Categories of integrable $sl(\infty)$-, $o(\infty)$-, $sp(\infty)$-modules

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Summary. We investigate several categories of integrable $\mathrm{sl}(\infty)$-, $o(\infty)$-, $sp(\infty)$-modules. In particular, we prove that the category of integrable $\mathrm{sl}(\infty)$-, $o(\infty)$-, $sp(\infty)$-modules with finite-dimensional weight spaces is semisimple. The most interesting category we study is the category $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ of tensor modules. Its objects M are defined as integrable modules of finite Loewy length such that the algebraic dual M^* is also integrable and of finite Loewy length.

We prove that the simple objects of $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ are precisely the simple tensor modules, i.e. the simple subquotients of the tensor algebra of the direct sum of the natural and conatural representations.

We also study injectives in $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ and compute the Ext^1's between simple modules. Finally, we characterize a certain subcategory $\mathrm{Tens}_{\mathfrak{g}}$ of $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ as the unique minimal abelian full subcategory of the category of integrable modules which contains a non-trivial module and is closed under tensor product and algebraic dualization.

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1. Introduction

The category of finite-dimensional representations of a Lie algebra is endowed with a natural contravariant involution

$$M \rightsquigarrow M^*, \tag{1}$$

where * indicates dual space. For categories of infinite-dimensional modules (1) is never an involution as $M \not\simeq M^{**}$. This is why one usually looks for a "restricted dual" or a "continuous dual" which might still yield a contravariant involution on a given category of infinite-dimensional modules. In this paper we study two categories of infinite-dimensional modules of certain infinite-dimensional Lie algebras and show, in particular, that there exists an interesting category $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ of infinite-dimensional representations on which the functor (1) of algebraic dualization is well-defined and preserves the property of a module to be of finite Loewy length.

More precisely, we study representations of locally finite Lie algebras, i.e. of direct limits of finite-dimensional Lie algebras. There are three well-known classical simple locally finite Lie algebras $sl(\infty), o(\infty), sp(\infty)$, each of them being defined by an obvious direct limit. None of these Lie algebras admits non-trivial finite-dimensional representations, and instead one studies integrable representations (the definition see in section 2 below). However, the category of integrable $\mathfrak{g}$-modules is vast (and "wild" in the technical sense), so it is reasonable to look for interesting subcategories.

One subcategory we study is the category of integrable weight modules with finite-dimensional weight spaces, and this is obviously an analog of the category of finite-dimensional representations of a classical finite-dimensional Lie algebra. It is less obvious that for $\mathfrak{g} = sl(\infty)$ this category contains some rather interesting simple modules which are not highest weight modules. The first main result of this paper is the proof of the semisimplicity of this category: an extension of Hermann Weyl's semisimplicity theorem to the classical Lie algebras $sl(\infty), o(\infty), sp(\infty)$.

The above category is clearly not the only reasonable generalization of the category of finite-dimensional representations, as for instance it does not contain the adjoint representation. Indeed, note that the adjoint representation has an infinite-dimensional weight space, the Cartan subalgebra itself. On the other hand, the adjoint representation is naturally a simple tensor module as defined in [PS]. More generally, we define the category $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ for $\mathfrak{g} \cong sl(\infty)$, $o(\infty)$, $sp(\infty)$ simply as the largest category of integrable $\mathfrak{g}$-modules which is closed under algebraic dualization and such that every object has finite Loewy length. This category is a (non-rigid) tensor category with respect to the usual tensor product.

The second main contribution of the present paper is the study of the category $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$. In particular, we study injectives in $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ and compute the Ext^1's between simple modules. We also give an alternative characterization of $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ by proving that an integrable $\mathfrak{g}$-module is an object of $\mathrm{Tens}_{\mathfrak{g}}$ if and only if it has finite Loewy length and admits only finitely many non-isomorphic simple subquotients each of which is a submodule of a suitable finite tensor product of natural and conatural modules. Finally, we describe a certain subcategory $\mathrm{Tens}_{\mathfrak{g}}$ of $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ as the unique minimal abelian full subcategory of the category of integrable modules which contains a non-trivial module and is closed under tensor product and algebraic dualization.

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2. Basic definitions

The ground field is $\mathbb{C}$ and $\otimes$ stands for $\otimes_{\mathbb{C}}$. If $\mathcal{C}$ is a category, $C \in \mathcal{C}$ indicates that C is an object of $\mathcal{C}$. If P is a set, we denote by 2^P the power set of P. We recall that the cardinal numbers $\beth_n$ are defined inductively: $\beth_0 = \mathrm{card}\,\mathbb{Z}$, $\beth_1 = \mathrm{card}\,2^{\mathbb{Z}}$, $\beth_n = \mathrm{card}\,2^{P_{n-1}}$, where P_{n-1} is a set of cardinality $\beth_{n-1}$.

In this paper $\mathfrak{g}$ stands for a *locally semisimple* (complex) Lie algebra. By definition, $\mathfrak{g} = \bigcup_{i\in\mathbb{Z}_{>0}} \mathfrak{g}_i$ where

$$\mathfrak{g}_1 \subset \mathfrak{g}_2 \subset \mathfrak{g}_3 \subset \dots \tag{2}$$

is a sequence of inclusions of semisimple finite-dimensional Lie algebras. We call the sequence (2) an *exhaustion* of $\mathfrak{g}$, and we will assume that it is fixed. A locally semisimple Lie algebra is *locally simple* if it admits an exhaustion (2) so that all $\mathfrak{g}_i$ are simple. It is clear that a locally simple Lie algebra is simple. If no restrictions on $\mathfrak{g}$ are clearly stated, in what follows $\mathfrak{g}$ is assumed to be an arbitrary locally semisimple Lie algebra.

A locally simple algebra $\mathfrak{g}$ is *diagonal* if an exhaustion (2) can be chosen so that all $\mathfrak{g}_i$ are classical simple Lie algebras and the natural representation V_i of $\mathfrak{g}_i$, when restricted to $\mathfrak{g}_{i-1}$, has the form $k_iV_{i-1} \oplus l_iV_{i-1}^* \oplus \mathbb{C}^{s_i}$ for some k_i, l_i and $s_i \in \mathbb{Z}_{\geq 0}$. Here V_{i-1} stands for the natural representation of $\mathfrak{g}_{i-1}$, $\mathbb{C}^{s_i}$ stands for the trivial module of dimension s_i, and k_iV_{i-1} (respectively, $l_iV_{i-1}^*$) denotes the direct sum of k_i(respectively, l_i) copies of V_{i-1} (respectively, V_{i-1}^*).

The three classical simple Lie algebras $sl(\infty), o(\infty)$ and $sp(\infty)$ (defined respectively as $sl(\infty) = \cup_i sl(i)$, $o(\infty) = \cup_i o(i)$, $sp(\infty) := \cup_i sp(2i)$ via the natural inclusions $sl(i) \subset sl(i+1)$) etc.) are clearly diagonal. Moreover, $sl(\infty), o(\infty), sp(\infty)$ are (up to isomorphism) the only finitary locally simple Lie algebras $\mathfrak{g}$; *finitary* means by definition that $\mathfrak{g}$ admits a faithful countable-dimensional $\mathfrak{g}$-module with a basis in which each element $g \in \mathfrak{g}$ acts through a finite matrix, [Ba1], [Ba3]. More generally, there exists also a classification of locally simple diagonal Lie algebras up to isomorphism, [BZh]. We do not use this classification in the present paper and present only the simplest example of a diagonal Lie algebra not isomorphic to $sl(\infty), o(\infty)$ or $sp(\infty)$. This is the Lie algebra $sl(2^\infty)$ defined as the direct limit $\lim\limits_{\to} sl(2^i)$ under the inclusions

$$sl(2^i) \to sl(2^{i+1}), A \to \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}.$$

A $\mathfrak{g}$-module M is *integrable* if $\dim \mathrm{span}\{m, g\cdot m, g\cdot m^2, \cdots\} < \infty$ for any $m \in M$ and $g \in \mathfrak{g}$. Since $\mathfrak{g}$ is locally semisimple, this is equivalent to the condition that, when restricted to any semisimple finite-dimensional subalgebra $\mathfrak{f}$ of $\mathfrak{g}$, M

is isomorphic to a (not necessarily countable) direct sum of finite-dimensional $\mathfrak{k}$-modules. We denote by $\mathrm{Int}_{\mathfrak{g}}$ the category of integrable $\mathfrak{g}$-modules; $\mathrm{Int}_{\mathfrak{g}}$ is a full subcategory of the category of $\mathfrak{g}$-modules $\mathfrak{g}$-mod.

Any countable-dimensional $\mathfrak{g}$-module $M \in \mathrm{Int}_{\mathfrak{g}}$ can be exhausted by finite dimensional $\mathfrak{g}_i$-modules M_i, i. e. there exists a chain of finite-dimensional $\mathfrak{g}_i$-submodules $M_1 \subset M_2 \subset \dots$ such that $M = \varinjlim M_i$. We call M *locally simple* if all M_i can be chosen to be simple modules. It is clear that a locally simple module is simple. Note also that if M is locally simple then any two exhaustions $\{M_i\}$ and $\{M'_i\}$ coincide from some point on: that follows from the fact that $M_i \cap M'_i \neq 0$ for some i and hence $M_j = M'_j = M_j \cap M'_j$ for any $j \geq i$. We say that a locally simple $\mathfrak{g}$-module $M = \varinjlim M_i$ is a *highest weight module* if there is a chain of nested Borel subalgebras $\mathfrak{b}_i$ of $\mathfrak{g}_i$ such that the $\mathfrak{b}_i$-highest weight space of M_i is mapped into the $\mathfrak{b}_{i+1}$-highest weight space of M_{i+1} under the inclusion $M_i \subset M_{i+1}$. The direct limit of highest weight spaces is then the $\mathfrak{b}$-*highest weight space of* M, where $\mathfrak{b} = \varinjlim \mathfrak{b}_i$.

By

$$\Gamma_{\mathfrak{g}} : \mathfrak{g} - \mathrm{mod} \rightsquigarrow \mathrm{Int}_{\mathfrak{g}},$$

$$M \mapsto \Gamma_{\mathfrak{g}}(M) := \{m \in M, \dim \mathrm{span}\{m, g \cdot m, g \cdot m^2, \cdots\} < \infty \quad \forall g \in \mathfrak{g}\}$$

we denote the *functor of* $\mathfrak{g}$-*integrable vectors*. It is an exercise to check that $\Gamma_{\mathfrak{g}}(M)$ is indeed a well-defined $\mathfrak{g}$-submodule of M; the fact that $\Gamma_{\mathfrak{g}}(M)$ is integrable is obvious. Furthermore, $\Gamma_{\mathfrak{g}}$ is a left-exact functor.

If $\mathfrak{g}$ is a diagonal (locally simple) Lie algebra, then one can define a *natural module* V of $\mathfrak{g}$. Indeed, the reader will verify that one can choose a subexhaustion of (2) such that the natural $\mathfrak{g}_i$-module V_i is a $\mathfrak{g}_i$-submodule of V_{i+1} for any i. Therefore, fixing arbitrary injective homomorphisms $V_i \to V_{i+1}$ of $\mathfrak{g}_i$-modules, we obtain a direct system and we set $V := \varinjlim V_i$. Note that V depends on the choice of the homomorphisms $V_i \to V_{i+1}$. If however, $\mathfrak{g} \cong sl(\infty)$, $o(\infty)$, $sp(\infty)$, then the homomorphisms $V_i \to V_{i+1}$ are unique up to proportionality, and one can prove that as a result V is unique up to isomorphism, i.e. in particular does not depend on the fixed exhaustion of $\mathfrak{g}$. In these latter cases we speak about *the natural representation*.

By choosing injective homomorphisms of $\mathfrak{g}_i$-modules $V_i^* \to V_{i+1}^*$, we obtain a direct system defining a *conatural representation* of $\mathfrak{g}$. We denote such a representation by V_*. For $\mathfrak{g} \cong sl(\infty)$, $o(\infty)$, $sp(\infty)$ V_* is unique up to isomorphism. In fact, $V \simeq V_*$ for $\mathfrak{g} \cong o(\infty)$, $sp(\infty)$.

