

On the δ -singularities of the electromagnetic field

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Abstract. The singularities of the electromagnetic field are derived to include all the point-like multipoles representing an electric charge and current distribution. We show that for higher orders, it is more efficient to have fields represented in terms of symmetric and trace free moments. In the static case, the delta-singularities are expressed for arbitrary multipole orders, while in the dynamic case we restrict ourselves to the lower orders. The algorithm we give can be easily extended to the next orders.

1. Introduction

|| In the cases of electrostatic and magnetostatic fields of point-like dipoles, one has the well-known procedure of introducing Dirac δ -function terms for obtaining correct expressions of the electric and magnetic fields defined on the entire space. The corresponding field expressions take the following form [1]:

$$\mathbf{E}_{\mathbf{p}}(\mathbf{r}) = -\frac{1}{3\varepsilon_0} \mathbf{p} \delta(\mathbf{r}) + \frac{1}{4\pi\varepsilon_0} \left(\frac{3(\boldsymbol{\nu} \cdot \mathbf{p})\boldsymbol{\nu} - \mathbf{p}}{r^3} \right)_{r \neq 0} = -\frac{1}{3\varepsilon_0} \mathbf{p} \delta(\mathbf{r}) + (\mathbf{E})_{r \neq 0}, \quad (1)$$

where $\boldsymbol{\nu} = \mathbf{r}/r$, and

$$\mathbf{B}_{\mathbf{m}}(\mathbf{r}) = \frac{2\mu_0}{3} \mathbf{m} \delta(\mathbf{r}) + \frac{\mu_0}{4\pi} \left(\frac{3(\boldsymbol{\nu} \cdot \mathbf{m})\boldsymbol{\nu} - \mathbf{m}}{r^3} \right)_{r \neq 0} = \frac{2\mu_0}{3} \mathbf{m} \delta(\mathbf{r}) + (\mathbf{B})_{r \neq 0}. \quad (2)$$

In these equations, by $(\dots)_{r \neq 0}$ we understand an expression in which the derivatives are calculated supposing $r \neq 0$, representing some well-known expressions of the fields. The expressions from equations (1) and (2) are introduced in Ref. [1] as conditions of compatibility with the average value of the electric or magnetic field over a spherical domain containing all the charges or currents inside. Another procedure for introducing equations (1) and (2) is based on an extension of the derivative $\partial_i \partial_j / (1/r)$ to the entire space [2]:

$$\partial_i \partial_j \frac{1}{r} = \left(\frac{3\nu_i \nu_j - \delta_{ij}}{r^3} \right)_{r \neq 0} - \frac{4\pi}{3} \delta_{ij} \delta(\mathbf{r}) \quad (3)$$

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|| This paper is a revisited, completed and finalized version of two previous articles: arXiv:0912.4684 and arXiv:1001.4114

and this procedure will be applied in the present paper. A more pedagogical and suitable approach for understanding the origin of the difference between the electric and magnetic cases is done in Ref. [3]. Refs. [4] and [5] contain generalizations of the equations (1) and (2) to the dynamic case for oscillating electric and magnetic dipoles.

Section 2 is dedicated to the reader less used with the tensorial formalism in handling multipolar expansions. For the informed reader, the section can be seen as an introduction to the notation and formulas used along the article. In Sections 3 and 4 the results for the electric and magnetic fields are presented in the static case. The dynamic case is treated in Section 5, and the conclusions are outlined in Section 6.

The formalism employed in this paper is a purely algebraic one. With a good understanding of the definitions and notation presented in Section 2, we think the reader will be able to verify every detail of the calculation and, possibly, to search and find some more adequate versions.

2. Preliminaries

2.1. Definitions, notation and formulas from the tensorial formalism

Let us consider a n -th order Cartesian tensor denoted by $\mathbf{T}^{(n)}$ and characterized by the components $T_{i_1 \dots i_n}$, $i_q = 1, 2, 3$. These components are, in fact, the components of a vector in the n tensorial product of the Euclidean space \mathbb{R}^3 :

$$\mathbf{T}^{(n)} = T_{i_1 \dots i_n} \mathbf{e}_{i_1} \otimes \dots \otimes \mathbf{e}_{i_n} = T_{i_1 \dots i_n} \mathbf{e}_{i_1 \dots i_n} ,$$

where \mathbf{e}_i are the unit vectors of a Cartesian basis in \mathbb{R}^3 and $\mathbf{e}_{i_1 \dots i_n}$ are the unit vectors of the tensorial product space. Some vectors from the tensorial product of spaces can be represented by the tensorial products of n vectors from the different space factors \mathbb{R}^3 . Particularly, in the case of identical factors, we employ the usual notation

$$\mathbf{a}^n = a_{i_1} \dots a_{i_n} \mathbf{e}_{i_1 \dots i_n} .$$

The vector \mathbf{a} can be the differential operator $\nabla = \mathbf{e}_i \partial_i$ and, in this case,

$$(\nabla^n) = \mathbf{e}_{i_1 \dots i_n} \partial_{i_1} \dots \partial_{i_n} .$$

Because of this simplified notation, for avoiding confusions, the Laplace operator in the space \mathbb{R}^3 is symbolized by Δ , and not by the frequent ∇^2 .

We employ the following notation for the tensorial contractions:

$$(\mathbf{A}^{(n)} || \mathbf{B}^{(m)})_{i_1 \dots i_{|n-m|}} = \begin{cases} A_{i_1 \dots i_{n-m} j_1 \dots j_m} B_{j_1 \dots j_m} & , n > m \\ A_{j_1 \dots j_n} B_{j_1 \dots j_n} & , n = m \\ A_{j_1 \dots j_n} B_{j_1 \dots j_n i_1 \dots i_{m-n}} & , n < m \end{cases} .$$

A fully symmetric tensor $\mathbf{S}^{(n)}$, which here is called simply "symmetric", has a projection on the subspace of symmetric and trace free (**STF**) tensors. Up to a numerical factor, this projection will be represented by the tensor $\mathcal{T}(\mathbf{S}^{(n)}) \equiv \mathcal{S}^{(n)}$. For $n = 2$, for example, we can write

$$S_{ij} = \mathcal{S}_{ij} + \delta_{ij} \mathbf{\Lambda}(\mathbf{S}^{(2)}) . \quad (4)$$

The condition $\mathcal{S}_{ii} = 0$ gives

$$\Lambda = \frac{1}{3}\mathcal{S}_{qq} . \quad (5)$$

In the case $n = 3$:

$$\mathcal{S}_{ijk} = \mathcal{S}_{ijk} + \delta_{\{ij}\Lambda_k\}(\mathbf{S}^{(3)}) ,$$

where by the symbol $A_{\{i_1\dots i_n\}}$ is denoted the sum over all distinct permutations of the indexes. The condition $\mathcal{S}_{iik} = 0$ for $k = 1, 2, 3$ implies

$$\Lambda_i(\mathbf{S}^{(3)}) = \frac{1}{5}\mathcal{S}_{qqi} .$$

Though, maybe, only for a theoretical interest, let us consider the general case for the **STF** projection of the symmetric tensor $\mathbf{S}^{(n)}$ defined up to a numerical factor by the equation

$$\mathcal{T}_{i_1\dots i_n}(\mathbf{S}^{(n)}) \equiv \mathcal{S}_{i_1\dots i_n} = \mathcal{S}_{i_1\dots i_n} - \delta_{\{i_1 i_2 \Lambda_{i_3\dots i_n}\}}(\mathbf{S}^{(n)}) .$$

$\mathbf{\Lambda}^{(n-2)} \equiv \mathbf{\Lambda}(\mathbf{S}^{(n)})$ is a symmetric tensor and is defined by the condition that $\mathcal{T}^{(n)}$ is a trace-free tensor. For low values of n , the ones of really practical interest, the components $\Lambda_{i_1\dots i_{n-2}}$ can be calculated directly from the equation system representing the vanishing relations of all the partial traces of the tensor $\mathcal{T}^{(n)}$. However, we mention here a general formula known from literature [6, 7] which, with the notation from the present paper, is written as

$$[\mathcal{T}[\mathbf{S}^{(n)}]]_{i_1\dots i_n} = \sum_{m=0}^{\lfloor n/2 \rfloor} \frac{(-1)^m (2n-1-2m)!!}{(2n-1)!!} \delta_{\{i_1 i_2 \dots i_{2m-1} i_{2m} \mathcal{S}_{i_{2m+1} \dots i_n}^{(n:m)}\}} .$$

The symbol $[\alpha]$ represents the integer part of α and $\mathcal{S}_{i_{2m+1} \dots i_n}^{(n:m)}$ denotes the components of the $(n-2m)$ -th order tensor obtained from $\mathbf{S}^{(n)}$ by contracting m pairs of symbols i . This equation is known as the *detracer theorem* [7]. As a consequence of this theorem, the components of the tensor $\mathbf{\Lambda}^{(n-2)}$ are written as

$$\Lambda_{i_1\dots i_{n-2}}[\mathbf{S}^{(n)}] = \sum_{m=0}^{\lfloor n/2-1 \rfloor} \frac{(-1)^m [2n-1-2(m+1)]!!}{(m+1)(2n-1)!!} \delta_{\{i_1 i_2 \dots i_{2m-1} i_{2m} \mathcal{S}_{i_{2m+1} \dots i_{n-2}}^{(n:m+1)}\}} .$$

These formulas are useful for defining and processing the multipole expansions of the electrodynamic potentials and fields.

2.2. Multipole expansion of the electromagnetic field in Cartesian coordinates

The multipole expansions of the potentials in the Lorenz gauge are written as [8, 9, 10, 11]:

$$\Phi(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \sum_{n \geq 0} \frac{(-1)^n}{n!} \nabla^n \left\| \frac{\mathbf{P}^{(n)}(\tau)}{r} \right\| \quad (6)$$

and

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \sum_{n \geq 1} \frac{(-1)^{n-1}}{n!} \left[\nabla \times \left(\nabla^{n-1} \left\| \frac{\mathbf{M}^{(n)}(\tau)}{r} \right\| \right) + \nabla^{n-1} \left\| \frac{\dot{\mathbf{P}}^{(n)}(\tau)}{r} \right\| \right] . \quad (7)$$

The dot symbolizes the time derivative and $\tau = t - r/c$ is the retarded time with respect to the origin O of the Cartesian axes in the point corresponding to the vector \mathbf{r} . The origin O is a point from the support of the electric charge and current distribution. The tensors $\mathbf{P}^{(n)}(t)$ and $\mathbf{M}^{(n)}(t)$ are the electric and magnetic moments of the electric charges and currents distributions $\rho(\mathbf{r}, t)$ and $\mathbf{J}(\mathbf{r}, t)$, the support of these distributions being included in the domain \mathcal{D} :

$$\mathbf{P}^{(n)}(t) = \int_{\mathcal{D}} d^3x \mathbf{r}^n \rho(\mathbf{r}, t),$$

and

$$\mathbf{M}^{(n)}(t) = \frac{n}{n+1} \int_{\mathcal{D}} d^3x \mathbf{r}^n \times \mathbf{J}(\mathbf{r}, t). \quad (8)$$

In the last equation, a tensorial contraction via the Levi-Civita pseudo-tensor ε_{ijk} is employed:

$$\{\mathbf{T}^{(n)}, \mathbf{a}\} \rightarrow \mathbf{T}^{(n)} \times \mathbf{a} = \varepsilon_{inqs} T_{i_1 \dots i_{n-1} q} a_s \mathbf{e}_{i_1 \dots i_n},$$

which in the particular case of $\mathbf{T}^{(n)} = \mathbf{b}^n$ becomes

$$\mathbf{b}^n \times \mathbf{a} = b_{i_1} \dots b_{i_{n-1}} (\mathbf{b} \times \mathbf{a})_{i_n} \mathbf{e}_{i_1 \dots i_n}.$$

The expansions (6) and (7) are running in the exterior of the minimal radius sphere including the support of ρ and \mathbf{J} .

