

RADIAL OSCILLATION OF HARMONIC FUNCTIONS IN THE KORENBLUM CLASS

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ABSTRACT. We study radial behavior of harmonic functions in the unit disk belonging to the Korenblum class. We prove that functions which admit two-sided Korenblum estimate either oscillate or have slow growth along almost all radii.

1. INTRODUCTION AND MAIN RESULTS

We study radial behavior of functions harmonic in the unit disk \mathbf{D} . Let $\text{Harm}(\mathbf{D})$ stay for the set of all real-valued functions harmonic in \mathbf{D} and $|E|$ for the normalized linear Lebesgue measure of a set E on the unit circle \mathbf{T} . It follows from the classical result of Lusin and Privalov see e.g. [16], that there exist functions $u \in \text{Harm}(\mathbf{D})$ which tend to infinity along almost all radii, while non-tangential growth may occur only on subsets of the unit circle having zero linear measure.

An important generalization of this result is due to Kahane and Katznelson [11]. They proved that for any function $v(r)$, $0 < r < 1$ such that $v(r) \nearrow \infty$ as $r \nearrow 1$ there exists a function $u \in \text{Harm}(\mathbf{D})$ such that $u(re^{i\theta}) \rightarrow \infty$ as $r \nearrow 1$ and also

$$(1.1) \quad |u(z)| < v(|z|), \quad z \in \mathbf{D}.$$

It is known (see [6, 9]), that for a wide class of majorants v relation (1.1) yields

$$\liminf_{r \nearrow 1} \frac{u(re^{i\phi})}{v(r)} \leq 0, \quad \text{and} \quad \limsup_{r \nearrow 1} \frac{u(re^{i\phi})}{v(r)} \geq 0$$

for almost all values of $\phi \in (-\pi, \pi)$. In other words, for almost all ϕ the values $u(re^{i\phi})$ "oscillate" between $\pm v(r)$. In this article we study this oscillation in more details.

We restrict ourselves for definiteness to functions which belong to the classical (harmonic) Korenblum class \mathcal{K} consisting of all real functions $u \in$

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Harm(\mathbf{D}) satisfying

$$(1.2) \quad u(z) \leq \log \frac{e}{1-|z|}, \quad z \in \mathbf{D}.$$

This class was introduced and studied in [13]. We refer the reader to [10, 17, 15, 5] for further properties of functions in \mathcal{K} , including their behavior near the boundary.

One of the typical examples here is the function

$$(1.3) \quad u_A(z) = \Re \sum_n A^n z^{2^{A^n}},$$

where $A \geq 2$ is an integer. It is easy to see that $|u_A(z)| \geq c|\log(1-|z|)|$ on a large portion of the unit disk, and along almost each radius this function oscillates between $c|\log(1-|z|)|$ and $-c|\log(1-|z|)|$. We give a quantitative description of this oscillation and prove that such oscillation occurs for every harmonic function u such that $u, -u \in \mathcal{K}$.

More precisely for a function $u \in \mathcal{K}$, we consider the weighted average

$$(1.4) \quad I_u(R, \varphi) = \int_{1/2}^R \frac{u(re^{i\varphi})}{(1-r) \left(\log \frac{1}{1-r}\right)^2} dr, \quad R \in (0, 1), \quad \varphi \in (-\pi, \pi).$$

A straightforward calculation shows that (1.2) gives

$$I_u(R, \varphi) \lesssim C \log \log \frac{1}{1-R}.$$

Here and in the sequel we use notation $a \lesssim b$ for the statement: there exists an absolute constant c such that $a \leq cb$; we also write $a \simeq b$ if $a \lesssim b$ and $b \lesssim a$.

If both $u \in \mathcal{K}$ and $-u \in \mathcal{K}$, the behavior of $I_u(R, \varphi)$ is controlled by the law of the iterated logarithm:

Theorem 1.1. *There exists K such that if u is a harmonic function in \mathbf{D} satisfying*

$$(1.5) \quad |u(z)| \leq \log \frac{e}{1-|z|},$$

then

$$(1.6) \quad \limsup_{R \nearrow 1} I_u(R, \phi) \left(\log \log \frac{1}{1-R} \log_4 \frac{1}{1-R} \right)^{-1/2} \leq K$$

for almost every $\phi \in (-\pi, \pi]$, (here $\log_4 x = \log \log \log \log x$) and also

$$(1.7) \quad \int_{-\pi}^{\pi} I_u(R, \phi)^2 d\phi \leq K \log \log \frac{1}{1-R}.$$

This result is sharp in the following sense: there exists $\alpha > 0$ such that the function $u = u_2$ defined by (1.3) with $A = 2$ satisfies

$$(1.8) \quad \limsup_{R \nearrow 1} I_u(R, \phi) \left(\log \log \frac{1}{1-R} \log_4 \frac{1}{1-R} \right)^{-1/2} > \alpha$$

for almost every $\phi \in (-\pi, \pi]$.

Theorem 1.1 allows one to obtain more precise estimates of the radial growth for the case when both u and $-u$ belong to \mathcal{K} .

Proposition 1.1. *Let $u \in \mathcal{K}$ and $-u \in \mathcal{K}$ then for any $a < 1/2$*

$$(1.9) \quad \liminf_{r \rightarrow 1} \frac{u(re^{i\phi})(\log |\log(1-r)|)^a}{|\log(1-r)|} \leq 0,$$

for almost every $\phi \in (-\pi, \pi]$.

The proof of Theorem 1.1 is based on approximation of $I_u(R, \phi)$ by the sum of discrete martingales and then application of the law of the iterated logarithm for martingales. Our approach is based on the ideas developed by D.L. Burkholder and R.F. Gundy [7], C.Y. Chang, J.M. Wilson, and T.H. Wolf [8], N. Makarov [14], J.M. Anderson, L.D. Pitt [1], and R. Bañuelos, I. Klemes, and C.N. Moore [3], see also the references therein. We also refer the reader to the monographs [2, 4]. However approximation of harmonic functions by martingales, which became classical by now, does not yield the desired approximation of the weighted average $I_u(R, \phi)$. In particular the law of the iterated logarithm for trigonometric series does not say much about oscillations of our model function (1.3). Instead we consider the martingale approximation of the function $I_u(R, \phi)$ itself. We use "super-dyadic" martingales corresponding to the algebras generated by intervals of length 2^{-2^n} which seem to be more appropriate for our purposes. One of possible ways to think about this construction is to consider a suitable martingale transform of the "classical" martingale of the harmonic function u thinned out to the "super-dyadic" algebras.

