

# Embedding mapping-class groups of orientable surfaces with one boundary component

Lluís Bacardit\*

## Abstract

Let  $S_{g,1,p}$  be an orientable surface of genus  $g$  with one boundary component and  $p$  punctures. Let  $\mathcal{M}_{g,1,p}$  be the mapping-class group of  $S_{g,1,p}$  relative to the boundary. We construct homomorphisms  $\mathcal{M}_{g,1,p} \rightarrow \mathcal{M}_{g',1,(b-1)}$ , where  $g' \geq 0$  and  $b \geq 1$ . We prove that the constructed homomorphisms  $\mathcal{M}_{g,1,p} \rightarrow \mathcal{M}_{g',1,(b-1)}$  are injective. One of these embeddings for  $g = 0$  is classic.

2000 *Mathematics Subject Classification*. Primary: 20F34; Secondary: 20E05, 20E36, 57M99.

*Key words*. Mapping-class group. Automorphisms of free groups. Ordering. Ends of groups.

## 1 General Notation

Let  $\mathbb{N}$  denote the set of finite cardinals,  $\{0, 1, 2, \dots\}$ .

Throughout, we fix elements  $g, p$  of  $\mathbb{N}$ .

Given two sets  $A$  and  $B$ , we denote by  $A \vee B$  the disjoint union of  $A$  and  $B$ .

Let  $G$  be a multiplicative group. For elements  $a, b$  of  $G$ , we write  $\bar{a} := a^{-1}$ ,  $a^b := \bar{b}ab$ ,  $[a] := \{a^g \mid g \in G\}$ , the conjugacy class of  $a$  in  $G$ . We let  $\text{Aut}(G)$  denote the group of all automorphisms of  $G$ , acting on  $G$  on the right with exponent notation.

An *ordering* of a set will mean a *total* ordering for the set.

We will make frequent use of sequences, usually with vector notation. We shall use the language of sequences to introduce indexed symbols and to realize free monoids. Formally, we define a *sequence* as a set endowed with a specified

---

\*The research was funded by Conseil Régional de Bourgogne and the MIC (Spain) through Project MTM2008-01550.

listing of its elements. Thus a sequence has an underlying set; with vector notation, the coordinates are the elements of (the underlying set of) the sequence. For two sequences  $A, B$ , their concatenation will be denoted  $A \vee B$ . By a sequence  $A$  in a given set  $X$ , we mean a sequence endowed with a specified map of sets  $A \rightarrow X$ ; to avoid extra notation, we shall use the same symbol to denote an element of  $A$  and its image in  $X$  even when the map is not injective. We often treat  $A$  as an element in the free monoid on  $X$  with concatenation as binary operation, and then the elements of  $A$  are its atomic factors.

Let  $i, j \in \mathbb{Z}$ . We write

$$[i \uparrow j] := \begin{cases} (i, i+1, \dots, j-1, j) \in \mathbb{Z}^{j-i+1} & \text{if } i \leq j, \\ () \in \mathbb{Z}^0 & \text{if } i > j. \end{cases}$$

Also,  $[i \uparrow \infty[ := (i, i+1, i+2, \dots)$ . We define  $[j \downarrow i]$  to be the reverse of the sequence  $[i \uparrow j]$ ,  $(j, j-1, \dots, i+1, i)$ .

Let  $v$  be a symbol. For each  $k \in \mathbb{Z}$ , we let  $v_k$  denote the ordered pair  $(v, k)$ . We let

$$v_{[i \uparrow j]} := \begin{cases} (v_i, v_{i+1}, \dots, v_{j-1}, v_j) & \text{if } i \leq j, \\ () & \text{if } i > j. \end{cases}$$

Also,  $v_{[i \uparrow \infty[ := (v_i, v_{i+1}, v_{i+2}, \dots)$ . We define  $v_{[j \downarrow i]}$  to be the reverse of the sequence  $v_{[i \uparrow j]}$ .

Suppose there is specified a set-map  $v_{[i \uparrow j]} \rightarrow A$ . We treat the elements of  $v_{[i \uparrow j]}$  as elements of  $A$  (possibly with repetitions), and, we say that  $v_{[i \uparrow j]}$  is a *sequence in  $A$* .

If  $v_{[i \uparrow j]}$  is a sequence in a multiplicative group  $G$ , we let

$$\begin{aligned} \Pi v_{[i \uparrow j]} &:= \begin{cases} v_i v_{i+1} \cdots v_{j-1} v_j \in G & \text{if } i \leq j, \\ 1 \in G & \text{if } i > j. \end{cases} \\ \Pi v_{[j \downarrow i]} &:= \begin{cases} v_j v_{j-1} \cdots v_{i+1} v_i \in G & \text{if } j \geq i, \\ 1 \in G & \text{if } j < i. \end{cases} \end{aligned}$$

Let  $F$  be the free group on a set  $X$ . Consider an element  $w$  of  $F$  and a sequence  $a_{[1 \uparrow k]}$  in  $X \vee \overline{X}$ . If  $\Pi a_{[1 \uparrow k]} = w$  in  $F$ , we say that  $a_{[1 \uparrow k]}$  is a *monoid expression* for  $w$  in  $X \vee \overline{X}$  of *length  $k$* . We say that  $a_{[1 \uparrow k]}$  is *reduced* if, for all  $j \in [1 \uparrow (k-1)]$ ,  $a_{j+1} \neq \overline{a_j}$  in  $X \vee \overline{X}$ . Each element of  $F$  has a unique reduced expression, called the *normal form*. Suppose that  $a_{[1 \uparrow k]}$  is the normal form for  $w$ . We define the *length* of  $w$  to be  $|w| := k$ .

For  $p$  an element of  $\mathbb{N} \cup \{\infty\}$ ,  $p \neq 0$ , let  $C_p$  be the *cyclic group of order  $p$*  with multiplicative notation. For  $q \in \mathbb{N}$ , let  $C_p^{*q}$  denote the free product of  $q$  copies of  $C_p$ .

## 2 Introduction and main results

Recall  $g, p$  are elements of  $\mathbb{N}$ . Let  $b$  be an element of  $\mathbb{N}$ ,  $b \geq 1$ . Let  $S_{g,b,p}$  be an orientable surface of genus  $g$  with  $b$  boundary components and  $p$  punctures.

Let  $\Sigma_{g,b,p}$  be the rank  $2g + b - 1 + p$  free group with generating set  $x_{[1\uparrow g]} \vee y_{[1\uparrow g]} \vee z_{[1\uparrow(b-1)]} \vee t_{[1\uparrow p]}$ . We view  $\Sigma_{g,b,p}$  as a presentation of  $\pi_1(S_{g,b,p}, *)$ , the fundamental group of  $S_{g,b,p}$  based at a point  $*$  in the  $b$ -th boundary component. In addition, for every  $l \in [1\uparrow(b-1)]$ ,  $z_l$  represents a loop around the  $l$ -th boundary component; for every  $k \in [1\uparrow p]$ ,  $t_k$  represents a loop around the  $k$ -th puncture, and  $(\prod_{i \in [1\uparrow g]} [x_i, y_i] \prod_{z \in [1\uparrow(b-1)]} z \prod_{t \in [1\uparrow p]} t)^{-1}$  represents a loop around the  $b$ -th boundary component. Note that, if  $p = 0$ , there is no puncture in  $S_{g,b,p} = S_{g,b,0}$ ,  $t_{[1\uparrow p]}$  is the empty sequence, and  $\prod_{t \in [1\uparrow p]} t = 1$ .

Let  $\mathcal{AM}_{g,b,p}$  denote the subgroup of  $\text{Aut}(\Sigma_{g,b,p} * \langle e_{[1\uparrow(b-1)]} \mid \rangle)$  consisting of all the automorphisms of  $\Sigma_{g,b,p} * \langle e_{[1\uparrow(b-1)]} \mid \rangle$  which map  $\Sigma_{g,b,p}$  to itself and respect the sets

$$\{\prod_{i \in [1\uparrow g]} [x_i, y_i] \prod_{z \in [1\uparrow(b-1)]} z \prod_{t \in [1\uparrow p]} t\}, \{\bar{z}_1^{e_1}\}, \{\bar{z}_2^{e_2}\}, \dots, \{\bar{z}_{(b-1)}^{e_{(b-1)}}\}, \{[\bar{t}_k]\}_{k \in [1\uparrow p]}.$$

We call  $\mathcal{AM}_{g,b,p}$  the *algebraic mapping-class group* of a surface of genus  $g$  with  $b$  boundary components and  $p$  punctures,  $S_{g,b,p}$ .

For  $b = 0$ , the (orientation-preserving) mapping-class group of  $S_{g,0,p}$ , denoted  $\mathcal{M}_{g,0,p}$ , is defined as the group of orientation-preserving homeomorphisms of  $S_{g,0,p}$  modulo isotopy. Let  $f$  be a homeomorphism of  $S_{g,0,p}$ , then  $f$  induces an automorphism of  $\pi_1(S_{g,0,p})$  which respects the set of conjugacy classes of  $t_{[1\uparrow p]}$ . Since  $f$  is not forced to fix the base point of  $\pi_1(S_{g,0,p})$ , the isotopy class of  $f$  defines an automorphism of  $\pi_1(S_{g,0,p})$  up to conjugation, hence an element of  $\text{Out}(\pi_1(S_{g,0,p}))$ . By the Dehn-Nielsen-Baer theorem, this correspondence is an isomorphism onto the subgroup of  $\text{Out}(\pi_1(S_{g,0,p}))$  which respects the set of conjugacy classes of  $t_{[1\uparrow p]}$ , [11, Theorem 3.6], [12, Theorem 2.9.A]. In particular,  $\mathcal{M}_{g,0,p} \leq \text{Out}(\pi_1(S))$ .

If  $b \geq 1$ , that is,  $S_{g,b,p}$  has non-empty boundary; we restrict ourselves to homeomorphisms and isotopies of  $S_{g,b,p}$  which fix the boundary pointwise. These homeomorphisms preserve the orientation of  $S_{g,b,p}$ . In this case, we take the base point of  $S_{g,b,p}$  in the  $b$ -th boundary component of  $S_{g,b,p}$ . We convert a boundary component of  $S_{g,b,p}$  into a puncture by identifying via a homeomorphism the boundary component with the boundary of a once punctured disc. If  $b \geq 1$ , by converting all the boundary components into punctures we can deduce  $\mathcal{M}_{g,b,p} \simeq \mathcal{AM}_{g,b,p}$  from the Dehn-Nielsen-Baer theorem, [10, Theorem 9.6]. See [10] for a background on algebraic mapping-class groups, with some changes of notation. From now on, we will deal with  $\mathcal{AM}_{g,b,p}$  and, mostly, in the case  $b = 1$ .

For  $p \geq 1$  and  $q \in [1\uparrow p]$ , we denote by  $\mathcal{AM}_{g,b,q \perp (p-q)}$  the subgroup of  $\mathcal{AM}_{g,1,p}$  consisting of all the automorphisms which respect the sets  $\{[\bar{t}_k]\}_{k \in [1\uparrow q]}$  and  $\{[\bar{t}_k]\}_{k \in [(q+1)\uparrow p]}$ . Let  $N$  be the normal closure in  $\Sigma_{g,b,p}$  of  $t_{[(q+1)\uparrow p]}$ . Notice  $\Sigma_{g,b,q} \simeq \Sigma_{g,b,p}/N$ . We say that we have eliminated the last  $(p - q)$

punctures. Since  $N$  is  $\mathcal{AM}_{g,b,q\perp(p-q)}$ -invariant, we can define a homomorphism  $\mathcal{AM}_{g,b,q\perp(p-q)} \rightarrow \mathcal{AM}_{g,b,q}$  which eliminates the last  $(p-q)$  punctures, see [10, Section 11].

Suppose  $b \geq 2$ . Let  $\phi \in \mathcal{AM}_{g,b,p}$  and  $l \in [1\uparrow(b-1)]$ . Since  $(\bar{z}_l^{e_l})^\phi = \bar{z}_l^{e_l}$ , we see  $\bar{z}_l^\phi = \bar{z}_l^{w_l}$  for some  $w_l \in \Sigma_{g,b,p} * \langle e_{[1\uparrow(b-1)]} \rangle$ . Since  $\phi$  maps  $\Sigma_{g,b,p}$  to itself and  $z_l \in \Sigma_{g,b,p}$ , we see  $w_l \in \Sigma_{g,b,p}$ . Since  $\bar{e}_l \bar{z}_l e_l = \bar{z}_l^{e_l} = (\bar{z}_l^{e_l})^\phi = \bar{e}_l^\phi \bar{z}_l^\phi e_l^\phi = \bar{e}_l^\phi (\bar{w}_l \bar{z}_l w_l) e_l^\phi$ , we see  $(e_l^\phi \bar{e}_l) \bar{z}_l (e_l \bar{e}_l^\phi) = \bar{w}_l \bar{z}_l w_l$  and  $e_l^\phi \bar{e}_l \in \bar{w}_l \langle z_l \rangle$ . Hence,  $e_l^\phi \in \bar{w}_l \langle z_l \rangle e_l$ .

In  $\mathcal{M}_{g,b,p}$ , the difference between a puncture and a boundary component is that the Dehn twist with respect to a loop around a puncture is trivial in  $\mathcal{M}_{g,b,p}$  and the Dehn twist with respect to a loop around a boundary component is not trivial in  $\mathcal{M}_{g,b,p}$ . In  $\mathcal{AM}_{g,b,p}$ , this fact is captured by the fact that for  $l \in [1\uparrow(b-1)]$  the map

$$\begin{cases} e_l & \mapsto z_l e_l, \\ a & \mapsto a, \quad a \in x_{[1\uparrow g]} \vee y_{[1\uparrow g]} \vee t_{[1\uparrow p]} \vee z_{[1\uparrow(b-1)]} \vee e_{[1\uparrow(l-1)]} \vee e_{[(l+1)\uparrow(b-1)]}. \end{cases}$$

defines an element of  $\mathcal{AM}_{g,b,p}$ . We can see  $e_l$  as an arc from the base point in the  $b$ -th boundary component to a point in the  $l$ -th boundary component.

Consider the homomorphism  $\Sigma_{g,b,p} * \langle e_{[1\uparrow(b-1)]} \rangle \rightarrow \Sigma_{g,b-1,p+1} * \langle e_{[1\uparrow(b-2)]} \rangle$  such that  $e_{(b-1)} \mapsto 1$ ,  $z_{(b-1)} \mapsto t_{p+1}$  and identifies all the other generators. This homomorphism corresponds to converting the  $(b-1)$ -th boundary component to a puncture. This homomorphism induces a homomorphism  $\mathcal{AM}_{g,b,p} \rightarrow \mathcal{AM}_{g,b-1,p+1}$  which forgets  $e_{b-1}$ , see [10, Section 9].

For  $g = 0$ ,  $b = 1$  and  $p \geq 1$ ,  $\mathcal{AM}_{0,1,p}$  is isomorphic to the  $p$ -string braid group. We have  $\mathcal{AM}_{0,1,p} = \langle \sigma_{[1\uparrow(p-1)]} \rangle$ , where for all  $i \in [1\uparrow(p-1)]$ ,  $\sigma_i \in \text{Aut}(\Sigma_{0,1,p})$  is defined by

$$(2.0.1) \quad \sigma_i := \begin{cases} t_i & \mapsto t_{i+1}, \\ t_{i+1} & \mapsto t_i^{t_{i+1}}, \\ t_k & \mapsto t_k \quad \text{if } k \in [1\uparrow(i-1)] \vee [(i+2)\uparrow p]. \end{cases}$$

Let  $d \in \mathbb{N}$ ,  $d \geq 2$ .