3. Injective modules in $\mathrm{Int}_{\mathfrak{g}}$ and semisimplicity of the category $\mathrm{Int}^{\mathrm{wt}}_{\mathfrak{g},\mathfrak{h}}$

Proposition 3.1. $\mathrm{Ext}^1_{\mathfrak{g}}(X, M^*) = 0$ *for any* $X, M \in \mathrm{Int}_{\mathfrak{g}}$.

Proof. We use that

$$\mathrm{Ext}^1_{\mathfrak{g}}(X, M^*) = \mathrm{Ext}^1_{\mathfrak{g}}(\mathbb{C}, \mathrm{Hom}_{\mathbb{C}}(X, M^*)) \simeq H^1(\mathfrak{g}, \mathrm{Hom}_{\mathbb{C}}(X, M^*)) = H^1(\mathfrak{g}, (X \otimes M)^*),$$

see for instance [W]. Therefore it suffices to show that $H^1(\mathfrak{g}, R^*) = 0$ for any integrable $\mathfrak{g}$-module R. Consider the standard complex for the cohomology of $\mathfrak{g}$ with coefficients in R^*:

$$0 \to R^* \to (\mathfrak{g} \otimes R)^* \to (\Lambda^2(\mathfrak{g}) \otimes R)^* \to \dots \tag{3}$$

It is dual to the standard homology complex

$$0 \leftarrow R \leftarrow \mathfrak{g} \otimes R \leftarrow \Lambda^2(\mathfrak{g}) \otimes R \leftarrow \dots,$$

which is the direct limit of complexes

$$0 \leftarrow R \leftarrow \mathfrak{g}_i \otimes R \leftarrow \Lambda^2(\mathfrak{g}_i) \otimes R \leftarrow \dots.$$

Since $H_1(\mathfrak{g}_i, R) = 0$ for each i, we get $H_1(\mathfrak{g}, R) = 0$. Therefore the dual complex (3) has trivial first cohomology, i.e. $H^1(\mathfrak{g}, R^*) = 0$. □

Proposition 3.2. *For any $M \in \mathrm{Int}_{\mathfrak{g}}$, $\Gamma_{\mathfrak{g}}(M^*)$ is an injective object of $\mathrm{Int}_{\mathfrak{g}}$.*

Proof. Let $X \in \mathrm{Int}_{\mathfrak{g}}$. The exact sequence of $\mathfrak{g}$-modules

$$0 \to \Gamma_{\mathfrak{g}}(M^*) \to M^* \to M^*/\Gamma_{\mathfrak{g}}(M^*) \to 0$$

induces an exact sequence of vector spaces

$$0 \to \mathrm{Hom}_{\mathbb{C}}(X, \Gamma_{\mathfrak{g}}(M^*)) \stackrel{\varphi}{\to} \mathrm{Hom}_{\mathbb{C}}(X, M^*) \to \mathrm{Hom}_{\mathbb{C}}(X, M^*/\Gamma_{\mathfrak{g}}(M^*)) \to$$

$$\to \mathrm{Ext}^1_{\mathfrak{g}}(X, \Gamma_{\mathfrak{g}}(M^*)) \stackrel{\psi}{\to} \mathrm{Ext}^1_{\mathfrak{g}}(X, M^*) = 0.$$

Since $\mathrm{Hom}_{\mathbb{C}}(X, M^*/\Gamma_{\mathfrak{g}}(M^*)) = 0$ (this follows from the facts that a quotient of an integrable $\mathfrak{g}$-module is again an integrable $\mathfrak{g}$-module and that $\mathrm{Int}_{\mathfrak{g}}$ is closed with respect to extensions) we conclude that ψ is an isomorphism, i.e. that $\mathrm{Ext}^1_{\mathfrak{g}}(X, \Gamma_{\mathfrak{g}}(M^*)) = 0$. □

Corollary 3.3. $\mathrm{Int}_{\mathfrak{g}}$ *has enough injectives.*

Proof. Let $M \in \mathrm{Int}_{\mathfrak{g}}$. Then $M \subset M^{**}$. By the very definition of $\Gamma_{\mathfrak{g}}$, $M \subset \Gamma_{\mathfrak{g}}(M^{**})$, and $\Gamma_{\mathfrak{g}}(M^{**})$ is an injective object of $\mathrm{Int}_{\mathfrak{g}}$ by Proposition 3.2. □

Note that there is a simpler proof of Corollary 3.3 not referring to Proposition 3.2. Indeed it is enough to notice that the functor $\Gamma_{\mathfrak{g}} : \mathfrak{g}\text{-mod} \rightsquigarrow \mathrm{Int}_{\mathfrak{g}}$ is right adjoint to the inclusion functor $\mathrm{Int}_{\mathfrak{g}} \subset \mathfrak{g}$-mod. Then the equality

$$\mathrm{Hom}_{\mathfrak{g}}(M, J_M) = \mathrm{Hom}_{\mathfrak{g}}(M, \Gamma_{\mathfrak{g}}(J_M))$$

allows us to conclude that, if $i : M \to J_M$ is an injective homomorphism of $M \in \mathrm{Int}_{\mathfrak{g}}$ into an injective $\mathfrak{g}$-module, then $\Gamma_{\mathfrak{g}}(J_M)$ is an injective object of $\mathrm{Int}_{\mathfrak{g}}$ and i factors through the inclusion $\Gamma_{\mathfrak{g}}(J_M) \subset J_M$. In particular, this argument allows to reduce the existence of injective hulls in $\mathrm{Int}_{\mathfrak{g}}$ to the well-known existence of injective hulls in $\mathfrak{g}$-mod.

With this in mind, we can view Propositions 3.1 and 3.2 as yielding an explicit construction of an injective module $\Gamma_{\mathfrak{g}}(M^*)$ associated to any $M \in \mathrm{Int}_{\mathfrak{g}}$.

In the rest of this section we assume that $\mathfrak{g}$ admits a splitting Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, i.e. an abelian subalgebra $\mathfrak{h} \subset \mathfrak{g}$ such that $\mathfrak{g}$ decomposes as

$$\mathfrak{h} \oplus \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}^{\alpha},$$

where

$$\mathfrak{g}^{\alpha} = \{g \in \mathfrak{g} | [h, g] = \alpha(h) g \text{ for any } h \in \mathfrak{h}\}.$$

It is well-known that in this case $\mathfrak{g}$ is isomorphic to a direct sum of copies of $sl(\infty), o(\infty), sp(\infty)$ and finite-dimensional simple Lie algebras, see [PStr].

We define the category $\mathrm{Int}^{\mathrm{wt}}_{\mathfrak{g},\mathfrak{h}}$ as the full subcategory of $\mathrm{Int}_{\mathfrak{g}}$ which consists of *weight modules* M, i.e. objects $M \in \mathrm{Int}_{\mathfrak{g}}$ which admit a decomposition

$$M = \bigoplus_{\alpha \in \mathfrak{h}^*} M^{\alpha}, \tag{4}$$

where

$$M^{\alpha} = \{m \in M | h \cdot m = \alpha(h) m \text{ for any } h \in \mathfrak{h}\}.$$

Note that (4) is automatically a decomposition of $\mathfrak{h}$-modules. It is also clear that there is a left exact functor

$$\Gamma^{\mathrm{wt}}_{\mathfrak{h}} : \mathrm{Int}_{\mathfrak{g}} \rightsquigarrow \mathrm{Int}^{\mathrm{wt}}_{\mathfrak{g},\mathfrak{h}}, \ M \mapsto \bigoplus_{\alpha \in \mathfrak{h}^*} M^{\alpha}.$$

By $\Gamma^{\mathrm{wt}}_{\mathfrak{g},\mathfrak{h}}$ we denote the composition

$$\Gamma^{\mathrm{wt}}_{\mathfrak{h}} \circ \Gamma_{\mathfrak{g}} : \mathfrak{g}\text{-mod} \rightsquigarrow \mathrm{Int}^{\mathrm{wt}}_{\mathfrak{g},\mathfrak{h}}.$$

Lemma 3.4. *If X is an injective object of $\mathrm{Int}_{\mathfrak{g}}$, then $\Gamma^{\mathrm{wt}}_{\mathfrak{h}}(X)$ is an injective object of $\mathrm{Int}^{\mathrm{wt}}_{\mathfrak{g},\mathfrak{h}}$.*

Proof. It suffices to note that $\Gamma^{\mathrm{wt}}_{\mathfrak{g}}$ is a right adjoint to the inclusion functor $\mathrm{Int}^{\mathrm{wt}}_{\mathfrak{g},\mathfrak{h}} \subset \mathrm{Int}_{\mathfrak{g}}$. □

Example 3.5. Let $\mathfrak{g} = sl(\infty)$ and $M = V \otimes V_*$. Consider the $\mathfrak{g}$-module M^*. Let's think of $M^* = (V \otimes V_*)^*$ as the space of all infinite matrices $B = (b_{ij}), i, j \in \mathbb{Z}_{>0}$, and of M as the space of finitary infinite matrices $A = (a_{ij}), i, j \in \mathbb{Z}_{>0}$, where $B(A) = \sum_{i,j} b_{ij} a_{ji}$. Then $\mathfrak{g}$ is identified with the subspace $F \subset (V \otimes V_*)^*$ of finitary matrices with trace zero, and the $\mathfrak{g}$-module structure on M^* is given by $A \cdot B = [A, B]$. We fix the Cartan subalgebra $\mathfrak{h}$ to be the algebra of finitary diagonal matrices, and we claim that $\Gamma^{\mathrm{wt}}_{\mathfrak{h}}(M^*) = F + D$ where D is the subspace of diagonal matrices. Indeed, clearly D equals the $\mathfrak{h}$-weight space $(M^*)^0$ of weight 0. Furthermore, any non-zero eigenspace of $\mathfrak{h}$ is the span of an elementary non-diagonal matrix, hence $\Gamma^{\mathrm{wt}}_{\mathfrak{h}}(M^*) = F + D$. Note also that we have a non-splitting exact sequence of $\mathfrak{g}$-modules

$$0 \to \mathfrak{g} \to \Gamma^{\mathrm{wt}}_{\mathfrak{h}}(M^*) \to T \to 0,$$

where $T = D/D \cap F$ is a trivial $\mathfrak{g}$-module of dimension $\beth_1$.

Corollary 3.6. *For any* $M \in \mathrm{Int}_{\mathfrak{g}}$, $\Gamma^{\mathrm{wt}}_{\mathfrak{g},\mathfrak{h}}(M^*)$ *is an injective object of* $\mathrm{Int}^{\mathrm{wt}}_{\mathfrak{g},\mathfrak{h}}$.

Define now $\mathrm{Int}^{\mathrm{fin}}_{\mathfrak{g},\mathfrak{h}}$ as the full subcategory of $\mathrm{Int}^{\mathrm{wt}}_{\mathfrak{g},\mathfrak{h}}$ consisting of $\mathfrak{h}$-weight modules $M = \bigoplus_{\alpha\in\mathfrak{h}^*} M^\alpha$ such that $\dim M^\alpha < \infty$ for any $\alpha \in \mathfrak{h}^*$.