These results can be directly applied to lacunary series. We use a result of J.-P. Kahane, M. Weiss, and G. Weiss, [12] in order to describe all functions u which are represented by the series

$$(1.10) \quad u(z) = \Re \sum_{k=1}^{\infty} a_{n_k} z^{n_k}, \quad n_{k+1}/n_k > \lambda > 1 \text{ for each } k,$$

and belong to \mathcal{K} in terms of the coefficient sequence $\{a_{n_k}\}$. It follows from this description that if a function u of the form (1.10) belongs to \mathcal{K} then $-u \in \mathcal{K}$.

Corollary. *Let $u \in \mathcal{K}$ be represented by a lacunary series (1.10). Then (1.6) holds almost everywhere, here K depends only on λ from the gap condition and on C in (1.2).*

The situation changes drastically when we consider functions $u \in \mathcal{K}$ that satisfy only one-sided estimate (1.2) instead of the two-sided estimate (1.5). This follows from the statement below

Theorem 1.2. *The series*

$$(1.11) \quad u(z) = \Re \sum_{n>1} 2^n \frac{z^{2^{2^n}}}{z^{2^{2^n}} - 1}$$

converges uniformly on compact sets in \mathbf{D} to a function in \mathcal{K} , and, for almost every $\phi \in (-\pi, \pi]$,

$$(1.12) \quad \liminf_{R \nearrow 1} I_u(R, \phi) \left(\log \log \frac{1}{1-R} \right)^{-1} > 0$$

The article is organized as follows. In the next Section we collect some results on premeasures and (discrete) martingales that are used later. Sections 3 and 4 deal with the martingale approximation of $I_u(R, \phi)$ and contain the proof of Theorem 1.1. An example showing that Theorem 1.1 is sharp is given in Section 5, there we also describe harmonic functions in the Korablum class given by lacunary series. Theorem 1.2 is proved in the last section.

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2. PRELIMINARIES

2.1. Representation of functions from \mathcal{K} . We refer the reader to the classical article [13] and to the monograph [10]. Let \mathcal{R} be the set of all (open, closed, half-closed) arcs on the unit circle \mathbf{T} . A function $\mu : \mathcal{R} \rightarrow \mathbf{R}$ is a *premeasure* if it satisfies

- $\mu(I_1 \cup I_2) = \mu(I_1) + \mu(I_2)$ when $I_1 \cap I_2 = \emptyset$ and $I_1 \cup I_2 \in \mathcal{R}$,
- $\mu(\mathbf{T}) = 0$
- $\lim_{n \rightarrow \infty} \mu(I_n) = 0$ whenever $I_1 \supset I_2 \supset \dots$ and $\bigcap_n I_n = \emptyset$.

We say that μ is κ -*bounded from above*, if in addition

$$(2.1) \quad \mu(I) \lesssim |I| \log \frac{e}{|I|}$$

for every $I \in \mathcal{R}$.

Everywhere below we assume (for simplicity) that $u(0) = 0$. Let also

$$P(re^{i\phi}) = \frac{1-r^2}{|e^{i\phi} - r|^2}$$

be the standard Poisson kernel. It is proved in [13] that each $u \in \mathcal{K}$ can be represented as the Poisson integral:

$$(2.2) \quad u(re^{i\phi}) = \int_{-\pi}^{\pi} P(re^{i(\phi-\theta)}) d\mu(\theta)$$

with respect to some κ -bounded from above premeasure μ . The integral in the right-hand side should be understood as

$$(2.3) \quad \int_{-\pi}^{\pi} P(re^{i(\phi-\theta)})d\mu(\theta) = \int_{-\pi}^{\pi} \left(P(re^{i(\phi-\theta)}) \right)'_{\theta} \mu((e^{i\theta}, 0))d\theta,$$

here $(e^{i\theta}, 0)$ stays for the arc of \mathbf{T} which connects 0 and $e^{i\theta}$.

In the case when u satisfies the two-sided estimate (1.5) the corresponding premeasure is κ -bounded from above and below:

$$(2.4) \quad |\mu(I)| \lesssim |I| \log \frac{e}{|I|}.$$

2.2. Martingales. In this part we recall the basic notions and facts about (super-dyadic) martingales (on the unit circle) which will be used in the sequel. We follow mainly [18], see also [19].

Given $n > 0$ let \mathcal{E}_n be the set of all super-dyadic intervals (on \mathbf{T}) of length 2^{-2^n} and \mathcal{F}_n be the σ -algebra generated by \mathcal{E}_n . Respectively we denote $\mathcal{E} = \cup \mathcal{E}_n$. A function $f : \mathbf{T} \rightarrow \mathbf{C}$ is measurable with respect to \mathcal{F}_n if it is constant on each $I \in \mathcal{E}_n$. Given any $f : \mathbf{T} \rightarrow \mathbf{C}$ we define its expectation with respect to $\mathbf{E}(f|\mathcal{F}_n)$ as the measurable (with respect to \mathcal{F}_n) function

$$(2.5) \quad \mathbf{E}(f|\mathcal{F}_n) = \sum_{I \in \mathcal{E}_n} \left(\frac{1}{|I|} \int_I f(t)dt \right) \mathbf{1}_I,$$

here $\mathbf{1}_I$ denotes the characteristic function of I . A sequence of function $\{f_n\}$, $f_n : \mathbf{T} \rightarrow \mathbf{C}$ is called a *martingale* if

- f_n is \mathcal{F}_n measurable
- $\mathbf{E}(f_n|\mathcal{F}_{n-1}) = f_{n-1}$

The *martingale differences* and the *square function* of a martingale $\{f_n\}$ are defined by

$$(2.6) \quad d_j = f_j - f_{j-1} \text{ and } s_n = \left(\sum_{j=1}^n \mathbf{E}(d_j^2|\mathcal{F}_{j-1}) \right)^{1/2},$$

respectively, see e.g. [18]. Remark, that our martingales are super-dyadic so the formula for s_n differs from one usually used for dyadic martingales, we refer the reader to [4] for related discussion. We also denote

$$(2.7) \quad u_n = (2 \log \log s_n^2)^{1/2}.$$

The following statement is a special case of Theorems 1 and 2 in [18].

Theorem B. *Let $\mathcal{O} = \{\phi \in (-\pi, \pi) : s_n(\phi) \rightarrow \infty \text{ as } n \rightarrow \infty\}$ and $|d_n| \lesssim 1$. Then*

$$(2.8) \quad \limsup_{n \rightarrow \infty} \frac{f_n(\phi)}{s_n(\phi)u_n(\phi)} \leq 1,$$

for almost all $\phi \in \mathcal{O}$.

We always identify ϕ and the point $e^{i\phi} \in \mathbf{T}$.