From equations (6) and (7) one obtains the following expansions of the fields $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$:

$$\begin{aligned} \mathbf{E}(\mathbf{r}, t) = & -\nabla\Phi(\mathbf{r}, t) - \frac{\partial \mathbf{A}(\mathbf{r}, t)}{dt} = \frac{1}{4\pi\varepsilon_0} \sum_{n \geq 0} \frac{(-1)^{n-1}}{n!} \nabla^{n+1} \left\| \frac{\mathbf{P}^{(n)}(\tau)}{r} \right. \\ & \left. - \frac{\mu_0}{4\pi} \sum_{n \geq 1} \frac{(-1)^{n-1}}{n!} \left[\nabla \times \left(\nabla^{n-1} \left\| \frac{\dot{\mathbf{M}}^{(n)}(\tau)}{r} \right\| \right) + \nabla^{n-1} \left\| \frac{\ddot{\mathbf{P}}^{(n)}(\tau)}{r} \right\| \right] \right. \end{aligned} \quad (9)$$

and

$$\begin{aligned} \mathbf{B}(\mathbf{r}, t) = & \nabla \times \mathbf{A}(\mathbf{r}, t) \\ = & \frac{\mu_0}{4\pi} \sum_{n \geq 1} \frac{(-1)^{n-1}}{n!} \left[\nabla^{n+1} \left\| \frac{\mathbf{M}^{(n)}(\tau)}{r} \right\| - \nabla^{n-1} \left\| \Delta \frac{\mathbf{M}^{(n)}(\tau)}{r} \right\| + \nabla \times \left(\nabla^{n-1} \left\| \frac{\dot{\mathbf{P}}^{(n)}(\tau)}{r} \right\| \right) \right] \end{aligned} \quad (10)$$

For the magnetic moments (8), it is also possible to introduce **STF** moments $\mathcal{M}^{(n)} = \mathcal{T}(\mathbf{M}^{(n)})$ but, this time, there are two steps required in order to complete the objective. The tensor $\mathbf{M}^{(n)}$ is symmetric only in the first $n-1$ indexes and satisfies the property

$$M_{i_1 \dots i_{n-2} q q} = 0.$$

In the first step we must obtain the symmetric projection (up to a numerical factor) $\overset{\leftrightarrow}{\mathbf{M}}^{(n)}$ of the tensor $\mathbf{M}^{(n)}$. We begin with the first simple example corresponding to $n=2$. Let us write the identity

$$M_{ij} = \frac{1}{2} (M_{ij} + M_{ji}) + \frac{1}{2} (M_{ij} - M_{ji}) = \overset{\leftrightarrow}{M}_{ij} + \frac{1}{2} \varepsilon_{ijk} N_k(\mathbf{M}^{(2)}),$$

where $\overset{\leftrightarrow}{\mathbf{M}}^{(2)}$ is the symmetric part of $\mathbf{M}^{(2)}$ and

$$\mathbf{N}_i(\mathbf{M}^{(2)}) = \varepsilon_{ijk} M_{jk} = \frac{2}{3} \int_{\mathcal{D}} d^3x [\mathbf{r} \times (\mathbf{r} \times \mathbf{J})] .$$

In this case ($n = 2$), $\mathcal{M}^{(2)} = \overset{\leftrightarrow}{\mathbf{M}}^{(2)}$ and, consequently, corresponds to the **STF** projection. Therefore,

$$M_{ij} = \mathcal{M}_{ij} + \frac{1}{2} \varepsilon_{ijk} N_k(\mathbf{M}^{(2)}) . \quad (11)$$

For $n \geq 3$, we can generalize this result writing the identity:

$$\begin{aligned} M_{i_1 \dots i_n} &= \frac{1}{n} (M_{i_1 \dots i_n} + M_{i_n i_2 \dots i_{n-1} i_1} + \dots + M_{i_1 \dots i_n i_{n-1}}) \\ &+ \frac{1}{n} [(M_{i_1 \dots i_n} - M_{i_n \dots i_{n-1} i_1}) + \dots (M_{i_1 \dots i_n} - M_{i_1 \dots i_{n-2} i_n i_{n-1}})] \\ &= \overset{\leftrightarrow}{M}_{i_1 \dots i_n} + \frac{1}{n} \sum_{\lambda=1}^{n-1} \varepsilon_{i_\lambda i_n q} N_{(i_1 \dots i_{n-1} q)}^{(\lambda)}(\mathbf{M}^{(n)}) , \end{aligned} \quad (12)$$

where by $N_{(i_1 \dots i_{n-1} i_n)}^{(\lambda)}$ we understand the component without the index i_λ and the tensor $\mathbf{N}^{(n-1)} = \mathbf{N}(\mathbf{M}^{(n)})$ is given by

$$\mathbf{N}_{i_1 \dots i_{n-1}}(\mathbf{M}^{(n)}) = \varepsilon_{i_{n-1} p q} M_{i_1 \dots i_{n-2} p q} = \frac{n}{n+1} \int_{\mathcal{D}} d^3x x_{i_1} \dots x_{i_{n-2}} [\mathbf{r} \times (\mathbf{r} \times \mathbf{J})]_{i_{n-1}} .$$

It is a tensor of the same type as $\mathbf{M}^{(n-1)}$, i.e. symmetric in the first $n-2$ indices and with $n-1$ vanishing traces ($N_{i_1 \dots i_{n-3} q q} = 0$). Therefore, the **STF** moment $\mathcal{M}^{(n)}$ is given by the components

$$\mathcal{M}_{i_1 \dots i_n} = \overset{\leftrightarrow}{M}_{i_1 \dots i_n} - \delta_{\{i_1 i_2} \Lambda_{i_3 \dots i_n\}}(\overset{\leftrightarrow}{\mathbf{M}}^{(n)}) .$$

As it will be seen in the dynamic case, we have to express even the symmetric projection of the tensor $\mathbf{N}(\mathbf{M}^{(n)})$. For this, it is useful to introduce the operator \mathcal{N} defining the correspondence

$$\mathbf{N}^{(n)} \rightarrow \mathcal{N}(\mathbf{N}^{(n)}) : (\mathcal{N}(\mathbf{N}^{(n)}))_{i_1 \dots i_{n-1}} = \varepsilon_{i_{n-1} p s} \mathbf{N}_{i_1 \dots i_{n-2} p s} .$$

Repeating this operation, we obtain:

$$\begin{aligned} \mathcal{N}^{2k}(\mathbf{M}^{(n)}) &= \frac{(-1)^k n}{n+1} \int_{\mathcal{D}} d^3x r^{2k} \mathbf{r}^{n-2k} \times \mathbf{J} , \\ \mathcal{N}^{2k+1}(\mathbf{M}^{(n)}) &= \frac{(-1)^k n}{n+1} \int_{\mathcal{D}} d^3x r^{2k} \mathbf{r}^{n-2k-1} \times (\mathbf{r} \times \mathbf{J}) . \end{aligned}$$

2.3. Delta-function identities and multipole singularities

Usually, the multipole expansions are considered, term by term, as functions of \mathbf{r} defined on \mathbb{R}^3 excepting a singular point which is chosen as the origin O of the Cartesian axes. Actually, these multipole terms are mathematical distributions (or generalized functions) considered firstly as regular ones having as support the entire space except the origin point O . The observable quantities are expressed as weighted averages on

spatial regions or as surface integrals of functions of field variables. No problems appear when in these regions $r \neq 0$, but when we have to calculate for example the interaction of a distribution (ρ, \mathbf{J}) with an external field (\mathbf{E}, \mathbf{B}) ,

$$W_{int} = \int d^3x (\rho \Phi - \mathbf{J} \cdot \mathbf{A})$$

and the system associated with the external field (\mathbf{E}, \mathbf{B}) is represented by a point-like multipole system placed in O , the singularities of the potentials or fields having as support this point become unavoidable. As done in [2], these singularities are determined starting from some δ -function identities associated with the extension of multiple spatial derivatives of the functions of the type $1/r$ to the entire space. Having in mind the dynamic case, too, we generalize such identities to the derivatives of the function $f(\tau)/r$. Representing the corresponding derivatives as functions of \mathbf{r} and t for $r \neq 0$, we can write their expressions in the form

$$D_{i_1 \dots i_n}(f) \equiv \partial_{i_1} \dots \partial_{i_n} \frac{f(\tau)}{r} = \sum_{l=0}^n \frac{1}{c^{n-l} r^{l+1}} C_{i_1 \dots i_n}^{(n,l)} \frac{d^{n-l} f(\tau)}{dt^{n-l}}, \quad (13)$$

where $C_{i_1 \dots i_n}^{(n,l)}$ are fully symmetric in the indexes $i_1 \dots i_n$. The general form of the coefficients C is a simple consequence of the symmetry properties and of the derivative rules:

$$C_{i_1 \dots i_n}^{(n,l)} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} K_k^{(n,l)} \delta_{\{i_1 i_2 \dots i_{2k-1} i_{2k} \nu_{2k+1} \dots \nu_n\}}. \quad (14)$$

For lower n the coefficients C and K can be easily calculated by successive derivative operations. As one can see in the following, for obtaining general results for the δ -type singularities of the multipole terms, it will be necessary to know only the coefficients

$$K_0^{(n,n)} = (-1)^n (2n-1)!! . \quad (15)$$

In Appendix A we enumerate the coefficients $C_{i_1 \dots i_n}^{(n,l)}$ for the first five values of n .

Considering the distribution representing the derivative (13) extended to the entire space, we understand by $(\partial_{i_1} \dots \partial_{i_n} (f(\tau)/r))_{(0)}$ the singular part having as support the point O . In the following, we are interested in writing explicitly this contribution.

A correct procedure for extracting this singular part is that employed in [2] defining

$$\begin{aligned} & \left\langle \left(\partial_{i_1} \dots \partial_{i_n} \frac{f(\tau)}{r} \right)_{(0)}, \phi \right\rangle = \lim_{\varepsilon \rightarrow 0} \int_{\mathcal{D}_\varepsilon} d^3x \partial_{i_1} \dots \partial_{i_n} \frac{f(\tau)}{r} \phi(\mathbf{r}) \\ & = \lim_{\varepsilon \rightarrow 0} \left[\oint_{\Sigma_\varepsilon} dS \nu_{i_1} \partial_{i_2} \dots \partial_{i_n} \frac{f(\tau)}{r} - \int_{\mathcal{D}_\varepsilon} d^3x \partial_{i_2} \dots \partial_{i_n} \frac{f(\tau)}{r} \partial_{i_1} \phi(\mathbf{r}) \right]. \quad (16) \end{aligned}$$

In this equation, $\phi(\mathbf{r})$ is supposed an element of the domain of distributions i.e. an infinitely differentiable function. As it can be seen from equation (13), similar restrictive properties are considered for $f(\tau)$. The domain \mathcal{D}_ε is the spherical region delimited by the spherical surface Σ_ε with radius ε . A basic hypothesis is the existence of the limits considered here, but the general demonstration of this property is a purely mathematical

problem and bypasses the objectives of the present paper. Therefore, the result of the limit considered is independent of the domains \mathcal{D}_ε defined only by the condition $\mathcal{D}_\varepsilon \rightarrow O$. Finally, for the calculation process in the present paper, the essential property is the possibility to represent the functions f and ϕ in the domain \mathcal{D}_ε by their Taylor series upon the origin O :

$$f(\tau) = \sum_{\lambda=0}^{\infty} \frac{(-1)^\lambda r^\lambda}{c^\lambda \lambda!} \left(\frac{d^\lambda f(\tau)}{dt^\lambda} \right)_{r=0},$$

$$\phi(\mathbf{r}) = \sum_{\alpha=0}^{\infty} \frac{1}{\alpha!} x_{i_1} \dots x_{i_\alpha} (\partial_{i_1} \dots \partial_{i_\alpha} \phi(\mathbf{r}))_{r=0} = \sum_{\alpha=0}^{\infty} \frac{r^\alpha}{\alpha!} \nu_{i_1} \dots \nu_{i_\alpha} (\partial_{i_1} \dots \partial_{i_\alpha} \phi(\mathbf{r}))_{r=0}. \quad (17)$$

Equation (16) can be written as

$$\left\langle (D_{i_1 \dots i_n}(f))_{(0)}, \phi \right\rangle = \lim_{\varepsilon \rightarrow 0} \oint_{\Sigma_\varepsilon} dS \nu_{i_1} \partial_{i_2} \dots \partial_{i_n} \frac{f(\tau)}{r} - \left\langle (D_{i_2 \dots i_n}(f))_{(0)}, \partial_{i_1} \phi \right\rangle$$

generating a recursive calculation for a given n .