Let  $\Sigma_{g,1,p^{(d)}}$  denote the group  $\langle x_{[1\uparrow g]} \vee y_{[1\uparrow g]} \vee \tau_{[1\uparrow p]} \mid \tau_1^d, \tau_2^d, \dots, \tau_p^d \rangle$ . Hence,  $\Sigma_{g,1,p^{(d)}} \simeq \Sigma_{g,1,0} * C_d^{*p}$ . Notice  $\Sigma_{g,1,p^{(d)}} = \Sigma_{g,1,p}$ , if  $p = 0$ .

Let  $\mathcal{AM}_{g,1,p^{(d)}}$  denote the group of all automorphisms of  $\Sigma_{g,1,p^{(d)}}$  that respect the sets

$$\{\Pi_{i \in [1\uparrow g]} [x_i, y_i] \Pi_{\tau_{[1\uparrow p]}}\}, \quad \{[\bar{\tau}_k]\}_{k \in [1\uparrow p]}.$$

Let  $\mathcal{P}_{[1\uparrow p]}$  be a set of  $p$  interior points of  $S_{g,1,0}$ . For each  $k \in [1\uparrow p]$ , let  $D(\mathcal{P}_k)$  be an open disc in the interior of  $S_{g,1,0}$  and centered at  $\mathcal{P}_k$ . For each  $k \in [1\uparrow p]$ , let  $D_k$  be a copy of the closed disc  $\{z \in \mathbb{C} \mid |z| \leq 1\}$ . We define

$$S_{g,1,p^{(d)}} := ((S_{g,1,0} - \vee_{k \in [1\uparrow p]} D(\mathcal{P}_k)) \vee (\vee_{k \in [1\uparrow p]} D_k)) / \sim$$

where  $\sim$  is the following identification. For every  $k \in [1\uparrow p]$ , we identify the boundary of  $D(\mathcal{P}_k)$ , denoted  $\partial D(\mathcal{P}_k)$ , with a copy of  $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ , and the boundary of  $D_k$ , denoted  $\partial D_k$ , with another copy of  $S^1$ . We define  $f_k : \partial D_k \rightarrow \partial D(\mathcal{P}_k)$ ,  $z \mapsto z^d$ . Now,  $\sim$  is defined by identifying  $z$  and  $f_k(z)$  for every  $z \in \partial D_k$  and every  $k \in [1\uparrow p]$ .

It can be seen that  $\Sigma_{g,1,p^{(d)}}$  is the fundamental group of  $S_{g,1,p^{(d)}}$ .

By analogy,  $\mathcal{AM}_{g,1,p^{(d)}}$  will be called the (*algebraic*) *mapping class group* of  $S_{g,1,p^{(d)}}$ .

Since the elements of  $\mathcal{AM}_{g,1,p}$  respect the set  $\{[\bar{t}_k]\}_{k \in [1\uparrow p]}$ , the natural homomorphism  $\Sigma_{g,1,p} \rightarrow \Sigma_{g,1,p^{(d)}}$  induces a natural homomorphism

$$\psi : \mathcal{AM}_{g,1,p} \rightarrow \mathcal{AM}_{g,1,p^{(d)}}.$$

If  $p = 0$ , then  $\Sigma_{g,1,p} = \Sigma_{g,1,p^{(d)}}$  and  $\psi$  is the identity.

**2.1 Theorem.** *The homomorphism  $\psi : \mathcal{AM}_{g,1,p} \rightarrow \mathcal{AM}_{g,1,p^{(d)}}$  is injective for all  $p \in \mathbb{N}$ .*

Let  $\kappa : S_{g',b,0} \rightarrow S_{g,1,0}$  be an index  $m \in \mathbb{N}$  branched regular cover with  $p$  branching points in the interior of  $S_{g,1,0}$  which lift to  $q$  points in  $S_{g',b,0}$ . Notice that  $q = 0$  if and only if  $p = 0$ . Let  $\kappa' : S_{g',b,q} \rightarrow S_{g,1,p}$  be the corresponding unbranched cover. We identify  $\Sigma_{g',b,q} = \pi_1(S_{g',b,q}, \hat{*})$  with  $\kappa'_*(\Sigma_{g',b,q})$ , where  $\hat{*}$  is a point in the  $b$ -th boundary component of  $S_{g',b,q}$ . Notice that  $\Sigma_{g',b,q}$  is a normal subgroup of  $\Sigma_{g,1,p}$  of index  $m$ . We set  $G := \Sigma_{g,1,p}/\Sigma_{g',b,q}$  the group of deck transformations.

We put  $\varrho = \prod_{i \in [1\uparrow g]} [x_i, y_i] \prod_{t \in [1\uparrow p]} \Sigma_{g',b,q} \in G$ . Let  $c$  be the order of  $\varrho$  in  $G$ . Since  $\varrho^c = 1$  in  $G$ , we see that  $(\prod_{i \in [1\uparrow g]} [x_i, y_i] \prod_{t \in [1\uparrow p]})^c \in \Sigma_{g',b,q}$ . Notice that  $(\prod_{i \in [1\uparrow g]} [x_i, y_i] \prod_{t \in [1\uparrow p]})^{-c}$  represents a loop around the  $b$ -th boundary component. We take a basis  $\hat{x}_{[1\uparrow g']} \vee \hat{y}_{[1\uparrow g']} \vee \hat{z}_{[1\uparrow(b-1)]} \vee \hat{t}_{[1\uparrow q]}$  of  $\Sigma_{g',b,q} = \pi_1(S_{g',b,q}, \hat{*})$  such that

$$\prod_{i \in [1\uparrow g']} [\hat{x}_i, \hat{y}_i] \prod_{\hat{z}_{[1\uparrow(b-1)]}} \prod_{\hat{t}_{[1\uparrow q]}} = (\prod_{i \in [1\uparrow g]} [x_i, y_i] \prod_{t \in [1\uparrow p]})^c.$$

Recall  $G$  has cardinality  $m$ . The subgroup  $\langle \varrho \rangle \leq G$  has index  $b = m/c$ . For every  $l \in [1\uparrow(b-1)]$ , let  $w_l \in \Sigma_{g,1,p} - \Sigma_{g',b,q}$  such that

$$\hat{z}_l = \bar{w}_l (\prod_{i \in [1\uparrow g]} [x_i, y_i] \prod_{t \in [1\uparrow p]})^{-c} w_l.$$

We put  $\rho_l = w_l \Sigma_{g',b,q} \in G$ . Then  $G = \langle \varrho \rangle \rho_1 \vee \langle \varrho \rangle \rho_2 \cdots \vee \langle \varrho \rangle \rho_{(b-1)} \vee \langle \varrho \rangle$ . That is, the boundary components of  $S_{g',b,p}$  are image by deck transformations of the  $b$ -th boundary component.

For every  $k \in [1\uparrow p]$  we put  $\varrho_k = t_k \Sigma_{g',b,q} \in G$ . Let  $d_k$  be the order of  $\varrho_k$  in  $G$ . Since  $t_k$  corresponds to a branching point,  $t_k \notin \Sigma_{g',b,q}$  and  $d_k \geq 2$ . Since  $\varrho_k^{d_k} = 1$  in  $G$ , we see that  $t_k^{d_k} \in \Sigma_{g',b,q}$ . Notice that  $t_k^{d_k}$  represents a loop around a lift of the  $k$ -th puncture of  $S_{g,1,p}$ . The subgroup  $\langle \varrho_k \rangle$  has index  $m_k = m/d_k$  in  $G$ . Hence,  $G = \langle \varrho_k \rangle \rho_{1,k} \vee \langle \varrho_k \rangle \rho_{2,k} \cdots \vee \langle \varrho_k \rangle \rho_{m_k,k}$ , where  $\rho_{i,k} = u_{i,k} \Sigma_{g',b,q} \in G$

for all  $i \in [1 \uparrow m_k]$ . Notice that  $(t_k^{d_k})^{u_{1,k}}, (t_k^{d_k})^{u_{2,k}}, \dots, (t_k^{d_k})^{u_{m_k,k}}$  represent loops around the  $m_k$  lifts of the  $k$ -th puncture. We choose  $u_{1,k}, u_{2,k}, \dots, u_{m_k,k}$  such that  $\{(t_k^{d_k})^{u_{1,k}}, (t_k^{d_k})^{u_{2,k}}, \dots, (t_k^{d_k})^{u_{m_k,k}}\} \subseteq \{\hat{t}_1, \hat{t}_2, \dots, \hat{t}_q\}$ . Then

$$\{\hat{t}_1, \hat{t}_2, \dots, \hat{t}_q\} = \bigvee_{k \in [1 \uparrow p]} \{(t_k^{d_k})^{u_{1,k}}, (t_k^{d_k})^{u_{2,k}}, \dots, (t_k^{d_k})^{u_{m_k,k}}\}.$$

Suppose, now, that  $\Sigma_{g',b,q}$  is  $\mathcal{AM}_{g,1,p}$ -invariant. It is easy to see that  $d_1 = d_k \geq 2$  for all  $k \in [1 \uparrow p]$ . Let  $d = d_1$ . Every  $\phi \in \mathcal{AM}_{g,1,p}$  induces an automorphism of  $\Sigma_{g',b,q}$  by restriction. Hence, we have a homomorphism  $\mathcal{AM}_{g,1,p} \rightarrow \text{Aut}(\Sigma_{g',b,q})$  given by restriction. Since, in  $\Sigma_{g,1,p}$ ,  $\Pi_{i \in [1 \uparrow g']}[\hat{x}_i, \hat{y}_i] \Pi_{\hat{z}_{[1 \uparrow (b-1)]}} \Pi_{\hat{t}_{[1 \uparrow q]}} = (\Pi_{i \in [1 \uparrow g]}[x_i, y_i] \Pi_{\hat{t}_{[1 \uparrow p]}})^c$ ,  $\hat{z}_l$  is conjugate to  $(\Pi_{i \in [1 \uparrow g]}[x_i, y_i] \Pi_{\hat{t}_{[1 \uparrow p]}})^{-c}$  for all  $l \in [1 \uparrow (b-1)]$ , and  $\hat{t}_k$  is conjugate to  $t_j^d$ , where  $j \in [1 \uparrow p]$ , for all  $k \in [1 \uparrow q]$ , we have that the image of the homomorphism  $\mathcal{AM}_{g,1,p} \rightarrow \text{Aut}(\Sigma_{g',b,q})$  lies inside  $\mathcal{AM}_{g',1,(b-1) \perp q}$ .

Since  $\Sigma_{g',b,q}$  is  $\mathcal{AM}_{g,1,p}$ -invariant, every homeomorphism of  $S_{g,1,p}$  lifts to a homeomorphism of  $S_{g',b,q}$  which fixes the  $b$ -th boundary component pointwise, but this lift may not fix the first  $(b-1)$  boundary components pointwise. If we convert the first  $(b-1)$  boundary components of  $S_{g',b,p}$  into punctures, this is not a problem. If we want to conserve the first  $(b-1)$  boundary components, we have to restrict ourselves to homeomorphisms of  $S_{g,1,p}$  whose lifts fix the boundaries pointwise. Algebraically, if we want to have a homomorphism inside  $\mathcal{AM}_{g',b,q}$  we have to define the image of  $\hat{e}_{[1 \uparrow (b-1)]}$ . To do this, we need to restrict ourselves to the following subgroup of  $\mathcal{AM}_{g,1,p}$  (since  $\Sigma_{g',b,q}$  is  $\mathcal{AM}_{g,1,p}$ -invariant, every element of  $\mathcal{AM}_{g,1,p}$  induces an automorphism of  $G = \Sigma_{g,1,p}/\Sigma_{g',b,q}$ ).

$$\mathcal{AM}_{g,1,p}^G := \{\phi \in \mathcal{AM}_{g,1,p} \mid \phi \text{ induces the identity of } G\}.$$

**2.2 Theorem.** *With the above notation, if  $\Sigma_{g',b,q}$  is  $\mathcal{AM}_{g,1,p}$ -invariant and  $(g, p, d) \neq (0, 2, 2)$  then the composition*

$$(2.2.1) \quad \mathcal{AM}_{g,1,p} \rightarrow \mathcal{AM}_{g',1,(b-1) \perp q} \rightarrow \mathcal{AM}_{g',1,(b-1)}$$

*where the first homomorphism is given by restriction and the second homomorphism is given by eliminating the last  $q$  punctures, is injective.*

Theorem 2.2 gives an algebraic proof of the following theorem, which is an analog for surfaces with one boundary component of a theorem of Birman and Hilden [4, Theorem 2] (the hypothesis that  $\kappa'_*(\Sigma_{g',b,q})$  is  $\mathcal{AM}_{g,1,p}$ -invariant can be removed, but then we need extra notation).

**2.3 Theorem.** *Let  $\kappa : S_{g',b,0} \rightarrow S_{g,1,0}$  be a finite index regular cover with  $p$  branching points in  $S_{g,1,0}$ . Let  $\hat{f}$  be an homeomorphism of  $S_{g',b,0}$  which fixes the  $b$ -th boundary component pointwise and preserves the fibers of  $\kappa : S_{g',b,0} \rightarrow S_{g,1,0}$ . Then  $\hat{f}$  induces an homeomorphism  $f$  of  $S_{g,1,0}$  such that  $\kappa \hat{f} = f \kappa$ . Let*

$\kappa' : S_{g',b,q} \rightarrow S_{g,1,p}$  be the corresponding unbranched cover. Suppose  $\kappa'_*(\Sigma_{g',b,q})$  is  $\mathcal{AM}_{g,1,p}$ -invariant and  $(g,p) \neq (0,2)$ . If  $\hat{f}$  is isotopic to the identity relative to the  $b$ -th boundary component, then  $f$  is isotopic to the identity relative to the boundary.

*Proof.* (using Theorem 2.2). It is a general fact that if  $\hat{f}$  preserves the fibers of  $\kappa : S_{g',b,0} \rightarrow S_{g,1,0}$ , then  $\hat{f}$  induces an homeomorphism  $f$  of  $S_{g,1,0}$  such that  $\kappa\hat{f} = f\kappa$ . In particular,  $f$  sends branching points to branching points.

