{V'_\lambda}$ have the same Jordan type. So $N|_{V_\lambda}$ and $N''|_{V_\lambda}$ have the same Jordan type. Therefore, there exists $D_+ \in \mathcal{L}(V_\lambda)$ such that $N''|_{V_\lambda} = D_+N|_{V_\lambda}D_+^{-1}$. By Lemma 6.8, there exists $D(\lambda) \in \mathcal{O}(V_\lambda \oplus V_{-\lambda})$ such that

$$N''|_{V_\lambda \oplus V_{-\lambda}} = D_+(\lambda)N|_{V_\lambda \oplus V_{-\lambda}}D_+(\lambda)^{-1}.$$

We define $D \in \mathcal{O}(2p)$ by $D|_{V_\lambda \oplus V_{-\lambda}} = D(\lambda)$, for all $\lambda \in \Lambda$. Then $N'' = DND^{-1}$ and D commutes with S . Then $S' = (UD)S(UD)^{-1}$ and $N' = (UD)N(UD)^{-1}$ and we conclude

$$C' = (UD)C(UD)^{-1}.$$

□

The classification of $\widehat{\mathcal{S}}_{\text{inv}}(2p+2)$ can be deduced from the classification of the set of orbits $\widetilde{\mathcal{I}}(2p)$ by \mathcal{I}_p as follows: introduce an action of the multiplicative group $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ on \mathcal{I}_p by

$$\text{for all } \mu \in \mathbb{C}^*, \mu \cdot (\Lambda, m, d) = (\mu\Lambda, m', d'), \forall (\Lambda, m, d) \in \mathcal{I}_p, \lambda \in \Lambda,$$

where $m'(\mu\lambda) = m(\lambda)$, $d'(\mu\lambda) = d(\lambda)$, $\forall \lambda \in \Lambda$. We have $i(\mu C) = \mu i(C)$, for all $C \in \mathcal{I}(2p)$ and $\mu \in \mathbb{C}^*$. Hence, there is a bijection $\widehat{i}: \mathbb{P}^1(\widetilde{\mathcal{I}}(2p)) \rightarrow \mathcal{I}_p/\mathbb{C}^*$ given by $\widehat{i}([C]) = [i(C)]$, if $[C]$ is the class of $C \in \mathcal{I}(2p)$ and $[(\Lambda, m, d)]$ is the class of $(\Lambda, m, d) \in \mathcal{I}_p$.

Proposition 6.11. *The set $\widehat{\mathcal{S}}_{\text{inv}}(2p+2)$ is in bijection with $\mathcal{I}_p/\mathbb{C}^*$.*

Proof. By Proposition 4.10, there is a bijection between $\widehat{\mathcal{S}}_{\text{s}}^i(2p+2)$ and $\mathbb{P}^1(\widetilde{\mathfrak{o}}(2p))$. By restriction, that induces a bijection between $\widehat{\mathcal{S}}_{\text{inv}}^i(2p+2)$ and $\mathbb{P}^1(\widetilde{\mathcal{I}}(2p))$. By Lemma 5.9, we have $\widehat{\mathcal{S}}_{\text{inv}}^i(2p+2) = \widehat{\mathcal{S}}_{\text{inv}}(2p+2)$. Then, the result follows: given $\mathfrak{g} \in \mathcal{S}_{\text{inv}}(2p+2)$ and an associated $\overline{C} \in \mathcal{I}(2p)$, the bijection maps $\widetilde{\mathfrak{g}}$ to $[i(\overline{C})]$ where $\widetilde{\mathfrak{g}}$ is the isomorphism class of \mathfrak{g} . □

Remark 6.12. Any $\mathfrak{g} \in \mathcal{S}(n+2)$ can be decomposed as an amalgamated product of its Fitting components, $\mathfrak{g} = \mathfrak{g}_N \times_{\mathfrak{a}} \mathfrak{g}_I$ (Remark 6.2). Also, $\mathfrak{g} \simeq \mathfrak{g}'$ if, and only if, $\mathfrak{g}_N \simeq \mathfrak{g}'_N$ and $\mathfrak{g}_I \simeq \mathfrak{g}'_I$. Remark that $\mathfrak{g}_N \in \mathcal{N}(k+2)$ for some $k \leq n$ and $\mathfrak{g}_I \in \mathcal{S}_{\text{inv}}(2\ell+2)$ for some ℓ with $2\ell \leq n$ and $k+2\ell = n$. Up to isomorphism (or the equivalent notion of i-isomorphism, see Proposition 6.4), the classification of $\mathcal{N}(k+2)$ is known (Proposition 5.5) and the classification of $\mathcal{S}_{\text{inv}}(2\ell+2)$ is known as well (Proposition 6.11). The decomposition of \mathfrak{g}_N and \mathfrak{g}_I as amalgamated products of Jordan-type Lie algebras is obtained in Propositions 5.5 and 6.7 and that allows us to write explicitly the commutation rules of \mathfrak{g} . So, the complete description and classification (up to isomorphism or i-isomorphism) of $\mathcal{S}_{\text{s}}(n+2)$ is achieved.

Remark that aside the singular quadratic Lie algebras context, we can completely solve the problem of the classification of $O(n)$ -adjoint orbits in $\mathfrak{o}(n)$ as follows: for $C \in \mathfrak{o}(n)$, consider its Fitting components C_N and C_I . They belong respectively to $\mathcal{N}(k)$, $k \leq n$ and to $\mathcal{I}(2\ell)$, $\ell \leq n$ with $k+2\ell = n$. Moreover, C and C' are conjugate if, and only if, C_N, C'_N and C_I, C'_I are conjugate (it results from the proof of Proposition 6.4). But C_N is nilpotent and the classification of nilpotent orbits is known (see Lemma 5.4). For the invertible C_I , the classification is given in Proposition 6.10. A Jordan-type decomposition of C can be then deduced (see (5.2) and the proof of Proposition 6.7). This gives an explicit description and classification of $O(n)$ -adjoint orbits in $\mathfrak{o}(n)$.