Let us express the δ -singularities for lower order derivatives. Even in these simple cases, it becomes obvious the importance of the angular average of a function $g(\boldsymbol{\nu})$ defined as

$$\langle g(\boldsymbol{\nu}) \rangle = \frac{1}{4\pi} \int g(\boldsymbol{\nu}) d\Omega(\boldsymbol{\nu}).$$

For this average, we have the well known formula [6]:

$$\langle \nu_{i_1} \dots \nu_{i_n} \rangle = \begin{cases} 0, & n = 2k + 1, \\ \frac{1}{(n+1)!!} \delta_{\{i_1 i_2 \dots i_{n-1} i_n\}}, & n = 2k, \quad k = 0, 1, \dots \end{cases} \quad (18)$$

The first δ -singularity is obtaining for $n = 2$ calculating the limit

$$\begin{aligned} \left\langle (D_{ij}(f))_{(0)}, \phi \right\rangle &= \lim_{\varepsilon \rightarrow 0} \left[\oint_{\Sigma_\varepsilon} dS \nu_i \left(\partial_j \frac{f(\tau)}{r} \right) \phi(\mathbf{r}) - \int_{\mathcal{D}_\varepsilon} d^3x \left(\partial_j \frac{f(\tau)}{r} \right) \partial_i \phi(\mathbf{r}) \right] \\ &= \lim_{\varepsilon \rightarrow 0} \left[\oint_{\Sigma_\varepsilon} dS \nu_i \left(\partial_j \frac{f(\tau)}{r} \right) \phi(\mathbf{r}) - \oint_{\Sigma_\varepsilon} \frac{dS}{r} \nu_j f(\tau) \partial_i \phi(\mathbf{r}) + \int_{\mathcal{D}_\varepsilon} \frac{d^3x}{r} f(\tau) \partial_i \partial_j \phi(\mathbf{r}) \right]. \end{aligned}$$

The first surface integral limit can be written as

$$\mathbf{L}_{1\sigma} = \lim_{\varepsilon \rightarrow 0} \oint d\Omega(\boldsymbol{\nu}) \varepsilon^2 \nu_i \left[-\frac{\nu_j}{c\varepsilon} \dot{f}(\tau_\varepsilon) - \frac{\nu_j}{\varepsilon^2} f(\tau_\varepsilon) \right] [\phi(0) + \varepsilon \nu_k (\partial_k \phi)_0 + \dots],$$

where $\tau_\varepsilon = t - \varepsilon/c$. Since $f(\tau_\varepsilon) = f(t) + \mathcal{O}(\varepsilon)$, and all the terms containing as factors positive powers of ε vanish at the limit $\varepsilon \rightarrow 0$, we can write

$$\mathbf{L}_{1\sigma} = -4\pi \langle \nu_i \nu_j \rangle f(t) \phi(0) = -\frac{4\pi}{3} \delta_{ij} f(t) \phi(0).$$

The second surface integral limit

$$\mathbf{L}_{2\sigma} = \lim_{\varepsilon \rightarrow 0} \oint d\Omega(\boldsymbol{\nu}) \varepsilon \nu_j f(\tau_\varepsilon) [(\partial_i \phi)_0 + \varepsilon \nu_k (\partial_i \partial_k \phi)_0 + \dots] = 0,$$

since all the terms contain as factors positive powers of ε . As the volume integral limit also vanishes, we can finally write:

$$\left(\partial_i \partial_j \frac{f(\tau)}{r} \right)_{(0)} = -\frac{4\pi}{3} f(t) \delta_{ij} \delta(\mathbf{r}). \quad (19)$$

Including the expression of the derivative for $r \neq 0$, the above result written for $f(t) = 1$ verifies equation (3).

Let us consider the next order derivative and calculate

$$\begin{aligned} \left\langle (D_{ijk}(f))_{(0)}, \phi \right\rangle &= \lim_{\varepsilon \rightarrow 0} \left[\oint_{\Sigma_\varepsilon} dS \nu_i \left(\partial_j \partial_k \frac{f(\tau)}{r} \right) \phi(\mathbf{r}) - \int_{\mathcal{D}_\varepsilon} d^3x \left(\partial_j \partial_k \frac{f(\tau)}{r} \right) \partial_i \phi(\mathbf{r}) \right] = \\ &= \lim_{\varepsilon \rightarrow 0} \left\{ \oint d\Omega(\boldsymbol{\nu}) \varepsilon^2 \left[\frac{1}{c^2 \varepsilon} \nu_i \nu_j \nu_k \ddot{f}(\tau_\varepsilon) + \frac{1}{c \varepsilon^2} \nu_i (3\nu_j \nu_k - \delta_{jk}) \dot{f}(\tau_\varepsilon) + \frac{1}{\varepsilon^3} \nu_i (3\nu_j \nu_k - \delta_{jk}) f(\tau_\varepsilon) \right] \right. \\ &\quad \left. \times (\phi(0) + \varepsilon \nu_l (\partial_l \phi)_0 + \dots) + \frac{4\pi}{3} \int_{\mathcal{D}_\varepsilon} d^3x f(t) \delta_{jk} \delta(\mathbf{r}) \partial_i \phi(\mathbf{r}) \right\}, \end{aligned}$$

where equation (19) is employed in the volume integral. In the surface integral limit, as $\varepsilon \rightarrow 0$, the terms containing as factor the second order time derivative \ddot{f} vanish because of the positive powers of ε . The terms containing the first order time derivative \dot{f} are proportional to positive powers of ε for the term corresponding to $\phi(0)$, but in this case the corresponding limit cancels since it contains also as factors the null averages $\langle \nu_i \nu_j \nu_k \rangle$ and $\langle \nu_i \rangle$. The only terms giving a non vanishing limit are provided by the product $f(\tau_\varepsilon) (\partial_l \phi)_0$ such that we can write

$$\begin{aligned} \left\langle (D_{ijk}(f))_{(0)}, \phi \right\rangle &= 4\pi \langle 3\nu_i \nu_j \nu_k \nu_l - \delta_{jk} \nu_i \nu_l \rangle f(t) (\partial_l \phi)_0 + \frac{4\pi}{3} f(t) \delta_{jk} (\partial_i \phi)_0 \\ &= 4\pi \left(\frac{1}{5} \delta_{\{ij} \delta_{kl\}} - \frac{1}{3} \delta_{jk} \delta_{il} \right) f(t) (\partial_l \phi)_0 + \frac{4\pi}{3} f(t) \delta_{jk} (\partial_i \phi)_0. \end{aligned}$$

Finally:

$$\left(\partial_i \partial_j \partial_k \frac{f(\tau)}{r} \right)_{(0)} = -\frac{4\pi}{5} f(t) \delta_{\{ij} \partial_k \} \delta(\mathbf{r}). \quad (20)$$

For $f(t) = 1$, equation (20) becomes the δ -singularity corresponding to equation (13) from Ref. [2]. Equations (19) and (20) will be applied in the static case for $f(t) = 1$.

3. Singularities of the electrostatic field

Let us consider the multipole expansion of the electrostatic field derived from the potential expansion (6) in the static case:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \sum_{n \geq 0} \frac{(-1)^{n-1}}{n!} \nabla^{n+1} \left\| \frac{\mathbf{P}^{(n)}}{r} \right\| = \frac{1}{4\pi\varepsilon_0} \sum_{n \geq 0} \frac{(-1)^{n-1}}{n!} \mathbf{P}^{(n)} \left\| \nabla^{n+1} \frac{1}{r} \right\|.$$

In the dipolar case ($n = 1$), the potential has no δ -singularities. For the electric field, we apply equation (19) obtaining the known result (1).

A first interesting feature of the δ -singularities problem appears beginning with the electric quadrupole potential and field. The singularity of the potential is obtained employing equation (19):

$$\Phi^{(2)}(\mathbf{r}) = \left(\frac{1}{8\pi\varepsilon_0} \nabla^2 \left\| \frac{\mathbf{P}^{(2)}}{r} \right\| \right)_{r \neq 0} - \frac{1}{6\varepsilon_0} \mathbf{P}_{qq} \delta(\mathbf{r}).$$

The first term from the right-hand side of this equation is invariant to the substitution $\mathbf{P}^{(2)} \rightarrow \mathcal{P}^{(2)} = \mathcal{T}(\mathbf{P}^{(2)})$ and, employing equations (4) and (5) with $\mathbf{S} = \mathbf{P}^{(2)}$, we can write

$$\Phi^{(2)}(\mathbf{r}) = \left(\frac{1}{8\pi\epsilon_0} \nabla^2 \left\| \frac{\mathcal{P}^{(2)}}{r} \right\| \right)_{r \neq 0} - \frac{1}{2\epsilon_0} \Lambda \delta(\mathbf{r}) .$$

Usually, since in the exterior of the charge distribution the potential $\Phi(\mathbf{r})$ is solution of the Laplace equation, it is represented by a series of spherical functions:

$$\Phi(\mathbf{r}) = \sum_{l=0}^{\infty} \frac{1}{r^{l+1}} \sum_{m=-l}^l Q_{lm} Y_{lm}(\theta, \varphi) . \quad (21)$$

The $2l + 1$ spherical moments corresponding to a given l are linear combinations of the $2l + 1$ components of the l -th order **STF** tensor $\mathcal{P}^{(l)}$. For the above example of the 4-polar potential, it is obvious that using the expansion representing the particular case of the equation (21), the *delta*-singularity of the potential is lost. The conclusion can be extended to the higher-order multipolar potentials. For the electrostatic field $\mathbf{E}(\mathbf{r})$, the employment of the corresponding expansion in spherical functions leads to an incomplete description of the δ -form singularities.

Based on this observation, in our opinion, some care is necessary when passing to the representations by spherical functions since it is equivalent to working only in a subspace of the mathematical objects used correctly in the theory. The direct substitution $\mathbf{P} \rightarrow \mathcal{P}$ or passing to the spherical tensorial representations is not always justified. This observation is pointed out in another context in [12], too.

Searching the δ -singularities of $\mathbf{E}^{(2)}(\mathbf{r})$, instead of a straightforward processing of the corresponding expression in terms of the primitive moment $\mathbf{P}^{(2)}$, we can introduce from the beginning the **STF** moment $\mathcal{P}^{(2)}$ writing

$$\begin{aligned} (\mathbf{E}^{(2)}(\mathbf{r}))_{(0)} &= -\frac{1}{8\pi\epsilon_0} \left(\nabla^3 \left\| \frac{\mathbf{P}^{(2)}}{r} \right\| \right)_{(0)} = -\frac{1}{8\pi\epsilon_0} \left(\nabla^3 \left\| \frac{\mathcal{P}^{(2)}}{r} + \mathbf{e}_i \partial_i \Delta \frac{\Lambda}{r} \right\| \right)_{(0)} \\ &= -\frac{1}{8\pi\epsilon_0} \left(\nabla^3 \left\| \frac{\mathcal{P}^{(2)}}{r} \right\| \right)_{(0)} + \frac{1}{2\epsilon_0} \Lambda \delta(\mathbf{r}) , \end{aligned}$$

since $\Delta(1/r) = -4\pi\delta(\mathbf{r})$. Equation (20) finally gives

$$\mathbf{E}^{(2)}(\mathbf{r}) = (\mathbf{E}^{(2)}(\mathbf{r}))_{r \neq 0} + \frac{1}{5\epsilon_0} \mathcal{P}^{(2)} \left\| \nabla \delta(\mathbf{r}) \right\| + \frac{1}{2\epsilon_0} \Lambda \nabla \delta(\mathbf{r}), \quad \Lambda = \Lambda(\mathbf{P}^{(2)}) , \quad (22)$$

where

$$(\mathbf{E}^{(2)}(\mathbf{r}))_{r \neq 0} = -\frac{1}{8\pi\epsilon_0} \left(\nabla^3 \left\| \frac{\mathbf{P}^{(2)}}{r} \right\| \right)_{r \neq 0} = -\frac{1}{8\pi\epsilon_0} \left(\nabla^3 \left\| \frac{\mathcal{P}^{(2)}}{r} \right\| \right)_{r \neq 0} .$$

Even from this simple example it becomes obvious that this calculation version is convenient for higher n since some contractions of $\mathcal{P}^{(n)}$ with the Kronecker symbols vanish.