Let  $\kappa' : S_{g',b,q} \rightarrow S_{g,1,p}$  be the corresponding unbranched cover. We identify  $\Sigma_{g',b,q}$  and  $\kappa'_*(\Sigma_{g',b,q})$ . Since  $f$  sends branching points to branching points,  $f$  restricts to a homeomorphism  $g$  of  $S_{g,1,p}$ . Since  $\Sigma_{g',b,q}$  is  $\mathcal{AM}_{g,1,p}$ -invariant, there are induced automorphisms  $g_* \in \mathcal{AM}_{g,1,p}$  and  $\hat{f}_* \in \mathcal{AM}_{g',1,(b-1)}$  such that  $g_* \mapsto \hat{f}_*$  by the composition (2.2.1). If  $\hat{f}$  is isotopic to the identity relative to the  $b$ -th boundary component, then  $\hat{f}_* = 1$ . Hence, by Theorem 2.2  $g_* = 1$ . Then  $g$  is isotopic to the identity relative to the boundary. Then  $f$  is isotopic to the identity relative to the boundary.  $\square$

### 3 Examples

We fix  $g,p$  such that  $(g,p) \neq (0,2)$ . Let  $\hat{S}$  be the universal cover of  $S_{g,1,p}$ . The fundamental group of  $S_{g,1,p}$ , denoted  $\Sigma_{g,1,p}$ , acts on  $\hat{S}$ . Let  $H$  be a subgroup of  $\Sigma_{g,1,p}$  of index  $m \in \mathbb{N}$ . Suppose  $H$  is  $\mathcal{AM}_{g,1,p}$ -invariant. The quotient space  $\hat{S}/H$  is an orientable surface, denoted  $S_{g',b,q}$ . We identify the fundamental group of  $S_{g',b,q}$ , denoted  $\Sigma_{g',b,q}$ , with  $H$ . The cover  $\hat{S} \rightarrow S_{g,1,p}$  induces a cover  $S_{g',b,q} \rightarrow S_{g,1,p}$  with group of deck transformation  $G := \Sigma_{g,1,p}/\Sigma_{g',b,q}$ . If  $t_k \notin \Sigma_{g',b,q}$  for all  $k \in [1\uparrow p]$ , then the corresponding cover  $S_{g',b,0} \rightarrow S_{g,1,0}$  has  $p$  branching points in  $S_{g,1,0}$  which lift to  $q$  points in  $S_{g',b,0}$ . By Theorem 2.2, we have an embedding  $\mathcal{AM}_{g,1,p} \hookrightarrow \mathcal{AM}_{g',1,(b-1)}$ . By choosing an appropriated basis of  $H$ , we can compute elements in the image of this embedding from elements of  $\mathcal{AM}_{g,1,p}$ .

The first example is classical. In the second example, we give a basis of  $H$  and compute elements in the image of the embedding.

**Example 1.** Let  $H$  be the kernel of the homomorphism  $\Sigma_{0,1,p} \rightarrow \langle \tau \mid \tau^2 \rangle$  such that  $t_k \mapsto \tau$  for all  $k \in [1\uparrow p]$ . It is standard to see that  $H$  is a free group of rank  $2p - 1$  with basis  $t_1^2, t_1t_2, t_1t_3, \dots, t_1t_p, t_1\bar{t}_2, t_1\bar{t}_3, \dots, t_1\bar{t}_p$ . It is easy to see that  $H$  is invariant by the generators of  $\mathcal{AM}_{0,1,p}$  given in (2.0.1). For  $k \in [1\uparrow p]$ , notice that  $\varrho_k = t_kH$  has order 2 in  $G := \Sigma_{0,1,p}/H \simeq C_2$ . Hence,  $\langle \varrho_k \rangle$  has index 1 in  $G$  and the  $k$ -th puncture in  $S_{g,1,p}$  lifts to one puncture in  $S_{g',b,q}$ . Thus,  $q = p$ .

- (a). If  $p$  is even, then  $\Pi t_{[1\uparrow p]} \in H$  and  $\varrho = \Pi t_{[1\uparrow p]}H$  has order 1 in  $G$ . Hence,  $\langle \varrho \rangle$  has index 2 in  $G$  and we have  $b = 2$ . Since  $\Sigma_{g',b,q}$  has rank  $2g' + b - 1 + q$  and  $H$  has rank  $2p - 1$ , we have  $2g' + 2 - 1 + p = 2p - 1$  and  $g' = (p - 2)/2$ . Hence,  $\mathcal{AM}_{0,1,p} \hookrightarrow \mathcal{AM}_{(p-2)/2,1,1}$ , if  $p$  is even.

- (b). If  $p$  is odd, then  $\Pi t_{[1\uparrow p]} \notin H$  and  $\varrho = \Pi t_{[1\uparrow p]} H$  has order 2 in  $G$ . Hence,  $\langle \varrho \rangle$  has index 1 in  $G$  and  $b = 1$ . Since  $\Sigma_{g',b,q}$  has rank  $2g' + b - 1 + q$  and  $H$  has rank  $2p - 1$ , we have  $2g' + 1 - 1 + p = 2p - 1$  and  $g' = (p - 1)/2$ . Hence,  $\mathcal{AM}_{0,1,p} \hookrightarrow \mathcal{AM}_{(p-1)/2,1,0}$ , if  $p$  is odd.

**Example 2.** Let  $\Sigma_{1,1,0} = \langle x, y \mid \rangle$ . Let  $H$  be the kernel of the homomorphism  $\Sigma_{1,1,0} \rightarrow \langle \tau_1 \mid \tau_1^2 \rangle \times \langle \tau_2 \mid \tau_2^2 \rangle$  such that  $x \mapsto \tau_1$ ,  $y \mapsto \tau_2$ . It is standard to see that  $H$  is a free group of rank 5. It can be shown that  $H$  is a characteristic subgroup of  $\Sigma_{1,1,0}$ . Notice that  $\varrho = \bar{x} \bar{y} x y H$  has order 1 in  $G := \Sigma_{1,1,0}/H \simeq C_2 \times C_2$ . Hence,  $\langle \varrho \rangle$  has index 4 in  $G$  and  $b = 4$ . We have  $p = 0$  and  $q = 0$ . Since  $\Sigma_{g',b,q}$  has rank  $2g' + b - 1 + q$  and  $H$  has rank 5, we have  $2g' + 4 - 1 + 0 = 5$  and  $g' = 1$ . Hence,  $\mathcal{AM}_{1,1,0} \hookrightarrow \mathcal{AM}_{1,1,3}$ . We take the following basis of  $\Sigma_{1,1,3}$ :  $\hat{x} = x^2$ ,  $\hat{y} = y^2$ ,  $\hat{t}_1 = (\bar{y} \bar{x} y x)^{\bar{x} \bar{y}^2 x^2 y^2}$ ,  $\hat{t}_2 = (\bar{y} \bar{x} y x)^y$ ,  $\hat{t}_3 = (\bar{y} \bar{x} y x)^{xy}$ . It is well-known that  $\mathcal{AM}_{1,1,0} = \langle \alpha, \beta \mid \alpha \beta \alpha = \beta \alpha \beta \rangle$ , where

$$\alpha := \begin{cases} x & \mapsto \bar{y}x, \\ y & \mapsto y, \end{cases} \quad \beta := \begin{cases} x & \mapsto x, \\ y & \mapsto xy. \end{cases}$$

A straightforward computation shows that the image of  $\alpha$  and  $\beta$  in  $\mathcal{AM}_{1,1,3}$ , denoted  $\hat{\alpha}$  and  $\hat{\beta}$ , are

$$\hat{\alpha} := \begin{cases} \hat{x} & \mapsto \hat{y}^{-1} \hat{x} \hat{y} \hat{t}_2 \hat{t}_3 \hat{t}_2^{-1} \hat{y}^{-1}, \\ \hat{y} & \mapsto \hat{y}, \\ \hat{t}_1 & \mapsto \hat{t}_3 \hat{t}_2^{-1} \hat{y}^{-1} \hat{x}^{-1} \hat{y}^{-1} \hat{x} \hat{y} \hat{t}_2 \hat{t}_3 \hat{t}_2^{-1}, \\ \hat{t}_2 & \mapsto \hat{t}_2, \\ \hat{t}_3 & \mapsto \hat{t}_1 \hat{t}_2 \hat{t}_3, \end{cases} \quad \hat{\beta} := \begin{cases} \hat{x} & \mapsto \hat{x}, \\ \hat{y} & \mapsto \hat{x} \hat{y} \hat{t}_2, \\ \hat{t}_1 & \mapsto \hat{t}_1 \hat{y}^{-1} \hat{x}^{-1} \hat{y} \hat{t}_2^{-1} \hat{y} \hat{x} \hat{y} \hat{t}_2, \\ \hat{t}_2 & \mapsto \hat{t}_3, \\ \hat{t}_3 & \mapsto \hat{t}_2 \hat{y}^{-1} \hat{x} \hat{y} \hat{t}_2 \hat{t}_3. \end{cases}$$

**Example 3.** Let  $F_3 := \langle a_{[1\uparrow 3]} \mid \rangle$ . Let  $H$  be the kernel of the homomorphism  $F_3 \rightarrow \langle \tau_1 \mid \tau_1^2 \rangle \times \langle \tau_2 \mid \tau_2^2 \rangle \times \langle \tau_3 \mid \tau_3^2 \rangle$  such that  $a_k \mapsto \tau_k$  for all  $k \in [1\uparrow 3]$ . It is standard to see that  $H$  is a free group of rank 17. It can be shown that  $H$  is a characteristic subgroup of  $F_3$ .

- (a). We identify  $\Sigma_{0,1,3}$  and  $F_3$  by putting  $a_k \leftrightarrow t_k$  for all  $k \in [1\uparrow 3]$ . Notice that  $\varrho = t_1 t_2 t_3 H$  has order 2 in  $G := \Sigma_{0,1,3}/H \simeq C_2 \times C_2 \times C_2$ . Hence,  $\langle \varrho \rangle$  has index 4 in  $G$  and  $b = 4$ . On the other hand, for all  $k \in [1\uparrow 3]$ ,  $\varrho_k = t_k H$  has order 2 in  $G$ . Hence, for all  $k \in [1\uparrow 3]$ ,  $\langle \varrho_k \rangle$  has index 4 in  $G$  and the  $k$ -th puncture in  $S_{0,1,3}$  lifts to 4 punctures in  $S_{g',b,q}$ . Thus,  $q = 12$ . Since  $\Sigma_{g',b,q}$  has rank  $2g' + b - 1 + q$  and  $H$  has rank 17, we have  $2g' + 4 - 1 + 12 = 17$  and  $g' = 1$ . Hence,  $\mathcal{AM}_{0,1,3} \hookrightarrow \mathcal{AM}_{1,1,3}$ .
- (b). We identify  $\Sigma_{1,1,1}$  and  $F_3$  by putting  $a_1 \leftrightarrow x_1$ ,  $a_2 \leftrightarrow y_1$  and  $a_3 \leftrightarrow t_1$ . Notice that  $\varrho = [x_1, y_1] t_1 H$  has order 2 in  $G := \Sigma_{1,1,1}/H \simeq C_2 \times C_2 \times C_2$ . Hence,  $\langle \varrho \rangle$  has index 4 in  $G$  and  $b = 4$ . On the other hand,  $\varrho_1 = t_1 H$  has order 2 in  $G$ . Hence,  $\langle \varrho_1 \rangle$  has index 4 in  $G$  and the puncture in  $S_{1,1,1}$  lifts to 4 punctures in  $S_{g',b,q}$ . Thus,  $q = 4$ . Since  $\Sigma_{g',b,q}$  has rank  $2g' + b - 1 + q$  and  $H$  has rank 17, we have  $2g' + 4 - 1 + 4 = 17$  and  $g' = 5$ . Hence,  $\mathcal{AM}_{1,1,1} \hookrightarrow \mathcal{AM}_{5,1,3}$ .



## 4 Proofs of Theorem 2.1 and Theorem 2.2

**4.1 Definition.** An element of  $\Sigma_{g,1,p}$  is said to be *t-squarefree* if, in its reduced expression, no two consecutive terms in  $t_{[1\uparrow p]} \vee \bar{t}_{[1\uparrow p]}$  are equal; for example:  $x_1x_1t_2t_3$  is *t-squarefree*;  $x_1t_2t_2y_1$  is non-*t-squarefree*.

To proof Theorem 2.1 we need the following theorem.

**4.2 Theorem.** For every  $\phi \in \mathcal{AM}_{g,1,p}$ , the elements of  $x_{[1\uparrow g]}^\phi \vee y_{[1\uparrow g]}^\phi \vee t_{[1\uparrow p]}^\phi$  are *t-squarefree*.

*Proof.* (of Theorem 2.1) If  $p = 0$ , then  $\psi$  is the identity and nothing needs to be said.

Suppose  $p \geq 1$ . Let  $a \in x_{[1\uparrow g]} \vee y_{[1\uparrow g]} \vee t_{[1\uparrow p]}$ . If  $\phi$  is an element of the kernel of  $\psi : \mathcal{AM}_{g,1,p} \rightarrow \mathcal{AM}_{g,1,p^{(a)}}$ , then  $a^\phi$  and  $a$  have the same image in  $\Sigma_{g,1,p^{(a)}}$ . On the other hand, by Theorem 4.2,  $a^\phi$  is *t-squarefree*. Hence,  $a^\phi$  has the same normal form in  $\Sigma_{g,1,p}$  as in  $\Sigma_{g,1,p^{(a)}}$ . Thus,  $a^\phi = a$ .  $\square$

Let  $N_d$  be the normal closure of  $t_1^d, t_2^d, \dots, t_p^d$  in  $\Sigma_{g,1,p}$ . Then

$$\Sigma_{g,1,p}/N_d = \Sigma_{g,1,p^{(a)}} = \langle x_{[1\uparrow g]} \vee y_{[1\uparrow g]} \vee \tau_{[1\uparrow p]} \mid \tau_1^d, \tau_2^d, \dots, \tau_p^d \rangle.$$

Let  $H \leq \Sigma_{g,1,p}$  be a normal subgroup of finite index such that  $N_d \leq H$ . Notice  $H/N_d \leq \Sigma_{g,1,p^{(a)}}$ . We set

$$\mathcal{AM}_{g,1,p}(H) = \{\phi \in \mathcal{AM}_{g,1,p} \mid H^\phi = H\},$$

and

$$\mathcal{AM}_{g,1,p^{(a)}}(H/N_d) = \{\tilde{\phi} \in \mathcal{AM}_{g,1,p^{(a)}} \mid (H/N_d)^{\tilde{\phi}} = H/N_d\}.$$

**4.3 Proposition.** With the above notation suppose  $(g, p, d) \neq (0, 2, 2)$ . Let  $\phi \in \mathcal{AM}_{g,1,p}(H)$ . Then  $\psi(\phi) \in \mathcal{AM}_{g,1,p^{(a)}}(H/N_d)$ . If  $\psi(\phi)|_{H/N_d} = 1$ , then  $\phi = 1$ .

*Proof.* Since  $N_d$  and  $H$  are  $\phi$ -invariant, the restriction of  $\psi(\phi) \in \mathcal{AM}_{g,1,p^{(a)}}$  to  $H/N_d$  is an element of  $\text{Aut}(H/N_d)$ . Hence,  $\psi(\phi) \in \mathcal{AM}_{g,1,p^{(a)}}(H/N_d)$ .