Example. Let a premeasure μ satisfy (2.4). Denote

$$(2.9) \quad g_n = \sum_{I \in \mathcal{F}_n} \mathbf{1}_I \frac{\mu(I)}{|I|},$$

$$(2.10) \quad d_j = 2^{-j}(g_j - g_{j-1}) \text{ and } f_n = \sum_{j=1}^n d_j.$$

Then the martingale $\{f_n\}$ meets the conditions of Theorem A.

Remark. The function u in Theorem 1.1 admits representation (2.2), so for each arc $I = (e^{i(\theta-\delta)}, e^{i(\theta+\delta)}) \subset \mathbf{T}$ the average $|I|^{-1}\mu(I)$ can be considered as an approximation of $u((1-\delta)e^{i\theta})$. Taking this into account one can observe that the functions f_n from the above example can be viewed as integral sums for $I_u(1-2^{-2^n}, \cdot)$. This martingale can be used as a hint in order to guess how Theorem 1.1 should be formulated. However this prove Theorem 1.1 we need a more developed martingale construction.

3. ATOMIC DECOMPOSITION

Let a function $u \in \text{Harm}(\mathbf{D})$ satisfy (1.5) and $u(0) = 0$. In this section we decompose the function $I_u(R, \phi)$ into sum of atoms. Such decomposition (following for example the scheme from [4]) will lead us to a martingale approximation of $I_u(R, \phi)$.

3.1. Preliminary decomposition. Denote

$$(3.1) \quad r_j = 1 - 2^{-2^j}, \quad j = 0, 1, \dots$$

Let $r_n \leq R < r_{n+1}$. We than have

$$(3.2) \quad \begin{aligned} I_u(R, \phi) &= \int_{1/2}^R \frac{u(re^{i\phi})dr}{(1-r)(\log(1-r))^2} = \\ &= \sum_{j=1}^n \underbrace{\int_{r_{j-1}}^{r_j} \frac{u(re^{i\phi})}{(1-r)(\log(1-r))^2} dr}_{v_j(\phi)} + \int_{r_n}^R \frac{u(re^{i\phi})dr}{(1-r)(\log(1-r))^2} = \\ &= \sum_{j=1}^{n-1} \underbrace{2v_{j+1}(\phi) - v_j(\phi)}_{w_j(\phi)} + \underbrace{\left(2v_1(\phi) - v_n(\phi) + \int_{r_n}^R \frac{u(re^{i\phi})dr}{(1-r)|\log(1-r)|^2} \right)}_{q_n(\phi)}. \end{aligned}$$

It follows from (1.5) that $|q_n(\phi)| \lesssim 1$. Therefore in order to prove Theorem 1.1 it suffices to consider

$$(3.3) \quad J_n(\phi) = \sum_1^{n-1} w_j(\phi).$$

We will approximate the sequence $\{J_n\}$ by the sum of super-dyadic martingales.

Using representation (2.2) we obtain

$$(3.4) \quad v_j(\phi) = \int_{-\pi}^{\pi} A_j(\phi - \theta) d\mu(\theta), \quad w_j(\phi) = \int_{-\pi}^{\pi} B_j(\phi - \theta) d\mu(\theta),$$

where

$$(3.5) \quad A_j(\psi) = \int_{r_{j-1}}^{r_j} \frac{P(re^{i\psi}) dr}{(1-r)(\log(1-r))^2}, \quad B_j(\psi) = 2A_j(\psi) - A_{j-1}(\psi).$$

Clearly

$$(3.6) \quad \int_{-\pi}^{\pi} A_j(\phi) d\phi = 2^{-j} \quad \text{and} \quad \int_{-\pi}^{\pi} B_j(\phi) d\phi = 0,$$

so the kernels B_j possess the cancellation property.

3.2. Construction of atoms. The functions $B_j(\phi)$ are concentrated mainly in the intervals around zero whose length is of order $2^{-2^{j-4}}$. We approximate them by functions $\tilde{B}_j(\phi)$ which are supported in the corresponding intervals. The following (technical) lemma estimates the error of such approximation.

Lemma 3.1. *For every j there exists an even function $\tilde{B}_j(\psi)$ such that*

$$\begin{aligned} \text{supp}(\tilde{B}_j) \subset \left(-\frac{1}{64}2^{-2^{j-4}}, \frac{1}{64}2^{-2^{j-4}}\right) =: J_j \quad \text{and} \quad B_j = \tilde{B}_j \quad \text{on} \quad \frac{1}{2}J_j, \\ \int_{-\pi}^{\pi} \tilde{B}_j = \int_{\mathbf{T}} B_j = 0, \quad |(B_j - \tilde{B}_j)'| \lesssim 2^{-2^j}. \end{aligned}$$

Moreover, the following estimates hold

$$(3.7) \quad (a) |\tilde{B}'| \lesssim 2^{-2^j} 2^{2 \cdot 2^j}, \quad (b) |\tilde{B}''| \lesssim 2^{-2^j} 2^{3 \cdot 2^j}, \quad (c) |\tilde{B}_j(\alpha) \alpha \log \frac{1}{\alpha}| \lesssim 2^{-j};$$

$$(3.8) \quad \int_{-\pi}^{\pi} |\tilde{B}'_j(\alpha) \alpha \log \frac{1}{\alpha}| d\alpha \lesssim 1.$$

Proof. Let r_j be given by (3.1). Elementary calculations show that

$$A_j(\phi) \lesssim \begin{cases} 2^{-2^j} 2^{2^j}, & \phi \leq 1 - r_j, \\ 2^{-2^j} \phi^{-1}, & 1 - r_j < \phi \leq 1 - r_{j-1}, \\ 2^{-2^j} \phi^{-2} 2^{-2^{j-1}}, & 1 - r_{j-1} < \phi. \end{cases}$$

Further, we obtain

$$|A'_j(\phi)| \lesssim \begin{cases} 2^{-2^j} \phi 2^{3 \cdot 2^j}, & \phi \leq 1 - r_j, \\ 2^{-2^j} \phi^{-2}, & 1 - r_j < \phi \leq 1 - r_{j-1}, \\ 2^{-2^j} \phi^{-3} 2^{-2^{j-1}}, & 1 - r_{j-1} < \phi; \end{cases}$$

and

$$|A''_j(\phi)| \lesssim \begin{cases} 2^{-2^j} 2^{3 \cdot 2^j}, & \phi \leq 1 - r_j, \\ 2^{-2^j} \phi^{-3}, & 1 - r_j < \phi \leq 1 - r_{j-1}, \\ 2^{-2^j} \phi^{-4} 2^{-2^{j-1}}, & 1 - r_{j-1} < \phi. \end{cases}$$