In case $n = 3$, the advantage of inserting from the very beginning the **STF** moments is more evident. Instead of searching the singularities of the fourth-order derivative

$\partial_i \partial_j \partial_k \partial_l (1/r)$, we search the singularities of the contraction of this derivative with the **STF** moment $\mathcal{P}^{(3)}$ writing

$$\left\langle \left(\nabla^4 \parallel \frac{\mathcal{P}^{(3)}}{r} \right)_{(0)}, \phi \right\rangle = \mathbf{e}_i \lim_{\varepsilon \rightarrow 0} \left[\oint_{\Sigma_\varepsilon} dS \nu_i \partial_j \partial_k \partial_l \frac{\mathcal{P}_{jkl}}{r} \phi(\mathbf{r}) - \int_{\mathcal{D}_\varepsilon} d^3x \partial_j \partial_k \partial_l \frac{\mathcal{P}_{jkl}}{r} \partial_i \phi(\mathbf{r}) \right].$$

For the surface integral limit, we apply equation (13) particularized to the static case. Denoting by \mathbf{L}_σ this limit,

$$\mathbf{L}_\sigma = \mathbf{e}_i \lim_{\varepsilon \rightarrow 0} \oint_{\Sigma_\varepsilon} \frac{d\Omega(\boldsymbol{\nu})}{\varepsilon^2} \nu_i C_{jkl}^{(3,3)} \mathcal{P}_{jkl} \phi(\mathbf{r})$$

and, inserting equation (A.4), one obtains

$$\mathbf{L}_\sigma = \mathbf{e}_i \lim_{\varepsilon \rightarrow 0} \oint_{\Sigma_\varepsilon} \frac{d\Omega(\boldsymbol{\nu})}{\varepsilon^2} (-15 \nu_i \nu_j \nu_k \nu_l + 3 \nu_i \delta_{\{jk} \nu_l\}) \mathcal{P}_{jkl} \phi(\mathbf{r}).$$

Since $\delta_{\{jk} \nu_l\} \mathcal{P}_{jkl} = 0$, we get

$$\begin{aligned} \mathbf{L}_\sigma &= -15 \mathbf{e}_i \lim_{\varepsilon \rightarrow 0} \oint_{\Sigma_\varepsilon} \frac{d\Omega(\boldsymbol{\nu})}{\varepsilon^2} \nu_i \nu_j \nu_k \nu_l \mathcal{P}_{jkl} \left(\phi(0) + \varepsilon \nu_q (\partial_q \phi)_0 + \frac{1}{2} \varepsilon^2 \nu_q \nu_s (\partial_q \partial_s \phi)_0 \right) \\ &= -4\pi \times 15 \mathbf{e}_i \lim_{\varepsilon \rightarrow 0} \left\langle \frac{1}{\varepsilon^2} \nu_i \nu_j \nu_k \nu_l \mathcal{P}_{jkl} \left(\phi(0) + \varepsilon \nu_q (\partial_q \phi)_0 + \frac{1}{2} \varepsilon^2 \nu_q \nu_s (\partial_q \partial_s \phi)_0 \right) \right\rangle. \end{aligned}$$

Denoting by α the power of ε in the Taylor series of $\phi(\mathbf{r})$, we easily see that for $\alpha \geq 3$ all the terms from the last equation vanish for $\varepsilon \rightarrow 0$. For $\alpha = 0$, the corresponding term vanishes since $\langle \nu_i \nu_j \nu_k \nu_l \rangle \mathcal{P}_{jkl} = 0$, as it is seen by inserting equation (18). For $\alpha = 1$, the related term contains the average of an odd number of factors ν yielding a null result, too. The only non vanishing result corresponds to $\alpha = 2$ such that

$$\mathbf{L}_\sigma = -4\pi \times \frac{15}{2} \mathbf{e}_i \langle \nu_i \nu_j \nu_k \nu_l \nu_q \nu_s \rangle \mathcal{P}_{jkl}.$$

In the last equation we have the contraction $\langle \nu_{i_1} \nu_{i_2} \nu_{i_3} \nu_{i_4} \nu_{i_5} \nu_{i_6} \rangle \mathcal{P}_{i_4 i_5 i_6}$. It is not necessary to write explicitly the angular average of the six factors ν since, considering the general form of this expression given in equation (18), we see that only the $3! = 6$ terms of the form $\delta_{i_1 i_j} \delta_{i_2 i_k} \delta_{i_3 i_l}$ with the indexes j, k, l covering all the permutations of (4, 5, 6) give non vanishing limits. Consequently,

$$\mathbf{L}_\sigma = -4\pi \frac{15 \times 3!}{2 \times 7!!} \mathbf{e}_i \mathcal{P}_{ijk} (\partial_j \partial_k \phi)_0 = -\frac{24\pi}{14} \mathcal{P}^{(3)} \parallel (\nabla^2 \phi)_0. \quad (23)$$

The volume integral limit is zero since

$$\left(\nabla^3 \parallel \frac{\mathcal{P}^{(3)}}{r} \right)_{(0)} = 0,$$

as it can be easily verified. Equation (23) implies

$$\left(\nabla^4 \parallel \frac{\mathcal{P}^{(3)}}{r} \right)_{(0)} = -\frac{24}{14} \mathcal{P}^{(3)} \parallel \nabla^2 \delta(\mathbf{r}). \quad (24)$$

In Appendix B this equality is proven for the general case. Let us write

$$(\mathbf{E}^{(3)}(\mathbf{r}))_{(0)} = \frac{1}{24\pi\varepsilon_0} \left(\nabla^4 \parallel \frac{\mathbf{P}^{(3)}}{r} \right)_{(0)} = -\frac{1}{14\varepsilon_0} \mathcal{P}^{(3)} \parallel \nabla^2 \delta(\mathbf{r}) - \frac{1}{2\varepsilon_0} \boldsymbol{\Lambda} \parallel \nabla^2 \delta(\mathbf{r})$$

or

$$\mathbf{E}^{(3)}(\mathbf{r}) = (\mathbf{E}^{(3)}(\mathbf{r}))_{r \neq 0} - \frac{1}{14\epsilon_0} \mathcal{P}^{(3)} \|\nabla^2 \delta(\mathbf{r}) - \frac{1}{2\epsilon_0} \mathbf{\Lambda} \|\nabla^2 \delta(\mathbf{r}) ,$$

where

$$(\mathbf{E}^{(3)}(\mathbf{r}))_{r \neq 0} = \frac{1}{24\pi\epsilon_0} \nabla^4 \|\frac{\mathbf{P}^{(3)}}{r} = \frac{1}{24\pi\epsilon_0} \nabla^4 \|\frac{\mathcal{P}^{(3)}}{r} .$$

Even if only for the mathematical interest, we mention the possibility of expressing the multipole electrostatic field for arbitrary n . In Appendix B it is introduced the general relation corresponding to equation (24):

$$\left(\nabla^{n+1} \|\frac{\mathcal{P}^{(n)}}{r} \right)_{(0)} = -\frac{4\pi n}{2n+1} \mathcal{P}^{(n)} \|\nabla^{n-1} \delta(\mathbf{r})$$

such that

$$\begin{aligned} (\mathbf{E}^{(n)}(\mathbf{r}))_{(0)} &= \frac{(-1)^{n-1}}{4\pi\epsilon_0 n!} \left(\nabla^{n+1} \|\frac{\mathcal{P}^{(n)}}{r} + \frac{n(n-1)}{2} \nabla^{n-1} \|\Delta \frac{\mathbf{\Lambda}^{(n-2)}}{r} \right)_{(0)} \\ &= \frac{(-1)^n n}{\epsilon_0} \left(\frac{1}{n!(2n+1)} \mathcal{P}^{(n)} + \frac{n-1}{2n!} \mathbf{\Lambda}^{(n-2)} \right) \|\nabla^{n-1} \delta(\mathbf{r}) . \end{aligned} \quad (25)$$

We point out that for $n \geq 4$, for obtaining the complete singular structure of the field, we have to introduce the irreducible tensors in the expression from equation (25) corresponding to the δ -singularities . This implies a recursive calculation introducing in equation (25) the **STF** projections of $\mathbf{\Lambda}^{(n-2)}$, $\mathbf{\Lambda}^{(n-4)}$, \dots . For example, in the case $n = 4$, we have to reduce the tensor $\mathbf{\Lambda}^{(2)} = \mathbf{\Lambda}(\mathbf{P}^{(4)})$ writing:

$$\mathbf{\Lambda}^{(2)} \|\nabla^3 \delta(\mathbf{r}) = \mathcal{T}(\mathbf{\Lambda}^{(2)}) \|\nabla^3 \delta(\mathbf{r}) + \mathbf{\Lambda}(\mathbf{\Lambda}^{(2)}) \nabla \Delta \delta(\mathbf{r}) ,$$

where

$$\begin{aligned} \mathbf{\Lambda}(\mathbf{\Lambda}^{(2)}) &= \frac{1}{30} \mathbf{P}_{qqss} , \\ \mathcal{T}_{ij}(\mathbf{\Lambda}^{(2)}) &= \frac{1}{7} \left(\mathbf{P}_{qqij} - \frac{1}{3} \mathbf{P}_{qqss} \delta_{ij} \right) . \end{aligned}$$

4. Singularities of the magnetostatic field

Let us write the multipole expansions of the vector potential $\mathbf{A}(\mathbf{r})$ and of the magnetic field $\mathbf{B}(\mathbf{r})$ in the static case:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \sum_{n \geq 1} \frac{(-1)^{n-1}}{n!} \nabla \times (\nabla^{n-1} \|\mathbf{M}^{(n)})$$

and, respectively,

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \sum_{n \geq 1} \frac{(-1)^{n-1}}{n!} \left[\nabla \left(\nabla^n \|\frac{\mathbf{M}^{(n)}}{r} \right) - \Delta \left(\nabla^{n-1} \|\frac{\mathbf{M}^{(n)}}{r} \right) \right] . \quad (26)$$

For $r \neq 0$, the multipole magnetic field $\mathbf{B}(\mathbf{r})$ is expressed, as in the electric case, only by a scalar potential since the second term from the right-hand side of equation

(26) vanishes. However, the δ -singularity of the magnetic field is not similar to the corresponding singularity of the electric field, this second term giving the difference. The difference becomes easily obvious if one considers the fictitious magnetic shells (or sheets) employed in the Ampère formalism. It suffices to consider the case of the point-like magnetic dipole which can be taken as the limit of a current loop of infinitesimal size. For finite dimensions, the field of the loop is derived from a scalar potential Φ_m which is defined by an integral on the corresponding sheet and having a “jump” in all their points. Just this jump generates the δ -singularity corresponding to the second term from equation (26). An explicit calculation of this limit when the loop concentrates in a point is given in Ref. [13].

In the dipolar case ($n = 1$), retaining the singular term, we have

$$(\mathbf{B}^{(1)}(\mathbf{r}))_{(0)} = \frac{\mu_0}{4\pi} \left(\mathbf{e}_i \partial_i \partial_j \frac{m_j}{r} \right)_{(0)} + 4\pi \mathbf{m} \delta(\mathbf{r}) .$$

Processing the first term as in the electrostatic case, we obtain the well-known expression (2). No δ -singularity is present in the dipolar vector potential.