Since  $H$  has finite index in  $\Sigma_{g,1,p}$ , there exists  $r \in \mathbb{Z}$ ,  $r \neq 0$ , such that

$$(\prod_{i \in [1\uparrow g]} [x_i, y_i] \prod t_{[1\uparrow p]})^r \in H.$$

Fix  $k \in [1\uparrow p]$ . Since  $H$  is normal in  $\Sigma_{g,1,p}$ , we see

$$\bar{t}_k (\prod_{i \in [1\uparrow g]} [x_i, y_i] \prod t_{[1\uparrow p]})^r t_k \in H.$$

If  $\psi(\phi)|_{H/N_d} = 1$ , in  $\Sigma_{g,1,p^{(a)}}$ ,

$$\begin{aligned} & \bar{\tau}_k (\prod_{i \in [1\uparrow g]} [x_i, y_i] \prod \tau_{[1\uparrow p]})^r \tau_k \\ &= (\bar{\tau}_k (\prod_{i \in [1\uparrow g]} [x_i, y_i] \prod \tau_{[1\uparrow p]})^r \tau_k)^{\psi(\phi)} \\ &= \bar{\tau}_k^{\psi(\phi)} (\prod_{i \in [1\uparrow g]} [x_i, y_i] \prod \tau_{[1\uparrow p]})^r \tau_k^{\psi(\phi)}. \end{aligned}$$

Hence, in  $\Sigma_{g,1,p^{(d)}}$ ,  $\tau_k^{\psi(\phi)}\bar{\tau}_k$  commutes with  $(\prod_{i \in [1 \uparrow g]} [x_i, y_i] \prod \tau_{[1 \uparrow p]})^r$ . Then  $\tau_k^{\psi(\phi)}\bar{\tau}_k \in \langle \prod_{i \in [1 \uparrow g]} [x_i, y_i] \prod \tau_{[1 \uparrow p]} \rangle$ , and,

$$(4.3.1) \quad \tau_k^{\psi(\phi)} = (\prod_{i \in [1 \uparrow g]} [x_i, y_i] \prod \tau_{[1 \uparrow p]})^{r'} \tau_k,$$

for some  $r' \in \mathbb{Z}$ . Recall  $[\tau_k^{\psi(\phi)}] = [\tau_{k'}]$ , for some  $k' \in [1 \uparrow p]$ . If  $(g, p) \neq (0, 1)$ , and if  $(g, p, d) \neq (0, 2, 2)$ , then (4.3.1) implies  $r' = 0$  and  $\tau_k^{\psi(\phi)} = \tau_k$ .

Recall  $\Sigma_{g,1,p^{(d)}} = \Sigma_{g,1,0} * C_d^{*p}$ . Fix  $a \in x_{[1 \uparrow g]} \vee y_{[1 \uparrow g]}$ . Since  $H$  has finite index in  $\Sigma_{g,1,p}$ , there exists  $s \in \mathbb{Z}$  such that  $a^s \in H$ . If  $\psi(\phi)|_{H/N_d} = 1$ , then  $(a^s)^{\psi(\phi)} = a^s$ , and,  $a^{\psi(\phi)} = a$ .

Since  $\Sigma_{g,1,p^{(d)}} = \Sigma_{g,1,0} * C_d^{*p}$ ,  $a^{\psi(\phi)} = a$  for all  $a \in x_{[1 \uparrow g]} \vee y_{[1 \uparrow g]}$ , and,  $\tau_k^{\psi(\phi)} = \tau_k$  for all  $k \in [1 \uparrow p]$ , we see  $\psi(\phi) = 1$ . By Theorem 2.1,  $\phi = 1$ .  $\square$

**4.4 Lemma.** *With the notation above Theorem 2.2, if  $\Sigma_{g',b,q}$  is  $\mathcal{AM}_{g,1,p}$ -invariant then the normal closure of  $t_1^d, t_2^d, \dots, t_p^d$  in  $\Sigma_{g,1,p}$  equals the normal closure of  $\hat{t}_1, \hat{t}_2, \dots, \hat{t}_q$  in  $\Sigma_{g',b,q}$ .*

*Proof.* Recall

$$(4.4.1) \quad \{\hat{t}_1, \hat{t}_2, \dots, \hat{t}_q\} = \bigvee_{k \in [1 \uparrow p]} \{(t_k^d)^{u_{1,k}}, (t_k^d)^{u_{2,k}}, \dots, (t_k^d)^{u_{m/d,k}}\}.$$

By (4.4.1), the normal closure of  $\hat{t}_1, \hat{t}_2, \dots, \hat{t}_q$  in  $\Sigma_{g',b,q}$  is a subgroup of the normal closure of  $t_1^d, t_2^d, \dots, t_p^d$  in  $\Sigma_{g,1,p}$ .

Let  $k \in [1 \uparrow p]$  and  $w \in \Sigma_{g,1,p}$ . By (4.4.1), it is enough to prove  $(t_k^d)^w = (t_k^d)^{u_{j,k}v}$  for some  $j \in [1 \uparrow (m/d)]$  and  $v \in \Sigma_{g',b,q}$ . Recall  $G = \Sigma_{g,1,p} / \Sigma_{g',b,q}$ ,  $\varrho_k = t_k \Sigma_{g',b,q} \in G$  and  $G = \langle \varrho_k \rangle \rho_{1,k} \vee \langle \varrho_k \rangle \rho_{2,k} \cdots \vee \langle \varrho_k \rangle \rho_{m/d,k}$ , where  $\rho_{j,k} = u_{j,k} \Sigma_{g',b,q} \in G$  for all  $j \in [1 \uparrow (m/d)]$ . Let  $j \in [1 \uparrow (m/d)]$  such that  $w \Sigma_{g',b,q} \in \langle \varrho_k \rangle \rho_{j,k}$ . Let  $r \in [1 \uparrow d]$  such that  $w \Sigma_{g',b,q} = \varrho_k^r \rho_{j,k} = t_k^r u_{j,k} \Sigma_{g',b,q}$ . Then  $w = t_k^r u_{j,k} v$ , for some  $v \in \Sigma_{g',b,q}$  and  $(t_k^d)^w = (t_k^d)^{t_k^r u_{j,k} v} = (t_k^d)^{u_{j,k} v}$ .  $\square$

*Proof. (of Theorem 2.2)* Recall  $N_d$  is the normal closure in  $\Sigma_{g,1,p}$  of  $t_1^d, t_2^d, \dots, t_p^d$ . By Lemma 4.4,  $N_d$  is the normal closure in  $\Sigma_{g',b,q}$  of  $\hat{t}_1, \hat{t}_2, \dots, \hat{t}_q$ . Hence,  $\Sigma_{g',b,0} = \Sigma_{g',b,q} / N_d$ . We identify  $\Sigma_{g',b,0}$  with  $\Sigma_{g',1,b-1}$  by identifying  $z_k$  with  $t_k$  for all  $k \in [1 \uparrow (b-1)]$ . Hence,  $\Sigma_{g',1,(b-1)} = \Sigma_{g',b,q} / N_d$ . Since  $\Sigma_{g,1,p^{(d)}} = \Sigma_{g,1,p} / N_d$ , the natural homomorphism  $\Sigma_{g,1,p} \rightarrow \Sigma_{g,1,p^{(d)}}$  restricts to the natural homomorphism  $\Sigma_{g',b,q} \rightarrow \Sigma_{g',1,(b-1)}$ .

Let  $\phi \in \mathcal{AM}_{g,1,p}$ . Since  $\psi : \mathcal{AM}_{g,1,p} \rightarrow \mathcal{AM}_{g,1,p^{(d)}}$  is given by the natural homomorphism  $\Sigma_{g,1,p} \rightarrow \Sigma_{g,1,p^{(d)}}$ , we see  $\psi(\phi) : \Sigma_{g,1,p^{(d)}} \rightarrow \Sigma_{g,1,p^{(d)}}$  completes the following commutative square

$$\begin{array}{ccc} \Sigma_{g,1,p} & \xrightarrow{\phi} & \Sigma_{g,1,p} \\ \downarrow & & \downarrow \\ \Sigma_{g,1,p^{(d)}} & \xrightarrow{\psi(\phi)} & \Sigma_{g,1,p^{(d)}} \end{array}$$

where the vertical arrows are the natural homomorphisms. Since  $\Sigma_{g',1,(b-1)} = \Sigma_{g',b,q}/N_d$  and  $\Sigma_{g',b,q}$  is  $\mathcal{AM}_{g,1,p}$ -invariant, by Proposition 4.3, there exists the restriction  $\psi(\phi)|_{\Sigma_{g',1,(b-1)}} : \Sigma_{g',1,(b-1)} \rightarrow \Sigma_{g',1,(b-1)}$ . Notice  $\psi(\phi)|_{\Sigma_{g',1,(b-1)}} : \Sigma_{g',1,(b-1)} \rightarrow \Sigma_{g',1,(b-1)}$  completes the following commutative square

$$\begin{array}{ccc} \Sigma_{g',b,q} & \xrightarrow{\phi|_{\Sigma_{g',b,q}}} & \Sigma_{g',b,q} \\ \downarrow & & \downarrow \\ \Sigma_{g',1,(b-1)} & \xrightarrow{\psi(\phi)|_{\Sigma_{g',1,(b-1)}}} & \Sigma_{g',1,(b-1)} \end{array}$$

where the vertical arrows are the natural homomorphisms. Since the second homomorphism of (2.2.1) is given by eliminating the last  $q$  punctures; that is, by the natural homomorphism  $\Sigma_{g',b,q} \rightarrow \Sigma_{g',1,(b-1)}$ , in the composition of (2.2.1) we have

$$\phi \mapsto \phi|_{\Sigma_{g',b,q}} \mapsto \psi(\phi)|_{\Sigma_{g',1,(b-1)}}.$$

By Proposition 4.3, if  $\psi(\phi)|_{\Sigma_{g',1,(b-1)}} = 1$ , then  $\phi = 1$ .  $\square$

The rest of the paper is dedicated to proof Theorem 4.2. The proof is similar to [3, 7.6 Corollary]. Notice Theorem 4.2 is trivial if  $p = 0$ .

## 5 McCool's Groupoid

For the rest of the paper we suppose  $p \geq 1$ .

Let  $n := 2g + p$ , and, let  $F_n$  be the free group on  $X$ , where  $X$  is a set with  $n$  elements.

**5.1 Notation.** Let  $w \in F_n$ . In this section we will denote by  $[w]$  the cyclic word of  $w$ .

**5.2 Definitions.** Let  $T$  be a set of words and cyclic words of  $F_n$ . Suppose the elements of  $T$  are reduced and cyclically reduced, respectively. We define the *Whitehead graph of  $T$*  as the graph with vertex set  $X \vee \overline{X}$ , and, one edge from  $a \in X \vee \overline{X}$  to  $b \in X \vee \overline{X}$  for every subword  $\overline{a}b$  which appears in  $w$  or  $[u]$ , where  $w$  and  $[u]$  are elements of  $T$ . We say that  $a$  is the initial vertex and  $b$  is the terminal vertex of the edges corresponding to the subword  $\overline{a}b$ . Repetitions have to be considered. For example, since the subword  $\overline{a}b$  appears twice in  $\overline{a}b\overline{a}b$ , the Whitehead graph of  $\{\overline{a}b\overline{a}b\}$  has 2 edges from  $a$  to  $b$  (and one edge from  $\overline{b}$  to  $\overline{a}$ ). Notice that the cyclic word  $[a]$  produces an edge from  $\overline{a}$  to  $a$  in the Whitehead graph.

We say that  $T$  is a *surface word set* if the Whitehead graph of  $T$  is an oriented segment, that is, the Whitehead graph of  $T$  is connected with exactly  $2n - 1$  edges, every vertex but one is the *initial vertex* of exactly one edge, and, every vertex but one is the *terminal vertex* of exactly one edge.

**5.3 Example.** Let  $F_4 := \langle a, b, c, d \mid \rangle$ .

(i). Let  $T := \{\overline{adc\bar{b}}, [\overline{db}], [\overline{ca}]\}$ . The Whitehead graph of  $T$  is

$$\bar{a} \rightarrow \bar{c} \rightarrow \bar{b} \rightarrow \bar{d} \rightarrow c \rightarrow a \rightarrow d \rightarrow b.$$

Hence,  $T$  is a surface word set.

(ii). Let  $T := \{\overline{adc\bar{b}}, \overline{db}, [\overline{ca}]\}$ . The Whitehead graph of  $T$  is

$$\bar{a} \rightarrow \bar{c} \rightarrow \bar{b} \quad \bar{d} \rightarrow c \rightarrow a \rightarrow d \rightarrow b.$$

Hence,  $T$  is not a surface word set.

(iii). Let  $T := \{\overline{adc\bar{b}}, dc, [\overline{db}], [\overline{ca}]\}$ . The Whitehead graph of  $T$  is

$$\bar{a} \rightarrow \bar{c} \rightarrow \bar{b} \rightarrow \bar{d} \rightrightarrows c \rightarrow a \rightarrow d \rightarrow b.$$

Hence,  $T$  is not a surface word set.

We illustrate the following remarks with examples in  $F_4 = \langle a, b, c, d \mid \rangle$ .

**5.4 Remarks.** Let  $T$  be a surface word set.

(i) The Whitehead graph of  $T$  defines a sequence  $a_{[1\uparrow 2n]}$  with underlying set  $X \vee \overline{X}$  such that for all  $i \in [1\uparrow(2n-1)]$ , the Whitehead graph of  $T$  has exactly one edge with initial vertex  $a_i$  and terminal vertex  $a_{i+1}$ , equivalently,  $\overline{a_i a_{i+1}}$  is a subword of exactly one element of  $T$ . We say that  $a_{[1\uparrow 2n]}$  is the *associated sequence* of  $T$ .

In Example 5.3(i), the associated sequence of  $T$  is  $(\bar{a}, \bar{c}, \bar{b}, \bar{d}, c, a, d, b)$ .

(ii) We can recover  $T$  from the associated sequence of  $T$ . The process to recover  $T$  from its associated sequence is the inverse process to construct the Whitehead graph. We give two examples below. From this process, it is easy to see that  $T$  has exactly one word, and, all other elements of  $T$  are cyclic words.

In  $F_4$ , from the sequence  $(a, b, c, d, \bar{a}, \bar{b}, \bar{c}, \bar{d})$  we have the surface word set  $\{\overline{abcd\bar{a}\bar{b}\bar{c}\bar{d}}\}$ , and, from the sequence  $(a, b, c, d, \bar{d}, \bar{c}, \bar{b}, \bar{a})$  we have the surface word set  $\{a, [\overline{b\bar{a}}], [\overline{c\bar{b}}], [\overline{d\bar{c}}], [\overline{d}]\}$ .

(iii) Let  $p$  be the cardinality of  $T$  minus one. We say that  $T$  is a  $(g, p)$ -surface word set, where  $g = (n - p)/2$ . By induction on  $n$ , it can be seen that  $n \geq p$  and  $n - p$  is even. Hence,  $g \in \mathbb{N}$ .

**5.5 Definition.** Let  $\phi \in \text{Aut}(F_n)$ .

We say that  $\phi$  is a *type-1 Nielsen automorphism* if  $\phi$  restricts to a permutation of  $X \vee \overline{X}$ .

We say that  $\phi$  is a *type-2 Nielsen automorphism* if there exists  $a, b \in X \vee \overline{X}$  such that  $a \neq b, \bar{b}$  and

$$\phi := \begin{cases} a & \mapsto ab, \\ c & \mapsto c \end{cases} \text{ for all } c \in X, c \neq a^{\pm 1}.$$

We denote  $\phi$  by  $(a \mapsto ab)$  or  $(\bar{a} \mapsto \bar{b}\bar{a})$ .

**5.6 Definition.** Let  $\mathcal{G}_{g,p}$  be the groupoid with objects  $(g, p)$ -surface word sets, and, given  $T_1, T_2$  two  $(g, p)$ -surface word sets

$$\text{Hom}(T_1, T_2) := \{\phi \in \text{Aut}(F_n) \mid T_1^\phi = T_2\},$$

where  $T_1^\phi := \{w^\phi, [u^\phi] \mid w, [u] \in T_1\}$ . Here,  $w^\phi$  is reduced and  $[u^\phi]$  is cyclically reduced. Hence,  $[v] = [u^\phi]$  means that  $v$  and  $u^\phi$  are conjugated.