Theorem 3.7. *The category* $\mathrm{Int}^{\mathrm{fin}}_{\mathfrak{g},\mathfrak{h}}$ *is semisimple.*

Proof. Let $M \in \mathrm{Int}^{\mathrm{fin}}_{\mathfrak{g},\mathfrak{h}}$ be simple. Then there is an $\mathfrak{h}$-module isomorphism

$$M = \oplus_{\alpha\in\mathfrak{h}^*} M^\alpha.$$

Therefore $M^* = \prod_{\alpha\in\mathfrak{h}^*}(M^\alpha)^*$. A non-difficult computation shows that $\Gamma^{\mathrm{wt}}_{\mathfrak{h}}(M^*)$ is isomorphic to $\oplus_{\alpha\in\mathfrak{h}^*}(M_\alpha)^*$. Moreover, using the fact that $\dim M^\alpha < \infty$ for all α, it is easy to check that $M_* := \oplus_{\alpha\in\mathfrak{h}^*}(M_\alpha)^*$ is a simple integrable $\mathfrak{g}$-module. Hence $M_* = \Gamma^{\mathrm{wt}}_{\mathfrak{g},\mathfrak{h}}(M^*)$. Applying $\Gamma^{\mathrm{wt}}_{\mathfrak{g},\mathfrak{h}}$ again, we see that

$$\Gamma^{\mathrm{wt}}_{\mathfrak{g},\mathfrak{h}}(\Gamma^{\mathrm{wt}}_{\mathfrak{g},\mathfrak{h}}(M^*)^*) = M.$$

Therefore M is injective in $\mathrm{Int}^{\mathrm{wt}}_{\mathfrak{g},\mathfrak{h}}$, and thus also in $\mathrm{Int}^{\mathrm{fin}}_{\mathfrak{g},\mathfrak{h}}$, by Corollary 3.6. $\square$

Example 3.8.

a) Let $\mathfrak{g} = sl(\infty)$. One checks immediately that all tensor powers $V^{\otimes k}$, V being the natural module, are objects of $\mathrm{Int}^{\mathrm{fin}}_{\mathfrak{g},\mathfrak{h}}$. The same applies to the tensor powers of the conatural module V_*. However, the category $\mathrm{Int}^{\mathrm{fin}}_{\mathfrak{g},\mathfrak{h}}$ contains also more interesting modules as the following one: $M = \varinjlim S^i(V_i)$, V_i being the natural representation of $sl(i)$. The module M has 1-dimensional weight spaces, but is not a highest weight module, see [DP1, Example 3]. Note also that the adjoint representation is not an object of $\mathrm{Int}^{\mathrm{fin}}_{\mathfrak{g},\mathfrak{h}}$.

b) Let $\mathfrak{g} = o(\infty)$ and let $\mathfrak{g}$ be exhausted by $\mathfrak{g}_i = o(2i)$, $i \geq 3$. Denote by S^1_i and S^2_i the two non-isomorphic spinor $\mathfrak{g}_i$-modules. Then S^1_i and S^2_i are both isomorphic to $S^1_{i-1} \oplus S^2_{i-1}$ as $\mathfrak{g}_{i-1}$-modules. Therefore there is an injective homomorphism of $\mathfrak{g}_{i-1}$-modules $\varphi^{ks}_{i-1} : S^k_{i-1} \to S^s_i$ for $k, s \in \{1, 2\}$, and moreover φ^{ks}_{i-1} is unique up to proportionality. Any sequence $\{t_i\}_{i\geq 3}$ of elements in $\{1, 2\}$ defines a direct system

$$S^{t_3}_3 \xrightarrow{\varphi^{t_3,t_4}_3} S^{t_4}_4 \xrightarrow{\varphi^{t_4,t_5}_4} S^{t_5}_5 \xrightarrow{\varphi^{t_5,t_6}_5} \dots$$

and hence a simple $\mathfrak{g}$-module $S(\{t_i\})$. Using the fact that $S(\{t_i\})$ is locally simple, it is easy to see that $S(\{t_i\}) = S(\{t'_i\})$ if and only if the "tails" of the sequence $\{t_i\}$ and $\{t'_i\}$ coincide, i.e. $t_i = t'_i$ for large enough i.

The modules $S(\{t_i\})$ are weight modules with 1-dimensional spaces for any Cartan subalgebra $\mathfrak{h}$ of the form $\mathfrak{h} = \cup_i\mathfrak{h}_i$ where $\mathfrak{h}_3 \subset \mathfrak{h}_4 \subset \dots$ are nested Cartan subalgebras of $\mathfrak{g}_3 = o(6) \subset \mathfrak{g}_4 = o(8) \subset \dots$. In particular, $S(\{t_i\}) \in \mathrm{Int}^{\mathrm{fin}}_{\mathfrak{g},\mathfrak{h}}$.

4. On the integrability of M^* for $M \in \mathrm{Int}_{\mathfrak{g}}$

Lemma 4.1. *Let $M \in \mathrm{Int}_{\mathfrak{g}}$. Then $M^* \in \mathrm{Int}_{\mathfrak{g}}$ if and only if for any $i > 0$ $\mathrm{Hom}_{\mathfrak{g}_i}(N, M) \neq 0$ only for finitely many non-isomorphic simple $\mathfrak{g}_i$-modules N.*

Proof. Fix i. Let Λ_i be the set of integral dominant weights of $\mathfrak{g}_i$ (for some fixed Borel subalgebra $\mathfrak{b}_i$ of $\mathfrak{g}_i$ with fixed Cartan subalgebra $\mathfrak{h}_i \subset \mathfrak{b}_i$) and V^i_λ be the simple $\mathfrak{g}_i$-module with highest weight λ. Denote by $\Lambda_i(M)$ the set of all $\lambda \in \Lambda_i$ such that $\mathrm{Hom}_{\mathfrak{g}_i}(V^i_\lambda, M) \neq 0$. Since M is a semisimple $\mathfrak{g}_i$-module, we can write M as

$$M = \oplus_{\lambda \in \Lambda_i(M)} M^\lambda \otimes V^i_\lambda,$$

where $M^\lambda := \mathrm{Hom}_{\mathfrak{g}_i}(V^i_\lambda, M)$ is a trivial $\mathfrak{g}_i$-module. We have

$$M^* = \prod_{\lambda \in \Lambda_i(M)} (V^i_\lambda)^* \otimes (M^\lambda)^*.$$

Suppose that $\Lambda_i(M)$ is finite. Then for any fixed $g \in \mathfrak{g}_i$ there is a polynomial $p_\lambda(z)$ such that $p_\lambda(g) \cdot (V^i_\lambda)^* = 0$. Set $p(z) := \prod_{\lambda \in \Lambda_i(M)} p_\lambda(z)$. Then $p(g) \cdot M^* = 0$. Hence g acts integrably on M^*, i.e. M^* is integrable over $\mathfrak{g}_i$.

Now let $\Lambda_i(M)$ be infinite. Let v_λ be a non-zero vector of weight $-\lambda$ in $(V^i_\lambda)^* \otimes (M^\lambda)^*$. One can choose h in the Cartan subalgebra of $\mathfrak{g}_i$ such that $\lambda(h) \neq \mu(h)$ for any $\mu \neq \lambda \in \Lambda_i(M)$. Let $v := \prod_{\lambda \in \Lambda_i(M)} (v_\lambda) \in \prod_{\lambda \in \Lambda_i(M)} (V^i_\lambda)^* \otimes (M^\lambda)^*$. Then $\dim(\mathbb{C}[h] \cdot v) = \infty$, and M^* is not $\mathfrak{g}_i$-integrable. □

Corollary 4.2. *Let $M, M' \in \mathrm{Int}_{\mathfrak{g}}$. If $M^*, (M')^* \in \mathrm{Int}_{\mathfrak{g}}$, then $(M \otimes M')^* \in \mathrm{Int}_{\mathfrak{g}}$ and $M^{**} \in \mathrm{Int}_{\mathfrak{g}}$.*

Proposition 4.3. *Let $\mathfrak{g}$ be a locally simple Lie algebra. There exists a non-trivial module $M \in \mathrm{Int}_{\mathfrak{g}}$ such that M^* is integrable if and only if $\mathfrak{g}$ is diagonal.*

Proof. First of all, if $\mathfrak{g}$ is diagonal, then any natural module $V = \varinjlim V_n$ satisfies the finiteness condition of Lemma 4.1, hence V^* is integrable.

Before we prove the other direction, note that, by passing to a subexhaustion, we can always assume that $\mathfrak{g}$ is exhausted by classical simple Lie algebras $\mathfrak{g}_i$ of the same type (A, B, C or D). Let now $M \in \mathrm{Int}_{\mathfrak{g}}$ be a non-trivial and M^* be integrable. We will show that $\mathfrak{g}$ is diagonal. Since M satisfies the finiteness condition of Lemma 4.1, $\mathrm{End}_{\mathbb{C}} M$ and its submodules satisfy this condition too. The adjoint module $\mathfrak{g}$ is a submodule of $\mathrm{End}_{\mathbb{C}} M$, hence this implies that for each i the number of $\mathfrak{g}_i$-isotypic components in $\mathfrak{g}_{i+k}$ is uniformly bounded for all $k > 0$. Since the adjoint module of $\mathfrak{g}_i$ is isomorphic to $(V_i \otimes V_i^*)/\mathbb{C}$ in the type A case, to $S^2(V_i)$ in type C, and to $\Lambda^2(V_i)$ in types B or D, one can easily check that for each i the number of $\mathfrak{g}_i$-isotypic components in V_{i+k} is also uniformly bounded by for all $k > 0$. Our goal is to show that for all sufficiently large i, V_{i+1} restricted to $\mathfrak{g}_i$ is isomorphic to a direct sum of copies of V_i, V_i^* and $\mathbb{C}$.

Let us start with the type A case. Pick an $sl(2)$-subalgebra in $\mathfrak{g}_n$ for some n. The set of $sl(2)$-weights in V is finite. Thus we can let $k \in \mathbb{Z}_{>0}$ be the maximal weight in this set and fix i such that k is a weight of V_i. Note that $sl(2) \subset \mathfrak{g}_i$. Then we have an isomorphism of $\mathfrak{g}_i$-modules

$$V_{i+1} = T_{\lambda_1}(V_i) \oplus \cdots \oplus T_{\lambda_s}(V_i),$$

where each λ_j is a Young diagram and $T_{\lambda_j}(V_i)$ is the image of the corresponding Young projector in the appropriate tensor power of V_i. Since V_{i+1} does not have any weight greater than k, each diagram λ_j has only one column. Indeed, otherwise we can put a vector of weight k in each box of the first row and put other weight vectors in all other boxes of λ_j so that the total sum of all weights of vectors is greater than k, which contradicts the fact that k is the maximal weight. Next we claim that the length of this column equals $0, 1, \dim V_i$, or $\dim V_i - 1$. Indeed, if we put in the boxes of λ_i linearly independent vectors of maximal possible sum of weights, the total sum is not greater than k only in these four cases. Hence each simple $\mathfrak{g}_i$-constituent of V_{i+1} is isomorphic to V_i, V_i^* or $\mathbb{C}$ (the numbers 0 and $\dim V_i$ correspond both to the trivial 1-dimensional $\mathfrak{g}_i$-module).

If each $\mathfrak{g}_i$ is of type B or C, D, let $\mathfrak{s}_i \subset \mathfrak{g}_i$ be a maximal root subalgebra of type A. Notice that by the previous argument the restriction of V_{i+1} on $\mathfrak{s}_i$ is a sum of natural, conatural and trivial modules. That is only possible if the restriction of V_{i+1} to $\mathfrak{g}_i$ is a sum of natural and trivial modules. □

Proposition 4.3 follows also from Corollary 3.9 in [Ba2].

Example 4.4.

a) Let $\mathfrak{g} = sl(\infty)$, and let $M = \varinjlim S^i(V_i)$ be as in Example 3.8, a). Then $\mathrm{Hom}_{\mathfrak{g}_i}(S^k(V_i), S^j(V_j)) \neq 0$ for all i, $k \leq j$. Hence $\mathrm{Hom}_{\mathfrak{g}_i}(S^k(V_i), M) \neq 0$ for all $k > 0$, and by Lemma 4.1 M^* is not an object of $\mathrm{Int}_{\mathfrak{g}}$.

b) Consider the case $\mathfrak{g} = o(\infty)$ and let $S(\{t_i\})$ be the $\mathfrak{g}$-module defined in Example 3.8, b). Then if N is a simple $\mathfrak{g}_i$-module, $\mathrm{Hom}_{\mathfrak{g}_i}(N, S(\{t_i\})) \neq 0$ iff $N \simeq S_i^1$ or $N \simeq S_i^2$. Hence $S(\{t_i\})^* \in \mathrm{Int}_{\mathfrak{g}}$ by Lemma 4.1. Moreover, $S(\{t_i\})^*$ is injective by Proposition 3.2.

c) Let $\mathfrak{g} = sl(\infty)$ and let M be as in Example 3.5. Then $\mathrm{Hom}_{\mathfrak{g}_i}(N, M) \neq 0$ if N is isomorphic to one of the following simple $\mathfrak{g}_i$-modules: trivial, natural, conatural, adjoint. Therefore M^* is $\mathfrak{g}$-integrable and injective in $\mathrm{Int}_{\mathfrak{g}}$. Furthermore, $M^* \cong \mathbb{C} \oplus \mathfrak{g}^*$.