Let us consider the quadrupole term from equation (26) and the corresponding singularities:

$$(\mathbf{B}^{(2)}(\mathbf{r}))_{(0)} = -\frac{\mu_0}{8\pi} \left(\nabla^3 \parallel \frac{\mathbf{M}^{(2)}}{r} - \nabla \parallel \left(\Delta \frac{\mathbf{M}^{(2)}}{r} \right) \right)_{(0)} .$$

By introducing the **STF** moment $\mathcal{M}^{(2)} \stackrel{\leftrightarrow}{=} \mathbf{M}^{(2)}$ defined by equation (11),

$$\begin{aligned} (\mathbf{B}^{(2)}(\mathbf{r}))_{(0)} &= -\frac{\mu_0}{8\pi} \left(\nabla^3 \parallel \frac{\mathcal{M}^{(2)}}{r} + \frac{1}{2} \mathbf{e}_i \varepsilon_{jkl} \partial_i \partial_j \partial_k \frac{N_k}{r} - \mathbf{e}_i \partial_j \Delta \frac{\mathcal{M}_{ji}}{r} + \frac{1}{2} \varepsilon_{ijl} \partial_j \Delta \frac{N_l}{r} \right)_{(0)} \\ &= -\frac{3\mu_0}{10} \mathcal{M}^{(2)} \parallel \nabla \delta(\mathbf{r}) - \frac{\mu_0}{4} \mathbf{N} \times \nabla \delta(\mathbf{r}) , \end{aligned}$$

where $\mathbf{N} = \mathbf{e}_i N_i$. It follows that

$$\mathbf{B}^{(2)}(\mathbf{r}) = (\mathbf{B}^{(2)}(\mathbf{r}))_{r \neq 0} - \frac{3\mu_0}{10} \mathcal{M}^{(2)} \parallel \nabla \delta(\mathbf{r}) - \frac{\mu_0}{4} \mathbf{N} \times \nabla \delta(\mathbf{r}) \quad (27)$$

where

$$(\mathbf{B}^{(2)}(\mathbf{r}))_{r \neq 0} = \left(-\frac{\mu_0}{8\pi} \nabla^3 \parallel \frac{\mathcal{M}^{(2)}}{r} \right)_{r \neq 0} = \left(-\frac{\mu_0}{8\pi} \nabla^3 \parallel \frac{\mathbf{M}^{(2)}}{r} \right)_{r \neq 0} .$$

Let us consider the quadrupolar vector potential $\mathbf{A}^{(2)}(\mathbf{r})$ with equation (11) already inserted:

$$\begin{aligned} \mathbf{A}^{(2)}(\mathbf{r}) &= -\frac{\mu_0}{8\pi} \mathbf{e}_i \varepsilon_{ijk} \partial_j \partial_l \frac{M_{lk}}{r} = -\frac{\mu_0}{8\pi} \nabla \times \left(\nabla \parallel \frac{\mathcal{M}^{(2)}}{r} \right) \\ &\quad + \frac{\mu_0}{16\pi} \nabla \left(\nabla \cdot \frac{\mathbf{N}}{r} \right) - \frac{\mu_0}{16\pi} \mathbf{N} \Delta \frac{1}{r} . \end{aligned} \quad (28)$$

The term proportional to $\Delta(1/r)$ vanishes for $r \neq 0$. Whereas $\mathbf{B}^{(2)}$ is invariant to the substitution $\mathbf{P}^{(2)} \rightarrow \mathcal{P}^{(2)}$, this property is not verified by the vector potential $\mathbf{A}^{(2)}$. This substitution has as result an additional gradient term, i.e. a gauge transformation of the potential. Though, by the extension to the entire space, the gradient term

from expression (28) generates a δ -singularity, this term gives no contribution to the singularities of the magnetic field. The singularities of \mathbf{B} are, by definition, only the singularities corresponding to the curl of the vector potential. It remains still the problem of independence of the physical results on the \mathbf{A} δ -singularities corresponding to different gauges. Maybe, the presence of such singularities in the expression of an interaction Hamiltonian corresponding to the density term $\mathbf{J} \cdot \mathbf{A}$ can generate the problem of proving the gauge independence of physical results as in the case of the Aharonov-Bohm effect.

We can give, as in the electrostatic case, the general formula for the singularities of the magnetic field writing

$$(\mathbf{B}^{(n)}(\mathbf{r}))_{(0)} = \frac{(-1)^{n-1} \mu_0}{4\pi n!} \left(\nabla^{n+1} \left\| \frac{\mathbf{M}^{(n)}}{r} - \nabla^{n-1} \left\| \Delta \frac{\mathbf{M}^{(n)}}{r} \right\|_{(0)} \right). \quad (29)$$

The introduction of the symmetric moment $\overset{\leftrightarrow}{\mathbf{M}}^{(n)}$, equation (12), gives

$$\nabla^{n+1} \left\| \frac{\mathbf{M}^{(n)}}{r} \right. = \nabla^{n+1} \left\| \frac{\overset{\leftrightarrow}{\mathbf{M}}^{(n)}}{r} + \frac{1}{n} \mathbf{e}_i \partial_i \partial_{i_1} \dots \partial_{i_n} \sum_{\lambda=1}^{n-1} \varepsilon_{i_\lambda i_n q} \frac{\mathbf{N}_{(i_1 \dots i_{n-1} q)}^{(\lambda)}}{r} = \nabla^{n+1} \left\| \frac{\overset{\leftrightarrow}{\mathbf{M}}^{(n)}}{r} \right.,$$

since in all terms of the sum there are the contractions $\varepsilon_{i_\lambda i_n q} \partial_{i_\lambda} \partial_{i_n}$ for $l = 1, \dots, n-1$ which vanish. From equation (25) we can write a corresponding result for the magnetic field with the substitutions $\mathbf{P}^{(n)} \rightarrow \overset{\leftrightarrow}{\mathbf{M}}^{(n)}$, $\mathcal{P}^{(n)} \rightarrow \mathcal{M}^{(n)}$, $\varepsilon_0 \rightarrow 1/\mu_0$:

$$\begin{aligned} \frac{(-1)^{n-1} \mu_0}{4\pi n!} \left(\nabla^{n+1} \left\| \frac{\overset{\leftrightarrow}{\mathbf{M}}^{(n)}}{r} \right\|_{(0)} \right) &= (-1)^n \mu_0 \left(\frac{n}{n! (2n+1)} \mathcal{M}^{(n)} \left\| \nabla^{n-1} \delta(\mathbf{r}) \right. \right. \\ &\quad \left. \left. + \frac{n(n-1)}{2n!} \tilde{\mathbf{\Lambda}}^{(n-2)} \left\| \nabla^{n-1} \delta(\mathbf{r}) \right\| \right), \end{aligned}$$

where $\tilde{\mathbf{\Lambda}}^{(n-2)} = \mathbf{\Lambda}(\overset{\leftrightarrow}{\mathbf{M}}^{(n)})$. The second term from equation (29) is processed as follows:

$$\begin{aligned} \nabla^{n-1} \left\| \Delta \frac{\mathbf{M}^{(n)}}{r} \right. &= \nabla^{n-1} \left\| \Delta \frac{\overset{\leftrightarrow}{\mathbf{M}}^{(n)}}{r} + \frac{1}{n} \mathbf{e}_i \partial_{i_1} \dots \partial_{i_{n-1}} \sum_{\lambda=1}^{n-1} \varepsilon_{i_\lambda i q} \Delta \frac{\mathbf{N}_{(i_1 \dots i_{n-1} q)}^{(\lambda)}}{r} \right. \\ &= \nabla^{n-1} \left\| \Delta \frac{\overset{\leftrightarrow}{\mathbf{M}}^{(n)}}{r} + \frac{n-1}{n} \mathbf{e}_i \varepsilon_{i q i_{n-1}} \mathbf{N}_{i_1 \dots i_{n-2} q} \partial_{i_1} \dots \partial_{i_{n-1}} \Delta \frac{1}{r} \right. \end{aligned}$$

The term containing $\overset{\leftrightarrow}{\mathbf{M}}^{(n)}$ can be further transformed to:

$$\begin{aligned} \overset{\leftrightarrow}{\mathbf{M}}^{(n)} \left\| \nabla^{n-1} \delta(\mathbf{r}) \right. &= \mathcal{M}^{(n)} \left\| \nabla^{n-1} \delta(\mathbf{r}) \right. + \mathbf{e}_i \partial_{i_1} \dots \partial_{i_{n-1}} \delta(\mathbf{r}) \delta_{\{i_1 i_2 \tilde{\mathbf{\Lambda}}_{i_3 \dots i_{n-1}} i\}} \\ &= \mathcal{M}^{(n)} \left\| \nabla^{n-1} \delta(\mathbf{r}) \right. + \frac{(n-1)(n-2)}{2} \mathbf{e}_i \partial_{i_1} \dots \partial_{i_{n-3}} \Delta \delta(\mathbf{r}) \tilde{\mathbf{\Lambda}}_{i_1 \dots i_{n-3} i} \\ &\quad + (n-1) \mathbf{e}_i \partial_i \partial_{i_1} \dots \partial_{i_{n-2}} \delta(\mathbf{r}) \tilde{\mathbf{\Lambda}}_{i_1 \dots i_{n-2}} \\ &= \mathcal{M}^{(n)} \left\| \nabla^{n-1} \delta(\mathbf{r}) \right. + \frac{(n-1)(n-2)}{2} \tilde{\mathbf{\Lambda}}^{n-2} \left\| \nabla^{n-3} \Delta \delta(\mathbf{r}) \right. + (n-1) \tilde{\mathbf{\Lambda}}^{n-2} \left\| \nabla^{n-1} \delta(\mathbf{r}) \right. \end{aligned}$$

Finally,

$$\begin{aligned} (\mathbf{B}^{(n)}(\mathbf{r}))_{(0)} &= \frac{(-1)^n \mu_0}{n!} \left[\left(-\frac{n+1}{2n+1} \mathcal{M}^{(n)} + \frac{(n-1)(n-2)}{2} \tilde{\mathbf{\Lambda}}^{(n-2)} \right) \|\nabla^{n-1} \delta(\mathbf{r}) \right. \\ &\quad \left. - \frac{n-1}{n} \mathbf{N}^{(n-1)} \mid \times \|\nabla^{n-1} \delta(\mathbf{r}) - \frac{(n-1)(n-2)}{2} \tilde{\mathbf{\Lambda}}^{(n-2)} \|\nabla^{n-3} \Delta \delta(\mathbf{r}) \right] \end{aligned} \quad (30)$$

where the notation

$$(\mathbf{A}^{(n)} \mid \times \mathbf{B}^{(n)})_{i_1 \dots i_n} = \varepsilon_{inqs} \mathbf{A}_{i_1 \dots i_{n-1} q} \mathbf{B}_{i_1 \dots i_{n-1} s}$$

is introduced. The general expression is written as

$$\mathbf{B}^{(n)}(\mathbf{r}) = (\mathbf{B}^{(n)}(\mathbf{r}))_{r \neq 0} + (\mathbf{B}^{(n)}(\mathbf{r}))_{(0)}(\mathbf{r}),$$

where

$$(\mathbf{B}^{(n)}(\mathbf{r}))_{r \neq 0} = \frac{(-1)^{n-1} \mu_0}{4\pi n!} \nabla^{n+1} \|\frac{\mathcal{M}^{(n)}}{r} = \frac{(-1)^{n-1} \mu_0}{4\pi n!} \nabla^{n+1} \|\frac{\mathbf{M}^{(n)}}{r},$$

with the derivatives calculated for $r \neq 0$.

Here is the place for the same observation as at the end of Section (3), but in the singular term containing $\mathbf{N}^{(n-1)}$ the reduction to the **STF** tensor $\mathbf{T}^{(n-1)}$ begins from $n = 3$. Particularly,

$$\mathcal{T}(\mathbf{N}^{(2)}) = \overset{\langle - \rangle}{\mathbf{N}}^{(2)}, \quad \overset{\langle - \rangle}{\mathbf{N}}_{ij} = \frac{1}{2} (\mathbf{N}_{ij} + \mathbf{N}_{ji}),$$

and

$$\mathbf{N}^{(2)} \mid \times \|\nabla^2 \delta(\mathbf{r}) = \overset{\langle - \rangle}{\mathbf{N}}^{(2)} \mid \times \|\nabla^2 \delta(\mathbf{r}) + \frac{1}{2} \mathcal{N}(\mathbf{N}^{(2)}) \|\nabla^2 \delta(\mathbf{r}) - \frac{1}{2} \mathcal{N}(\mathbf{N}^{(2)}) \Delta \delta(\mathbf{r}).$$

Obviously, the analysis of the higher order singular terms is more complicated for the magnetic field than for the electric one, but it is purely a technical problem.