We say that  $(T_1, T_2, \phi) \in \text{Hom}(T_1, T_2)$  is a *type-1 Nielsen* of  $\mathcal{G}_{g,p}$  if  $\phi$  is a type-1 Nielsen automorphism. Similarly, for type-2 Nielsen automorphisms. We say that  $(T_1, T_2, \phi) \in \text{Hom}(T_1, T_2)$  is a Nielsen if it is either a type-1 Nielsen or a type-2 Nielsen.

We illustrate the following remarks with examples in  $F_4 = \langle a, b, c, d \mid \rangle$ .

**5.7 Remark.** Let  $(T_1, T_2, \phi)$  be a Nielsen of  $\mathcal{G}_{g,p}$ .

- (i) If  $(T_1, T_2, \phi)$  is a type-1 Nielsen, then the associated sequence of  $T_2$  is obtained from the associated sequence of  $T_1$  by applying the permutation  $\phi$  to every element of the sequence.

In  $F_4$ , let  $T_1 = \{a\bar{d}\bar{b}c, [\bar{a}b], [\bar{c}d]\}$ . Notice the associated sequence of  $T_1$  is  $(a, b, c, d, \bar{b}, \bar{a}, \bar{d}, \bar{c})$ . If  $\phi := (a \mapsto \bar{b}, b \mapsto c, c \mapsto \bar{a}, d \mapsto \bar{d})$ , then the associated sequence of  $T_2$  is  $(\bar{b}, c, \bar{a}, \bar{d}, \bar{c}, b, d, a)$ .

- (ii) Suppose  $(T_1, T_2, \phi)$  is a type-2 Nielsen. Then  $\phi = (a_i \mapsto ba_i)$  for some  $i \in [1 \uparrow 2n]$ ,  $b \in X \vee \overline{X}$  such that  $a_i \neq b, \bar{b}$ . Since in the Whitehead graph of  $T$  there are exactly  $2n - 1$  edges, there exists  $w \in T_1$  or  $[u] \in T_1$  such that applying  $\phi$  to  $w$  or  $[u]$  produces a cancellation. If the cancellation appears from the subword  $\bar{a}_{i-1}a_i$ , then  $b = a_{i-1}$ . If the cancellation appears from the subword  $\bar{a}_i a_{i+1}$ , then  $b = a_{i+1}$ . Hence, either  $\phi = (a_i \mapsto a_{i-1}a_i)$  for some  $i \in [2 \uparrow 2n]$ ,  $a_i \neq \bar{a}_{i-1}$ ; or  $\phi = (\bar{a}_i \mapsto \bar{a}_i \bar{a}_{i+1})$  for some  $i \in [1 \uparrow (2n - 1)]$ ,  $a_i \neq \bar{a}_{i+1}$ . In the former case the associated sequence of  $T_2$  is obtained from the associated sequence of  $T_1$  by moving  $a_i$  from immediately after  $a_{i-1}$  to immediately before  $\bar{a}_{i-1}$ . In the later case the associated sequence of  $T_2$  is obtained from the associated sequence of  $T_1$  by moving  $a_i$  from immediately before  $a_{i+1}$  to immediately after  $\bar{a}_{i+1}$ .

In  $F_4$ , let  $T_1 = \{a\bar{b}\bar{c}\bar{d}\bar{a}\bar{b}\bar{c}\bar{d}\}$ . Notice the associated sequence of  $T_1$  is  $(a, b, c, d, \bar{a}, \bar{b}, \bar{c}, \bar{d})$ . If  $\phi := (b \mapsto ab)$ , then the associated sequence of  $T_2$  is  $(a, c, d, b, \bar{a}, \bar{b}, \bar{c}, \bar{d})$ . In fact  $(a\bar{b}\bar{c}\bar{d}\bar{a}\bar{b}\bar{c}\bar{d})^{(b \mapsto ab)} = a\bar{b}\bar{a}\bar{c}\bar{d}\bar{b}\bar{c}\bar{d}$ . If  $\phi := (\bar{a} \mapsto \bar{a}\bar{b})$ , then the associated sequence of  $T_2$  is  $(b, c, d, \bar{a}, \bar{b}, a, \bar{c}, \bar{d})$ . In fact  $(a\bar{b}\bar{c}\bar{d}\bar{a}\bar{b}\bar{c}\bar{d})^{(\bar{a} \mapsto \bar{a}\bar{b})} = b\bar{a}\bar{b}\bar{c}\bar{d}\bar{a}\bar{c}\bar{d}$ .

**5.8 Remark.** It is easy to see  $\{\Pi_{i \in [1 \uparrow g]}[x_i, y_i] \Pi_{j \in [1 \uparrow p]}t_j, [\bar{t}_1], [\bar{t}_2], \dots, [\bar{t}_p]\}$  is a  $(g, p)$ -surface word set of  $\Sigma_{g,1,p}$ . Its associated sequence is

$$(\bar{x}_1, y_1, x_1, \bar{y}_1, \bar{x}_2, y_2, x_2, \bar{y}_2, \dots, \bar{x}_g, y_g, x_g, \bar{y}_g, t_1, \bar{t}_1, t_2, \bar{t}_2, \dots, t_p, \bar{t}_p).$$

We say that  $\{\Pi_{i \in [1 \uparrow g]}[x_i, y_i] \Pi_{j \in [1 \uparrow p]}t_j, [\bar{t}_1], [\bar{t}_2], \dots, [\bar{t}_p]\}$  is the standard  $(g, p)$ -surface word set of  $\Sigma_{g,1,p}$ .

**5.9 Remark.**  $\mathcal{AM}_{g,1,p} = \text{Hom}(T, T)$ , where  $T$  is the standard  $(g, p)$ -surface word set of  $\Sigma_{g,1,p}$ .

**5.10 Theorem** (McCool [15],[9]).  $\mathcal{G}_{g,p}$  is generated by Nielsen elements.

## 6 Ends of free group

Let  $n := 2g + p$  and  $F_n$  is the free group on  $X$ , where  $|X| = n$ .

**6.1 Notation.** Let  $a_{[1 \uparrow k]}$  be the normal form for  $w \in F_n$ ; in particular,  $w = \Pi a_{[1 \uparrow k]}$ . The set of elements of  $F_n$  whose normal forms have  $a_{[1 \uparrow k]}$  as an initial segment is denoted  $(w\star)$ ; and, the set of elements of  $F_n$  whose normal forms have  $a_{[1 \uparrow k]}$  as a terminal segment is denoted  $(\star w)$ . The elements of  $(w\star)$  are said to *begin with*  $w$ , and the elements of  $(\star w)$  are said to *end with*  $w$ .

**6.2 Review.** An *end* of  $F_n$  is a sequence  $a_{[1 \uparrow \infty]}$  in  $X \vee \bar{X}$  such that, for each  $i \in [1 \uparrow \infty[$ ,  $a_{i+1} \neq \bar{a}_i$ . We represent  $a_{[1 \uparrow \infty]}$  as a formal right-infinite reduced product,  $a_1 a_2 \cdots$  or  $\Pi a_{[1 \uparrow \infty]}$ .

We denote the set of ends of  $F_n$  by  $\partial F_n$ .

For each  $w \in F_n$ , we define the *shadow* of  $w$  in  $\partial F_n$  to be

$$(w\blacktriangleleft) := \{a_{[1 \uparrow \infty]} \in \partial F_n \mid \Pi a_{[1 \uparrow |w|]} = w\}.$$

Thus, for example,  $(1\blacktriangleleft) = \partial F_n$ .

**6.3 Definition.** Let  $T$  be a surface word set. We now give  $\partial F_n$  an ordering,  $<_T$ , with respect to  $T$  as follows. Let  $a_{[1 \uparrow 2n]}$  be the associated sequence of  $T$ . Recall  $a_{[1 \uparrow 2n]}$  is a listing of the elements of  $X \vee \bar{X}$ . For each  $w \in F_n$ , we assign an ordering,  $<_T$ , to a partition of  $(w\blacktriangleleft)$  into  $2n$  or  $2n - 1$  subsets, depending as  $w = 1$  or  $w \neq 1$ , as follows. We set

$$(a_1\blacktriangleleft) <_T (a_2\blacktriangleleft) <_T (a_3\blacktriangleleft) <_T \cdots <_T (a_{2n-1}\blacktriangleleft) <_T (a_{2n}\blacktriangleleft).$$

If  $i \in [1 \uparrow n]$  and  $w \in (\star \bar{a}_i)$ , then we set

$$\begin{aligned} (wa_{i+1} \blacktriangleleft) &<_T (wa_{i+2} \blacktriangleleft) <_T (wa_{i+3} \blacktriangleleft) <_T \cdots \\ \cdots &<_T (wa_{2n-1} \blacktriangleleft) <_T (wa_{2n} \blacktriangleleft) <_T (wa_1 \blacktriangleleft) <_T (wa_2 \blacktriangleleft) <_T (wa_3 \blacktriangleleft) <_T \cdots \\ \cdots &<_T (wa_{i-2} \blacktriangleleft) <_T (wa_{i-1} \blacktriangleleft). \end{aligned}$$

Hence, for each  $w \in F_n$ , we have an ordering  $<_T$  of a partition of  $(w \blacktriangleleft)$  into  $2n$  or  $2n - 1$  subsets.

If  $b_{[1 \uparrow \infty]}$  and  $c_{[1 \uparrow \infty]}$  are two different ends, then there exists  $i \in \mathbb{N}$  such that  $b_{[1 \uparrow i]} = c_{[1 \uparrow i]}$  and  $b_{i+1} \neq c_{i+1}$ . Let  $w = \Pi b_{[1 \uparrow i]} = \Pi c_{[1 \uparrow i]}$  in  $F_n$ . Then  $b_{[1 \uparrow \infty]}$  and  $c_{[1 \uparrow \infty]}$  lie in  $(w \blacktriangleleft)$ , but lie in different elements of the partition of  $(w \blacktriangleleft)$  into  $2n$  or  $2n - 1$  subsets. We then order  $b_{[1 \uparrow \infty]}$  and  $c_{[1 \uparrow \infty]}$  using the order of the elements of the partition of  $(w \blacktriangleleft)$  that they belong to. This completes the definition of the ordering  $<_T$  of  $\partial F_n$ .

Let  $w$  be the non-cyclic element of  $T$ . We remark that in  $(\partial F_n, <_T)$  the smallest element is  $w^\infty$  and the largest element is  $\bar{w}^\infty$ .

For example, in  $F_4 = \langle a, b, c, d \mid \rangle$  we take  $T = \{a\bar{d} \bar{b}c, [\bar{a}b], [\bar{c}d]\}$ . The associated sequence of  $T$  is  $(a, b, c, d, \bar{b}, \bar{a}, \bar{d}, \bar{c})$ . In  $(\partial F_4, <_T)$ , the smallest element is  $(a\bar{d} \bar{b}c)^\infty$ , and, the largest element is  $(\bar{c}b\bar{d}\bar{a})^\infty$ .

**6.4 Notation.** We denote by  $<$  the order on  $\partial \Sigma_{g,1,p}$  with respect to the standard  $(g, p)$ -surface word set of  $\Sigma_{g,1,p}$ .

**6.5 Review.** Let  $\hat{S}$  be the universal cover of  $S_{g,1,p}$ . Suppose  $S_{g,1,p}$  has negative Euler characteristic, that is,  $2g + p \geq 2$ . Then  $\hat{S}$  can be identified with the hyperbolic plane. Let  $\partial \hat{S}$  be the boundary of  $\hat{S}$ . It is well-known that  $\partial \hat{S}$  can be identified with  $\mathbb{R} \vee \{\infty\}$ . Let  $*$  be the point in  $\partial \hat{S}$  corresponding to  $\infty$  by this identification. The identification between  $\partial \hat{S}$  and  $\mathbb{R} \vee \{\infty\}$  restricts to an identification between  $\partial \hat{S} - \{*\}$  and  $\mathbb{R}$ . By work of Nielsen-Thurston [5], [16], there is an action of  $\mathcal{M}_{g,1,p}$  on  $\partial \hat{S}$  with a fixed point, which we can suppose to be  $*$  in  $\partial \hat{S}$ . Hence, an action of  $\mathcal{M}_{g,1,p}$  on  $\mathbb{R}$ . By [16], this action preserves the usual order of  $\mathbb{R}$ . Remark 5.9 and Proposition 6.6 give the analog statement for  $\mathcal{AM}_{g,1,p}$  and  $\partial \Sigma_{g,1,p}$ .

Let  $\phi \in \text{Aut}(F_n)$ . It is proved in [5] that  $(\Pi a_{[1 \uparrow \infty]})^\phi = \lim_{k \rightarrow \infty} (\Pi a_{[1 \uparrow k]})^\phi$  defines a map  $\partial F_n \rightarrow \partial F_n$ , which we still denote by  $\phi$ .

**6.6 Proposition.** *Let  $T_1, T_2$  be surface word sets of  $F_n$  and  $(T_1, T_2, \phi) \in \text{Hom}(T_1, T_2)$ . Then  $\phi : (\partial F_n, \leq_{T_1}) \rightarrow (\partial F_n, \leq_{T_2})$  respects the orderings.*

*Proof.* By Theorem 5.10, we can restrict ourselves to the case where  $(T_1, T_2, \phi)$  is a Nielsen.

By Remark 5.7(i), the result is clear if  $(T_1, T_2, \phi)$  is a type-1 Nielsen. Hence, we suppose  $(T_1, T_2, \phi)$  is a type-2 Nielsen.

Let  $a_{[1\uparrow 2n]}$  be the associated sequence of  $T_1$ . Then either  $\phi = (a_i \mapsto a_{i-1}a_i)$  for some  $i \in [2\uparrow 2n]$ ,  $a_i \neq \bar{a}_{i-1}$ , or,  $\phi = (\bar{a}_i \mapsto \bar{a}_i\bar{a}_{i+1})$  for some  $i \in [1\uparrow(2n-1)]$ ,  $a_i \neq \bar{a}_{i+1}$ .

Suppose  $\phi = (a_i \mapsto a_{i-1}a_i)$  for some  $i \in [2\uparrow 2n]$ ,  $a_i \neq \bar{a}_{i-1}$ .