Let α and β be even smooth functions, $0 \leq \alpha \leq 1$, such that

$$\begin{aligned} \alpha &= 1, \quad \text{on } (-2^{-7}2^{-2^{j-4}}, 2^{-7}2^{-2^{j-4}}); \\ \text{supp}(\alpha) &\subset [-2^{-6}2^{-2^{j-4}}, 2^{-6}2^{-2^{j-4}}], \quad |\alpha'| \lesssim 2^{2^{j-4}}, \quad \text{and } |\alpha''| \lesssim 2^{2^{j-3}}; \\ \int_{-\pi}^{\pi} \beta &= 1, \quad \text{supp}(\beta) \subset [-2^{-6}2^{-2^{j-4}}, -2^{-7}2^{-2^{j-4}}] \cup [2^{-7}2^{-2^{j-4}}, 2^{-6}2^{-2^{j-4}}]; \\ |\beta| &\lesssim 2^{2^{j-4}}, \quad |\beta'| \lesssim 2^{2^{j-3}}, \quad \text{and } |\beta''| \lesssim 2^{2^{j-2}}. \end{aligned}$$

Then we define

$$\tilde{B}_j = \alpha B_j + \beta \int_{-\pi}^{\pi} (B_j - \alpha A_j).$$

Clearly \tilde{B}_j is an even function. The required estimates now follows by a direct inspection. \square

Now let

$$(3.9) \quad \tilde{w}_j(\phi) = \int_{-\pi}^{\pi} \tilde{B}_j(\phi - \theta) d\mu(\theta).$$

Lemma 3.2. *The following inequality holds*

$$|w_j(\phi) - \tilde{w}_j(\phi)| \lesssim 2^{-2^j}$$

for any j and $\phi \in (-\pi, \pi)$.

Proof. Clearly (1.5) implies that $|\mu(I)| \lesssim 1$ for any interval I on the circle. Since $B_j - \tilde{B}_j$ is an even function and $\mu(\mathbf{T}) = 0$ we have

$$\begin{aligned} |w_j(\phi) - \tilde{w}_j(\phi)| &= \left| \int_{-\pi}^{\pi} (B_j - \tilde{B}_j)(\phi - \theta) d\mu(\theta) \right| \\ &= \left| \int_0^{\pi} (B_j - \tilde{B}_j)'(\theta) \mu(\phi - \theta, \phi + \theta) \right| \lesssim 2^{-2^j}. \end{aligned}$$

\square

This lemma implies

$$(3.10) \quad \left| \sum_1^n w_j(\phi) - \sum_1^n \tilde{w}_j(\phi) \right| \lesssim 1.$$

We divide the circle into $64 \cdot 2^{2^k}$ non-overlapping arcs of length $\frac{1}{64}2^{-2^k}$ and define by \mathcal{H}_k the collection of these arcs. Let also $\mathcal{H} = \cup_k \mathcal{H}_k$. We obtain

$$(3.11) \quad \tilde{w}_j(\phi) = \int_{-\pi}^{\pi} \tilde{B}_j(\phi - \theta) d\mu(\theta) = \sum_{I \in \mathcal{H}_{j-4}} \int_I \tilde{B}_j(\phi - \theta) d\mu(\theta) = \sum_{I \in \mathcal{H}_{j-4}} \lambda_I(\phi).$$

Functions λ_I are our atoms. For each $I \in \mathcal{H}_k$, $k > 6$ we have

$$\text{supp} \lambda_I \subset 3I;$$

and

$$(3.12) \quad \int_{3I} \lambda_I = \int_{-\pi}^{\pi} \lambda_I(\phi) d\phi = \int_{-\pi}^{\pi} \int_I \tilde{B}_j(\phi - \theta) d\mu(\theta) d\phi = 0.$$

4. FROM ATOMS TO MARTINGALES

4.1. Construction of martingales. Relation (3.10) gives a decomposition of $\sum_{j=1}^n \tilde{w}_j$ into the sum of atoms λ_I , $I \in \mathcal{H}$. It may happen that $\text{supp} \lambda_{I_1} \cap \text{supp} \lambda_{I_2} \neq \emptyset$ for (neighboring) $I_1, I_2 \in \mathcal{H}$, $|I_1| = |I_2|$.

Given $\omega \in \mathbf{T}$ and an arc $I \in \mathbf{T}$ we denote $\omega I = \{\omega\zeta; \zeta \in I\}$. Respectively $\omega\mathcal{E} = \{\omega I; I \in \mathcal{E}\}$, $\omega\mathcal{E}_n = \{\omega I; I \in \mathcal{E}_n\}$. The lemma below follows from a more general statement Lemma 2.1.2 in [4], see also [8] (we adjust the formulation for our setting.)

Lemma 4.1. *There exists a finite partition $\mathcal{H} = \cup_{s=1}^N V^{(s)}$ and a set of points $\{\omega_s\}_{s=1}^N \subset \mathbf{T}$ such that $V^{(s)} \cap V^{(t)} = \emptyset$, $s \neq t$ and, for each $I \in V^{(s)} \cap \mathcal{H}_k$, there exists $I' \in \omega_s \mathcal{E}_k$ for which $\text{supp} \lambda_I \subset 3I \subset I'$. In addition if, for some s and k , $I_1, I_2 \in V^{(s)} \cap \mathcal{H}_k$, $I_1 \neq I_2$ then $I'_1 \cap I'_2 = \emptyset$.*

Now for each $s = 1, 2, \dots, N$ and $n \geq 1$ we define

$$(4.1) \quad \Lambda_n^{(s)}(\phi) = \sum_{I \in V^{(s)}, 64|I| \geq 2^{-2n}} \lambda_I(\phi).$$

Then

$$(4.2) \quad \sum_{j=1}^n \tilde{w}_j(\phi) = \sum_{s=1}^N \Lambda_{n-4}^{(s)}(\phi).$$

We consider the corresponding martingales with the sequence of (shifted) super-dyadic σ -algebras

$$(4.3) \quad f_n^{(s)} = E(\Lambda_n^s | \omega_s \mathcal{F}_n).$$

4.2. Estimate of the martingale approximation. Our first aim is to estimate the error

$$|f_n^{(s)} - \Lambda_n^{(s)}| \leq \sum_{I \in V^{(s)}, 64|I| \geq 2^{-2n}} |E(\lambda_I | \omega_s \mathcal{F}_n) - \lambda_I|.$$

Lemma 4.2. *The following inequality holds*

$$(4.4) \quad |f_n^{(s)}(\phi) - \Lambda_n^{(s)}(\phi)| \lesssim 1.$$

Proof. First we prove that

$$(4.5) \quad |\lambda_I(\phi)| \lesssim 1 \quad \text{and} \quad |\lambda'_I(\phi)| \leq |I|^{-64}$$

for any $I \in \mathcal{H}$ and any $\phi \in (-\pi, \pi]$;