7. QUADRATIC DIMENSION OF REDUCED SINGULAR QUADRATIC LIE ALGEBRAS AND INVARIANCE OF $\text{dup}(\mathfrak{g})$

7.1. Let (\mathfrak{g}, B) be a quadratic Lie algebra. It is shown in [BB07] that the space of invariant symmetric bilinear forms on \mathfrak{g} and the space generated by non-degenerated ones are the same. Let us call it $\mathcal{B}(\mathfrak{g})$. The dimension of $\mathcal{B}(\mathfrak{g})$ is the *quadratic dimension* of \mathfrak{g} , denote it by $d_q(\mathfrak{g})$. Obviously, $d_q(\mathfrak{g}) = 1$ if \mathfrak{g} is simple. If \mathfrak{g} is reductive, but neither simple, nor one-dimensional, then

$$d_q(\mathfrak{g}) = s(\mathfrak{g}) + \frac{\dim(\mathcal{Z}(\mathfrak{g}))(1 + \dim(\mathcal{Z}(\mathfrak{g})))}{2},$$

where $\mathcal{Z}(\mathfrak{g})$ is the center of \mathfrak{g} and $s(\mathfrak{g})$ is the number of simple ideals of a Levi factor of \mathfrak{g} [BB07]. A general formula for $d_q(\mathfrak{g})$ is not known. Here, we give a formula for reduced singular quadratic Lie algebras. To any symmetric bilinear form B' on \mathfrak{g} , there is an associated symmetric map $D : \mathfrak{g} \rightarrow \mathfrak{g}$ satisfying

$$B'(X, Y) = B(D(X), Y), \forall X, Y \in \mathfrak{g}.$$

The following Lemma is straightforward.

Lemma 7.1. *Let (\mathfrak{g}, B) be a quadratic Lie algebra, B' be a bilinear form on \mathfrak{g} and $D \in \mathcal{L}(\mathfrak{g})$ its associated symmetric map. Then:*

(1) *B' is invariant if, and only if, D satisfies*

$$(I) \quad D([X, Y]) = [D(X), Y] = [X, D(Y)], \forall X, Y \in \mathfrak{g}.$$

(2) *B' is non-degenerate if, and only if, D is invertible.*

A symmetric map D satisfying I is called a *centromorphism* of \mathfrak{g} . The space of centromorphisms and the space generated by invertible centromorphisms are the same, denote it by $\mathcal{C}(\mathfrak{g})$. We have $d_q(\mathfrak{g}) = \dim(\mathcal{C}(\mathfrak{g}))$.

Proposition 7.2. *Let \mathfrak{g} be a reduced singular quadratic Lie algebra and $D \in \mathcal{L}(\mathfrak{g})$ be a symmetric map. Then:*

(1) *D is a centromorphism if, and only if, there exists $\mu \in \mathbb{C}$ and a symmetric map $Z : \mathfrak{g} \rightarrow \mathcal{Z}(\mathfrak{g})$ such that $Z|_{[\mathfrak{g}, \mathfrak{g}]} = 0$ and $D = \mu \text{Id} + Z$. Moreover D is invertible if, and only if, $\mu \neq 0$.*

(2)

$$d_q(\mathfrak{g}) = 1 + \frac{\dim(\mathcal{Z}(\mathfrak{g}))(1 + \dim(\mathcal{Z}(\mathfrak{g})))}{2}.$$

Proof.

- (1) If $\mathfrak{g} = \mathfrak{o}(3)$, with $B = \lambda \kappa$ and κ the Killing form, the two results are obvious. So, we examine the case where \mathfrak{g} is solvable, and then \mathfrak{g} can be realized as a double extension: $\mathfrak{g} = (\mathbb{C}X_1 \oplus \mathbb{C}Y_1) \overset{\perp}{\oplus} \mathfrak{q}$, with corresponding bilinear form \overline{B} on \mathfrak{q} , $C = \text{ad}(Y_1)$, $\overline{C} = C|_{\mathfrak{q}} \in \mathfrak{o}(\mathfrak{q})$.

Let D be an invertible centromorphism. One has $D \circ \text{ad}(X) = \text{ad}(X) \circ D$, for all $X \in \mathfrak{g}$ and that implies $DC = CD$. Using formula (1) of Lemma 4.2 and $CD = DC$, from $[D(X), Y_1] = [X, D(Y_1)]$, we find $D(C(X)) = B(D(X_1), Y_1)C(X)$. Let $\mu = B(D(X_1), Y_1)$. Since D is invertible, one has $\mu \neq 0$ and $C(D - \mu \text{Id}) = 0$. Since $\ker(C) = \mathbb{C}X_1 \oplus \ker(\overline{C}) \oplus \mathbb{C}Y_1 = \mathcal{Z}(\mathfrak{g}) \oplus \mathbb{C}Y_1$, there exists a map $Z : \mathfrak{g} \rightarrow \mathcal{Z}(\mathfrak{g})$ and $\varphi \in \mathfrak{g}^*$ such that $D - \mu \text{Id} = Z + \varphi \otimes Y_1$. But D maps $[\mathfrak{g}, \mathfrak{g}]$ into itself, so $\varphi|_{[\mathfrak{g}, \mathfrak{g}]} = 0$. One has $[\mathfrak{g}, \mathfrak{g}] = \mathbb{C}X_1 \oplus \text{Im}(\overline{C})$. If $X \in \text{Im}(\overline{C})$, let $X = C(Y)$. Then $D(X) = D(C(Y)) = \mu C(Y)$, so $D(X) = \mu X$. For Y_1 , $D([Y_1, X]) = DC(X) = \mu C(X)$ for all $X \in \mathfrak{g}$. But also, $D([Y_1, X]) = [D(Y_1), X] = \mu C(X) + \varphi(Y_1)C(X)$, hence $\varphi(Y_1) = 0$.