5. The dynamic case

Considering the n -th order multipole electric and magnetic fields, writing separately the contributions of the electric and magnetic multipoles,

$$\begin{aligned} \mathbf{E}^{(n,p)}(\mathbf{r}, t) &= \frac{(-1)^{n-1}}{4\pi \varepsilon_0 n!} \left(\nabla^{n+1} \|\frac{\mathbf{P}^{(n)}(\tau)}{r} - \frac{1}{c^2} \nabla^{n-1} \|\frac{\ddot{\mathbf{P}}^{(n)}(\tau)}{r} \right), \\ \mathbf{E}^{(n,m)}(\mathbf{r}, t) &= \frac{(-1)^n}{4\pi \varepsilon_0 c^2 n!} \nabla \times \left(\nabla^{n-1} \|\frac{\dot{\mathbf{M}}^{(n)}(\tau)}{r} \right) \end{aligned} \quad (31)$$

and

$$\begin{aligned} \mathbf{B}^{(n,p)}(\mathbf{r}, t) &= \frac{(-1)^{n-1} \mu_0}{4\pi n!} \nabla \times \left(\nabla^{n-1} \|\frac{\dot{\mathbf{P}}^{(n)}(\tau)}{r} \right), \\ \mathbf{B}^{(n,m)}(\mathbf{r}, t) &= \frac{(-1)^{n-1} \mu_0}{4\pi n!} \left(\nabla^{n+1} \|\frac{\mathbf{M}^{(n)}(\tau)}{r} - \nabla^{n-1} \|\Delta \frac{\mathbf{M}^{(n)}(\tau)}{r} \right), \end{aligned} \quad (32)$$

we search the corresponding δ -singularities. Processing the different terms from equation (31) and (32) for reducing the moment tensors to the **STF** ones, we deal with the Laplace operator Δ applied to functions of the type $f(\tau)/r$. Since the equation verified by these functions is

$$\Delta \frac{f(\tau)}{r} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \frac{f(\tau)}{r} = -4\pi f(t) \delta(\mathbf{r}), \quad (33)$$

the processing is more complicated in the dynamic case compared with the static one. For the electric field of the electric dipole we write the singular part as

$$(\mathbf{E}^{(1,p)}(\mathbf{r}, t))_{(0)} = \frac{1}{4\pi\epsilon_0} \left[\nabla \left(\nabla \cdot \frac{\mathbf{p}(\tau)}{r} \right) - \frac{1}{c^2} \frac{\ddot{\mathbf{p}}(\tau)}{r} \right]_{(0)} = \frac{1}{4\pi\epsilon_0} \mathbf{e}_i \left(\partial_i \partial_j \frac{p_j(\tau)}{r} \right)_{(0)}.$$

Note that the second term proportional to $\ddot{\mathbf{p}}$ has no δ -singularity. Equation (19) gives a result similar to that from the static case:

$$(\mathbf{E}^{(1,p)}(\mathbf{r}, t))_{(0)} = -\frac{1}{3\epsilon_0} \mathbf{p}(t) \delta(\mathbf{r}).$$

The magnetic field $\mathbf{B}^{(1,p)}(\mathbf{r}, t)$ of the electric dipole has no δ -singularity.

Obviously, the electric field $\mathbf{E}^{(1,m)}(\mathbf{r}, t)$ has no δ -singularity either. Writing the singular part of $\mathbf{B}^{(1,m)}(\mathbf{r}, t)$,

$$(\mathbf{B}^{(1,m)}(\mathbf{r}, t))_{(0)} = \frac{\mu_0}{4\pi} \left[\nabla \left(\nabla \cdot \frac{\mathbf{m}(\tau)}{r} \right) - \frac{1}{c^2} \frac{\ddot{\mathbf{m}}(\tau)}{r} + 4\pi \mathbf{m}(t) \delta(\mathbf{r}) \right]_{(0)},$$

one obtains the result similar to that from the static case:

$$(\mathbf{B}^{(1,m)}(\mathbf{r}, t))_{(0)} = \frac{2\mu_0}{3} \mathbf{m}(t) \delta(\mathbf{r}).$$

Let us consider the electric field $\mathbf{E}^{(2,p)}(\mathbf{r}, t)$ and search the corresponding δ -singularities by introducing firstly the **STF** moment $\mathcal{P}^{(2)}$:

$$\begin{aligned} (\mathbf{E}^{(2,p)}(\mathbf{r}, t))_{(0)} &= -\frac{1}{8\pi\epsilon_0} \left[\nabla^3 \parallel \frac{\mathbf{P}^{(2)}(\tau)}{r} - \frac{1}{c^2} \nabla \parallel \frac{\ddot{\mathbf{P}}^{(2)}(\tau)}{r} \right]_{(0)} \\ &= -\frac{1}{8\pi\epsilon_0} \left[\nabla^3 \parallel \frac{\mathcal{P}^{(2)}(\tau)}{r} - \frac{1}{c^2} \nabla \parallel \frac{\ddot{\mathcal{P}}^{(2)}(\tau)}{r} + \nabla \left(\Delta \frac{\Lambda(\tau)}{r} \right) - \frac{1}{c^2} \nabla \frac{\ddot{\Lambda}(\tau)}{r} \right]_{(0)}. \end{aligned}$$

The insertion of equation (33) gives

$$\mathbf{E}_{(0)}^{(2,p)} = -\frac{1}{8\pi\epsilon_0} \left[\nabla^3 \parallel \frac{\mathcal{P}^{(2)}(\tau)}{r} - \frac{1}{c^2} \nabla \parallel \frac{\ddot{\mathcal{P}}^{(2)}(\tau)}{r} - 4\pi \Lambda(t) \nabla \delta(\mathbf{r}) \right]_{(0)}.$$

Expressing the singularity of the first term from the parenthesis by the insertion of equation (20), we can finally write

$$\mathbf{E}^{(2,p)}(\mathbf{r}, t) = (\mathbf{E}^{(2,p)}(\mathbf{r}, t))_{r \neq 0} + \frac{1}{5\epsilon_0} \mathcal{P}^{(2)}(t) \parallel \nabla \delta(\mathbf{r}) + \frac{1}{2\epsilon_0} \Lambda(t) \delta(\mathbf{r}), \quad (34)$$

where

$$(\mathbf{E}^{(2,p)}(\mathbf{r}, t))_{r \neq 0} = -\frac{1}{8\pi\epsilon_0} \left[\nabla^3 \parallel \frac{\mathcal{P}^{(2)}(\tau)}{r} - \frac{1}{c^2} \nabla \parallel \frac{\ddot{\mathcal{P}}^{(2)}(\tau)}{r} \right]_{r \neq 0}.$$

The result (34) is similar to the result (22) from the static case. For the electric 4-polar term, the electric field expression for $r \neq 0$ is invariant to the substitution $\mathbf{P}^{(2)} \rightarrow \mathcal{P}^{(2)}$ as in the static case but, as we will see in the following, such property is not yet verified for higher order multipoles in the dynamical case.

Searching the singularities corresponding to the electric field $\mathbf{E}^{(2,m)}$ of the magnetic quadrupole, the difference from the static case is obvious. Let us express this field with the help of the **STF** magnetic moment $\mathcal{M}^{(2)} = \overset{\leftrightarrow}{\mathbf{M}}^{(2)}$:

$$\begin{aligned} \mathbf{E}^{(2,m)}(\mathbf{r}, t) &= \frac{1}{8\pi\epsilon_0 c^2} \nabla \times \left(\nabla \parallel \frac{\dot{\mathbf{M}}^{(2)}}{r} \right) = \frac{1}{8\pi\epsilon_0 c^2} \left[\nabla \times \left(\nabla \parallel \frac{\dot{\mathcal{M}}^{(2)}}{r} \right) \right. \\ &+ \left. \frac{1}{2} \mathbf{e}_i \epsilon_{ijk} \epsilon_{lkq} \partial_j \partial_l \frac{\dot{M}_q(\tau)}{r} \right] = \frac{1}{8\pi\epsilon_0 c^2} \left[\nabla \times \left(\nabla \parallel \frac{\dot{\mathcal{M}}^{(2)}}{r} \right) - \frac{1}{2} \nabla^2 \parallel \frac{\dot{\mathbf{N}}(\tau)}{r} + \frac{1}{2} \Delta \frac{\dot{\mathbf{N}}(\tau)}{r} \right]. \end{aligned}$$

We employed the same notation as in equation (27). The singularities added by the extension of this expression to the entire space are the following:

$$\begin{aligned} \left[\nabla \times \left(\nabla \parallel \frac{\dot{\mathcal{M}}^{(2)}}{r} \right) \right]_{(0)} &= \mathbf{e}_i \epsilon_{ijk} \left[\partial_j \partial_l \frac{\dot{M}_{lk}}{r} \right]_{(0)} = -\frac{4\pi}{3} \mathbf{e}_i \epsilon_{ijk} \dot{M}_{jk}(t) \delta(\mathbf{r}) = 0, \\ \left[\nabla^2 \parallel \frac{\dot{\mathbf{N}}(\tau)}{r} \right]_{(0)} &= -\frac{4\pi}{3} \dot{\mathbf{N}}(t) \delta(\mathbf{r}). \end{aligned} \quad (35)$$

For the last term from the expression of $\mathbf{E}^{(2,m)}$, we have to consider the equation (33), i.e.

$$\Delta \frac{\dot{\mathbf{N}}(\tau)}{r} = \frac{1}{c^2} \ddot{\mathbf{N}}(\tau) - 4\pi \dot{\mathbf{N}}(t) \delta(\mathbf{r}),$$

such that we can write

$$\begin{aligned} \mathbf{E}^{(2,m)}(\mathbf{r}, t) &= \frac{1}{4\pi\epsilon_0} \left[\frac{1}{2c^2} \nabla \times \left(\nabla \parallel \frac{\dot{\mathcal{M}}^{(2)}(\tau)}{r} \right) - \left(\frac{1}{4c^2} \nabla^2 \parallel \frac{\dot{\mathbf{N}}(\tau)}{r} - \frac{1}{4c^4} \frac{\ddot{\mathbf{N}}(\tau)}{r} \right) \right]_{r \neq 0} \\ &+ \frac{1}{12\epsilon_0 c^2} \dot{\mathbf{N}}(t) \delta(\mathbf{r}) - \frac{1}{4\epsilon_0 c^2} \dot{\mathbf{N}}(t) \delta(\mathbf{r}). \end{aligned} \quad (36)$$

In the above result one can notice that for $r \neq 0$, the expression of the electric field is not invariant to the substitution $\mathbf{P}^{(2)} \rightarrow \mathcal{P}^{(2)}$. The additional term which appears as a consequence of this substitution is exactly the expression of the electric field corresponding to an electric dipole with the moment

$$\delta \mathbf{p}' = -\frac{1}{4c^2} \dot{\mathbf{N}}. \quad (37)$$

In equation (36), we have written separately the term which represents the δ -singularity corresponding to this dipole.