The following correspondence by the action of  $(a_i \mapsto a_{i-1}a_i)$  is clear.

$$\begin{array}{ccc}
& (a_i \mapsto a_{i-1}a_i) & \\
(\star \bar{a}_i a_{i-1}) & \longrightarrow & (\star \bar{a}_i), \\
[(\star a_{i-1}) - (\star \bar{a}_i a_{i-1})] & \longrightarrow & (\star a_{i-1}), \\
(\star a_i) & \longrightarrow & (\star a_i), \\
(\star a_k) & \longrightarrow & (\star a_k), \quad a_k \neq a_{i-1}^{\pm 1}, a_i^{\pm 1}, \\
(\star \bar{a}_{i-1}) & \longrightarrow & (\star \bar{a}_{i-1}), \\
(\star \bar{a}_i) & \longrightarrow & (\star \bar{a}_i \bar{a}_{i-1}).
\end{array}$$

The following correspondence by the action of  $(a_i \mapsto a_{i-1}a_i)$  is clear.

$$\begin{array}{ccc}
& (a_i \mapsto a_{i-1}a_i) & \\
(a_{i-1} \blacktriangleleft) & \longrightarrow & (a_{i-1} \blacktriangleleft) - (a_{i-1}a_i \blacktriangleleft), \\
(a_i \blacktriangleleft) & \longrightarrow & (a_{i-1}a_i \blacktriangleleft), \\
(a_k \blacktriangleleft) & \longrightarrow & (a_k \blacktriangleleft), \quad a_k \neq a_{i-1}^{\pm 1}, a_i^{\pm 1}, \\
(\bar{a}_{i-1}a_i \blacktriangleleft) & \longrightarrow & (a_i \blacktriangleleft), \\
[(\bar{a}_{i-1} \blacktriangleleft) - (\bar{a}_{i-1}a_i \blacktriangleleft)] & \longrightarrow & (\bar{a}_{i-1} \blacktriangleleft), \\
(\bar{a}_i \blacktriangleleft) & \longrightarrow & (\bar{a}_i \blacktriangleleft).
\end{array}$$

From the first row of the first table and the second table we deduce the following table.

$$\begin{array}{ccc}
& (a_i \mapsto a_{i-1}a_i) & \\
(\star \bar{a}_i a_{i-1})(a_{i-1} \blacktriangleleft) & \longrightarrow & (\star \bar{a}_i)[(a_{i-1} \blacktriangleleft) - (a_{i-1}a_i \blacktriangleleft)], \\
(\star \bar{a}_i a_{i-1})(a_i \blacktriangleleft) & \longrightarrow & (\star \bar{a}_i)(a_{i-1}a_i \blacktriangleleft), \\
(\star \bar{a}_i a_{i-1})(a_k \blacktriangleleft) & \longrightarrow & (\star \bar{a}_i)(a_k \blacktriangleleft), \quad a_k \neq a_{i-1}^{\pm 1}, a_i^{\pm 1}, \\
(\star \bar{a}_i a_{i-1})(\bar{a}_i \blacktriangleleft) & \longrightarrow & (\star \bar{a}_i)(\bar{a}_i \blacktriangleleft).
\end{array}$$

Notice the cases  $(\star \bar{a}_i a_{i-1})(\bar{a}_{i-1}a_i \blacktriangleleft)$  and  $(\star \bar{a}_i a_{i-1})[(\bar{a}_{i-1} \blacktriangleleft) - (\bar{a}_{i-1}a_i \blacktriangleleft)]$  do not have to be considered since they are not in reduced form.

Let  $\mathbf{e}, \mathbf{f} \in \partial F_n$  such that  $\mathbf{e} = (w\bar{a}_i a_{i-1})\mathbf{e}'$ ,  $\mathbf{f} = (w\bar{a}_i a_{i-1})\mathbf{f}'$  and the first letter of  $\mathbf{e}'$  is different from the first letter of  $\mathbf{f}'$ . Let  $j \in [1\uparrow 2n]$  such that  $a_j = \bar{a}_{i-1}$ . By the third table,  $\mathbf{e}^{(a_i \mapsto a_{i-1}a_i)} = (u\bar{a}_i)\mathbf{e}''$ ,  $\mathbf{f}^{(a_i \mapsto a_{i-1}a_i)} = (u\bar{a}_i)\mathbf{f}''$  in reduced form. Let  $b_{[1\uparrow 2n]}$  be the associated sequence of  $T_2$ . Recall  $b_{[1\uparrow 2n]}$  is obtained from  $a_{[1\uparrow 2n]}$  by moving  $a_i$  from immediately after  $a_{i-1}$  to immediately before  $a_j = \bar{a}_{i-1}$ . There are two cases according to  $j < i-1$  or  $i-1 < j$ .



If  $j < i - 1$ , then

$$\begin{aligned} b_{[1\uparrow(j-1)]} &= a_{[1\uparrow(j-1)]}, \\ b_j &= a_i, \\ b_{[(j+1)\uparrow i]} &= a_{[j\uparrow(i-1)]}, \\ b_{[(i+1)\uparrow 2n]} &= a_{[(i+1)\uparrow 2n]}. \end{aligned}$$

The partition with respect to  $a_{[1\uparrow 2n]}$  of  $(\bar{a}_j \blacktriangleleft) = (a_{i-1} \blacktriangleleft)$  is  $(a_{j+1} \blacktriangleleft), (a_{j+2} \blacktriangleleft), \dots, (a_{i-1} \blacktriangleleft), (a_i \blacktriangleleft), (a_{i+1} \blacktriangleleft), \dots, (a_{2n} \blacktriangleleft), (a_1 \blacktriangleleft), (a_2 \blacktriangleleft), \dots, (a_{j-1} \blacktriangleleft)$ . The partition with respect to  $b_{[1\uparrow 2n]}$  of  $(\bar{a}_i \blacktriangleleft)$  is  $(a_j \blacktriangleleft), (a_{j+1} \blacktriangleleft), \dots, (a_{i-1} \blacktriangleleft), (a_{i+1} \blacktriangleleft), (a_{i+2} \blacktriangleleft), \dots, (a_{2n} \blacktriangleleft), (a_1 \blacktriangleleft), (a_2 \blacktriangleleft), \dots, (a_{j-1} \blacktriangleleft)$ . By the third table,

$$\begin{array}{ccc} & (a_i \mapsto a_{i-1}a_i) & \\ (w\bar{a}_i a_{i-1})(a_{j+1} \blacktriangleleft) & \longrightarrow & (u\bar{a}_i)(a_{j+1} \blacktriangleleft), \\ (w\bar{a}_i a_{i-1})(a_{j+2} \blacktriangleleft) & \longrightarrow & (u\bar{a}_i)(a_{j+2} \blacktriangleleft), \\ & \vdots & \\ (w\bar{a}_i a_{i-1})(a_{i-2} \blacktriangleleft) & \longrightarrow & (u\bar{a}_i)(a_{i-2} \blacktriangleleft), \\ (w\bar{a}_i a_{i-1})(a_{i-1} \blacktriangleleft) & \longrightarrow & (u\bar{a}_i)[(a_{i-1} \blacktriangleleft) - (a_{i-1}a_i \blacktriangleleft)], \\ (w\bar{a}_i a_{i-1})(a_i \blacktriangleleft) & \longrightarrow & (u\bar{a}_i)(a_{i-1}a_i \blacktriangleleft), \\ (w\bar{a}_i a_{i-1})(a_{i+1} \blacktriangleleft) & \longrightarrow & (u\bar{a}_i)(a_{i+1} \blacktriangleleft), \\ & \vdots & \\ (w\bar{a}_i a_{i-1})(a_{2n} \blacktriangleleft) & \longrightarrow & (u\bar{a}_i)(a_{2n} \blacktriangleleft), \\ (w\bar{a}_i a_{i-1})(a_1 \blacktriangleleft) & \longrightarrow & (u\bar{a}_i)(a_1 \blacktriangleleft), \\ (w\bar{a}_i a_{i-1})(a_2 \blacktriangleleft) & \longrightarrow & (u\bar{a}_i)(a_2 \blacktriangleleft), \\ & \vdots & \\ (w\bar{a}_i a_{i-1})(a_{j-1} \blacktriangleleft) & \longrightarrow & (u\bar{a}_i)(a_{j-1} \blacktriangleleft). \end{array}$$

Since  $a_j = \bar{a}_{i-1}$ , the first column is ordered with respect to  $T_1$ . On the other hand,  $a_j = \bar{a}_{i-1}$  implies that the partition of  $(u\bar{a}_i)(a_{i-1} \blacktriangleleft)$  with respect to  $T_2$  ends with  $(u\bar{a}_i)(a_{i-1}a_i \blacktriangleleft)$ . Then, the second column of this table is ordered with respect to  $T_2$ . Hence, if  $(w\bar{a}_i a_{i-1})\mathfrak{e}' <_{T_1} (w\bar{a}_i a_{i-1})\mathfrak{f}'$  then  $(u\bar{a}_i)\mathfrak{e}'' <_{T_2} (u\bar{a}_i)\mathfrak{f}''$ .

If  $i - 1 < j$ , then

$$\begin{aligned} b_{[1\uparrow(i-1)]} &= a_{[1\uparrow(i-1)]} \\ b_{[i\uparrow(j-2)]} &= a_{[(i+1)\uparrow(j-1)]} \\ b_{j-1} &= a_i \\ b_{[j\uparrow 2n]} &= a_{[j\uparrow 2n]} \end{aligned}$$

The partition with respect to  $a_{[1\uparrow 2n]}$  of  $(\bar{a}_j \blacktriangleleft) = (a_{i-1} \blacktriangleleft)$  is  $(a_{j+1} \blacktriangleleft), (a_{j+2} \blacktriangleleft), \dots, (a_{2n} \blacktriangleleft), (a_1 \blacktriangleleft), (a_2 \blacktriangleleft), \dots, (a_{i-1} \blacktriangleleft), (a_i \blacktriangleleft), (a_{i+1} \blacktriangleleft), \dots, (a_{j-1} \blacktriangleleft)$ . The partition with respect to  $b_{[1\uparrow 2n]}$  of  $(\bar{a}_i \blacktriangleleft)$  is  $(a_j \blacktriangleleft), (a_{j+1} \blacktriangleleft), \dots, (a_{2n} \blacktriangleleft), (a_1 \blacktriangleleft), (a_2 \blacktriangleleft), \dots, (a_{i-1} \blacktriangleleft), (a_{i+1} \blacktriangleleft), (a_{i+2} \blacktriangleleft), \dots, (a_{j-1} \blacktriangleleft)$ . By the third table,

$$\begin{array}{ccc}
& & (a_i \mapsto a_{i-1}a_i) \\
(w\bar{a}_i a_{i-1})(a_{j+1} \blacktriangleleft) & \longrightarrow & (u\bar{a}_i)(a_{j+1} \blacktriangleleft), \\
(w\bar{a}_i a_{i-1})(a_{j+2} \blacktriangleleft) & \longrightarrow & (u\bar{a}_i)(a_{j+2} \blacktriangleleft), \\
& \vdots & \\
(w\bar{a}_i a_{i-1})(a_{2n} \blacktriangleleft) & \longrightarrow & (u\bar{a}_i)(a_{2n} \blacktriangleleft), \\
(w\bar{a}_i a_{i-1})(a_1 \blacktriangleleft) & \longrightarrow & (u\bar{a}_i)(a_1 \blacktriangleleft), \\
(w\bar{a}_i a_{i-1})(a_2 \blacktriangleleft) & \longrightarrow & (u\bar{a}_i)(a_2 \blacktriangleleft), \\
& \vdots & \\
(w\bar{a}_i a_{i-1})(a_{i-2} \blacktriangleleft) & \longrightarrow & (u\bar{a}_i)(a_{i-2} \blacktriangleleft), \\
(w\bar{a}_i a_{i-1})(a_{i-1} \blacktriangleleft) & \longrightarrow & (u\bar{a}_i)[(a_{i-1} \blacktriangleleft) - (a_{i-1}a_i \blacktriangleleft)], \\
(w\bar{a}_i a_{i-1})(a_i \blacktriangleleft) & \longrightarrow & (u\bar{a}_i)(a_{i-1}a_i \blacktriangleleft), \\
(w\bar{a}_i a_{i-1})(a_{i+1} \blacktriangleleft) & \longrightarrow & (u\bar{a}_i)(a_{i+1} \blacktriangleleft), \\
& \vdots & \\
(w\bar{a}_i a_{i-1})(a_{j-1} \blacktriangleleft) & \longrightarrow & (u\bar{a}_i)(a_{j-1} \blacktriangleleft).
\end{array}$$

Since  $a_j = \bar{a}_{i-1}$ , the first column is ordered with respect to  $T_1$ . On the other hand,  $a_j = \bar{a}_{i-1}$  implies that the partition of  $(u\bar{a}_i)(a_{i-1} \blacktriangleleft)$  with respect to  $T_2$  ends with  $(u\bar{a}_i)(a_{i-1}a_i \blacktriangleleft)$ . Then, the second column of this table is ordered with respect to  $T_2$ . Hence, if  $(w\bar{a}_i a_{i-1})\mathbf{e}' <_{T_1} (w\bar{a}_i a_{i-1})\mathbf{f}'$  then  $(u\bar{a}_i)\mathbf{e}'' <_{T_2} (u\bar{a}_i)\mathbf{f}''$ .

For every row of the first table, there is a case which needs to be considered. Similarly, in all these cases, it can be shown that if  $\mathbf{e} <_{T_1} \mathbf{f}$ , then  $\mathbf{e}^{(a_i \mapsto a_{i-1}a_i)} <_{T_2} \mathbf{f}^{(a_i \mapsto a_{i-1}a_i)}$ .

The case  $\phi = (\bar{a}_i \mapsto \bar{a}_i \bar{a}_{i+1})$  for some  $i \in [1 \uparrow (2n-1)]$ ,  $a_i \neq \bar{a}_{i+1}$ , is similar.  $\square$

## 7 $t$ -squarefreeness

Recall  $2g + p = n$  and  $\Sigma_{g,1,p}$  is the free group on  $x_{[1 \uparrow g]} \vee y_{[1 \uparrow g]} \vee t_{[1 \uparrow p]}$ .

The following definition extends Definition 4.1 to  $\Sigma_{g,1,p} \cup \partial \Sigma_{g,1,p}$ .

**7.1 Definition.** An element of  $\Sigma_{g,1,p} \cup \partial \Sigma_{g,1,p}$  is said to be  $t$ -squarefree if, in its reduced expression, no two consecutive terms in  $t_{[1 \uparrow p]} \vee \bar{t}_{[1 \uparrow p]}$  are equal.

**7.2 Notation.** In the standard surface word set, we denote

$$\bar{z}_1 = \prod_{i \in [1 \uparrow g]} [x_i, y_i] \prod t_{[1 \uparrow p]} \quad \text{and} \quad z_1 = \prod \bar{t}_{[p \downarrow 1]} \prod_{i \in [g \downarrow 1]} [y_i, x_i].$$

From the last comment of Definition 6.3, the smallest element of  $(\partial \Sigma_{g,1,p}, <)$  is  $\bar{z}_1^\infty$  and the largest element of  $(\partial \Sigma_{g,1,p}, <)$  is  $z_1^\infty$ . We denote by  $\min(\partial \Sigma_{g,1,p}) = \bar{z}_1^\infty$  and  $\max(\partial \Sigma_{g,1,p}) = z_1^\infty$  these facts.