Indeed let $\omega_\phi(\theta) = \int_\phi^\theta d\mu = \mu(\phi, \theta)$ and let $I = (\alpha, \beta) \in \mathcal{H}_{j-4}$. We have

$$|\lambda_I(\phi)| = \left| \int_I \tilde{B}_j(\phi - \theta) d\mu(\theta) \right| \leq$$

$$\left| \int_I \tilde{B}'_j(\phi - \theta) \omega_\phi(\theta) d\theta \right| + |\tilde{B}_j(\phi - \alpha) \omega_\phi(\alpha)| + |\tilde{B}_j(\phi - \beta) \omega_\phi(\beta)|.$$

Using inequalities (3.8) and (3.7 (c)), we obtain the first estimate in (4.5). To prove the second estimate we write

$$|\lambda'_I(\phi)| \leq \left| \int_I \tilde{B}''_j(\phi - \theta) \omega_\phi(\theta) d\theta \right| + |\tilde{B}'_j(\phi - \alpha) \omega_\phi(\alpha)| + |\tilde{B}'_j(\phi - \beta) \omega_\phi(\beta)|$$

and use the inequalities (3.7 (a)) and (3.7 (b)).

Now we have

$$|E(\lambda_I | \omega_s \mathcal{F}_n) - \lambda_I| \leq 2^{-2^n} \max |\lambda'_I|.$$

This inequality together with (4.5) imply

$$(4.6) \quad |f_n^{(s)}(\phi) - \Lambda_n^{(s)}(\phi)| \leq \sum_{k=1}^n \sum_{I \in V^s \cap \mathcal{H}_k, I' \ni \phi} |E(\lambda_I | \omega_s \mathcal{F}_n)(\phi) - \lambda_I(\phi)| \leq \\ \sum_{k=1}^{n-6} \sum_{I \in V^s \cap \mathcal{H}_k, I' \ni \phi} |E(\lambda_I | \omega_s \mathcal{F}_n)(\phi) - \lambda_I(\phi)| + \\ \sum_{k=n-5}^n \sum_{I \in V^s \cap \mathcal{H}_k, I' \ni \phi} (|E(\lambda_I | \omega_s \mathcal{F}_n)(\phi)| + |\lambda_I(\phi)|) \lesssim 2^{-2^n} \sum_{k=1}^{n-6} 2^{2^{k+6}} + 1 \lesssim 1,$$

and (4.4) now follows. \square

For each martingale $\{f_n^{(s)}\}$ we can now estimate its square function,

$$s_n^{(s)} = \left(\sum_{k=1}^n E(|f_k^{(s)} - f_{k+1}^{(s)}|^2 | \mathcal{F}_k) \right)^{1/2}.$$

Lemma 4.3. *For each $s = 1, \dots, N$ and $n \geq 1$*

$$(4.7) \quad |f_n^{(s)} - f_{n+1}^{(s)}| \lesssim 1 \quad \text{and} \quad s_n^{(s)} \lesssim \sqrt{n}.$$

Proof. The first inequality follows from Lemma 4.2 and (4.5). The second inequality is now straightforward. \square

4.3. Mean estimates. We first prove inequality (1.7) from Theorem 1.1, namely

$$\int_{-\pi}^{\pi} I_u(R, \phi)^2 d\phi \lesssim \log \log \frac{1}{1-R}.$$

Let, as before, r_n 's be given by (3.1) and $r_n \leq R < r_{n+1}$. Then (see (3.2), (3.10), (3.5), and (4.4))

$$\left| I_u(R, \phi) - \sum_{s=1}^N f_{n-4}^{(s)}(\phi) \right| \lesssim 1.$$

Therefore it suffices to prove that

$$(4.8) \quad \int_{-\pi}^{\pi} f_{n-4}^{(s)}(\phi)^2 d\phi \lesssim n \simeq \log \log \frac{1}{1-R}$$

for each $s = 1, 2, \dots, N$.

We use the fact that the martingale differences $f_j^{(s)}(\phi) - f_{j-1}^{(s)}(\phi)$ are pairwise orthogonal. Therefore

$$(4.9) \quad \int_{\mathbf{T}} |f_{n-4}^{(s)}(\phi)|^2 d\phi = \int_{\mathbf{T}} \left| \sum_{j=1}^{n-4} (f_j^{(s)}(\phi) - f_{j-1}^{(s)}(\phi)) \right|^2 d\phi = \\ \int_{\mathbf{T}} \sum_{j=1}^{n-4} |f_j^{(s)}(\phi) - f_{j-1}^{(s)}(\phi)|^2 d\phi = \int_{\mathbf{T}} \left(s_{n-4}^{(s)}(\phi) \right)^2 d\phi \lesssim n,$$

the last inequality follows from (4.7). This yields (1.7).

4.4. The law of the iterated logarithm. The relation (1.6) from Theorem 1.1 now follows easily from Theorem A and the estimate (4.7) of the square function. Indeed, since $|I_u(R, \phi) - \sum_{s=1}^N f_{n-4}^{(s)}(\phi)| \lesssim 1$ it suffices to prove that for each s

$$\limsup_n \frac{f_n^{(s)}}{n \log \log n} \lesssim 1$$

almost everywhere. In case $s_n^{(s)}(\phi) \rightarrow \infty$ this follows from relations (2.8) and (4.7). Otherwise $s_n^{(s)}(\phi)$ stays bounded and

$$|f_n^{(s)}(\phi)| \leq \sum_{j=1}^n |f_j^{(s)}(\phi) - f_{j-1}^{(s)}(\phi)| \leq n^{1/2} s_n^{(s)}(\phi)^{1/2}.$$

This completes the proof of Theorem 1.1. Proposition 1.1 formulated in the introduction follows readily.

5. LACUNARY SERIES

5.1. Example. We begin with an example showing that Theorem 1.1 is sharp. Let

$$u(z) = \Re \sum_{n=1}^{\infty} 2^n z^{2^{2^n}}, \quad \text{and} \quad I_u(R, \phi) = \int_{1/2}^R \frac{u(re^{i\phi})}{(1-r)(\log(1-r))^2} dr.$$

It is proved in [6] that $u \in \mathcal{K}$. We will show that for some $a > 0$

$$\limsup_{R \rightarrow 1} I_u(R, \phi) \left(\log \log \frac{1}{1-R} \log_4 \frac{1}{(1-R)} \right)^{-1/2} \geq a,$$

for almost all $\phi \in (-\pi, \pi]$. Let

$$c_n = 2^n \int_{1/2}^1 \frac{r^{2^{2^n}}}{(1-r)(\log(1-r))^2} dr,$$

and $r_k = 1 - 2^{-2^k}$. Suppose that $R \in (r_k, r_{k+1})$. We prove first that

$$(5.1) \quad |I_u(R, \phi) - \sum_{j=1}^k c_j \cos(2^{2^j} \phi)| \lesssim 1.$$

Indeed,

$$(5.2) \quad |I_u(R, \phi) - I_u(r_k, \phi)| \lesssim \int_{r_k}^{r_{k+1}} \frac{dr}{(1-r)|\log(1-r)|} \lesssim 1.$$