Assume we have shown that $D(X_1) = \mu X_1$. Then if $X \in \mathfrak{q}$, $B(D(X_1), X) = \mu B(X_1, X) = 0$. Moreover, $B(D(X_1), X) = B(X_1, D(X))$, so $\varphi(X) = 0$. Thus, to prove (1), we must prove that $D(X_1) = \mu X_1$. We decompose \mathfrak{q} respectively to \overline{C} as in Appendix 1. Let $\mathfrak{l} = \ker(\overline{C})$. Then:

$$\mathfrak{q} = (\mathfrak{l} \oplus \mathfrak{l}') \overset{\perp}{\oplus} (\mathfrak{u} \oplus \mathfrak{u}')$$

and C is an isomorphism from $\mathfrak{l}' \overset{\perp}{\oplus} (\mathfrak{u} \oplus \mathfrak{u}')$ onto $\mathfrak{l} \overset{\perp}{\oplus} (\mathfrak{u} \oplus \mathfrak{u}')$. Or

$$\mathfrak{q} = (\mathfrak{l} + \mathfrak{l}') \overset{\perp}{\oplus} \mathbb{C}T \overset{\perp}{\oplus} (\mathfrak{u} \oplus \mathfrak{u}')$$

and C is an isomorphism from $\mathfrak{l}' \overset{\perp}{\oplus} \mathbb{C}T \overset{\perp}{\oplus} (\mathfrak{u} \oplus \mathfrak{u}')$ onto $\mathfrak{l} \overset{\perp}{\oplus} \mathbb{C}T \overset{\perp}{\oplus} (\mathfrak{u} \oplus \mathfrak{u}')$.

If $\mathfrak{u} \oplus \mathfrak{u}' \neq \{0\}$, there exist $X', Y' \in \mathfrak{u} \oplus \mathfrak{u}'$ such that $B(X', Y') = -1$ and $X, Y \in \mathfrak{l}' \overset{\perp}{\oplus} (\mathfrak{u} \oplus \mathfrak{u}')$ (resp. $\mathfrak{l}' \overset{\perp}{\oplus} \mathbb{C}T \overset{\perp}{\oplus} (\mathfrak{u} \oplus \mathfrak{u}')$) such that $X' = C(X)$, $Y' = C(Y)$. It follows that $[C(X), Y] = X_1$ and then $D(X_1) = [DC(X), Y] = \mu[C(X), Y] = \mu X_1$.

If $\mathfrak{u} \oplus \mathfrak{u}' = \{0\}$, then either $\mathfrak{q} = (\mathfrak{l} + \mathfrak{l}') \overset{\perp}{\oplus} \mathbb{C}T$ or $\mathfrak{q} = \mathfrak{l} + \mathfrak{l}'$. The first case is similar to the situation above, setting $X' = Y' = \frac{T}{i}$ and $X, Y \in \mathfrak{l}' \overset{\perp}{\oplus} \mathbb{C}T$.

In the second case, $\mathfrak{l} = \text{Im}(\overline{C})$ is totally isotropic and C is an isomorphism from \mathfrak{l}' onto \mathfrak{l} . For any non-zero $X \in \mathfrak{l}'$, choose a non-zero $Y \in \mathfrak{l}'$ such that $B(C(X), Y) = 0$. Then $D([X, Y]) = D(B(C(X), Y)X_1) = 0$. But this is also equal to $[D(X), Y] = \mu[X, Y] + \varphi(X)C(Y)$. Since D is invertible, $[X, Y] = 0$ and we conclude that $\varphi(X) = 0$. Therefore $\varphi|_{\mathfrak{l}'} = 0$. There exist $L, L' \in \mathfrak{l}'$ such that $X_1 = [L, L']$ and then $D(X_1) = \mu X_1$.

Finally, $\mathcal{C}(\mathfrak{g})$ is generated by invertible centromorphism, so the necessary condition of (1) follows. The sufficiency is a simple verification.

- (2) As in (1), we can restrict to a double extension and follow the same notation. By (1), D is a centromorphism if, and only if, $D(X) = \mu X + Z(X)$, for all $X \in \mathfrak{g}$ with $\mu \in \mathbb{C}$ and Z is a symmetric map from \mathfrak{g} into $\mathcal{Z}(\mathfrak{g})$ satisfying $Z|_{[\mathfrak{g}, \mathfrak{g}]} = 0$. To compute $d_q(\mathfrak{g})$, we use Appendix 1. Assume $\dim(\mathfrak{q})$ is even and write $\mathfrak{q} = (\mathfrak{l} \oplus \mathfrak{l}') \oplus (\mathfrak{u} \oplus \mathfrak{u}')$ with $\mathfrak{l} = \ker(\overline{C})$, $\mathcal{Z}(\mathfrak{g}) = \mathbb{C}X_1 \oplus \mathfrak{l}$, $\text{Im}(\overline{C}) = \mathfrak{l} \oplus (\mathfrak{u} \oplus \mathfrak{u}')$ and $[\mathfrak{g}, \mathfrak{g}] = \mathbb{C}X_1 \oplus \text{Im}(\overline{C})$. Let us define $Z : \mathfrak{l}' \oplus \mathbb{C}Y_1 \rightarrow \mathfrak{l} \oplus \mathbb{C}X_1$: set basis $\{X_1, X_2, \dots, X_r\}$ of $\mathfrak{l} \oplus \mathbb{C}X_1$ and $\{Y'_1 = Y_1, Y'_2, \dots, Y'_r\}$ of $\mathfrak{l}' \oplus \mathbb{C}Y_1$ such that $B(Y'_i, X_j) = \delta_{ij}$. Then Z is completely defined by

$$Z \left(\sum_{j=1}^r \mu_j Y'_j \right) = \sum_{i=1}^r \left(\sum_{j=1}^r v_{ij} \mu_j \right) X_i$$

with $v_{ij} = v_{ji} = B(Y'_i, Z(Y'_j))$ and the formula follows. The case of $\dim(\mathfrak{q})$ odd is completely similar. □