5. On the Loewy length of $\Gamma_{\mathfrak{g}}(M^*)$ for $M \in \mathrm{Int}_{\mathfrak{g}}$

Recall that the *socle*, $\mathrm{soc}(M)$, of a $\mathfrak{g}$-module M is the largest semisimple submodule of M. The *socle filtration* of M is the filtration of $\mathfrak{g}$-modules

$$0 \subset \mathrm{soc}(M) \subset \mathrm{soc}^1(M) \subset \cdots \subset \mathrm{soc}^i(M) \subset \ldots,$$

where $\mathrm{soc}^i(M) = p_i^{-1}(\mathrm{soc}(M/\mathrm{soc}^{i-1}(M))$ and $p_i : M \to M/\mathrm{soc}^{i-1}(M)$ is the natural projection. We say the the socle filtration of M is *exhaustive* if $M = \varinjlim(\mathrm{soc}^i(M))$. We say that M has *finite Loewy length* if the socle filtration of

M is finite and exhaustive. The *Loewy length* of M equals $k+1$ where $k = \min\{r \mid \mathrm{soc}^r(M) = M\}$.

Proposition 5.1. *Let $M \in \mathrm{Int}_{\mathfrak{g}}$ be a simple $\mathfrak{g}$-module such that $\Gamma_{\mathfrak{g}}(M^*)$ has finite Loewy length. Then there exist $n \in \mathbb{Z}_{>0}$ and a direct system M_i of simple finite-dimensional $\mathfrak{g}_i$-modules such that $M = \varinjlim M_i$ and $\dim \mathrm{Hom}_{\mathfrak{g}_i}(M_i, M_j) = 1$ for all $j > i > n$.*

We first prove several lemmas.

Lemma 5.2. *Let $Q = \varinjlim Q_i \in \mathrm{Int}_{\mathfrak{g}}$, where Q_i are finite-dimensional, not necessarily simple, $\mathfrak{g}_i$-modules. Assume that for all sufficiently large i there exists a simple $\mathfrak{g}_i$-submodule $X_i \subset Q_i$ such that $\dim \mathrm{Hom}_{\mathfrak{g}_i}(X_i, X_{i+1}) > 2$. Then there exists a locally simple module $X = \varinjlim X_i \in \mathrm{Int}_{\mathfrak{g}}$ and a non-trivial extension of $\mathfrak{g}$-modules*

$$0 \to Q \to Z \to X \to 0.$$

Proof. Fix a sequence of injective homomorphisms of $\mathfrak{g}_i$-modules $f_i : X_i \to X_{i+1}$ and set $X = \varinjlim X_i$. Let $Z_i := X_i \oplus Q_i$ and consider the injective homomorphisms of $\mathfrak{g}_i$-modules

$$a_i : Z_i \to Z_{i+1},\ a_i((x,q)) := (f_i(x), t_i(x) + e_i(q)),$$

where t_i are some injective homomorphisms $X_i \to Q_{i+1}$, $e_i : Q_i \to Q_{i+1}$ are the given inclusions, and $q \in Q_i$, $x \in X_i$. Put $Z := \varinjlim Z_i$.

Then, clearly, Q is a submodule of Z and the quotient Z/Q is isomorphic to X. Thus we have constructed an extension of X by Q. This extension splits if and only if for all sufficiently large i there exist non-zero homomorphisms $p_i : X_i \to Q_i$ such that $t_i = p_{i+1} \circ f_i - e_i \circ p_i$, see the following diagram:

$$\begin{array}{ccc} X_{i+1} & \overset{p_{i+1}}{\to} & Q_{i+1} \\ \uparrow f_i & t_i \nearrow & \uparrow e_i \\ X_i & \overset{p_i}{\to} & Q_i. \end{array}$$

Assume that for any choice of $\{t_i\}$ such a splitting exists. If $n_i := \dim \mathrm{Hom}_{\mathfrak{g}_i}(X_i, Q_i)$, this assumption implies

$$\dim \mathrm{Hom}_{\mathfrak{g}_i}(X_i, Q_{i+1}) \le n_i + n_{i+1}.$$

On the other hand, $\dim \mathrm{Hom}_{\mathfrak{g}_i}(X_i, Q_{i+1}) \ge k_i n_{i+1}$ where $k_i := \dim \mathrm{Hom}_{\mathfrak{g}_i}(X_i, X_{i+1})$. Since $k_i > 2$, we have $n_{i+1} < n_i$. As $n_i > 0$ for all i, we obtain a contradiction. □

Corollary 5.3. *Let $Q \in \mathrm{Int}_{\mathfrak{g}}$ be a simple $\mathfrak{g}$-module satisfying the assumption of Lemma 5.2. Then Q admits no non-zero homomorphism into an injective object of $\mathrm{Int}_{\mathfrak{g}}$ of finite Loewy length.*

Proof. For any $m > 0$ we will now construct an integrable module $Z^{(m)} \supset Q$ whose socle equals Q and whose Loewy length is greater than m. For $m = 1$ this was done in Lemma 5.2. Proceeding by induction, we set

$$Z_i^{(m)} := X_i \oplus Z_i^{(m-1)} = X_i \oplus (X_i \oplus Z_i^{(m-2)})$$

and define $a_i^{(m)} : Z_i^{(m)} \to Z_{i+1}^{(m)}$ by

$$a_i^{(m)}(x, x', z) = (f_i(x), r_i^{(m-1)}(x) + f_i(x'), t_i^{(m-2)}(x') + q_i^{(m-2)}(z)),$$

where now $\{t_i^{(m-2)}\}$ is a set of non-zero homomorphisms $t_i^{(m-2)} : X_i \to Z_{i+1}^{(m-2)}$ and $\{r_i^{(m-1)}\}$ is a set of non-zero homomorphisms $r_i^{(m-1)} : X_i \to X_{i+1}$. As in the proof of Lemma 5.2 one can choose $\{t_i^{(m-2)}\}$ and $\{r_i^{(m-1)}\}$ so that $Z^{(m)}$ is a non-split extension of X by $Z^{(m-1)}$, and $Z^{(m)}/Z^{(m-2)}$ is a non-split self-extension of X. Therefore the Loewy length of $Z^{(m)}$ is greater than m. The statement follows. $\square$

Lemma 5.4. *Let $Q = \varinjlim Q_i \in \mathrm{Int}_{\mathfrak{g}}$ be a simple $\mathfrak{g}$-module which admits a non-zero homomorphism into an injective object of $\mathrm{Int}_{\mathfrak{g}}$ of finite Loewy length. Then there exist $n \in \mathbb{Z}_{>0}$ and a direct system of simple $\mathfrak{g}_i$-submodules S_i of Q such that $Q = \varinjlim S_i$ and $\dim \mathrm{Hom}_{\mathfrak{g}_i}(S_i, S_j) = 1$ for all $j > i > n$.*

Proof. Decompose each Q_i into a direct sum of isotypic components, $Q_i = Q_i^1 \oplus \cdots \oplus Q_i^{l(i)}$. We define a directed graph Γ as follows. The set of vertices $V(\Gamma)$ is by definition $\{Q_i^j\}$, and $V(\Gamma) = \cup_{i>0} V(\Gamma)_i$, where $V(\Gamma)_i = \{Q_i^1, \dots, Q_i^{l(i)}\}$. An edge $A \to B$ belongs to Γ if $A \in V(\Gamma_i)$, $B \in V(\Gamma_{i+1})$ and $\mathrm{Hom}_{\mathfrak{g}_i}(A, B) \neq 0$.

Let $\Gamma_{>i}$ be the full subgraph of Γ whose set of vertices equals $\cup_{k>i} V(\Gamma)_k$. For any vertex A of Γ we denote by $V(A)$ the set of vertices B such that there is a directed path from A to B. Let $\Gamma(A)$ be the full subgraph of Γ whose set of vertices equals $V(A)$, and $\Gamma(A)_{>i}$ be the full subgraph of $\Gamma(A)$ whose set of vertices equals $\cup_{k>i}(V(\Gamma)_k \cap V(A))$. Note that the simplicity of Q implies that $\Gamma_{>i}$ and $\Gamma(A)_{>i}$ are connected. In particular, if $\Gamma(A)$ is a tree, then $\Gamma(A)$ is just a string.

We will now prove that there exists a vertex A such that $\Gamma(A)$ is a tree. Indeed, assume the contrary. This implies that one can find an infinite sequence of vertices $A_1 \in V(\Gamma)_{i_1}, A_2 \in V(\Gamma)_{i_2}, \dots$ such that the number of paths from A_n to A_{n+1} is greater than 2 for all n. Then $Q = \varinjlim Q_{i_k}$. In addition, one can easily see that Q satisfies the assumption of Lemma 5.2 and hence Q admits no non-zero homomorphism into an injective object of $\mathrm{Int}_{\mathfrak{g}}$ of finite Loewy length. Contradiction.

Fix now $A \in V(\Gamma)_i$ such that $\Gamma(A)$ is a tree. Then, as we mentioned above, $V(\Gamma)$ is necessarily a string $A_i = \{A \to A_{i+1} \to A_{i+2} \dots\}$. Let S_j be a simple submodule of A_j, $j \geq i$. Then by Lemma 5.2 there exists n, such that $\dim \mathrm{Hom}_{\mathfrak{g}_j}(S_j, S_k) = 1$ for any $k > j \geq n$. Fix $s \in S_n$ and set $S_j = U(\mathfrak{g}_j) \cdot s$ for all $j \geq n$. Then S_j are simple and $Q = \varinjlim S_j$ satisfies the condition in the lemma. $\square$

Lemma 5.5. *Let $Q = \varinjlim S_i \in \mathrm{Int}_{\mathfrak{g}}$, where S_i are simple $\mathfrak{g}_i$-modules such that, for some n, $\dim \mathrm{Hom}_{\mathfrak{g}_i}(S_i, S_j) = 1$ for all $j > i > n$. Then Q^* has a unique simple submodule Q_*, and $Q_* \in \mathrm{Int}_{\mathfrak{g}}$.*

Proof. The condition on Q implies that $\dim \mathrm{Hom}_{\mathfrak{g}_i}(S_i, Q) = 1$ for all sufficiently large i. Therefore $\dim \mathrm{Hom}_{\mathfrak{g}_i}(S_i^*, Q^*) = 1$ for all sufficiently large i. Note also that $Q_* = \varinjlim S_i^*$ is uniquely defined (as $\dim \mathrm{Hom}_{\mathfrak{g}_i}(S_i, S_{i+1}) = 1$) and is a simple integrable submodule of Q^*. Let S be some simple submodule of Q^*. Since $Q^* = \varprojlim S_i^*$ and $\mathrm{Hom}_{\mathfrak{g}}(S, Q^*) \neq 0$, we have $\mathrm{Hom}_{\mathfrak{g}_i}(S, S_i^*) \neq 0$ for some i. Therefore $S_i^* \subset S$ as the multiplicity of S_i^* in Q^* is 1. This implies $S = Q_*$. □

We are now ready to prove Proposition 5.1.

Proof of Proposition 5.1. Fix $0 \neq m \in M$ and put $M_i := U(\mathfrak{g}_i) \cdot m$. Then, by the simplicity of M, we have $M = \varinjlim M_i$. Since $\Gamma_{\mathfrak{g}}(M^*)$ has finite Loewy length, M^* has a simple submodule Q. By Lemma 5.4, Q satisfies the assumption of Lemma 5.5. The composition of the canonical injection $M \to (M^*)^*$ and the dual map $(M^*)^* \to Q^*$ defines an injective homomorphism $M \to Q^*$. By Lemma 5.5 $M \simeq Q_*$ and, since Q_* also satisfies the assumption of Lemma 5.5, we conclude that the claim of Proposition 5.1 holds for M.□

The following statement is a direct consequence of Proposition 5.1.