Further,

$$I_u(r_k, \phi) = \sum_{j=1}^{\infty} \int_{1/2}^{r_k} \frac{2^j r^{2^{2^j}}}{(1-r)(\log(1-r))^2} dr \cos(2^{2^j} \phi) = \sum_{j=1}^{\infty} c_{j,k} \cos(2^{2^j} \phi).$$

For $j > k$ we have

$$c_{j,k} = \int_{1/2}^{r_k} \frac{2^j r^{2^{2^j}}}{(1-r)(\log(1-r))^2} dr \lesssim 2^j (1 - 2^{-2^k})^{2^{2^j}} \lesssim 2^j \exp(-2^{2^j-2^k})$$

and $\sum_{j>k} c_{j,k} \lesssim 1$. Finally, for $j \leq k$ we get

$$|c_{j,k} - c_j| = 2^j \int_{r_k}^1 \frac{r^{2^{2^j}}}{(1-r)(\log(1-r))^2} dr \leq 2^j \int_{r_k}^1 \frac{dr}{(1-r)(\log(1-r))^2} \lesssim 2^{j-k}.$$

Therefore $\sum_{j \leq k} |c_{j,k} - c_j| \lesssim 1$. Inequality (5.1) is proved.

Now we apply the law of the iterated logarithm proved in [20] to the lacunary series

$$(5.3) \quad \sum_j c_j \cos(2^{2^j} \phi).$$

First note that

$$\begin{aligned} c_j &= 2^j \int_{1/2}^1 \frac{r^{2^{2^j}}}{(1-r)(\log(1-r))^2} dr \leq 2^j \int_{1/2}^{r_{j-1}} \frac{r^{2^{2^j}}}{(1-r)(\log(1-r))^2} dr + \\ & 2^j \int_{r_{j-1}}^1 \frac{dr}{(1-r)(\log(1-r))^2} \lesssim 2^j (1 - 2^{-2^{j-1}})^{2^{2^j}} + 2^j 2^{-j+1} \lesssim 1. \end{aligned}$$

and

$$c_j \geq 2^j r_j^{2^{2^j}} \int_{r_j}^1 \frac{dr}{(1-r)(\log(1-r))^2} \gtrsim 1.$$

Thus the coefficients in (5.3) are bounded from 0 and ∞ and

$$B_k^2 = \frac{1}{2} \sum_{j=1}^k c_j^2 \simeq k.$$

Finally

$$\limsup_{R \rightarrow 1} \frac{I_u(R, \phi)}{\left(\log \log \frac{1}{1-R} \log_4 \frac{1}{(1-R)}\right)^{1/2}} \gtrsim \limsup_{k \rightarrow \infty} \frac{\sum_{j=1}^k c_j \cos(2^{2^j} \phi)}{(k \log \log k)^{1/2}}.$$

The last upper limit is larger than a constant for almost all $\phi \in (-\pi, \pi]$ by the law of the iterated logarithm for trigonometric lacunary series.

5.2. Description of Korenblum harmonic functions represented by lacunary series. Using results from [13] and [12], we get the following description of lacunary series that represent Korenblum functions.

Proposition 5.1. *Let $\{n_k\}_{k=1}^\infty$ be a sequence of positive integers such that $n_{k+1} \geq \lambda n_k$ for each k , where $\lambda > 1$. Let*

$$(5.4) \quad u(z) = \Re \sum_k c_{n_k} z^{n_k}, \quad c_{n_k} \in \mathbf{C},$$

where the series converges in the unit disc. Then the following conditions are equivalent:

- (1) there exists γ_1 such that $u(z) \leq \gamma_1 \log \frac{e}{1-|z|}$ for any $z \in \mathbf{D}$;
- (2) there exists γ_2 such that $|u(z)| \leq \gamma_2 \log \frac{e}{1-|z|}$ for any $z \in \mathbf{D}$;
- (3) there exists γ_3 such that $\sum_{n_k \leq N} |c_{n_k}| \leq \gamma_3 \log N$ for any $N \geq 2$.

Proof. Let $u_r(e^{i\phi}) = u(re^{i\phi})$. We show first that (1) implies (3). It follows from [13, p.209] that $|c_{n_k}| \leq C_1 \log n_k$, where $C_1 = C_1(\gamma_1)$.

Let r_N be such that $r_N^N = 1/2$, we have

$$u(r_N e^{i\phi}) = \Re \sum_{n_k \leq N} c_{n_k} r_N^{n_k} e^{in_k \phi} + \Re \sum_{n_k > N} c_{n_k} r_N^{n_k} e^{in_k \phi} = s_N(\phi) + t_N(\phi).$$

The reminder term can be estimated as follows

$$|t_N(\phi)| \leq C_1 \sum_{n_k > N} \log n_k 2^{-n_k/N} \leq \beta(\gamma_1, \lambda) \log N.$$

By Theorem I in [12] there exists $\alpha = \alpha(\lambda)$ and $\phi \in (-\pi, \pi)$ such that

$$s_N(\phi) \geq \alpha \sum_{n_k \leq N} |c_{n_k} r_N^{n_k}|.$$

(This statement is elementary when $\lambda > 2$; the result in [12] is more general.)

Thus we have

$$\sum_{n_k \leq N} |a_{n_k}| \leq 2 \sum_{n_k \leq N} |c_{n_k} r_N^{n_k}| \leq 2\alpha^{-1}(u(re^{i\phi}) + |t_N(\phi)|) \leq \gamma_3 \log N.$$

Now assume that (3) holds and let $r_N < r \leq r_{N+1}$. Then (2) follows readily

$$|u(re^{i\phi})| \leq \sum_{n_k \leq N} |c_{n_k}| + \sum_{n_k > N} \gamma_3 \log n_k r^{n_k} \leq \gamma_2 \log \frac{e}{1-r}.$$

□

Thus the Korenblum functions represented by (5.4) satisfy the assumption of Theorem 1.1.

6. NON-OSCILLATION: EXAMPLE

6.1. Construction. In this section we prove Theorem 1.2.

Lemma 6.1. *The series*

$$(6.1) \quad u(z) = \Re \sum_{n \geq 1} 2^n \frac{z^{2^{2^n}}}{z^{2^{2^n}} - 1} = \Re \sum_{n \geq 1} 2^n a_n(z)$$

converges uniformly on compact sets in \mathbf{D} to a function in \mathcal{K} .