Given two ends  $\mathbf{e}, \mathbf{f} \in \partial \Sigma_{g,1,p}$ , we write

$$[\mathbf{e} \uparrow \mathbf{f}] := \{\mathbf{g} \in \partial \Sigma_{g,1,p} \mid \mathbf{e} \leq \mathbf{g} \leq \mathbf{f}\}.$$

**7.3 Lemma.** Let  $k_0 \in [1 \uparrow p]$ ,  $w \in \Sigma_{g,1,p} - (\star t_{k_0}) - (\star \bar{t}_{k_0})$  and  $i_0 \in [1 \uparrow g]$ . Then, in  $(\partial \Sigma_{g,1,p}, \leq)$ , the following hold:

(i).  $wt_{k_0} \bar{w}(\bar{z}_1^\infty) \leq wt_{k_0} ((\Pi t_{[k_0 \uparrow p]} \Pi_{i \in [1 \uparrow g]} [x_i, y_i] \Pi t_{[1 \uparrow (k_0-1)]})^\infty) = \min(wt_{k_0} t_{k_0} \blacktriangleleft);$

(ii).  $\max(wt_{k_0} t_{k_0} \blacktriangleleft) < \min(w\bar{t}_{k_0} \bar{t}_{k_0} \blacktriangleleft);$

(iii).  $\max(w\bar{t}_{k_0} \bar{t}_{k_0} \blacktriangleleft) = w\bar{t}_{k_0} ((\Pi \bar{t}_{[k_0 \downarrow 1]} \Pi_{i \in [g \downarrow 1]} [y_i, x_i] \Pi \bar{t}_{[p \downarrow (k_0+1)]})^\infty) \leq w\bar{t}_{k_0} \bar{w}(z_1^\infty);$

(iv).  $(wt_{k_0} t_{k_0} \blacktriangleleft) \cup (w\bar{t}_{k_0} \bar{t}_{k_0} \blacktriangleleft) \subseteq [wt_{k_0} \bar{w}(\bar{z}_1^\infty) \uparrow w\bar{t}_{k_0} \bar{w}(z_1^\infty)];$

(v). If  $2g + p \geq 3$ , then one of the following holds:

(a)  $\bar{t}_p(\bar{z}_1^\infty) > w\bar{t}_{k_0} \bar{w}(z_1^\infty);$

(b)  $\bar{t}_p(\bar{z}_1^\infty) < wt_{k_0} \bar{w}(\bar{z}_1^\infty);$

and, hence,  $\bar{t}_p(\bar{z}_1^\infty) \notin [wt_{k_0} \bar{w}(\bar{z}_1^\infty) \uparrow w\bar{t}_{k_0} \bar{w}(z_1^\infty)];$

(vi). If  $a \in \{x_{i_0}, \bar{x}_{i_0}, y_{i_0}, \bar{y}_{i_0}\}$ , then one of the following holds:

(a).  $a(z_1^\infty) > w\bar{t}_{k_0} \bar{w}(z_1^\infty);$

(b).  $a(z_1^\infty) < wt_{k_0} \bar{w}(\bar{z}_1^\infty);$

and, hence,  $a(z_1^\infty) \notin [wt_{k_0} \bar{w}(\bar{z}_1^\infty) \uparrow w\bar{t}_{k_0} \bar{w}(z_1^\infty)].$

*Proof.* Recall  $<$  is the ordering with respect to sequence the

$$(\bar{x}_1, y_1, x_1, \bar{y}_1, \bar{x}_2, y_2, x_2, \bar{y}_2, \dots, \bar{x}_g, y_g, x_g, \bar{y}_g, t_1, \bar{t}_1, t_2, \bar{t}_2, \dots, t_p, \bar{t}_p).$$

(i). It is straightforward to see that

$$wt_{k_0} ((\Pi t_{[k_0 \uparrow p]} \Pi_{i \in [1 \uparrow g]} [x_i, y_i] \Pi t_{[1 \uparrow (k_0-1)]})^\infty) = \min(wt_{k_0} t_{k_0} \blacktriangleleft).$$

Let  $a \in X \vee \bar{X}$  be such that  $\bar{w}((\Pi_{i \in [1 \uparrow g]} [x_i, y_i] \Pi t_{[1 \uparrow p]})^\infty) \in (a \blacktriangleleft)$ . Note  $a \neq \bar{t}_{k_0}$ .

If  $a \neq t_{k_0}$ , then  $(wt_{k_0} a \blacktriangleleft) < (wt_{k_0} t_{k_0} \blacktriangleleft)$ , and we have

$$wt_{k_0} \bar{w}(\bar{z}_1^\infty) = wt_{k_0} \bar{w}((\Pi_{i \in [1 \uparrow g]} [x_i, y_i] \Pi t_{[1 \uparrow p]})^\infty) < \min(wt_{k_0} t_{k_0}).$$

If  $a = t_{k_0}$ , then  $\bar{w}$  is completely canceled in  $\bar{w}((\Pi_{i \in [1 \uparrow g]} [x_i, y_i] \Pi t_{[1 \uparrow p]})^\infty)$ , and, moreover,

$$\begin{aligned} wt_{k_0} \bar{w}(\bar{z}_1^\infty) &= wt_{k_0} \bar{w}((\Pi_{i \in [1 \uparrow g]} [x_i, y_i] \Pi t_{[1 \uparrow p]})^\infty) \\ &= wt_{k_0} ((\Pi t_{[k_0 \uparrow p]} \Pi_{i \in [1 \uparrow g]} [x_i, y_i] \Pi t_{[1 \uparrow (k_0-1)]})^\infty) \\ &= \min(wt_{k_0} t_{k_0} \blacktriangleleft). \end{aligned}$$

(ii). It is clear.

(iii). It is straightforward to see that

$$w\bar{t}_{k_0}((\Pi\bar{t}_{[k_0\downarrow 1]}\Pi_{i\in[g\downarrow 1]}[y_i, x_i]\Pi\bar{t}_{[p\downarrow(k_0-1)]})^\infty) = \max(w\bar{t}_{k_0}\bar{t}_{k_0} \blacktriangleleft).$$

Let  $a \in X \vee \bar{X}$  be such that  $\bar{w}((\Pi\bar{t}_{[p\downarrow 1]}\Pi_{i\in[g\downarrow 1]}[y_i, x_i])^\infty) \in (a \blacktriangleleft)$ . Note  $a \neq t_{k_0}$ .

If  $a \neq \bar{t}_{k_0}$ , then  $(w\bar{t}_{k_0}\bar{t}_{k_0} \blacktriangleleft) < (w\bar{t}_{k_0}a \blacktriangleleft)$ , and we have

$$\max(w\bar{t}_{k_0}\bar{t}_{k_0} \blacktriangleleft) < w\bar{t}_{k_0}\bar{w}((\Pi\bar{t}_{[p\downarrow 1]}\Pi_{i\in[g\downarrow 1]}[y_i, x_i])^\infty) = w\bar{t}_{k_0}\bar{w}(z_1^\infty).$$

If  $a = \bar{t}_{k_0}$ , then  $\bar{w}$  is completely canceled in  $\bar{w}((\Pi\bar{t}_{[p\downarrow 1]}\Pi_{i\in[g\downarrow 1]}[y_i, x_i])^\infty)$ , and, moreover,

$$\begin{aligned} w\bar{t}_{k_0}\bar{w}(z_1^\infty) &= w\bar{t}_{k_0}\bar{w}((\Pi\bar{t}_{[p\downarrow 1]}\Pi_{i\in[g\downarrow 1]}[y_i, x_i])^\infty) \\ &= w\bar{t}_{k_0}((\Pi\bar{t}_{[k_0\downarrow 1]}\Pi_{i\in[g\downarrow 1]}[y_i, x_i]\Pi\bar{t}_{[p\downarrow(k_0+1)]})^\infty) \\ &= \max(w\bar{t}_{k_0}\bar{t}_{k_0} \blacktriangleleft). \end{aligned}$$

(iv). Follows from (i)-(iii).

(v). By (i)-(iii),

$$wt_{k_0}\bar{w}((\Pi_{i\in[1\uparrow g]}[x_i, y_i]\Pi t_{[1\uparrow p]})^\infty) < w\bar{t}_{k_0}\bar{w}((\Pi\bar{t}_{[p\downarrow 1]}\Pi_{i\in[g\downarrow 1]}[y_i, x_i])^\infty).$$

**Case 1.**  $w = 1$ . Since  $(\bar{t}_p\bar{x}_1 \blacktriangleleft) \cup (\bar{t}_p t_1 \blacktriangleleft) > (\bar{t}_{k_0}\bar{t}_p \blacktriangleleft)$ , we see

$$\bar{t}_p(\bar{z}_1^\infty) = \bar{t}_p((\Pi_{i\in[1\uparrow g]}[x_i, y_i]\Pi t_{[1\uparrow p]})^\infty) > \bar{t}_{k_0}((\Pi\bar{t}_{[p\downarrow 1]}\Pi_{i\in[g\downarrow 1]}[y_i, x_i])^\infty) = \bar{t}_{k_0}(z_1^\infty).$$

Thus, (a) holds.

**Case 2.**  $w \notin (\bar{t}_p\star) \cup \{1\}$ . Since  $(\bar{t}_p \blacktriangleleft) > (w\bar{t}_{k_0} \blacktriangleleft)$ , we see

$$\begin{aligned} \bar{t}_p(\bar{z}_1^\infty) &= \bar{t}_p((\Pi_{i\in[1\uparrow g]}[x_i, y_i]\Pi t_{[1\uparrow p]})^\infty) \\ &> w\bar{t}_{k_0}\bar{w}((\Pi\bar{t}_{[p\downarrow 1]}\Pi_{i\in[g\downarrow 1]}[y_i, x_i])^\infty) = w\bar{t}_{k_0}\bar{w}(z_1^\infty). \end{aligned}$$

Thus, (a) holds.

**Case 3.**  $w \in (\bar{t}_p\bar{t}_p\star)$ . Since  $(\bar{t}_p\bar{x}_1 \blacktriangleleft) \cup (\bar{t}_p t_1 \blacktriangleleft) > (w\bar{t}_{k_0} \blacktriangleleft)$ , we see

$$\begin{aligned} \bar{t}_p(\bar{z}_1^\infty) &= \bar{t}_p((\Pi_{i\in[1\uparrow g]}[x_i, y_i]\Pi t_{[1\uparrow p]})^\infty) \\ &> w\bar{t}_{k_0}\bar{w}((\Pi\bar{t}_{[p\downarrow 1]}\Pi_{i\in[g\downarrow 1]}[y_i, x_i])^\infty) = w\bar{t}_{k_0}\bar{w}(z_1^\infty). \end{aligned}$$

Thus, (a) holds.

**Case 4.**  $w \in (\bar{t}_p\star) - (\bar{t}_p\bar{t}_p\star)$ .

Here,

$$wt_{k_0}\bar{w}(\bar{z}_1^\infty) = wt_{k_0}\bar{w}((\Pi_{[1\uparrow g]}[x_i, y_i]\Pi t_{[1\uparrow p]})^\infty) \in (wt_{k_0} \blacktriangleleft) \subset (\bar{t}_p \blacktriangleleft) - (\bar{t}_p\bar{t}_p \blacktriangleleft).$$

Hence,

$$\begin{aligned}\bar{t}_p((\prod_{i \in [1 \uparrow g]} [x_i, y_i] \Pi t_{[1 \uparrow p]})^\infty) &= \min((\bar{t}_p \blacktriangleleft) - (\bar{t}_p \bar{t}_p \blacktriangleleft)) \\ &\leq wt_{k_0} \bar{w}((\prod_{i \in [1 \uparrow g]} [x_i, y_i] \Pi t_{[1 \uparrow p]})^\infty).\end{aligned}$$

To prove (b) holds, it remains to show that

$$\bar{t}_p((\prod_{i \in [1 \uparrow g]} [x_i, y_i] \Pi t_{[1 \uparrow p]})^\infty) \neq wt_{k_0} \bar{w}((\prod_{i \in [1 \uparrow g]} [x_i, y_i] \Pi t_{[1 \uparrow p]})^\infty),$$

that is,  $(\prod_{i \in [1 \uparrow g]} [x_i, y_i] \Pi t_{[1 \uparrow p]})^\infty \neq t_p wt_{k_0} \bar{w}((\prod_{i \in [1 \uparrow g]} [x_i, y_i] \Pi t_{[1 \uparrow p]})^\infty)$ , that is,  $t_p wt_{k_0} \bar{w} \notin \langle \prod_{i \in [1 \uparrow g]} [x_i, y_i] \Pi t_{[1 \uparrow p]} \rangle$ . We can write  $w = \bar{t}_p u$  where  $u \notin (t_p \star)$ . Then  $t_p wt_{k_0} \bar{w} = ut_{k_0} \bar{u} t_p$ , in normal form. Thus it suffices to show

$$ut_{k_0} \bar{u} t_p \notin \langle \prod_{i \in [1 \uparrow g]} [x_i, y_i] \Pi t_{[1 \uparrow p]} \rangle.$$

If  $u = 1$ , then  $ut_{k_0} \bar{u} t_p \notin \langle \prod_{i \in [1 \uparrow g]} [x_i, y_i] \Pi t_{[1 \uparrow p]} \rangle$ , since  $2g + p \geq 3$ .

If  $u \neq 1$ , then  $ut_{k_0} \bar{u} t_p \notin \langle \prod_{i \in [1 \uparrow g]} [x_i, y_i] \Pi t_{[1 \uparrow p]} \rangle$ , since  $ut_{k_0} \bar{u} t_p$  does not lie in the submonoid of  $\Sigma_{g,1,p}$  generated by  $\prod_{i \in [1 \uparrow g]} [x_i, y_i] \Pi t_{[1 \uparrow p]}$ , nor in the submonoid generated by  $\prod \bar{t}_{[p \downarrow 1]} \prod_{i \in [g \downarrow 1]} [y_i, x_i]$ .

In all four cases (v) holds.

(vi). Let  $a \in \{x_{i_0}, \bar{x}_{i_0}, y_{i_0}, \bar{y}_{i_0}\}$ .

**Case 1.**  $w = 1$ . Since  $(a \blacktriangleleft) < (t_{k_0} \blacktriangleleft)$ , we see

$$a(z_1^\infty) = a((\prod \bar{t}_{[p \downarrow 1]} \prod_{i \in [g \downarrow 1]} [y_i, x_i])^\infty) < t_{k_0}((\prod_{i \in [1 \uparrow g]} [x_i, y_i] \Pi t_{[1 \uparrow p]})^\infty) = t_{k_0}(\bar{z}_1^\infty).$$

Thus, (b) holds.

**Case 2.**  $w \notin (a \star) \cup \{1\}$ .

If  $(a \blacktriangleleft) > (w \blacktriangleleft)$ , then  $(a \blacktriangleleft) > (w \blacktriangleleft) \supset (w \bar{t}_{k_0} \blacktriangleleft)$  and

$$\begin{aligned}a(z_1^\infty) &= a((\prod \bar{t}_{[p \downarrow 1]} \prod_{i \in [g \downarrow 1]} [y_i, x_i])^\infty) \\ &> w \bar{t}_{k_0} \bar{w}((\prod \bar{t}_{[p \downarrow 1]} \prod_{i \in [g \downarrow 1]} [y_i, x_i])^\infty) = w \bar{t}_{k_0} \bar{w}(z_1^\infty).\end{aligned}$$

Thus, (a) holds.

If  $(a \blacktriangleleft) < (w \blacktriangleleft)$ , then  $(a \blacktriangleleft) < (w \blacktriangleleft) \supset (wt_{k_0} \blacktriangleleft)$  and

$$\begin{aligned}a(z_1^\infty) &= a((\prod \bar{t}_{[p \downarrow 1]} \prod_{i \in [g \downarrow 1]} [y_i, x_i])^\infty) \\ &< wt_{k_0} \bar{w}((\prod_{i \in [1 \uparrow g]} [x_i, y_i] \Pi t_{[1 \uparrow p]})^\infty) = wt_{k_0} \bar{w}(\bar{z}_1^\infty).\end{aligned}$$

Thus, (b) holds.