7.2. As a consequence of Proposition 7.2, we prove:

Proposition 7.3. *The dup-number is invariant under isomorphism, i.e. if \mathfrak{g} and \mathfrak{g}' are quadratic Lie algebras with $\mathfrak{g} \simeq \mathfrak{g}'$, then $\text{dup}(\mathfrak{g}) = \text{dup}(\mathfrak{g}')$.*

Proof. Assume that $\mathfrak{g} \simeq \mathfrak{g}'$. Since an i-isomorphism does not change $\text{dup}(\mathfrak{g}')$, we can assume that $\mathfrak{g} = \mathfrak{g}'$ as Lie algebras equipped with invariant bilinear forms B and B' . Thus, we have two dup-numbers, $\text{dup}_B(\mathfrak{g})$ and $\text{dup}_{B'}(\mathfrak{g})$.

We choose \mathfrak{z} such that $\mathcal{Z}(\mathfrak{g}) = (\mathcal{Z}(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}]) \oplus \mathfrak{z}$. Then $\mathfrak{z} \cap \mathfrak{z}^{\perp B} = \{0\}$, \mathfrak{z} is a central ideal of \mathfrak{g} and $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{z}$ with \mathfrak{l} a reduced quadratic Lie algebra. Then $\text{dup}_B(\mathfrak{g}) = \text{dup}_B(\mathfrak{l})$ (see (2.2)). Similarly, $\mathfrak{z} \cap \mathfrak{z}^{\perp B'} = \{0\}$, $\mathfrak{g} = \mathfrak{l}' \oplus \mathfrak{z}$ with \mathfrak{l}' a reduced quadratic Lie algebra and $\text{dup}_{B'}(\mathfrak{g}) = \text{dup}_{B'}(\mathfrak{l}')$. Now, \mathfrak{l} and \mathfrak{l}' are isomorphic to $\mathfrak{g}/\mathfrak{z}$, so $\mathfrak{l} \simeq \mathfrak{l}'$. Therefore, it is enough to prove the result for reduced quadratic Lie algebras to conclude that $\text{dup}_B(\mathfrak{l}) = \text{dup}_{B'}(\mathfrak{l})$ and then that $\text{dup}_B(\mathfrak{g}) = \text{dup}_{B'}(\mathfrak{g})$.

Consider \mathfrak{g} a reduced quadratic Lie algebra equipped with bilinear forms B and B' and associated 3-forms I and I' . (see (1.6)). We have $\text{dup}_B(\mathfrak{g}) = \dim(\mathcal{V}_I)$ and $\text{dup}_{B'}(\mathfrak{g}) = \dim(\mathcal{V}_{I'})$ with $\mathcal{V}_I = \{\alpha \in \mathfrak{g}^* \mid \alpha \wedge I = 0\}$ and $\mathcal{V}_{I'} = \{\alpha \in \mathfrak{g}^* \mid \alpha \wedge I' = 0\}$.

We start with the case $\text{dup}_B(\mathfrak{g}) = 3$. This is true if, and only if, $\dim([\mathfrak{g}, \mathfrak{g}]) = 3$ [PU07]. Then $\text{dup}_{B'}(\mathfrak{g}) = 3$.

If $\text{dup}_B(\mathfrak{g}) = 1$, then \mathfrak{g} is of type S_1 with respect to B . We apply Proposition 7.2 to obtain an invertible centromorphism $D = \mu \text{Id} + Z$ for a non-zero $\mu \in \mathbb{C}$, $Z = \mathfrak{g} \rightarrow \mathcal{Z}(\mathfrak{g})$ satisfying $Z|_{[\mathfrak{g}, \mathfrak{g}]} = 0$ and such that $B'(X, Y) = B(D(X), Y)$, for all $X, Y \in \mathfrak{g}$. Then $I'(X, Y, Z) = B'([X, Y], Z) = B([D(X), Y], Z) = \mu B([X, Y], Z) = \mu I(X, Y, Z)$, for all $X, Y, Z \in \mathfrak{g}$. So $I' = \mu I$ and $\text{dup}_{B'}(\mathfrak{g}) = \text{dup}_B(\mathfrak{g})$.

Finally, if $\text{dup}_B(\mathfrak{g}) = 0$, then from the previous cases, \mathfrak{g} cannot be of type S_3 or S_1 with respect to B' , so $\text{dup}_{B'}(\mathfrak{g}) = 0$.

□

8. APPENDIX 1

In this Appendix, we recall some facts on skew-symmetric maps used in the paper. Nothing here is new, but short proofs are given for the sake of completeness.

Throughout this section, (V, B) is a quadratic vector space and C is an element of $\mathfrak{o}(V)$. We recall the useful identity $\ker(C) = (\text{Im}(C))^\perp$.

Lemma 8.1. *There exist subspaces W and N of V such that:*

- (1) $N \subset \ker(C)$, $C(W) \subset W$ and $V = W \oplus N$.
- (2) Let $B_W = B|_{W \times W}$ and $C_W = C|_W$. Then B_W is non-degenerate, $C_W \in \mathfrak{o}(W, B_W)$ and $\ker(C_W) \subset \text{Im}(C_W) = \text{Im}(C)$.