Corollary 5.6. *Let $M \in \mathrm{Int}_{\mathfrak{g}}$ be a simple $\mathfrak{g}$-module such that $\Gamma_{\mathfrak{g}}(M^*)$ has finite Loewy length. Then for any sufficiently large i there exists a simple $\mathfrak{g}_i$-module N such that* $\dim \mathrm{Hom}_{\mathfrak{g}_i}(N, M) = 1$.

The next corollary is a direct consequence of Lemma 5.5 and Proposition 5.1.

Corollary 5.7. *Let $M \in \mathrm{Int}_{\mathfrak{g}}$ be a simple $\mathfrak{g}$-module such that $\Gamma_{\mathfrak{g}}(M^*)$ has finite Loewy length. Then M^* has a unique simple submodule M_*, and $M_* \in \mathrm{Int}_{\mathfrak{g}}$.*

Theorem 5.8. *Let $\mathfrak{g}$ be a locally simple algebra which has a non-trivial module M such that M^* is integrable and has finite Loewy length, then $\mathfrak{g}$ is isomorphic to $sl(\infty)$, $o(\infty)$ or $sp(\infty)$.*

Proof. By Proposition 4.3 we know that $\mathfrak{g}$ is diagonal. Assume that $\mathfrak{g}$ is not finitary and there exists M satisfying the conditions of the theorem. Also assume that in the restriction of V_i to $\mathfrak{g}_{i-1}$ there is no costandard module (for types B, C and D it is automatic). Let $\mathfrak{g} = \varinjlim \mathfrak{g}_i$. Fix n and let $\varphi_k : \mathfrak{g}_n \to \mathfrak{g}_{n+k}$ denote the inclusion defined by our fixed exhaustion of $\mathfrak{g}$. Since $\mathfrak{g}$ is diagonal, there exists a root subalgebra $\mathfrak{l}_k \subset \mathfrak{g}_{n+k}$ such that $\mathfrak{l}_k \simeq \mathfrak{g}_n \oplus \cdots \oplus \mathfrak{g}_n$ and $\varphi_k(\mathfrak{g}_n)$ is the diagonal subalgebra in $\mathfrak{l}_k$. Let a_k be the number of simple direct summands in $\mathfrak{l}_k$. Since $\mathfrak{g}$ is not finitary, $a_k \to \infty$.

By Corollary 5.6 $M = \varinjlim M_i$ is a direct limit of simple modules and, by possibly increasing n, we have $\dim \mathrm{Hom}_{\mathfrak{g}_n}(M_n, M_{n+k}) = 1$ for all k. Choose a set of Borel subalgebras $\mathfrak{b}_i \subset \mathfrak{g}_i$ such that $\varphi_k(\mathfrak{b}_n) \subset \mathfrak{b}_{n+k}$. Let h be the highest coroot of $\mathfrak{g}_n$ and let λ be the highest weight of some simple $\mathfrak{l}_k$-constituent L of M_{n+k}. Since M^* is integrable, Lemma 4.1 implies that $\lambda(\varphi_k(h))$ is bounded by some number t. If $h_1, \dots, h_{a_k}$ are the images of $\varphi_k(h)$ in the simple direct summands of $\mathfrak{l}_k$ under the natural projections, we have $\lambda(h_j) \neq 0$ for at most t direct summands. Therefore L isomorphic to an outer tensor product of at most t non-trivial simple $\mathfrak{g}_n$-modules. Since M_{n+k} is invariant under permutation of direct summands of $\mathfrak{l}_k$, we have at least $a_k - t$ simple constituents of M_{n+k} obtained from L by permutation of the simple direct summands of $\mathfrak{l}_k$. Note that all these simple constituents are isomorphic as $\varphi_k(\mathfrak{g}_n)$-modules. Thus the multiplicity of any simple $\varphi_{n+k}(\mathfrak{g}_n)$-module in M_{n+k} is at least $a_k - t$. Since $a_k \to \infty$, this contradicts Proposition 5.1.

The case when the restriction of V_n to $\mathfrak{g}_{n-1}$ contains a costandard simple constituent can be handled by a similar argument which we leave to the reader. $\square$

6. The category $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ for $\mathfrak{g} \simeq sl(\infty), o(\infty), sp(\infty)$

Define $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ as the largest full subcategory of $\mathrm{Int}_{\mathfrak{g}}$ which is closed under algebraic dualization and such that every object in it has finite Loewy length.

It is clear that $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ is closed with respect to finite direct sums, however $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ is not closed with respect to arbitrary direct sums (see Corollary 6.17 below). Note also that, if $\mathfrak{g}$ is finite-dimensional and semisimple, the objects of $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ are integrable modules which have finitely many isotypic components.

It follows from Theorem 5.8 that if $\mathfrak{g}$ is locally simple and $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ contains a non-trivial module, then $\mathfrak{g}$ is finitary. In the rest of this section we assume that $\mathfrak{g} \simeq sl(\infty)$, $o(\infty)$ or $sp(\infty)$.

Set $T^{p,q} := V^{\otimes p} \otimes (V_*)^{\otimes q}$, where V and V_* are respectively the natural and conatural $\mathfrak{g}$-modules ($V_* \simeq V$ when $\mathfrak{g} \simeq o(\infty), sp(\infty)$). The modules $T^{p,q}$ have been studied in [PS]; in particular, $T^{p,q}$ has finite length and is semisimple only if $pq = 0$ for $\mathfrak{g} = sl(\infty)$, and if $p + q \leq 1$ for $\mathfrak{g} = o(\infty), sp(\infty)$. Moreover, the Loewy length of $T^{p,q}$ equals $\min\{p, q\} + 1$ for $\mathfrak{g} = sl(\infty)$ and $[\frac{p+q}{2}] + 1$ for $\mathfrak{g} = o(\infty), sp(\infty)$. A simple module M is called a *simple tensor module* if it is a submodule (or, equivalently, a subquotient) of $T^{p,q}$ for some p, q.

It is well-known that there is a choice of nested Borel subalgebras $\mathfrak{b}_i \subset \mathfrak{g}_i$ such that all simple tensor modules are $\mathfrak{b}$-highest weight modules for $\mathfrak{b} = \varinjlim \mathfrak{b}_i$, see [PS]. (Moreover, the positive roots of any such $\mathfrak{b}$ are not generated by the simple roots of $\mathfrak{b}$. However, in the present paper we will make no further reference to this fact.)

Denote by Θ the set of all highest weights of simple tensor modules. If $\lambda \in \Theta$, by V_λ we denote the simple tensor module with highest weight λ, and, as in section 4, by V_λ^i we denote the simple $\mathfrak{g}_i$-highest weight module with highest weight λ (here λ is considered as a weight of $\mathfrak{g}_i$). It is easy to check (cf [PS]) that every $\lambda \in \Theta$ can be written in the form $\lambda = \sum a_i \gamma_i$ for some finite set $\gamma_1, ..., \gamma_s$ of linearly independent weights of V and some $a_i \in \mathbb{Z}$. We put $|\lambda| := \sum |a_i|$. It is not hard to see that for any k the set of all $|\mu| \leq k$ in Θ is finite. It follows from [PS] that all simple subquotients of $T^{p,q}$ are isomorphic to V_μ with $|\mu| \leq p + q$, and that if V_λ is a submodule in $T^{p,q}$ then $|\lambda| = p + q$.

Note that $(T^{p,q})^*, (T^{p,q})^{**}$, etc., are integrable modules. Indeed, it is easy to see (cf. [PS]) that for any fixed λ and any fixed $i > 0$ the non-vanishing of $\mathrm{Hom}_{\mathfrak{g}_i}(N, V_\lambda)$ for a simple $\mathfrak{g}_i$-module N implies $N \simeq V_\mu^i$ for $|\mu| \leq |\lambda|$. Hence the condition of Lemma 4.1 is satisfied for $T^{p,q}$ for fixed p, q. This shows that $(T^{p,q})^* \in \mathrm{Int}_{\mathfrak{g}}$. By Corollary 4.2, $(T^{p,q})^{**} \in \mathrm{Int}_{\mathfrak{g}}$, etc..

Lemma 6.1. *Fix $p, q \in \mathbb{Z}_{\geq 0}$.*

a) $(T^{p,q})^*$ has finite Loewy length, and all simple subquotients of $(T^{p,q})^*$ are tensor modules of the form V_λ for $|\lambda| \leq p+q$.

b) The direct product $\prod_{f\in\mathcal{F}} T_f^{p,q}$ of any family $\mathcal{F} = \{T_f^{p,q}\}$ of copies of $T^{p,q}$ has finite Loewy length, and all simple subquotients of $\prod_{f\in\mathcal{F}} T_f^{p,q}$ are tensor modules of the form V_λ for $|\lambda| \leq p+q$.

Proof. First we prove b) using induction in $p+q$. The case $p+q=0$ is trivial. If $p+q>0$, without loss of generality we can assume that $p>0$ (if $p=0$ and $q>0$ we replace V by V_* in the argument below). There is a canonical injective homomorphism $U \to \prod_{f\in\mathcal{F}} T_f^{p,q}$, where $U := V \otimes \prod_{f\in\mathcal{F}} T_f^{p-1,q}$, so we can consider U as a submodule of $\prod_{f\in\mathcal{F}} T_f^{p,q}$. By the induction assumption b) holds for $\prod_{f\in\mathcal{F}} T_f^{p-1,q}$. Since $T^{r,s}$ has finite length for all r, s, [PS], this implies that U has finite Loewy length and all simple subquotients of U are simple tensor modules of the form V_λ for $|\lambda| \leq p+q$. The quotient $(\prod_{f\in\mathcal{F}} T_f^{p,q})/U$ is isomorphic to a submodule of $R := \prod_{f\in\mathcal{F}} (V' \otimes T_f^{p-1,q})$, where V' is a copy of the vector space V with trivial $\mathfrak{g}$-module structure. Since $R \simeq \prod_{f\in\mathcal{F}} (\bigoplus_{i\in\mathbb{Z}} T_{f,i}^{p-1,q})$, by the induction assumption b) holds for R. Therefore b) holds for $\prod_{f\in\mathcal{F}} T_f^{p,q}$.

a) To prove that $(T^{p,q})^*$ has finite Loewy length, we consider $U' := V_* \otimes (T^{p-1,q})^*$ as a submodule of $(T^{p,q})^*$. By the induction assumption, U' has finite Loewy length. The quotient $(T^{p,q})^*/U'$ is a submodule of $R' = \prod_{i\in\mathbb{Z}} (T_i^{p-1,q})^*$. The latter $\mathfrak{g}$-module has finite Loewy length by induction assumption and b). The statement about the simple subquotients of $(T^{p,q})^*$ follows by an induction argument similar to the one in the proof of b). This proves a) for $(T^{p,q})^*$. $\square$

Example 6.2.

a) We start with the simplest example. Let $\mathfrak{g} = sl(\infty), o(\infty), sp(\infty)$ and $M = V^* = (T^{1,0})^*$. Then $M \in \widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ by Lemma 6.1. Furthermore, M is an injective object of $\mathrm{Int}_{\mathfrak{g}}$ by Proposition 3.2. It is easy to see that $\mathrm{soc}(M) = V_*$ and that $M/\mathrm{soc}(M) = V^*/V_*$ is a trivial module of cardinality $\beth_1$. Since $\mathrm{soc}(M)$ is simple, M is an injective hull of V_*.

b) Let $\mathfrak{g}$ be as in a) but let $M = V^{**} = (T^{1,0})^{**}$. The exact sequence $0 \to V_* \to V^* \to V^*/V_* \to 0$ yields an exact sequence

$$0 \to (V^*/V_*)^* \to M \to (V_*)^* \to 0. \tag{5}$$

Since $(V^*/V_*)^*$ is a trivial $\mathfrak{g}$-module (cf. a)), it is injective, and hence (5) splits. This yields an isomorphism $M = V^{**} = (V_*)^* \oplus T$, T being a trivial $\mathfrak{g}$-module of cardinality $\beth_2$.

c) Here is a more interesting example. We consider the $\mathfrak{g}$-module M^* where $\mathfrak{g} = \mathrm{sl}(\infty)$ and $M = V \otimes V_* = T^{1,1}$ as in Example 3.5. Recall the notation introduced in Example 3.5. In addition, let Sc be the one-dimensional space of scalar matrices, and F_r (respectively F_c) denote respectively the spaces of matrices with finitely many non-zero rows (resp., columns) (F has codimension 1 in $F_r \cap F_c$). It is important to notice that $\mathfrak{g} \cdot M^* \subset F_r + F_c$.