Proof. We split the unit disk into disjoint annuli

$$\mathbf{D} = \{z : |z| < 1/2\} \cup (\cup_n A_n), \quad A_n = \{z : 2^{-2^n} \geq 1 - |z| > 2^{-2^{n+1}}\}.$$

Let $z \in A_n$ and $k > n + 1$. Then (by a straightforward calculation)

$$(6.2) \quad \left| \frac{2^{k+1} a_{k+1}(z)}{2^k a_k(z)} \right| \lesssim 2^{-2^{2^k}}.$$

Choosing n sufficiently large we will see that the ratio of two consequent terms with numbers larger than n does not exceed some $q < 1$. This yields the uniform convergence of (6.1) on compact sets in \mathbf{D} .

A direct estimate shows that $|a_{n+1}(z)| \leq C$ when $z \in A_n$ and thus

$$\sum_{k > n} 2^k |a_k(z)| \leq C_q \log \frac{1}{1 - |z|}, \quad z \in A_n.$$

To complete the proof it suffices to estimate $\Re \sum_{k \leq n} 2^k a_k(z)$. Since

$$\Re(w/(w-1)) \leq 1/2 \quad \text{for } w \in \mathbf{D}$$

we obtain:

$$\Re \sum_{k \leq n} 2^k a_k(z) \leq \sum_{k \leq n} 2^k \leq C \log \frac{1}{1 - |z|}, \quad z \in A_n.$$

□

6.2. Remark. Since $u(0) = 0$ we have $\int_{-\pi}^{\pi} u(re^{i\theta}) d\theta = 0$ for all $r \in (0, 1)$. The summands in (6.1) have the form

$$(6.3) \quad a_n(z) = \Phi(z^{2^{2^n}}); \quad \Phi(w) = \frac{w}{w-1}, \quad w \in \mathbf{D}.$$

The function $\zeta = \Phi(w)$ maps the unit disk \mathbf{D} onto the half-plane $\{\zeta; \Re \zeta < 1/2\}$. The asymmetry in the distribution of $\Re \Phi(w)$ forces u to attain huge negative values on "small" parts of \mathbf{D} while being positive (or slightly negative) on the remaining part of \mathbf{D} thus maintaining zero average along the circles $|z| = r$. In this construction the oscillation along almost all radii disappears.

6.3. Main lemma. Let for brevity $N_n = 2^{2^n}$ so that $N_{n+1} = N_n^2$. Fix some $a > 2$ and let

$$\begin{aligned} E_n &= \{\phi \in (-\pi, \pi] : \cos N_n \phi > 1 - n^{-a}\}, \\ E_{n,m} &= \{\phi \in (-\pi, \pi] : \cos N_n \phi > 1 - N_m^{-1}\}, \\ F_m &= (\cup_{n \geq m} E_n) \cup (\cup_{n < m} E_{n,m}), \quad \text{and} \quad F = \cap_m F_m. \end{aligned}$$

Since $|E_n| \simeq n^{-a/2}$ and $\sum_{1 \leq n < m} |E_{n,m}| \rightarrow 0$ ($m \rightarrow \infty$), we have $|F| = 0$. Theorem 1.2 now follows from the lemma below:

Lemma 6.2. *For each $\phi \notin F$*

$$(6.4) \quad \liminf_{R \nearrow 1} \frac{I_u(R, \phi)}{\log \log \frac{1}{1-R}} > 0.$$

Proof. We fix $\phi \notin F$. There exists m such that $\phi \notin F_m$.

For each l we denote

$$(6.5) \quad I_l(\phi) = \int_{1-N_l^{-1}}^{1-N_l^{-2}} \frac{u(re^{i\phi})}{(1-r) \left(\log \frac{1}{1-r}\right)^2} dr.$$

Assume $R = 1 - 2^{-2^t}$, $t > m$, and let $[t]$ stay for the integer part of t . We then have

$$I_u(R, \phi) = \sum_{l < [t]} I_l(\phi) + \int_{1-N_{[t]}^{-1}}^R \frac{u(re^{i\phi})}{(1-r) \left(\log \frac{1}{1-r}\right)^2} dr.$$

We will prove that there exist constants $\gamma, \Gamma > 0$ and a number $l_0 > m$ such that, for $l > l_0$

$$(6.6) \quad I_l(\phi) > \gamma,$$

and

$$(6.7) \quad \int_{1-N_{[t]}^{-1}}^R \frac{u(re^{i\phi})}{(1-r) \left(\log \frac{1}{1-r}\right)^2} dr > -\Gamma, \quad \text{when } t > l_0.$$

Then (6.4) will follow readily. \square

6.4. Technical details. We have

$$I_l(\phi) = \sum_{j=1}^{\infty} \Re \int_{1-N_l^{-1}}^{1-N_l^{-2}} 2^j \frac{(re^{i\phi})^{N_j}}{(re^{i\phi})^{N_j} - 1} \frac{dr}{(1-r) \left(\log \frac{1}{1-r}\right)^2} = \sum_{j=1}^{\infty} i_{l,j}(\phi),$$

and similarly

$$\int_{1-N_{[t]}^{-1}}^R \frac{u(re^{i\phi})}{(1-r) \left(\log \frac{1}{1-r}\right)^2} dr = \sum_{j=1}^{\infty} \tilde{i}_j(\phi).$$

Inequalities (6.6) and (6.7) follow from the propositions below.

Proposition 6.1. *Let $\phi \notin F_m$. There exist $\gamma > 0$ and $l_0 > m$ such that for all $l > l_0$*

$$(6.8) \quad i_{l,j}(\phi) > 0, \text{ if } j < l,$$

$$(6.9) \quad i_{l,l}(\phi) > 2\gamma,$$

and

$$(6.10) \quad \sum_{j>l} |i_{l,j}(\phi)| < \gamma.$$

Proposition 6.2. *Let $\phi \notin F_m$. There exist $\Gamma > 0$ and $l_0 > m$ such that for $[t] > l_0$*

$$(6.11) \quad \tilde{i}_j(\phi) > 0 \text{ if } j < [t],$$

$$(6.12) \quad \tilde{i}_{[t]} > -\Gamma/2,$$

and

$$(6.13) \quad \sum_{j>[t]} |\tilde{i}_j(\phi)| < \Gamma/2, \text{ if } l > l_0.$$

Proof of (6.8) and (6.11). Denote $c_j(\phi) = \cos N_j \phi$. We have

$$i_{l,j}(\phi) = \int_{1-N_l^{-1}}^{1-N_l^{-2}} 2^j \frac{r^{N_j} (r^{N_j} - c_j(\phi))}{r^{2N_j} - 2r^{N_j} c_j(\phi) + 1} \frac{dr}{(1-r) \left(\log \frac{1}{1-r} \right)^2}.$$

Then, for $r > 1 - N_l^{-1}$, and $m < j < l$ we obtain

$$r^{N_j} \geq \left(1 - N_j^{-2}\right)^{N_j} > 1 - 2^{-2j} > 1 - j^{-a} > c_j(\phi), \text{ when } \phi \notin F_m.$$

For $j \leq m < l$ we have

$$r^{N_j} \geq \left(1 - N_m^{-2}\right)^{N_j} > 1 - N_m^{-1} > c_j(\phi)$$

for any $\phi \notin F_m$. Hence for all $j < l$ we have $\Re a_j(re^{i\phi}) > 0$ when $r > 1 - N_l^{-1}$, and $\phi \notin F_m$, which implies both (6.8) for all $l > m$ and (6.11).