Proof. We follow the proof of Proposition 2.4, given in [PU07]. Let $N_0 = \ker(C) \cap \text{Im}(C)$ and let N be a complementary subspace of N_0 in $\ker(C)$, $\ker(C) = N_0 \oplus N$. Since $\ker(C) = (\text{Im}(C))^\perp$, we have $B(N_0, N) = \{0\}$ and $N \cap N^\perp = \{0\}$. So, if $W = N^\perp$, one has $V = W \oplus N$. From $C(N) = \{0\}$, we deduce that $C(W) \subset W$.

It is clear that B is non-degenerate and that $C_W \in \mathfrak{o}(W)$. Moreover, since $C(W) \subset W$ and $C(N) = \{0\}$, then $\text{Im}(C) = \text{Im}(C_W)$. It is immediate that $\ker(C_W) = N_0$, so $\ker(C_W) \subset \text{Im}(C_W)$. □

Lemma 8.2. *Assume that $\ker(C) \subset \text{Im}(C)$. Denote $L = \ker(C)$. Let $\{L_1, \dots, L_r\}$ be a basis of L .*

- (1) *If $\dim(V)$ is even, there exist subspaces L' with basis $\{L'_1, \dots, L'_r\}$, U with basis $\{U_1, \dots, U_s\}$ and U' with basis $\{U'_1, \dots, U'_s\}$ such that $B(L_i, L'_j) = \delta_{ij}$, for all $1 \leq i, j \leq r$, L and L' are totally isotropic, $B(U_i, U'_j) = \delta_{ij}$, for all $1 \leq i, j \leq s$, U and U' are totally isotropic and*

$$V = (L \oplus L') \oplus (U \oplus U').$$

Moreover $\text{Im}(C) = L \oplus (U \oplus U')$ and $C : L' \oplus (U \oplus U') \rightarrow L \oplus (U \oplus U')$ is a bijection.

- (2) *If $\dim(V)$ is odd, there exist subspaces L' , U and U' as in (1) and $v \in V$ such that $B(v, v) = 1$ and*

$$V = (L \oplus L') \oplus \mathbb{C}v \oplus (U \oplus U').$$

Moreover $\text{Im}(C) = L \oplus \mathbb{C}v \oplus (U \oplus U')$ and $C : L' \oplus \mathbb{C}v \oplus (U \oplus U') \rightarrow L \oplus \mathbb{C}v \oplus (U \oplus U')$ is a bijection.

- (3) *In both cases, $\text{rank}(C)$ is even.*

Proof. Since $(\ker(C))^\perp = \text{Im}(C)$, L is isotropic.

- (1) If $\dim(V)$ is even, there exist maximal isotropic subspaces W_1 and W_2 such that $V = W_1 \oplus W_2$ [Bou59] and $L \subset W_1$. Let U be a complementary subspace of L in W_1 , $W_1 = L \oplus U$ and $\{U_1, \dots, U_s\}$ a basis of U . Consider the isomorphism $\Psi : W_2 \rightarrow W_1^*$ defined by $\Psi(w_2)(w_1) = B(w_2, w_1)$, for all $w_1 \in W_1, w_2 \in W_2$. Define $L'_i = \Psi^{-1}(L_i^*), 1 \leq i \leq r, L' = \text{span}\{L'_1, \dots, L'_r\}, U'_j = \Psi^{-1}(U_j^*), 1 \leq j \leq s, U' = \text{span}\{U'_1, \dots, U'_s\}$. Then $B(L_i, L'_j) = \delta_{ij}, 1 \leq i, j \leq r, L$ and L' are isotropic, $B(U_i, U'_j) = \delta_{ij}, 1 \leq i, j \leq s, U$ and U' are isotropic and

$$V = (L \oplus L') \overset{\perp}{\oplus} (U \oplus U').$$

Since $\text{Im}(C) = L^\perp$, we have $\text{Im}(C) = L \overset{\perp}{\oplus} (U \oplus U')$. Finally, if $v \in L' \overset{\perp}{\oplus} (U \oplus U')$ and $C(v) = 0$, then $v \in L$. So $v = 0$. Therefore C is one to one from $L' \overset{\perp}{\oplus} (U \oplus U')$ into $L \overset{\perp}{\oplus} (U \oplus U')$ and since the dimensions are the same, C is a bijection.

- (2) There exist maximal isotropic subspaces W_1 and W_2 such that $V = (W_1 \oplus W_2) \overset{\perp}{\oplus} \mathbb{C}v$, with $v \in V$ such that $B(v, v) = 1$ and $L \subset W_1$ [Bou59]. Then the proof is essentially the same as in (1).
- (3) Assume $\dim(V)$ even. Define a bilinear form Δ on $L' \overset{\perp}{\oplus} (U \oplus U')$ by $\Delta(v_1, v_2) = B(v_1, C(v_2))$, for all $v_1, v_2 \in L' \overset{\perp}{\oplus} (U \oplus U')$. Since $C \in \mathfrak{o}(V)$, Δ is skew-symmetric. Let $v_1 \in L' \overset{\perp}{\oplus} (U \oplus U')$ such that $\Delta(v_1, v_2) = 0$, for all $v_2 \in L' \overset{\perp}{\oplus} (U \oplus U')$. Then $B(v_1, w) = 0$, for all $w \in L \overset{\perp}{\oplus} (U \oplus U')$. It follows that $B(v_1, w) = 0$, for all $w \in V$, so $v_1 = 0$ and Δ is non-degenerate. So $\dim(L' \overset{\perp}{\oplus} (U \oplus U'))$ is even. Therefore $\dim(L') = \dim(L)$ is even and $\text{rank}(C)$ is even. If V is odd-dimensional, the proof is completely similar. \square

Corollary 8.3. *If $C \in \mathfrak{o}(V)$, then $\text{rank}(C)$ is even.*

Proof. By Lemma 8.1, $\text{Im}(C) = \text{Im}(C_W)$ and $\text{rank}(C_W)$ is even by the preceding Lemma. \square

For instance, if $C \in \mathfrak{o}(V)$ and C is invertible, then $\dim(V)$ must be even. But this can also be proved directly: when C is invertible, then the skew-symmetric form Δ_C on V defined by $\Delta_C(v_1, v_2) = B(v_1, C(v_2))$, for all $v_1, v_2 \in V$, is clearly non-degenerate.