Let us express the field $\mathbf{B}^{(2)}(\mathbf{r}, t)$. For the part $\mathbf{B}^{(2,p)}$, we insert the **STF** moment $\mathcal{P}^{(2)}$ which gives

$$\mathbf{B}^{(2,p)}(\mathbf{r}, t) = \frac{\mu_0}{8\pi} \nabla \times \left(\nabla \parallel \frac{\dot{\mathbf{P}}^{(2)}(\tau)}{r} \right) = \frac{\mu_0}{8\pi} \left[\nabla \times \left(\nabla \parallel \frac{\dot{\mathcal{P}}^{(2)}(\tau)}{r} \right) + \mathbf{e}_i \epsilon_{ijk} \partial_j \partial_l \frac{\Lambda(\tau)}{r} \delta_{lk} \right]$$

$$= \frac{\mu_0}{8\pi} \nabla \times \left(\nabla \parallel \frac{\dot{\mathcal{P}}^{(2)}(\tau)}{r} \right), \quad (38)$$

since $\varepsilon_{ijk} \partial_j \partial_l \delta_{lk} = \varepsilon_{ijk} \partial_j \partial_k = 0$. It follows that $\mathbf{B}^{(2,p)}$ is invariant to the substitution $\mathbf{P}^{(2)} \rightarrow \mathcal{P}^{(2)}$ and has no δ -singularities due to equation (35) written for $\mathcal{M}^{(2)} \rightarrow \mathcal{P}^{(2)}$.

The introduction of the **STF** magnetic moment $\mathcal{M}^{(2)} = \overset{\leftrightarrow}{\mathbf{M}}^{(2)}$ in the expression of $\mathbf{B}^{(2,m)}$ results in

$$\begin{aligned} \mathbf{B}^{(2,m)}(\mathbf{r}, t) &= -\frac{\mu_0}{8\pi} \left(\nabla^3 \parallel \frac{\mathbf{M}^{(2)}(\tau)}{r} - \nabla \parallel \Delta \frac{\mathbf{M}^{(2)}(\tau)}{r} \right) \\ &= -\frac{\mu_0}{8\pi} \left(\nabla^3 \parallel \frac{\mathcal{M}^{(2)}(\tau)}{r} - \nabla \parallel \Delta \frac{\mathcal{M}^{(2)}(\tau)}{r} + \frac{1}{2} \nabla \times \Delta \frac{\mathbf{N}(\tau)}{r} \right). \end{aligned}$$

Separating the δ -singularities with the help of equations (20) and (33), we obtain

$$\begin{aligned} \mathbf{B}^{(2,m)}(\mathbf{r}, t) &= -\frac{\mu_0}{8\pi} \left(\nabla^3 \parallel \frac{\mathcal{M}^{(2)}(\tau)}{r} - \frac{1}{c^2} \nabla \parallel \ddot{\mathcal{M}}^{(2)}(\tau) + \frac{1}{2c^2} \nabla \times \frac{\ddot{\mathbf{N}}(\tau)}{r} \right)_{r \neq 0} \\ &\quad - \frac{3\mu_0}{10} \mathcal{M}^{(2)}(t) \parallel \nabla \delta(\mathbf{r}) - \frac{\mu_0}{4} \mathbf{N}(t) \times \nabla \delta(\mathbf{r}). \end{aligned} \quad (39)$$

Since for $r \neq 0$ the magnetic field of the magnetic quadrupole can be written as

$$\left(\mathbf{B}^{(2,m)}(\mathbf{r}, t) \right)_{r \neq 0} = -\frac{\mu_0}{8\pi} \left(\nabla^3 \parallel \frac{\mathbf{M}^{(2)}(\tau)}{r} - \frac{1}{c^2} \nabla \parallel \frac{\ddot{\mathbf{M}}^{(2)}(\tau)}{r} \right)_{r \neq 0},$$

we can conclude that for $r \neq 0$ this field is not invariant to the substitution $\mathbf{M}^{(2)} \rightarrow \mathcal{M}^{(2)}$ and the additional term introduced by this substitution is equivalent to that of an electric dipole having the moment $\delta \mathbf{p}'$ given by equation (37). Obviously, the corresponding term has no δ -singularity such that it is not represented by a singular term in equation (39).

Let us go further to the field of the third order electric multipole. Beginning with the electric field, we have to extend to the entire space the expression

$$\mathbf{E}^{(3,p)}(\mathbf{r}, t) = \frac{1}{24\pi\varepsilon_0} \left(\nabla^4 \parallel \frac{\mathbf{P}^{(3)}(\tau)}{r} - \frac{1}{c^2} \nabla^2 \parallel \frac{\ddot{\mathbf{P}}^{(3)}(\tau)}{r} \right). \quad (40)$$

We consider separately each term from the last equation introducing the **STF** moment $\mathcal{P}^{(3)}$:

$$\nabla^4 \parallel \frac{\mathbf{P}^{(3)}(\tau)}{r} = \nabla^4 \parallel \frac{\mathcal{P}^{(3)}(\tau)}{r} + \mathbf{e}_i \partial_i \partial_{i_1} \partial_{i_2} \partial_{i_3} \frac{\delta_{\{i_1 i_2 \Lambda_k\}}(\tau)}{r} = \nabla^4 \parallel \frac{\mathcal{P}^{(3)}(\tau)}{r} + 3\mathbf{e}_i \partial_i \partial_j \frac{\Lambda_j(\tau)}{r}.$$

Inserting equation (B.8) for the δ -singularity of the first term from the right-hand side of the last equation and employing equation (33) for the second one, we can write

$$\begin{aligned} \nabla^4 \parallel \frac{\mathbf{P}^{(3)}(\tau)}{r} &= \left(\nabla^4 \parallel \frac{\mathcal{P}^{(3)}(\tau)}{r} + \frac{3}{c^2} \nabla^2 \parallel \frac{\ddot{\mathbf{\Lambda}}(\tau)}{r} \right)_{r \neq 0} \\ &\quad - \frac{12\pi}{7} \mathcal{P}^{(3)}(t) \parallel \nabla^2 \delta(\mathbf{r}) - \frac{4\pi}{c^2} \ddot{\mathbf{\Lambda}}(t) \delta(\mathbf{r}) - 12\pi \mathbf{\Lambda}(t) \parallel \nabla^2 \delta(\mathbf{r}), \end{aligned} \quad (41)$$

where $\mathbf{\Lambda} = \Lambda_i \mathbf{e}_i$. Regarding the second term from equation (40), we similarly obtain

$$\nabla^2 \left\| \frac{\ddot{\mathbf{P}}^{(3)}(\tau)}{r} \right\| = \left(\nabla^2 \left\| \frac{\ddot{\mathcal{P}}^{(3)}(\tau)}{r} \right\| + 2 \nabla^2 \left\| \frac{\ddot{\mathbf{\Lambda}}(\tau)}{r} \right\| + \frac{1}{c^2} \frac{\ddot{\mathbf{\Lambda}}(\tau)}{r} \right)_{r \neq 0} - \frac{20\pi}{3} \ddot{\mathbf{\Lambda}}(t) \delta(\mathbf{r}) \quad (42)$$

since the first term containing $\nabla^2 \left\| (\mathcal{P}^{(3)}(\tau)/r) \right\|$ has no δ -singularity. The insertion of equations (41) and (42) in equation (40) gives

$$\begin{aligned} \mathbf{E}^{(3,p)}(\mathbf{r}, t) &= \frac{1}{24\pi\epsilon_0} \left(\nabla^4 \left\| \frac{\mathcal{P}^{(3)}(\tau)}{r} \right\| - \frac{1}{c^2} \nabla^2 \left\| \frac{\ddot{\mathcal{P}}^{(3)}(\tau)}{r} \right\| \right)_{r \neq 0} \\ &+ \frac{1}{4\pi\epsilon_0} \left(\frac{1}{6c^2} \nabla^2 \left\| \frac{\ddot{\mathbf{\Lambda}}(\tau)}{r} \right\| - \frac{1}{6c^4} \frac{\ddot{\mathbf{\Lambda}}(\tau)}{r} \right)_{r \neq 0} \\ &- \frac{1}{14\epsilon_0} \mathcal{P}^{(3)}(t) \|\nabla^2 \delta(\mathbf{r}) - \frac{1}{2\epsilon_0} \mathbf{\Lambda}(t) \|\nabla^2 \delta(\mathbf{r}) + \frac{1}{9\epsilon_0} \ddot{\mathbf{\Lambda}}(t) \delta(\mathbf{r}) . \end{aligned}$$

From the above equation we can conclude that $\mathbf{E}^{(3,p)}(\mathbf{r}, t)$ is not invariant to the substitution $\mathcal{P}^{(3)} \rightarrow \mathbf{P}^{(3)}$. This substitution introduces an additional term which has precisely the expression of the electric field of an electric dipole of moment

$$\delta \mathbf{p}'' = \frac{1}{6c^2} \ddot{\mathbf{\Lambda}}(t) . \quad (43)$$

Let us express the magnetic field $\mathbf{B}^{(3,p)}$:

$$\mathbf{B}^{(3,p)}(\mathbf{r}, t) = \frac{\mu_0}{24\pi} \nabla \times \left(\nabla^2 \left\| \frac{\dot{\mathcal{P}}^{(3)}(\tau)}{r} \right\| \right) = \frac{\mu_0}{24\pi} \mathbf{e}_i \varepsilon_{ijk} \partial_j \partial_{i_1} \partial_{i_2} \frac{P_{i_1 i_2 k}(\tau)}{r} . \quad (44)$$

The introduction of the STF moment $\mathcal{P}^{(3)}$ has as result

$$\mathbf{B}^{(3,p)}(\mathbf{r}, t) = \frac{\mu_0}{24\pi} \nabla \times \left(\nabla^2 \left\| \frac{\dot{\mathcal{P}}^{(3)}(\tau)}{r} \right\| \right) + \frac{\mu_0}{24\pi} \mathbf{e}_i \varepsilon_{ijk} \partial_j \partial_{i_1} \partial_{i_2} \frac{\delta_{\{i_1 i_2 \Lambda_k\}}(\tau)}{r}$$

and since $\nabla \times (\nabla^2 \left\| (\dot{\mathcal{P}}^{(3)}(\tau)/r) \right\|)$ has no δ -singularities, we can write after simple algebraic calculation and after using equation (33) that

$$\begin{aligned} \mathbf{B}^{(3,p)}(\mathbf{r}, t) &= \left(\frac{\mu_0}{24\pi} \nabla \times (\nabla^2 \left\| \frac{\dot{\mathcal{P}}^{(3)}(\tau)}{r} \right\| + \frac{\mu_0}{4\pi} \frac{1}{6c^2} \nabla \times \frac{\ddot{\mathbf{\Lambda}}(\tau)}{r} \right)_{r \neq 0} \\ &+ \frac{\mu_0}{6c^2} \dot{\mathbf{\Lambda}}(t) \times \nabla \delta(\mathbf{r}) . \end{aligned}$$

For $r \neq 0$, the additional term introduced by the substitution $\mathbf{P}^{(3)} \rightarrow \mathcal{P}^{(3)}$ in equation (44) represents the magnetic field corresponding to the electric dipole with the moment $\delta \mathbf{p}''$ defined by equation (43).

Let us consider the electric moment up to $n = 3$, and the magnetic one up to $n = 2$ and the corresponding sums of fields representing one of the first approximations of a complex system assimilated with a point-like multipolar system. From the above results, as it was expected, we see that the substitutions of the primitive moments by

the corresponding **STF** ones introduce some additional terms. The sum of these terms is equivalent to the substitution

$$\mathbf{p} \rightarrow \tilde{\mathbf{p}} = \mathbf{p} + \delta\mathbf{p} ,$$

where

$$\delta\mathbf{p} = \delta\mathbf{p}' + \delta\mathbf{p}'' = -\frac{1}{c^2}\dot{\mathbf{t}},$$

and the vector \mathbf{t} is defined as

$$\mathbf{t} = \frac{1}{4}\mathbf{N} - \frac{1}{6}\dot{\mathbf{A}} = \frac{1}{10} \int_{\mathcal{D}} d^3x \left((\mathbf{r} \cdot \mathbf{J}) \mathbf{r} - 2r^2 \mathbf{J} \right) .$$

This vector is the so-called electric dipolar toroidal moment and represents a first term from a series of toroidal moments introduced by Dubovik *et al* [14]-[16] by generalizing Zeldovich's original idea that a closed toroidal current represents a certain new kind of dipole [17]. The corresponding δ -singularities can be easily identified in the expressions established above.