Proof of (6.9). We fix $l > m$ and choose $r^* \in (1 - N_l^{-1}, 1 - N_l^{-2})$ so that

$$(6.14) \quad (r^*)^{N_l} = 1 - \frac{1}{2}l^{-a}.$$

For $\phi \notin F_m$ we have $c_l(\phi) < c_l := 1 - l^{-a}$ and

$$\begin{aligned} i_{l,l}(\phi) &\geq \int_{1-N_l^{-1}}^{1-N_l^{-2}} 2^l \frac{r^{N_l} (r^{N_l} - c_l)}{r^{2N_l} - 2r^{N_l} c_l + 1} \frac{dr}{(1-r) \left(\log \frac{1}{1-r} \right)^2} = \\ &\left(\int_{1-N_l^{-1}}^{r^*} + \int_{r^*}^{1-N_l^{-2}} \right) 2^l \frac{r^{N_l} (r^{N_l} - c_l)}{(r^{N_l} - c_l)^2 + 1 - c_l^2} \frac{dr}{(1-r) \left(\log \frac{1}{1-r} \right)^2} \\ &= j_1 + j_2, \end{aligned}$$

Applying the inequality $x(x^2 + y^2)^{-1} \geq -(2y)^{-1}$, we have

$$\frac{r^{N_l} - c_l}{(r^{N_l} - c_l)^2 + (1 - c_l^2)} \geq -l^{a/2}.$$

We also have $\log 1/(1-r) \simeq 2^l$, for $r \in (1 - N_l^{-1}, 1 - N_l^{-2})$. Therefore

$$j_1 \gtrsim -2^{-l} l^{a/2} \int_{1-N_l^{-1}}^{r^*} \frac{dr}{1-r} \gtrsim -2^{-l} l^{a/2} a \log l.$$

The right-hand side here can be made arbitrary small by choosing sufficiently large l_0 .

To complete the proof of (6.9) it remains to show that

$$(6.15) \quad j_2 \simeq 1.$$

For $r > r^*$ and $c_l(\phi) < c_l$ we have

$$1 \geq \frac{r^{N_l} (r^{N_l} - c_l(\phi))}{r^{2N_l} - 2r^{N_l} c_l(\phi) + 1} > \frac{1 - \frac{1}{2} l^{-a}}{4 - \frac{3}{2} l^{-a}} > \frac{1}{4}.$$

Hence

$$j_2 \simeq 2^{-l} \int_{r^*}^{1-N_l^{-2}} \frac{dr}{(1-r)}.$$

Relation (6.15) now follows from

$$2^{-l} \int_{1-N_l^{-1}}^{1-N_l^{-2}} \frac{dr}{(1-r)} = \frac{\log 2}{2}; \quad 2^{-l} \int_{1-N_l^{-1}}^{r^*} \frac{dr}{(1-r)} \rightarrow 0, \text{ as } l \rightarrow \infty.$$

Proof of (6.12) can be done in a similar way if one chooses r^* so that $(r^*)^{N_{[t]}} = 1 - \frac{1}{2} [t]^{-a}$ and repeats the above reasonings.

Proof of (6.10) and (6.13). These inequalities can be proved in the same manner so we restrict ourselves to (6.10) only.

First we estimate the sum $\sum_{j>l+1} |i_{l,j}(\phi)|$. Inequality (6.2) implies that for $l > l_0$

$$\sum_{j>l+1} |i_{l,j}(\phi)| \lesssim 2^{-l}$$

and the contribution of this sum can be done arbitrary small by choosing l_0 large enough.

It remains to estimate

$$i_{l,l+1}(\phi) = \int_{1-N_l^{-1}}^{1-N_l^{-2}} 2^{l+1} \frac{r^{N_{l+1}} (r^{N_{l+1}} - c_{l+1}(\phi))}{r^{2N_{l+1}} - 2r^{N_{l+1}} c_{l+1}(\phi) + 1} \frac{dr}{(1-r) \left(\log \frac{1}{1-r} \right)^2}.$$

Since $r^{N_{l+1}} < 2^{-1}$ for $r \in (1 - N_l^{-1}, 1 - N_l^{-2})$ we have

$$|i_{l,l+1}(\phi)| \lesssim 2^{-l} \int_{1-N_l^{-1}}^{1-N_l^{-2}} \frac{r^{N_{l+1}}}{1-r} dr = \\ 2^{-l} \int_{1-N_l^{-1}}^{\rho_l} \frac{r^{N_{l+1}}}{1-r} dr + 2^{-l} \int_{\rho_l}^{1-N_l^{-2}} \frac{r^{N_{l+1}}}{1-r} dr,$$

here $\rho_l = 1 - \log \frac{1}{\varepsilon} N_l^{-2} \in (1 - N_l^{-1}, 1 - N_l^{-2})$. Then for all sufficiently large l we obtain $r^{N_{l+1}} < 2\varepsilon$ when $r \in (1 - N_l^{-1}, \rho_l)$ and thus the first integral is less than $\text{const } \varepsilon$. Finally,

$$2^{-l} \int_{\rho_l}^{1-N_l^{-2}} \frac{dr}{1-r} = 2^{-l} \log \log \frac{1}{\varepsilon} < \varepsilon,$$

when l is large enough. Choosing ε small enough and $l_0 = l_0(\varepsilon)$ large enough, we obtain the desired inequality (6.10).

This completes the proof of Theorem 1.2

6.5. Concluding remark. It is interesting to compare the behavior of the function defined by (6.1) with the results on radial growth obtained in [6]. Calculations of the last section show that there exists a set $F \subset [0, 1]$ such that

$$|F \cap [r, 1]| = O\left((1-r)^2 \log\left(\frac{1}{1-r}\right)^c\right) \quad (r \rightarrow 1)$$

and

$$\liminf_{r \nearrow 1, r \notin F} \frac{u(re^{i\phi})}{\log \frac{1}{1-r}} > 0$$

for almost each $\phi \in (-\pi, \pi]$. Thus function u growth as $\log \frac{1}{1-r}$ along almost every radius when we delete a system of very thin rings from the unit disc.

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