When C is semi-simple (i.e. diagonalizable), we have $V = \ker(C) \overset{\perp}{\oplus} \text{Im}(C)$ and $C|_{\text{Im}(C)}$ is invertible. So semi-simple elements are completely described by:

Lemma 8.4. *Assume C is semi-simple and invertible. Then there is a basis $\{e_1, \dots, e_p, f_1, \dots, f_p\}$ of V such that $B(e_i, e_j) = B(f_i, f_j) = 0, B(e_i, f_j) = \delta_{ij}, 1 \leq i, j \leq p$. For $1 \leq i \leq p$, there exist non-zero $\lambda_i \in \mathbb{C}$ such that $C(e_i) = \lambda_i e_i$ and $C(f_i) = -\lambda_i f_i$.*

Moreover, if Λ denotes the spectrum of C , then $\lambda \in \Lambda$ if, and only if, $-\lambda \in \Lambda$; λ and $-\lambda$ have the same multiplicity.

Proof. We prove the result by induction on $\dim(V)$. Assume $\dim(V) = 2$. Let $\{e_1, e_2\}$ be an eigenvector basis of V corresponding to eigenvalues λ_1 and λ_2 . We have $B(C(v), v') = -B(v, C(v'))$ and C is invertible, so $B(e_1, e_1) = B(e_2, e_2) = 0$, $B(e_1, e_2) \neq 0$ and $\lambda_2 = -\lambda_1$. Let $f_1 = \frac{1}{B(e_1, e_2)}e_2$, then the basis $\{e_1, f_1\}$ is a convenient basis.

Assume that the result is true for quadratic vector spaces of dimension n with $n \leq 2(p-1)$. Assume $\dim(V) = 2p$. Let $\{e_1, \dots, e_{2p}\}$ be an eigenvector basis with corresponding eigenvalues $\lambda_1, \dots, \lambda_{2p}$. As before, $B(e_i, e_i) = 0$, $1 \leq i \leq 2p$, so there exists j such that $B(e_1, e_j) \neq 0$. Then $\lambda_j = -\lambda_1$. Let $f_1 = \frac{1}{B(e_1, e_j)}e_j$. Then

$B|_{\text{span}\{e_1, f_1\}}$ is non-degenerate, so $V = \text{span}\{e_1, f_1\} \oplus V_1$, where $V_1 = \text{span}\{e_1, f_1\}^\perp$. But C maps V_1 into itself, so we can apply the induction assumption and the result follows. \square

As a consequence, we have this classical result, used in Section 5:

Lemma 8.5.

- (1) Let C be a semi-simple element of $\mathfrak{o}(n)$. Then C belongs to the $\text{SO}(n)$ -adjoint orbit of an element of the standard Cartan subalgebra of $\mathfrak{o}(n)$ (i.e., an element with matrix $\text{diag}_{2p}(\lambda_1, \dots, \lambda_p, -\lambda_1, \dots, -\lambda_p)$ if $n = 2p$ and $\text{diag}_{2p+1}(\lambda_1, \dots, \lambda_p, 0, -\lambda_1, \dots, -\lambda_p)$ if $n = 2p + 1$ in the canonical basis of \mathbb{C}^n).
- (2) Let C and C' be semi-simple elements of $\mathfrak{o}(n)$. Then C and C' are in the same $\text{O}(n)$ -adjoint orbit if, and only if, they have the same spectrum, with same multiplicities.

Proof.

- (1) We have $\mathbb{C}^n = \ker(C) \oplus \text{Im}(C)$ and $\text{rank}(C)$ is even. So $\dim(\ker(C))$ is even if $n = 2p$ and odd, if $n = 2p + 1$. Then apply Lemma 8.4 to $C|_{\text{Im}(C)}$ to obtain the result.
- (2) If C and C' have the same spectrum and their eigenvalues, same multiplicities, they are $\text{O}(n)$ -conjugate to the same element of the standard Cartan subalgebra. \square

Remark 8.6.

- (1) Attention: $\text{O}(n)$ -adjoint orbits are generally not the same as $\text{SO}(n)$ -adjoint orbits.
- (2) Lemma 8.5(1) is a particular case of a general and classical result on semi-simple Lie algebras: any semi-simple element of a semi-simple Lie algebra belongs to a Cartan subalgebra and all Cartan subalgebras are conjugate

under the adjoint action [Sam80]. Here, $\mathfrak{o}(n)$ is a semi-simple Lie algebra and the adjoint group is $SO(n)$.

9. APPENDIX 2

Here we prove:

Lemma 9.1. *Let (\mathfrak{g}, B) be a non-Abelian 5-dimensional quadratic Lie algebra. Then \mathfrak{g} is a singular quadratic Lie algebra.*

Proof.

- We assume \mathfrak{g} is not solvable and we write $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{r}$ with \mathfrak{s} semi-simple and \mathfrak{r} the radical of \mathfrak{g} [Bou71]. Then $\mathfrak{s} \simeq \mathfrak{sl}(2)$ and $B|_{\mathfrak{s} \times \mathfrak{s}} = \lambda \kappa$ where κ is the Killing form.