We first show that $\mathrm{soc}(M^*) = Sc \oplus F = \mathbb{C} \oplus \mathfrak{g}$. It is obvious that $Sc \oplus F \subset \mathrm{soc}(M^*)$. To see that $Sc \oplus F = \mathrm{soc}(M^*)$, let X be any non-trivial simple submodule of $\mathrm{soc}(M^*)$ not lying in $Sc \oplus F$. Consider $0 \neq x \in X$. Then $\mathfrak{g} \cdot x \subset F_r + F_c$. Furthermore, it is easy to check that for any $0 \neq y \in F_r + F_c$, there exists $A \in \mathfrak{g}$ such that $A \cdot y \in F$ and $A \cdot y \neq 0$. Hence $X = F$. Since it is clear that Sc is the largest trivial $\mathfrak{g}$-submodule of M^*, we have shown that $\mathrm{soc}(M^*) = Sc \oplus F$.

We now compute $\mathrm{soc}^1(M^*)$. We claim that $F_r + F_c \subset \mathrm{soc}^1(M^*)$. Since $BA \in F$ for $B \in F_r$, $A \in F$, the action of $\mathfrak{g}$ on F_r/F is simply left multiplication. Using this it is not difficult to establish an isomorphism of $\mathfrak{g}$-modules $F_r/F \simeq \bigoplus_{q \in Q} V_q$, where Q is a family of copies of V of cardinality $2^{\mathbb{Z}}$. Similarly, $F_c/F \simeq \bigoplus_{q \in Q} (V_*)_q$. (It is convenient to think here of V_* as the space of all row vectors each of which have finitely many non-zero entries.) This implies $F_r + F_c \subset \mathrm{soc}^1(M^*)$.

On the other hand $M^*/(F_r + F_c)$ is a trivial $\mathfrak{g}$-module as $\mathfrak{g} \cdot M^* \subset F_r + F_c$. In order to compute $\mathrm{soc}^1(M^*)$ we need to find all $z \in M^*$ such that $\mathfrak{g} \cdot z \subset Sc + F$. A direct computation shows that $\mathfrak{g} \cdot z \in Sc + F$ if and only $z \in J$, J denoting the set of matrices each row and each column of which have finitely many non-zero elements. (In fact, $g \cdot J \subset F$). Thus $\mathrm{soc}^1(M^*) = F_r + F_c + J$, and we obtain the socle filtration of M^*:

$$0 \subset Sc \oplus F \subset F_r + F_c + J \subset M^*.$$

In particular, the Loewy length of M^* equals 3, the irreducible subquotients of M^* up to isomorphism are $\mathbb{C}, V, V_*, \mathfrak{g}$, and all of them occur with multiplicity $2^{\mathbb{Z}}$, except $\mathfrak{g}$ which occurs with multiplicity 1.

Note that M^* is decomposable and is isomorphic to $\mathbb{C} \oplus \mathfrak{g}^*$. As the socle of $\mathfrak{g}^*$ is simple (being isomorphic to $\mathfrak{g}$), $\mathfrak{g}^*$ is indecomposable. Moreover $\mathfrak{g}^*$ is an injective hull of $F = \mathfrak{g}$.

d) We now give an example illustrating statement b) of Lemma 6.1. Let $\mathfrak{g} = sl(\infty), o(\infty), sp(\infty)$ and $M = \prod_{f \in \mathcal{F}} V_f$, $\mathcal{F}$ being an infinite family of copies of the natural module V. Set $M^{\mathrm{fin}} = \{\psi : \mathcal{F} \to V \,|\, \dim(\psi(\mathcal{F})) < \infty\}$. Then M^{fin} is a $\mathfrak{g}$-submodule of M, and $\mathfrak{g} \cdot M \subset M^{\mathrm{fin}}$. Hence M/M^{fin} is a trivial

$\mathfrak{g}$-module. Moreover, $M^{\mathrm{fin}} \simeq \bigoplus_{g\in 2^{\mathcal{F}}} V_g$, where $2^{\mathcal{F}}$ is the set of subsets of $\mathcal{F}$. Indeed, $M^{\mathrm{fin}} = \varinjlim(\prod_{f\in\mathcal{F}}(V^i)_f) = \varinjlim((\prod_{f\in\mathcal{F}}\mathbb{C}_f)\otimes V^i) \cong \varinjlim \bigoplus_{g\in 2^{\mathcal{F}}}(\mathbb{C}_g\otimes V^i) = \varinjlim(\bigotimes_{g\in 2^{\mathcal{F}}}(V^i)_g) = \bigoplus_{g\in 2^{\mathcal{F}}} V_g$.

This yields an exact sequence

$$0 \to \bigoplus_{g\in 2^{\mathcal{F}}} V_g \to M \to T \to 0, \tag{6}$$

T being trivial module of dimension $\operatorname{card} 2^{\mathcal{F}}$. Since M has no non-zero trivial submodules, (6) is in fact the socle filtration of M. Consequently the Loewy length of M equals 2.

Corollary 6.3. *Let $M \in \mathrm{Int}_{\mathfrak{g}}$ have finite Loewy length and all simple subquotients of M be isomorphic to V_λ where $|\lambda|$ is less or equal than a fixed $k \in \mathbb{Z}_{>0}$. Then*

a) for any family $\mathcal{F}$ $\prod_{f\in\mathcal{F}} M_f$ has finite Loewy length and all simple subquotients of $\prod_{f\in\mathcal{F}} M_f$ are isomorphic to V_λ with $|\lambda| \le k$;

b) M^* has finite Loewy length and all simple subquotients of M^* are isomorphic to V_λ with $|\lambda| \le k$;

c) $M \in \widetilde{\mathrm{Tens}}_{\mathfrak{g}}$.

Proof. a) The socle filtration of M induces a finite filtration on $\prod_{f\in\mathcal{F}} M_f$

$$0 \subset \prod_{f\in\mathcal{F}} \mathrm{soc}(M_f) \subset \cdots \subset \prod_{f\in\mathcal{F}} \mathrm{soc}^i(M_f) \subset \cdots \subset \prod_{f\in\mathcal{F}} M_f.$$

Furthermore,

$$\mathrm{soc}^i(M)/\mathrm{soc}^{i-1}(M) \simeq \bigoplus_{|\lambda|\le k} \bigoplus_{g\in\mathcal{F}_\lambda} (V_\lambda)_g \tag{7}$$

for some families $\mathcal{F}_\lambda$. Hence

$$\prod_{f\in\mathcal{F}} (\mathrm{soc}^i(M_f)/\mathrm{soc}^{i-1}(M_f)) \simeq \bigoplus_{|\lambda|\le k} \prod_{f\in\mathcal{F}} (\bigoplus_{g\in\mathcal{F}_\lambda} (V_\lambda)_g)_f.$$

Note that for each λ

$$\prod_{f\in\mathcal{F}} (\bigoplus_{g\in\mathcal{F}_\lambda} (V_\lambda)_g)_f \subset \prod_{(f,g)\in\mathcal{F}\times\mathcal{F}_\lambda} (V_\lambda)_{(f,g)}.$$

By Lemma 6.1 b), $\prod_{(f,g)\in\mathcal{F}\times\mathcal{F}_\lambda}(V_\lambda)_{(f,g)}$ has finite Loewy length and all its simple subquotients are isomorphic ot V_μ with $|\mu| \le |\lambda| \le k$. The same holds for $\prod_{f\in\mathcal{F}}(\mathrm{soc}^i(M_f)/\mathrm{soc}^{i-1}(M_f))$. Therefore a) holds.

b) Since all V_λ with $|\lambda| \le k$ satisfy the conditions of Lemma 4.1, M satisfies the condition of Lemma 4.1 and therefore $M^* \in \mathrm{Int}_{\mathfrak{g}}$.

The socle filtration of M induces a finite filtration on M^*

$$\cdots \subset (\mathrm{soc}^i(M))^* \subset (\mathrm{soc}^{i-1}(M))^* \subset \cdots .$$

Using (7) we get

$$(\mathrm{soc}^{i-1}(M))^*/(\mathrm{soc}^i(M))^* \simeq \bigoplus_{|\lambda|\leq k} \prod_{g\in\mathcal{F}_\lambda} (V_\lambda^*)_g.$$

By Lemma 6.1 b) V_λ^* has finite Loewy length and its simple subquotients are isomorphic to V_μ with $|\mu| \leq |\lambda|$, hence by a) the same holds for $\prod_{g\in\mathcal{F}_\lambda}(V_\lambda^*)_g$. This implies that b) holds.

c) Note that if M satisfies the assumptions of the corollary, then M^* and all higher duals M^{**} etc, satisfy the the assumptions of the corollary. Hence $M \in \widetilde{\mathrm{Tens}}_{\mathfrak{g}}$. □

Remarkably, there is following abstract characterization of simple tensor modules.

Theorem 6.4. *If $M \in \mathrm{Int}_{\mathfrak{g}}$ is simple and $\Gamma_{\mathfrak{g}}(M^*)$ has finite Loewy length, then M is a simple tensor module.*

Proof. By Proposition 5.1, $M = \varinjlim M_i$ for some $n \in \mathbb{Z}_+$ and simple nested $\mathfrak{g}_i$-submodules $M_i \subset M$ with $\dim \mathrm{Hom}_{\mathfrak{g}_i}(M_i, M) = 1$ for all $i \geq n$. If $\mathfrak{g} = sl(\infty)$, it is useful to consider M as a $gl(\infty)$-module by extending the $sl(i)$-module structure on M_i to a $gl(i)$-module structure in a way compatible with the injections $M_i \to M_{i+1}$. It is easy to see that the condition $\dim \mathrm{Hom}_{\mathfrak{g}_i}(M_i, M) = 1$ for all $i \geq n$ ensures the existence of such an extension. Note, furthermore, that $\dim \mathrm{Hom}_{gl(i)}(M_i, M) = 1$. This allows us to assume that $\mathfrak{g} = gl(\infty)$ and $\mathfrak{g}_i = gl(i)$.

Let now $\mathfrak{c}$ denote the derived subalgebra of the centralizer of $\mathfrak{g}_n$ in $\mathfrak{g}$. Then obviously $\mathfrak{c}$ is a simple finitary Lie algebra whose action on M induces a trivial action on M_n. Hence, as a $\mathfrak{c}$-module, M is isomorphic to a quotient of $U(\mathfrak{g}) \otimes_{U(\mathfrak{c}\oplus\mathfrak{g}_n)} M_n$, or equivalently to a quotient of $S^{\cdot}(\mathfrak{g}/(\mathfrak{c}\oplus\mathfrak{g}_n)) \otimes M_n$. Note that $\mathfrak{g}/(\mathfrak{c}\oplus\mathfrak{g}_n)$, considered as a $\mathfrak{c}$-module has finite length and that its simple subquotients are natural, conatural, and possibly 1-dimensional trivial $\mathfrak{c}$-modules. This implies that every simple $\mathfrak{c}$-subquotient of M is a simple tensor $\mathfrak{c}$-module. In addition, for $i \geq n$, the number of non-zero marks of the highest weight of any simple $\mathfrak{g}_i$-submodule of M is not greater than n plus the multiplicity of the non-trivial simple constituents of the $\mathfrak{g}_n$-module $\mathfrak{g}/(\mathfrak{c}\oplus\mathfrak{g}_n)$. In particular, if λ_i denotes the highest weight of M_i then λ_i has at most $3n$ non-zero marks.