**Case 3.**  $w \in (a \bar{t}_p \star)$ .

Since  $a((\prod \bar{t}_{[p \downarrow 1]} \prod_{i \in [g \downarrow 1]} [y_i, x_i])^\infty) = \max(a \bar{t}_p \blacktriangleleft)$ , we see

$$\begin{aligned}a(z_1^\infty) &= a((\prod \bar{t}_{[p \downarrow 1]} \prod_{i \in [g \downarrow 1]} [y_i, x_i])^\infty) \\ &\geq w \bar{t}_{k_0} \bar{w}((\prod \bar{t}_{[p \downarrow 1]} \prod_{i \in [g \downarrow 1]} [y_i, x_i])^\infty) = w \bar{t}_{k_0} \bar{w}(z_1^\infty).\end{aligned}$$

To prove (a) holds, it remains to show that

$$a((\prod_{[p\downarrow 1]} \bar{t} \prod_{i \in [g\downarrow 1]} [y_i, x_i])^\infty) \neq w \bar{t}_{k_0} \bar{w} ((\prod_{[p\downarrow 1]} \bar{t} \prod_{i \in [g\downarrow 1]} [y_i, x_i])^\infty),$$

that is,  $((\prod_{[p\downarrow 1]} \bar{t} \prod_{i \in [g\downarrow 1]} [y_i, x_i])^\infty) \neq \bar{a} w \bar{t}_{k_0} \bar{w} ((\prod_{[p\downarrow 1]} \bar{t} \prod_{i \in [g\downarrow 1]} [y_i, x_i])^\infty)$ , that is  $\bar{a} w \bar{t}_{k_0} \bar{w} \notin \langle \prod_{[p\downarrow 1]} \bar{t} \prod_{i \in [g\downarrow 1]} [y_i, x_i] n \rangle$ . We can write  $w = a \bar{t}_p u$  where  $u \notin (t_p \star)$ . Then  $\bar{a} w \bar{t}_{k_0} \bar{w} = \bar{t}_p u \bar{t}_{k_0} \bar{u} t_p \bar{a}$ , in normal form. Thus it suffices to show that

$$\bar{t}_p u \bar{t}_{k_0} \bar{u} t_p \bar{a} \notin \langle \prod_{[p\downarrow 1]} \bar{t} \prod_{i \in [g\downarrow 1]} [y_i, x_i] \rangle,$$

which is clear since  $\bar{t}_p u \bar{t}_{k_0} \bar{u} t_p \bar{a}$  does not lie in the submonoid of  $\Sigma_{g,1,p}$  generated by  $\prod_{[p\downarrow 1]} \bar{t} \prod_{i \in [g\downarrow 1]} [y_i, x_i]$ , nor in the submonoid generated by  $\prod_{i \in [1\uparrow g]} [x_i, y_i] \prod_{[1\uparrow p]}$ .

**Case 4.**  $w \in (a \star) - (a \bar{t}_p \star)$ ,  $|w| \geq 2$ .

If  $(a \bar{t}_p \blacktriangleleft) > (w \blacktriangleleft)$ , then  $(a \bar{t}_p \blacktriangleleft) > (w \blacktriangleleft) \supset (w \bar{t}_{k_0} \blacktriangleleft)$  and

$$\begin{aligned} a(z_1^\infty) &= a((\prod_{[p\downarrow 1]} \bar{t} \prod_{i \in [g\downarrow 1]} [y_i, x_i])^\infty) \\ &> w \bar{t}_{k_0} \bar{w} ((\prod_{[p\downarrow 1]} \bar{t} \prod_{i \in [g\downarrow 1]} [y_i, x_i])^\infty) = w \bar{t}_{k_0} \bar{w} (z_1^\infty). \end{aligned}$$

Thus, (a) holds.

If  $(a \bar{t}_p \blacktriangleleft) < (w \blacktriangleleft)$ , then  $(a \bar{t}_p \blacktriangleleft) < (w \blacktriangleleft) \supset (w t_{k_0} \blacktriangleleft)$  and

$$\begin{aligned} a(z_1^\infty) &= a((\prod_{[p\downarrow 1]} \bar{t} \prod_{i \in [g\downarrow 1]} [y_i, x_i])^\infty) \\ &< w t_{k_0} \bar{w} ((\prod_{i \in [1\uparrow g]} [x_i, y_i] \prod_{[1\uparrow p]})^\infty) = w t_{k_0} \bar{w} (\bar{z}_1^\infty). \end{aligned}$$

Thus, (b) holds.

**Case 5.**  $w = a$ .

Since  $a(z_1^\infty) = \max(a \bar{t}_p \blacktriangleleft, (a \bar{t}_p \blacktriangleleft) \supset (a \bar{t}_p \bar{y}_g \bar{x}_g \blacktriangleleft)$  and  $(a \bar{t}_p \bar{y}_g \bar{x}_g \blacktriangleleft) > (a \bar{t}_{k_0} \bar{a} \bar{t}_p \blacktriangleleft)$ , we see

$$\begin{aligned} a(z_1^\infty) &= a((\prod_{[p\downarrow 1]} \bar{t} \prod_{i \in [g\downarrow 1]} [y_i, x_i])^\infty) \\ &> w \bar{t}_{k_0} \bar{w} ((\prod_{[p\downarrow 1]} \bar{t} \prod_{i \in [g\downarrow 1]} [y_i, x_i])^\infty) = w \bar{t}_{k_0} \bar{w} (z_1^\infty). \end{aligned}$$

Thus, (a) holds.

In all five cases (vi) holds. □

**7.4 Theorem.** *If  $2g + p \geq 3$  then, for each  $\phi \in \mathcal{AM}_{g,1,p}$ ,*

(i).  $\bar{t}_p^\phi(\bar{z}_1^\infty)$  is a  $t$ -squarefree end,

(ii). for every  $i_0 \in [1\uparrow g]$  and every  $a \in \{x_{i_0}, \bar{x}_{i_0}, y_{i_0}, \bar{y}_{i_0}\}$ ,  $a^\phi(z_1^\infty)$  is a  $t$ -squarefree end.

*Proof.* (i). Recall  $\bar{z}_1 = \prod_{i \in [1 \uparrow g]} [x_i, y_i] \Pi t_{[1 \uparrow p]}$  and  $z_1 = \Pi \bar{t}_{[p \downarrow 1]} \prod_{i \in [g \downarrow 1]} [y_i, x_i]$ . Let us  $\cup [t]_{[1 \uparrow p]}$  denote  $\bigcup_{k \in [1 \uparrow p]} [t_k]$ . By Lemma 7.3(v),  $\bar{t}_p(\bar{z}_1^\infty)$  does not lie in

$$\bigcup_{u \in \cup [t]_{[1 \uparrow p]}} [(u(\bar{z}_1^\infty) \uparrow \bar{u}(z_1^\infty))] (= \bigcup_{k=1}^p \bigcup_{w \in \Sigma_{g,1,p} - (\star t_k) - (\star \bar{t}_k)} [(wt_k \bar{w}(\bar{z}_1^\infty) \uparrow (w \bar{t}_k \bar{w}(z_1^\infty)))]).$$

Notice that  $\phi$  permutes the elements of each of the following sets:

$$\cup [t]_{[1 \uparrow p]}; \quad \{\bar{z}_1^\infty\}; \quad \{z_1^\infty\}; \quad \text{and} \quad \bigcup_{u \in [t]_{[1 \uparrow p]}} [u(\bar{z}_1^\infty) \uparrow \bar{u}(z_1^\infty)].$$

Hence,  $(\bar{t}_p(\bar{z}_1^\infty))^\phi$  does not lie in  $\bigcup_{u \in [t]_{[1 \uparrow p]}} [u(\bar{z}_1^\infty) \uparrow \bar{u}(z_1^\infty)]$ . By Lemma 7.3(iv),

$$\bigcup_{u \in [t]_{[1 \uparrow p]}} [u(\bar{z}_1^\infty) \uparrow \bar{u}(z_1^\infty)] \supseteq \bigcup_{k=1}^p \bigcup_{w \in \Sigma_{g,1,p} - (\star t_k) - (\star \bar{t}_k)} ((wt_k t_k \blacktriangleleft) \cup (w \bar{t}_k \bar{t}_k \blacktriangleleft)).$$

Hence,  $(\bar{t}_p(\bar{z}_1^\infty))^\phi$  does not lie in the right-hand side set either, and, hence,  $(\bar{t}_p(\bar{z}_1^\infty))^\phi$  is a  $t$ -squarefree end. Since  $(\bar{t}_p(\bar{z}_1^\infty))^\phi = \bar{t}_p^\phi(\bar{z}_1^\infty)$ , the desired result holds.

(ii). The same proof as (i) using Lemma 7.3(vi) instead of Lemma 7.3(v).  $\square$

*Proof.* (of Theorem 4.2) Recall (2.0.1).  $\mathcal{AM}_{0,1,2} = \langle \sigma_1 \rangle$ , and

$$t_2^{\mathcal{AM}_{0,1,2}} = \{t_2^{\sigma_1^{2m}}, t_2^{\sigma_1^{2m+1}} \mid m \in \mathbb{Z}\} = \{t_2^{(t_1 t_2)^m}, t_1^{(t_1 t_2)^m} \mid m \in \mathbb{Z}\}$$

Thus, every element of  $t_2^{\mathcal{AM}_{0,1,2}}$  is  $t$ -squarefree.

Suppose, now,  $2g + p \geq 3$ . Let  $i_0 \in [1 \uparrow g]$  and  $a \in \{x_{i_0}, y_{i_0}\}$ . By Theorem 7.4(ii),  $a^\phi(z_1^\infty) = a^\phi((\Pi \bar{t}_{[p \downarrow 1]} \Pi [x, y]_{[g \downarrow 1]})^\infty)$  is a  $t$ -squarefree end. Hence, either  $a^\phi$  is  $t$ -squarefree or  $a^\phi = ut_k t_k v$  in normal form, and  $t_k v$  is canceled in  $a^\phi(z_1^\infty) = ut_k t_k v(z_1^\infty)$ ; moreover  $ut_k, t_k v$  are  $t$ -squarefree. By Theorem 7.4(ii),

$$\bar{a}^\phi(z_1^\infty) = \bar{a}^\phi((\Pi \bar{t}_{[p \downarrow 1]} \Pi [x, y]_{[g \downarrow 1]})^\infty)$$

is a  $t$ -squarefree end. Hence,  $\bar{a}^\phi \neq \bar{v} \bar{t}_k \bar{t}_k \bar{u}$ .

Since  $\phi$  permutes  $\bigcup_{k \in [1 \uparrow p]} [t_k]$ , we can write  $\bar{t}_p^\phi = \bar{t}_{p^\pi}^{w_p}$ , where  $\pi$  is a permutation of  $[1 \uparrow p]$  and  $w_p \in \Sigma_{g,1,p} - (t_{p^\pi} \star) - (\bar{t}_{p^\pi} \star)$ . It is not difficult to see that

$$\bar{t}_p^\phi(\bar{z}_1^\infty) = \bar{w}_p \bar{t}_{p^\pi} w_p ((\Pi_{i \in [1 \uparrow g]} [x_i, y_i] \Pi t_{[1 \uparrow p]})^\infty) \in (\bar{w}_p \blacktriangleleft).$$

By Theorem 7.4(i),  $\bar{t}_p^\phi(\bar{z}_1^\infty)$  is a  $t$ -squarefree end. Hence,  $\bar{w}_p$  is  $t$ -squarefree.

Since  $\bar{w}_p$  is  $t$ -squarefree,  $\bar{t}_p^\phi = \bar{w}_p \bar{t}_p w_p$  is also  $t$ -squarefree. Hence,  $t_p^\phi$  is  $t$ -squarefree.

Suppose, now,  $2g + p \geq 2$ . Let  $k \in [1 \uparrow p]$ . Since  $t_k$  is in the  $\mathcal{AM}_{g,1,p}$ -orbit of  $t_p$ ,  $t_k^\phi$  is  $t$ -squarefree for all  $\phi \in \mathcal{AM}_{g,1,p}$ .  $\square$

## Acknowledgments

The autor is grateful to Warren Dicks and Luis Paris for many interesting observations.

## References

- [1] Javier Aramayona, Christopher J. Leininger, Juan Souto *Injections of mapping class groups* Geom. and Top. **13**(2009), 2523–2541.
- [2] Norbert A’Campo, *Le groupe de monodromie du déploiement des singularités isolées de courbes planes I.* Math. Ann. **213**(1975), 1–32.
- [3] Lluís Bacardit and Warren Dicks, *Actions of the braid group, and new algebraic proofs of results of Dehornoy and Larue*, Groups – Complexity – Cryptology, **1**(2009), 77 – 129
- [4] Joan S. Birman and Hugh M. Hilden, *On isotopies of homeomorphisms of Riemann surfaces*, Ann. of Math. **97**(1973), 424–439.
- [5] Daryl Cooper, *Automorphisms of free groups have finitely generated fixed point sets*, J.algebra **111**(1987), 453–456.
- [6] John Crisp and Luis Paris, *Arin groups of type B and D*, Adv. Geom. **5**(2005), 607–636.
- [7] John Crisp and Luis Paris, *Representations of the braid group by automorphisms of groups, invariants of links, and Gar side groups*, Pacific J. Math. **221**(2005), 1–27.
- [8] Patrick Dehornoy, Ivan Baryshnikov, Dale Rolfsen and Bert West, *Why are braids orderable?*, Panoramas et Synthèses **14**, Soc. Math. France, Paris, 2002.
- [9] Warren Dicks and Edward Formanek, *Automorphism subgroups of finite index in algebraic mapping class groups*, J. Algebra **189**(1997), 58–89.
- [10] Warren Dicks and Edward Formanek, *Algebraic mapping-class groups of orientable surfaces with boundaries*, pp. 57–116, in: *Infinite groups: geometric, combinatorial and dynamical aspects* (eds. Laurent Bartholdi, Tulley Cherin-Silverstein, Tatiana Smirnov-Magnified, Andrej Auk), Progress in Mathematics **248**, Birkenstock Verlag, Basel, 2005.  
Errata and addenda at: <http://mat.ab.cat/~dicks/Boundaries.HTML>
- [11] B. Farb and D. Margalit, *A primer book on mapping class groups*, [http://www.mines.edu/fs\\_home/dlarue/papers/dml.pdf](http://www.mines.edu/fs_home/dlarue/papers/dml.pdf)
- [12] Nikolai V. Ivanov, *Mapping class groups*, p. 523 – 633 in Handbook of geometric topology, North-Holland, Amsterdam, 2002.
- [13] Roger C. Lyndon and Paul E. Schuss, *Combinatorial group theory*, Berget. Math. Grenada. **89**, Springer-Verlag, Berlin, 1977.
- [14] Wilhelm Magnus, *Über Automorphism en Von Fundamentalist Bernadette FlewäCheng*, Math. Ann. **109**(1934), 617–646.
- [15] James McCool, *Generating the mapping class group (an algebraic approach)*, Pu bl. Mat. **40**(1996/02), 457468.
- [16] Hamish Short and Bert Wiest, *Orderings of mapping class groups after Thurston*, Ensign. Math. **46**(2000), 279–312.

LLUÍS BACARDIT,  
INSTITUT DE MATHMATIQUES DE BOURGOGNE  
UNIVERSITÉ DE BOURGOGNE  
UMR 5584 DU CNRS, BP 47870  
21078 DIJON CEDEX  
FRANCE

*E-mail address:* [Lluís.Bacardit@u-bourgogne.fr](mailto:Lluís.Bacardit@u-bourgogne.fr)