If $\lambda = 0$, consider $\Psi : \mathfrak{s} \rightarrow \mathfrak{r}^*$ defined by $\Psi(S)(R) = B(S, R)$, for all $S \in \mathfrak{s}$, $R \in \mathfrak{r}$. Then Ψ is one-to-one and $\Psi(\text{ad}(X)(S)) = \check{\text{ad}}(X)(\Psi(S))$, for all $X, S \in \mathfrak{s}$. So Ψ must be a homomorphism from the representation $(\mathfrak{s}, \text{ad}|_{\mathfrak{s}})$ of \mathfrak{s} into the representation $(\mathfrak{r}^*, \check{\text{ad}}|_{\mathfrak{s}})$, so $\Psi = 0$, a contradiction.

So $\lambda \neq 0$. Then $B|_{\mathfrak{s} \times \mathfrak{s}}$ is non-degenerate. Therefore $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{s}^\perp$ and $\text{ad}(\mathfrak{s})|_{\mathfrak{s}^\perp}$ is an orthogonal 2-dimensional representation of \mathfrak{s} . Hence, $\text{ad}(\mathfrak{s})|_{\mathfrak{s}^\perp} = 0$ and $[\mathfrak{s}, \mathfrak{s}^\perp] = 0$. We have $B(X, [Y, Z]) = B([X, Y], Z) = 0$, for all $X \in \mathfrak{s}$, $Y \in \mathfrak{s}^\perp$, $Z \in \mathfrak{g}$. It follows that \mathfrak{s}^\perp is an ideal of \mathfrak{g} and therefore a quadratic 2-dimensional Lie algebra. So \mathfrak{s}^\perp is Abelian. Finally, $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{s}^\perp$ with \mathfrak{s}^\perp a central ideal of \mathfrak{g} , so $\text{dup}(\mathfrak{g}) = \text{dup}(\mathfrak{s}) = 3$.

- We assume that \mathfrak{g} is solvable and we write $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{z}$ with \mathfrak{z} a central ideal of \mathfrak{g} (Proposition 2.4). Then $\dim(\mathfrak{l}) \geq 3$. If $\dim(\mathfrak{l}) = 3$ or 4, then it is proved in Proposition 2.10 that \mathfrak{l} is singular, so \mathfrak{g} is singular. So we can assume that \mathfrak{g} is reduced, i.e. $\mathfrak{Z}(\mathfrak{g}) \subset [\mathfrak{g}, \mathfrak{g}]$. It results that $\dim(\mathfrak{Z}(\mathfrak{g})) = 1$ or 2 (Remark 2.3).

– If $\dim(\mathfrak{Z}(\mathfrak{g})) = 1$, $\mathfrak{Z}(\mathfrak{g}) = \mathbb{C}X_0$. Then $\dim([\mathfrak{g}, \mathfrak{g}]) = 4$ and $[\mathfrak{g}, \mathfrak{g}] = X_0^\perp$. We can choose Y_0 such that $B(X_0, Y_0) = 1$ and $B(Y_0, Y_0) = 0$. Let

$\mathfrak{q} = (\mathbb{C}X_0 \oplus \mathbb{C}Y_0)^\perp$. Then $\mathfrak{g} = (\mathbb{C}X_0 \oplus \mathbb{C}Y_0) \oplus \mathfrak{q}$. If $X, X' \in \mathfrak{q}$, then $B(X_0, [X, X']) = B([X_0, X], X') = 0$, so $[X, X'] \in X_0^\perp$. Write $[X, X'] = \lambda(X, X')X_0 + [X, X']_{\mathfrak{q}}$ with $[X, X']_{\mathfrak{q}} \in \mathfrak{q}$. Remark that $[X, [X', X'']] = \lambda(X, [X', X'']_{\mathfrak{q}})X_0 + [X, [X', X'']_{\mathfrak{q}}]_{\mathfrak{q}}$, for all $X, X', X'' \in \mathfrak{q}$. So $[\cdot, \cdot]_{\mathfrak{q}}$ satisfies the Jacobi identity. Moreover $B([X, X'], X'') = -B(X', [X, X'']_{\mathfrak{q}})$. But also $B([X, X'], X'') = B([X, X']_{\mathfrak{q}}, X'')$. So $(\mathfrak{q}, [\cdot, \cdot]_{\mathfrak{q}}, B|_{\mathfrak{q} \times \mathfrak{q}})$ is a 3-dimensional quadratic Lie algebra.

If \mathfrak{q} is an Abelian Lie algebra, then $[X, X'] \in \mathbb{C}X_0$, for all $X, X' \in \mathfrak{q}$. Write $B(Y_0, [X, X']) = B([Y_0, X], X')$ to obtain $[X, X'] = B(\text{ad}(Y_0)(X), X')X_0$, for all $X, X' \in \mathfrak{q}$. Since $\dim(\mathfrak{q}) = 3$ and $\text{ad}(Y_0)|_{\mathfrak{q}}$ is skew-symmetric, there exists $Q_0 \in \mathfrak{q}$ such that $\text{ad}(Y_0)(Q_0) = 0$. It follows that $Q_0 \in \mathfrak{Z}(\mathfrak{g})$ and that is a contradiction since $\dim(\mathfrak{Z}(\mathfrak{g})) = 1$.

Therefore $(\mathfrak{q}, [\cdot, \cdot]_{\mathfrak{q}}) \simeq \mathfrak{sl}(2)$. Consider

$$0 \rightarrow \mathbb{C}X_0 \rightarrow X_0^{\perp} \rightarrow \mathfrak{q} \rightarrow 0.$$

Then there is a section $\sigma : \mathfrak{q} \rightarrow X_0^{\perp}$ such that $\sigma([X, X']_{\mathfrak{q}}) = [\sigma(X), \sigma(X')]$, for all $X, X' \in \mathfrak{q}$ [Bou71]. Then $\sigma(\mathfrak{q})$ is a Lie subalgebra of \mathfrak{g} , isomorphic to $\mathfrak{sl}(2)$ and that is a contradiction since \mathfrak{g} is solvable.