Consider first the case when $\mathfrak{g} = gl(\infty)$. Then every weight λ_i can be written uniquely in the form

$$a_1^i\varepsilon_1 + \cdots + a_k^i\varepsilon_k + b_1^i\varepsilon_{n-k} + \cdots + b_k^i\varepsilon_n$$

for some fixed k, $a_1^i \geq a_2^i \geq \cdots \geq a_k^i \geq 0$ and $0 \geq b_1^i \geq \cdots \geq b_k^i$. We claim that for sufficiently large i the weight stabilizes, i.e. $a_j^i = a_j^{i+1} = \cdots = a_j^p = \ldots$ and $b_j^i = b_j^{i+1} = \cdots = b_j^p = \ldots$ for all j, $1 \leq j \leq k$. Indeed, assume the contrary. Let j be the smallest index such that the sequence $\{a_j^i\}$ does not stabilize. By the branching rule for $gl(m) \subset gl(m+1)$ (see for instance [GW]) the sequence $\{a_j^i\}$ is non-decreasing. Hence there is p such that $a_j^{p+1} > a_j^p$. Set $\mu = \lambda_p + \varepsilon_j$. Then the multiplicity of M_{p-1} in V^p_μ is not zero and the multiplicity of V^p_μ in M_{p+1} is not zero. Since $V^p_\mu \neq M_p$, this shows that the multiplicity of M_{p-1} in M_{p+1} is at least 2. Contradiction. Similarly the sequence $\{b^i_j\}$ stabilizes. As it is easy to see, this is sufficient to conclude that $M \simeq V_\lambda$ for some $\lambda \in \Theta$.

Let $\mathfrak{g} = o(\infty)$ or $sp(\infty)$. In the first case we assume that $\mathfrak{g}_i = o(2i+1)$. Then $\lambda_i = a^i_1\varepsilon_1 + \cdots + a^i_k\varepsilon_k$ for some fixed k and $a^i_1 \geq a^i_2 \geq \cdots \geq a^i_k \geq 0$. The sequence $\{a^i_j\}$ is non-decreasing for every fixed j as follows from the branching laws for the respective pairs $o(2m+1) \subset o(2m+3)$ and $sp(2n) \subset sp(2m+2)$, see [GW]. Then by repeating the argument in the previous paragraph we can prove that $\{a^i_j\}$ stabilizes, and consequently $M \simeq V_\lambda$ for some $\lambda \in \Theta$. □

Corollary 6.3 and Theorem 6.4 show that a simple module $M \in \mathrm{Int}_\mathfrak{g}$ is an object of $\widetilde{\mathrm{Tens}}_\mathfrak{g}$ if and only if $\Gamma_\mathfrak{g}(M^*)$ has finite Loewy length. Below we will use this fact to give an equivalent definition of $\widetilde{\mathrm{Tens}}_\mathfrak{g}$ (Corollary 6.13). Furthermore, it is easy to check (see also [PS]) that for sufficiently large i the simple $\mathfrak{g}_i$-module V^i_λ occurs in Y with multiplicity 1, and all other simple $\mathfrak{g}_i$-constituents have infinite multiplicity and are isomorphic to V^i_μ with $|\mu| < |\lambda|$. In what follows we call this unique $\mathfrak{g}_i$-constituent the *canonical* $\mathfrak{g}_i$*-constituent of* V_λ. Note also that by Corollary 5.7 for each simple object M of $\widetilde{\mathrm{Tens}}_\mathfrak{g}$, M_* is a well-defined simple object in $\widetilde{\mathrm{Tens}}_\mathfrak{g}$. Hence M_* is well defined also for any semisimple object M of $\widetilde{\mathrm{Tens}}_\mathfrak{g}$: if $M = \bigoplus_{\lambda\in\Theta} M^\lambda \otimes V_\lambda$ (M^λ being trivial $\mathfrak{g}$-modules), then $M_* = \bigoplus_{\lambda\in\Theta} M^\lambda \otimes (V_\lambda)_*$. It is clear that $M_* \cong M$ for $\mathfrak{g} \cong o(\infty)$, $sp(\infty)$.

Corollary 6.5. *The simple objects of* $\widetilde{\mathrm{Tens}}_\mathfrak{g}$ *are precisely the simple tensor modules.*

Lemma 6.6. *Let* $M \cong V_\lambda$ *be a simple tensor module. Then* $\mathrm{soc}((M_*)^*) \simeq M$. *If* V_μ *is a subquotient of* $(M_*)^*$ *and* $\mu \neq \lambda$, *then* $|\mu| < |\lambda|$.

Proof. The first statement follows from Corollary 5.7.

The second statement follows immediately from the fact that $\mathrm{Hom}_{\mathfrak{g}_i}(V^i_\mu, (M_*)^*) \neq 0$ implies $|\mu| < |\lambda|$. □

Corollary 6.7. *a) For any simple* $M \in \widetilde{\mathrm{Tens}}_\mathfrak{g}$, $(M_*)^*$ *is an injective hull of* M *in* $\mathrm{Int}_\mathfrak{g}$ *(and hence also in* $\widetilde{\mathrm{Tens}}_\mathfrak{g}$*).*

b) Any indecomposable injective object in $\widetilde{\mathrm{Tens}}_\mathfrak{g}$ *is isomorphic to* M^* *for some simple module* $M \in \widetilde{\mathrm{Tens}}_\mathfrak{g}$. *In particular, any indecomposable injective module is isomorphic to a direct summand of* $(T^{p,q})^*$ *for some* p, q.

c) For any $M \in \widetilde{\mathrm{Tens}}_\mathfrak{g}$, *any injective hull* I_M *of* M *in* $\mathrm{Int}_\mathfrak{g}$ *is an object of* $\widetilde{\mathrm{Tens}}_\mathfrak{g}$.

Proof. a) Follows directly from Proposition 3.2 and Lemma 6.6.

b) To derive b) from a) it suffices to note that an injective module in $\widetilde{\mathrm{Tens}}_\mathfrak{g}$ is indecomposable if and only if it has simple socle.

c) follows from the fact that I_M is isomorphic to a submodule of $\Gamma_{\mathfrak{g}}(M^{**})$, see Corollary 3.3 □

In what follows we set $I_\lambda := ((V_\lambda)_*)^*$.

Corollary 6.8. $\mathrm{End}_{\mathfrak{g}}(I_\lambda) = \mathbb{C}$.

Proof. If $\varphi \in \mathrm{End}_{\mathfrak{g}}(I_\lambda)$, then $\varphi|_{V_\lambda} = c\,\mathrm{Id}$ for $c \in \mathbb{C}$. Therefore $V_\lambda \subset \mathrm{Ker}(\varphi - c\,\mathrm{Id})$. Furthermore, any non-zero $\mathfrak{g}$-submodule of I_λ contains $\mathrm{soc}(I_\lambda) = V_\lambda$, hence $V_\lambda \subset \mathrm{Im}(\varphi - c\,\mathrm{Id})$. This implies $\varphi - c\,\mathrm{Id} = 0$, as otherwise V_λ would be isomorphic to a subquotient of I_λ/V_λ contrary to Lemma 6.6. □

Lemma 6.9. *Let $X, Y, Z, M \in \widetilde{\mathrm{Tens}}_{\mathfrak{g}}$. Assume furthermore that Y is simple, $Y = \mathrm{soc}(M)$, and there exists an exact sequence*

$$0 \to X \to Z \xrightarrow{p} Y \to 0.$$

Then there exists $\tilde{M} \in \mathrm{Int}_{\mathfrak{g}}$ such that $Z \subset \tilde{M}$ and $\tilde{M}/X \simeq M$.

Proof. Let Y_i be the canonical $\mathfrak{g}_i$-constituent of Y. Then $Y = \varinjlim Y_i$. Set $Z_i := p^{-1}(Y_i)$ and $Q_i := Z_i \cap X$. Then $Z_i = Y_i \oplus Q_i$ and there are injective homomorphisms $\varphi_i : Z_i \to Z_{i+1}$

$$\varphi_i(y, q) = (e_i(y), t_i(y) + f_i(q)),\ \ y \in Y_i, q \in Q_i$$

for some non-zero homomorphisms $e_i : Y_i \to Y_{i+1}$, $t_i : Y_i \to Q_{i+1}$ and $f_i : Q_i \to Q_{i+1}$. Clearly, $Z = \varinjlim Z_i$.

On the other hand, $M = \varinjlim M_i$ for some nested finite-dimensional $\mathfrak{g}_i$-submodules $M_i \subset M$ such that $Y_i \subset M_i$. Moreover, $\dim \mathrm{Hom}_{\mathfrak{g}_i}(Y_i, M_i) = 1$ by Lemma 6.6. Therefore, M_i has a unique $\mathfrak{g}_i$-module decomposition $M_i = R_i \oplus Y_i$. The inclusions $\psi_i : M_i \to M_{i+1}$ are given by

$$\psi_i(r, y) = (p_i(r), s_i(r) + e_i(y)),\ \ y \in Y_i, r \in R_i$$

for some non-zero homomorphisms $p_i : R_i \to R_{i+1}$ and $s_i : R_i \to Y_{i+1}$.

Define $\tilde{M}_i := R_i \oplus Y_i \oplus Q_i$ and let $\zeta_i : \tilde{M}_i \to \tilde{M}_{i+1}$ be given by the formula

$$\zeta(r, y, q) = (p_i(r), s_i(r) + e_i(y), t_i(y) + f_i(q)).$$

Set $\tilde{M} := \varinjlim \tilde{M}_i$. It is easy to check that $\tilde{M}$ satisfies the conditions of the lemma. □

Lemma 6.10. *If $\mathrm{Hom}_{\mathfrak{g}}(I_\lambda, I_\mu) \neq 0$, then $|\mu| \leq |\lambda|$. If I is any injective object of $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ and $0 \neq \varphi \in \mathrm{Hom}_{\mathfrak{g}}(I, I_\mu)$, then φ is surjective.*

Proof. The first statement follows immediately from Lemma 6.6.

To prove the second statement put $X = \mathrm{Ker}\varphi$, $Y = V_\mu$, $Z = \varphi^{-1}(Y)$ and $M = I_\mu$. Construct $\tilde{M}$ as in Lemma 6.9. By the injectivity of I, the injective homomorphism $Z \to \tilde{M}$ extends to a homomorphism $\tilde{M} \to I$. The latter induces a homomorphism $\eta : M = I_\mu \to I/X$.

Let now $\bar{\varphi} : I/X \to I_\mu$ denote the injective homomorphism induced by φ. Then it is obvious that $\bar{\varphi} \circ \eta(y) = y$ for any $y \in Y$. By Corollary 6.8, we have $\bar{\varphi} \circ \eta = \mathrm{Id}$. Hence $\bar{\varphi}$ is an isomorphism, i.e. φ is surjective. □

Proposition 6.11. *The Loewy length of I_λ equals $|\lambda| + 1$.*

Proof. By Lemma 6.6 we know that the Loewy length of I_λ is at most $|\lambda| + 1$. We prove equality by induction in $|\lambda|$. Fix $\mu \in \Theta$ such that $|\mu| = |\lambda| - 1$ and $\mathrm{Hom}_{\mathfrak{g}_i}(V^i_\mu, V^{i+1}_\lambda) \neq 0$. We claim that $\mathrm{Ext}^1(V_\mu, V_\lambda) \neq 0$. Indeed, consider non-zero homomorphisms $\varphi_i \in \mathrm{Hom}_{\mathfrak{g}_i}(V^i_\mu, V^{i+1}_\lambda)$. Set $X = \varinjlim X_i$, where $X_i = V^i_\mu \oplus V^i_\lambda$, $q_i : X_i \to X_{i+1}$ is given by $q_i(x, y) = (e_i(x), \varphi_i(x) + f_i(y))$ for $x \in V_\mu$, $y \in V_\lambda$, and $e_i : V^i_\mu \to V^{i+1}_\mu$ and $f_i : V^i_\lambda \to V^{i+1}_\lambda$ denote the fixed inclusions. It is easy to see that X is a non-trivial extension of V_μ by V_λ.