The procedure can be continued for the next orders, the technique being the same as in the case of the first orders treated above. Unlike in the static case, in the dynamic one it is more complicated to establish expressions of the singularities for an arbitrary n . The recursive character involving a general number of n steps makes such relations hard to derive and apply [11]. Furthermore, at the current level of applications, we believe that it is mandatory to have expressions up to $n = 3$.

6. Conclusion

In this paper we have expressed the δ -singularities of the electric and magnetic multipoles. The results were given for arbitrary multipole orders n in the static case, while in the dynamic one, the outcomes correspond to the lower orders. However, the calculation algorithm is presented in a manner that facilitates the processing of the results step by step to the next orders of the dynamic expressions. The central idea of the article is that, instead of employing the δ -singularities of the functions $f(\tau)/r$, it is more efficient for the higher orders to search directly such singularities for the multipole fields represented in terms of the **STF** moments. Working in Cartesian coordinates and employing a particular system of parameters and notations adapted to the technique of reducing the primitive Cartesian moments to the corresponding **STF** ones, we have stressed the significance of the first type of moments in the process of searching the field singularities.

In our opinion, a first field of application of these results is to extend the classical argumentation of the generalization of the Fermi contact term to arbitrary multipole-multipole interactions, useful in the studies of the hyperfine interaction, [18, 19].

Appendix A. Some derivatives of $f(\tau)/r$

$$\partial_{i_1} \dots \partial_{i_n} \frac{f(\tau)}{r} = \sum_{l=0}^n \frac{1}{c^{n-l} r^{l+1}} C_{i_1 \dots i_n}^{(n,l)} \frac{d^{n-l} f(\tau)}{dt^{n-l}}, \quad \tau = t - \frac{r}{c}.$$

$$C^{(0,0)} = 1. \quad (\text{A.1})$$

$$C_i^{(1,0)} = -\nu_i, \quad C_i^{(1,1)} = -\nu_i. \quad (\text{A.2})$$

$$C_{ij}^{(2,0)} = \nu_i \nu_j, \quad C_{ij}^{2,1} = 3\nu_i \nu_j - \delta_{ij}, \quad C_{ij}^{(2,2)} = 3\nu_i \nu_j - \delta_{ij}. \quad (\text{A.3})$$

$$\begin{aligned} C_{ijk}^{(3,0)} &= -\nu_i \nu_j \nu_k, \quad C_{ijk}^{(3,1)} = -6\nu_i \nu_j \nu_k + \delta_{\{ij\nu_k\}}, \\ C^{(3,2)} &= -15\nu_i \nu_j \nu_k + 3\delta_{\{ij\nu_k\}}, \quad C^{(3,3)} = -15\nu_i \nu_j \nu_k + 3\delta_{\{ij\nu_k\}}. \end{aligned} \quad (\text{A.4})$$

$$\begin{aligned} C_{ijkl}^{(4,0)} &= \nu_i \nu_j \nu_k \nu_l, \\ C_{ijkl}^{(4,1)} &= 10\nu_i \nu_j \nu_k \nu_l - \delta_{\{ij\nu_k\nu_l\}}, \\ C_{ijkl}^{(4,2)} &= 45\nu_i \nu_j \nu_k \nu_l - 6\delta_{\{ij\nu_k\nu_l\}} + \delta_{\{ij\delta_{kl}\}}, \\ C_{ijkl}^{(4,3)} &= 105\nu_i \nu_j \nu_k \nu_l - 15\delta_{\{ij\nu_k\nu_l\}} + 3\delta_{\{ij\delta_{kl}\}}, \\ C^{(4,4)} &= 105\nu_i \nu_j \nu_k \nu_l - 15\delta_{\{ij\nu_k\nu_l\}} + 3\delta_{\{ij\delta_{kl}\}}. \end{aligned} \quad (\text{A.5})$$

Appendix B. Some general formulas for STF tensors singularities

When searching the δ -singularities of the electromagnetic fields, we have to calculate singularities of some derivatives of **STF** electric and magnetic moments as

$$\nabla^{n+1} \left\| \frac{\mathbf{T}^{(n)}(\tau)}{r} \right\|, \quad \nabla \times \left(\nabla^{n-1} \left\| \frac{\mathbf{T}^{(n)}(\tau)}{r} \right\| \right), \quad (\text{B.1})$$

where $\mathbf{T}^{(n)}$ is one of the **STF** electric or magnetic moments.

Let us search the δ -singularities of the first expression for $n \geq 2$ starting from the definition

$$\begin{aligned} \left\langle \left(\nabla^{n+1} \left\| \frac{\mathbf{T}^{(n)}(\tau)}{r} \right\| \right)_{(0)}, \phi \right\rangle &= \lim_{\varepsilon \rightarrow 0} \mathbf{e}_i \left[\oint_{\Sigma_\varepsilon} dS \nu_i \partial_{i_1} \dots \partial_{i_n} \frac{T_{i_1 \dots i_n}(\tau)}{r} \phi(\mathbf{r}) \right. \\ &\quad \left. - \int_{\mathcal{D}_\varepsilon} d^3x \partial_{i_1} \dots \partial_{i_n} \frac{T_{i_1 \dots i_n}(\tau)}{r} \phi(\mathbf{r}) \partial_i \phi(\mathbf{r}) \right]. \end{aligned} \quad (\text{B.2})$$

Denoting \mathbf{L}_σ the limit corresponding to the surface integral from the last equation, the insertion of equations (13), (17) gives:

$$\begin{aligned} \mathbf{L}_\sigma &= \lim_{\varepsilon \rightarrow 0} \mathbf{e}_i \sum_{l=0}^n \sum_{\alpha \geq 0} \sum_{\lambda \geq 0} \frac{(-1)^{\lambda} \varepsilon^{\alpha-l+1+\lambda}}{c^{n-l+\lambda} \alpha! \lambda!} \\ &\quad \times \oint_{\Sigma_\varepsilon} d\Omega(\boldsymbol{\nu}) \nu_i C_{i_1 \dots i_n}^{(n,l)} \nu_{i_1} \nu_{j_1} \dots \nu_{j_\alpha} \frac{d^{n-l+\lambda}}{dt^{n-l+\lambda}} T_{i_1 \dots i_n}(t) (\partial_{j_1} \dots \partial_{j_\alpha} \phi)_0. \end{aligned} \quad (\text{B.3})$$

Considering the expressions (14) of the coefficients $C^{(n,l)}$, in equation (B.3) we can perform the substitutions

$$C_{i_1 \dots i_n}^{(n,l)} \rightarrow K_0^{(n,l)} \nu_{i_1} \dots \nu_{i_n}$$

without changing the result since the contractions of $\mathbb{T}_{i_1 \dots i_n}$ with all the terms from the coefficients $C_{i_1 \dots i_n}^{(n,l)}$ containing at least a symbol Kronecker vanish. Expressing the integral in this equation as angular average, we can write

$$\begin{aligned} \mathbf{L}_\sigma = 4\pi \mathbf{e}_i \lim_{\varepsilon \rightarrow 0} \sum_{l=0}^n \sum_{\alpha \geq 0} \sum_{\lambda \geq 0} \frac{(-1)^\lambda \varepsilon^{\alpha-l+1+\lambda} K_0^{(n,l)}}{c^{n-l+\lambda} \alpha! \lambda!} \\ \times \langle \nu_{i_1} \dots \nu_{i_n} \nu_i \nu_{j_1} \dots \nu_{j_\alpha} \rangle \frac{d^{n-l+\lambda}}{dt^{n-l+\lambda}} \mathbb{T}_{i_1 \dots i_n}(t) (\partial_{j_1} \dots \partial_{j_\alpha} \phi)_0 . \end{aligned} \quad (\text{B.4})$$

As we can see from equation (18), the contractions

$$\langle \nu_{i_1} \dots \nu_{i_n} \nu_i \nu_{j_1} \dots \nu_{j_\alpha} \rangle \mathbb{T}_{i_1 \dots i_n}(t)$$

are different from zero only if $\alpha + 1 + n \geq 2n$, i.e.

$$\alpha \geq n - 1 . \quad (\text{B.5})$$

Let $e = \alpha - l + 1 + \lambda$ be the power of ε in equation (B.4). Since $l \leq n$ and $\lambda \geq 0$, the inequality (B.5) implies $e \geq 0$, the equality holding only for $\alpha = n - 1$, $l = n$, $\lambda = 0$. $\mathbf{L}_\sigma \neq 0$ only in this case since for $e > 0$ the limit vanishes because of the positive values of the powers of ε . Therefore, equation (B.4) becomes

$$\mathbf{L}_\sigma = 4\pi \mathbf{e}_i \frac{K_0^{(n,n)}}{(n-1)!} \langle \nu_{i_1} \dots \nu_{i_n} \nu_i \nu_{j_1} \dots \nu_{j_{n-1}} \rangle \mathbb{T}_{i_1 \dots i_n}(t) (\partial_{j_1} \dots \partial_{j_{n-1}} \phi)_0 . \quad (\text{B.6})$$

As one can easily see from equation (18), the remaining contractions are different from zero only for the $n!$ terms of the form $\delta_{i_1 j_1} \dots \delta_{i_{n-1} j_{n-1}} \delta_{i_n i}$, and, inserting the value (15) of $K_0^{(n,n)}$ we can finally write

$$\mathbf{L}_\sigma = \mathbf{e}_i \frac{(-1)^n 4\pi n}{2n+1} \mathbb{T}_{i_1 \dots i_{n-1} i}(t) (\partial_{i_1} \dots \partial_{i_{n-1}} \phi)_0 = \frac{(-1)^n 4\pi n}{2n+1} \mathbf{T}^{(n)}(t) \|\nabla^{n-1} \phi\|_0 . \quad (\text{B.7})$$

The volume integral limit from equation (B.2) can be processed applying repeatedly the Gauss theorem. The new surface integral limit can be treated as in the previous case since it is represented by an equation similar to equation (B.4) with $n \rightarrow n - 1$. The contractions to be considered this time are

$$\langle \nu_{i_1} \dots \nu_{i_n} \nu_{j_1} \dots \nu_{j_\alpha} \rangle \mathbb{T}_{i_1 \dots i_n} .$$

They are different from zero only if $\alpha \geq n$ and, since $l \leq n - 1$, the exponents of ε are $e = \alpha - l + 1 + \alpha \geq 2$ such that the limit of this integral for $\varepsilon \rightarrow 0$ cancels. Therefore, equation (B.7) inserted in equation (B.2) has as final result

$$\left(\nabla^{n+1} \left\| \frac{\mathbf{T}^{(n)}(\tau)}{r} \right\|_{(0)} \right) = 4\pi \frac{(-1)^n n}{2n+1} \mathbf{T}^{(n)}(t) \|\nabla^{n-1} \delta(\mathbf{r})\| . \quad (\text{B.8})$$

The curl expression from equation (B.1) has no δ -singularities and this result is obtained applying the same procedure as in the case of the first expression from this equation.

Indeed, processing the corresponding surface integral limit, we can reduce it to terms having as factors the following contractions

$$\varepsilon_{ijk} \langle \nu_{i_1} \dots \nu_{i_{n-1}} \nu_j \nu_{j_1} \dots \nu_{j_{n-2}} \rangle \mathbb{T}_{i_1 \dots i_{n-1} k}(t) .$$

But

$$\varepsilon_{ijk} \langle \nu_{i_1} \dots \nu_{i_{n-1}} \nu_j \nu_{j_1} \dots \nu_{j_{n-2}} \rangle \mathbb{T}_{i_1 \dots i_{n-1} k}(t) \sim \varepsilon_{ijk} \mathbb{T}_{i_1 \dots i_{n-2} jk} = 0 ,$$

therefore,

$$\left[\nabla \times \left(\nabla^{n-1} \parallel \frac{\mathbf{T}^{(n)}(\tau)}{r} \right) \right]_{(0)} = 0 . \quad (\text{B.9})$$

Acknowledgments

The authors would like to thank Dr. Raluca Paiu for her help with the documentation. The work of RZ was supported by grant ID946 (no. 44/2007) of Romanian National Authority for Scientific Research.

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