- If $\dim(\mathcal{Z}(\mathfrak{g})) = 2$, then we choose a non-zero $X_0 \in \mathcal{Z}(\mathfrak{g})$ and $Y_0 \in \mathfrak{g}$ such that $B(X_0, Y_0) = 1$ and $B(Y_0, Y_0) = 0$. Let $\mathfrak{q} = (\mathbb{C}X_0 \oplus \mathbb{C}Y_0)^{\perp}$.

Then $\mathfrak{g} = (\mathbb{C}X_0 \oplus \mathbb{C}Y_0) \oplus^{\perp} \mathfrak{q}$ and as in the preceding case, $[X, X'] \in X_0^{\perp}$, for all $X, X' \in \mathfrak{q}$. Write $[X, X'] = \lambda(X, X')X_0 + [X, X']_{\mathfrak{q}}$ with $[X, X']_{\mathfrak{q}} \in \mathfrak{q}$. Same arguments as in the preceding case allow us to conclude that $[\cdot, \cdot]_{\mathfrak{q}}$ satisfies the Jacobi identity and that $B|_{\mathfrak{q} \times \mathfrak{q}}$ is invariant. So $(\mathfrak{q}, [\cdot, \cdot]_{\mathfrak{q}}, B|_{\mathfrak{q} \times \mathfrak{q}})$ is a 3-dimensional quadratic Lie algebra.

If $\mathfrak{q} \simeq \mathfrak{sl}(2)$, then apply the same reasoning as in the preceding case to obtain a contradiction with \mathfrak{g} solvable.

If \mathfrak{q} is an Abelian Lie algebra, then $[X, X'] \in \mathbb{C}X_0$, for all $X, X' \in \mathfrak{q}$. Again, as in the preceding case, $[X, X'] = B(\text{ad}(Y_0)(X), X')X_0$, for all $X, X' \in \mathfrak{q}$. Then it is easy to check that \mathfrak{g} is a double extension of the quadratic vector space \mathfrak{q} by $\overline{C} = \text{ad}(Y_0)|_{\mathfrak{q}}$. By Proposition 4.3, \mathfrak{g} is singular.

□

Remark 9.2. Let us give a list of all non-Abelian 5-dimensional quadratic Lie algebras:

- $\mathfrak{g} \stackrel{i}{\simeq} \mathfrak{o}(3) \oplus^{\perp} \mathbb{C}^2$ with \mathbb{C}^2 central, $\mathfrak{o}(3)$ equipped with bilinear form $\lambda \kappa$, $\lambda \in \mathbb{C}$, $\lambda \neq 0$ and κ the Killing form. We have $\text{dup}(\mathfrak{g}) = 3$.
- $\mathfrak{g} \stackrel{i}{\simeq} \mathfrak{g}_4 \oplus^{\perp} \mathbb{C}$ with \mathbb{C} central, \mathfrak{g}_4 the double extension of \mathbb{C} by $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, \mathfrak{g} is solvable, non-nilpotent and $\text{dup}(\mathfrak{g}) = 3$.
- $\mathfrak{g} \stackrel{i}{\simeq} \mathfrak{g}_5$, a double extension of \mathbb{C}^3 by $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$, \mathfrak{g} is nilpotent and $\text{dup}(\mathfrak{g}) = 3$.

See Proposition 4.4 for the definition of \mathfrak{g}_4 and \mathfrak{g}_5 . Remark that $\mathfrak{g}_4 \oplus^{\perp} \mathbb{C}$ is actually the double extension of \mathbb{C}^3 by $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

REFERENCES

- [BB07] I. Bajo and S. Benayadi, *Lie algebras with quadratic dimension equal to 2*, Journal of Pure and Applied Algebra **209** (2007), no. 3, 725 – 737. ↑5, 33

- [Bou58] N. Bourbaki, *Eléments de Mathématiques. Algèbre, Algèbre Multilinéaire*, Vol. Fasc. VII, Livre II, Hermann, Paris, 1958. ↑6
- [Bou59] ———, *Eléments de Mathématiques. Algèbre, Formes sesquilinéaires et formes quadratiques*, Vol. Fasc. XXIV, Livre II, Hermann, Paris, 1959. ↑7, 37
- [Bou71] ———, *Eléments de Mathématiques. Groupes et Algèbres de Lie*, Vol. Chapitre I, Algèbres de Lie, Hermann, Paris, 1971. ↑39, 40
- [CM93] D. H. Collingwood and W. M. McGovern, *Nilpotent Orbits in Semisimple Lie Algebras*, Van Nostrand Reinhold Mathematics Series, New York, 1993. ↑3, 19, 22, 23, 24, 25
- [FS87] G. Favre and L. J. Santharoubane, *Symmetric, invariant, non-degenerate bilinear form on a Lie algebra*, *Journal of Algebra* **105** (1987), 451–464. ↑2, 3, 15, 21
- [Kac85] V. Kac, *Infinite-dimensional Lie algebras*, Cambridge University Press, New York, 1985. ↑15
- [MR85] A. Medina and Ph. Revoy, *Algèbres de Lie et produit scalaire invariant*, *Ann. Sci. École Norm. Sup.* **4** (1985), 553 – 561. ↑2, 15
- [PU07] G. Pinczon and R. Ushirobira, *New Applications of Graded Lie Algebras to Lie Algebras, Generalized Lie Algebras, and Cohomology*, *Journal of Lie Theory* **17** (2007), no. 3, 633 – 668. ↑1, 7, 8, 9, 10, 11, 17, 18, 35, 36
- [Sam80] H. Samelson, *Notes on Lie algebras*, Universitext, Springer-Verlag, 1980. ↑39

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