Thus, we have a non-zero homomorphism $I_\lambda \to I_\mu$. By Lemma 6.10, it is surjective. Hence the Loewy length of I_λ is greater or equal to the Loewy length of I_μ plus 1. The statement follows. □

The following theorem strengthens the claim of Corollary 6.3.

Theorem 6.12. *Let $M \in \mathrm{Int}_{\mathfrak{g}}$. Then $M \in \widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ if and only if there exists a finite subset $\Theta_M \subset \Theta$ such that any simple subquotient of M is isomorphic to V_μ for $\mu \in \Theta_M$.*

Proof. Assume that $M \in \widetilde{\mathrm{Tens}}_{\mathfrak{g}}$. It is sufficient to prove the existence of Θ_M for a semisimple M since then the general case follows from Lemma 6.6. Without loss of generality we may assume that $M = \bigoplus_{j \in C} V_{\lambda_j}$, where V_{λ_j} are pairwise non-isomorphic. We claim that if C is infinite, then M^* does not have finite Loewy length. Indeed, M^* contains a submodule isomorphic to $\bigoplus_{j \in C} I_{\mu_j}$, where $V_{\mu_j} = (V_{\lambda_j})_*$. If C is infinite, then $|\mu_j| = |\lambda_j|$ is unbounded and the socle filtration of $\bigoplus_{j \in C} I_{\mu_j}$ is infinite. This proves one direction.

Now assume that M admits a finite set Θ_M as in the statement of the theorem. We claim first that if M' is a quotient of M and $\mathrm{Ext}^1_{\mathfrak{g}}(M', V_\lambda) \neq 0$ for some $\lambda \in \Theta$, then M has a subquotient isomorphic to V_μ for some $\mu < \lambda$. Indeed, by extending the sequence $0 \to V_\lambda \to I_\lambda$ to a minimal injective resolution $0 \to V_\lambda \to I_\lambda \xrightarrow{i} I^1_\lambda \to ...$, we see that there is a non-zero homomorphism $M' \xrightarrow{p} I^1_\lambda$. Furthermore, by the minimality of the resolution, we have $\mathrm{soc}(I^1_\lambda) \subset \mathrm{Im} i$. Hence by Lemma 6.9 every simple constituent of $\mathrm{soc}(I^1_\lambda)$ is of the form V_ν for $\nu < \lambda$. Since $(\mathrm{Im} p) \cap \mathrm{soc}(I^1_\lambda) \neq 0$, some simple constituent of $\mathrm{soc}(I'_\lambda)$ is isomorphic to a subquotient of M' and thus of M.

We show now that M has finite Loewy length. Consider a minimal (with respect to the order $\leq$) weight $\lambda \in \Theta$. The above argument shows that $\mathrm{Ext}^1_{\mathfrak{g}}(M', V_\lambda) = 0$ for any quotient M' of M. This implies that every subquotient of M isomorphic to V_λ is a quotient of M. Hence M admits a surjective homomorphism $\zeta : M \to M_\lambda$, where M_λ is isomorphic to a direct sum of copies of V_λ and $\Theta_{\ker \zeta} = \Theta_M \setminus \{\lambda\}$. By an induction argument we obtain that M has finite Loewy length. Therefore $M \in \widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ by Corollary 6.3 c). □

Corollary 6.13. *A $\mathfrak{g}$-module $M \in \mathrm{Int}_{\mathfrak{g}}$ is an object of $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ if and only if both M and $\Gamma_{\mathfrak{g}}(M^*)$ have finite Loewy length.*

Proof. In one direction the statement is trivial. We need to prove that, if $M \in \mathrm{Int}_{\mathfrak{g}}$ satisfies the above two conditions, then $M^* \in \mathrm{Int}_{\mathfrak{g}}$. For a semisimple M this follows directly from Theorem 6.12 (as we have already pointed out). The argument gets completed by induction on the Loewy length. Let $M \in \mathrm{Int}_{\mathfrak{g}}$ have Loewy length k, and $\Gamma_{\mathfrak{g}}(M^*)$ have finite Loewy length. Consider the homomorphism $\pi : M \to \mathrm{top}(M)$ onto the maximal semisimple quotient $\mathrm{top}(M)$ of M. Then $\Gamma_{\mathfrak{g}}((\mathrm{top}(M))^*) \subset \Gamma_{\mathfrak{g}}(M^*)$, hence $\mathrm{top}(M) \in \widetilde{\mathrm{Tens}}_{\mathfrak{g}}$, i.e. in particular $(\mathrm{top}(M))^* \in \mathrm{Int}_{\mathfrak{g}}$. Therefore there is an exact sequence

$$0 \to (\mathrm{top}(M))^* \to \Gamma_{\mathfrak{g}}(M^*) \to \Gamma_{\mathfrak{g}}((\mathrm{Ker}\pi)^*) \to 0,$$

implying that $\Gamma_{\mathfrak{g}}((\mathrm{Ker}\pi)^*)$ has finite Loewy length. Since the Loewy length of $\mathrm{Ker}\pi$ equals $k-1$, we can conclude that $(\mathrm{Ker}\pi)^* \in \mathrm{Int}_{\mathfrak{g}}$. Hence $\Gamma_{\mathfrak{g}}(M^*) = M^*$. □

Corollary 6.14. *$\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ is a tensor category with respect to $\otimes$.*

Proof. It suffices to show that $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ is closed with respect to $\otimes$. The fact that, if $M \in \widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ and $M' \in \widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ then $M \otimes M' \in \widetilde{\mathrm{Tens}}_{\mathfrak{g}}$, follows immediately from Theorem 6.12. □

The following theorem concerns the structure of injective modules in $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$.

Theorem 6.15. *Any injective module $I \in \widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ has a finite filtration $\{I_j\}$ such that, for each j, I_{j+1}/I_j is isomorphic to a direct sum of copies of I_{μ_j} for some $\mu_j \in \Theta$.*

Proof. We use induction on the length of the filtration. Assume that $0 = I_0 \subset I_1 \subset I_k$ is already constructed. Let $\mathrm{soc}(I/I_k) = \bigoplus_{f\in\mathcal{F}} Y_f$ for a family $\mathcal{F}$ of simple modules Y_f (there are only finitely many non-isomorphic modules among $\{Y_f\}_{f\in\mathcal{F}}$). Denoting by p the projection $\mu_f : I \to I/I_k$, set $X_f := p^{-1}(Y_f)$. By Lemma 6.9, there exists $\tilde{Y_f} \in \mathrm{Int}_{\mathfrak{g}}$ such that $I_k \subset X_f \subset \tilde{Y_f}$ and $\tilde{Y_f}/I_k \simeq I_{\mu_f}$, $\mu_f \in \Theta$ being the highest weight of Y_f. The inclusion $X_f \subset I$ induces a homomorphism $\psi_f : \tilde{Y_f} \to I$. Let $\psi_f : \tilde{Y_f}/I_k \tilde{\to} I_{\mu_f} \to I/I_k$ the corresponding homomorphism of quotients. Then $\bar{\psi} := \bigoplus_{f\in\mathcal{F}} \bar{\psi}_f : \bigoplus_{f\in\mathcal{F}} I_{\mu_f} \to I$ is injective since its restriction to $\mathrm{soc}(\bigoplus_{f\in\mathcal{F}} I_{\mu_f})$ is an isomorphism. This shows that if $I_{k+1} := p^{-1}(\bar{\psi}(\bigoplus_{f\in\mathcal{F}} I_{\mu_f}))$, there is an isomorphism $I_{k+1}/I_k \simeq \bigoplus_{f\in\mathcal{F}} I_{\mu_f}$.

The filtration terminates at a finite step as I has finite Loewy length. □

Example 6.16. *Let $\mathfrak{g} = sl(\infty), o(\infty), sp(\infty)$ and let M be a countable direct sum of copies of V, i.e. $M = \bigoplus_{f\in\mathcal{F}} V_f$, $\mathrm{card}\mathcal{F} = \beth_0$. Then $(M_*)^*$ can be identified with the set of all infinite matrices $\{b_{ij}\}_{i,j\in\mathbb{Z}_{>0}}$, the action of $\mathfrak{g}$ being left multiplication. The socle $\mathrm{soc}((M_*)^*)$ is the space of matrices F_r with finitely many non-zero rows and is isomorphic to $\bigoplus_{g\in 2^{\mathcal{F}}} V_g$. (Note that the module $\prod_{f\in\mathcal{F}} V_f$ considered in*

Example 6.2 d) is a submodule of $(M_*)^*$ *and has the same socle as* $(M_*)^*$*). We thus obtain the diagram*

$$\begin{array}{ccc} \bigoplus_{g\in 2^{\mathcal{F}}} V_g & \subset & (M_*)^* \\ \cup & & \cup \\ M & \subset & I_M \end{array},$$

I_M *being the injective hull of* M *within* $(M_*)^*$*. Moreover,* I_M *is the largest submodule of* $(M_*)^*$ *such that* $\mathfrak{g}\cdot I_M = M$*. A direct computation shows that* I_M *coincides with the space of all matrices with finite rows (i.e. each row has finitely many non-zero entries).*

Note that $I_M \not\simeq \bigoplus_{f\in\mathcal{F}}(I_{\varepsilon_1})_f$ *(*$\varepsilon_1 \in \Theta$ *is the highest weight of* V*). In fact* I_M *has the following filtration as in Theorem 6.15:* $0 \subset \bigoplus_{f\in\mathcal{F}}(I_{\varepsilon_1})_f \subset I_M$*. Here* $I_M/\bigoplus_{f\in\mathcal{F}}(I_{\varepsilon_1})_f$ *is a trivial module of cardinality* $2^{\mathcal{F}}$ *which is interpreted as a direct sum of* $2^{\mathcal{F}}$ *copies of* I_0*.*

For any $k \in \mathbb{Z}_{>0}$ we now define $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}^k$ be the subcategory of modules whose simple quotients are isomorphic to V_μ with $|\mu| \leq k$. Theorem 6.12 and Corollary 6.3 a) imply the following.

Corollary 6.17. *The category* $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}^k$ *is closed under direct products and direct sums.*

Corollary 6.18. *a) The category* $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ *equals the direct limit* $\varinjlim \widetilde{\mathrm{Tens}}_{\mathfrak{g}}^k$*.*

b) If $\{M_f\}_{f\in\mathcal{F}}$ *is an infinite family of objects of* $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$*, then* $\prod_{f\in\mathcal{F}} M_f \in \widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ *(equivalently,* $\bigoplus_{f\in\mathcal{F}} \in \widetilde{\mathrm{Tens}}_{\mathfrak{g}}$*) if and only if there is* k *such that* $M_f \in \widetilde{\mathrm{Tens}}_{\mathfrak{g}}^k$ *for all* $f \in \mathcal{F}$*.*

Proof. a) follows directly from Theorem 6.12.

Consider now $\prod_{f\in\mathcal{F}} M_f$. If $M_f \in \widetilde{\mathrm{Tens}}_{\mathfrak{g}}^k$ for some k, then $\prod_{f\in\mathcal{F}} M_f \in \widetilde{\mathrm{Tens}}_{\mathfrak{g}}^k$ (and thus also $\bigoplus_{f\in\mathcal{F}} M_f \in \widetilde{\mathrm{Tens}}_{\mathfrak{g}}^k$) by Corollary 6.3 a). If no such k exists, then $\bigoplus_{f\in\mathcal{F}} M_f \notin \widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ by Theorem 6.12, hence also $\prod_{f\in\mathcal{F}} M_f \notin \widetilde{\mathrm{Tens}}_{\mathfrak{g}}$. □

Corollary 6.19. *Every object in* $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ *has a finite injective resolution.*

We now introduce the following partial order on Θ: we set $\mu \leq \lambda$ if for any sufficiently large i there exists $j > i$ such that $\mathrm{Hom}_{\mathfrak{g}_i}(V^i_\mu, V^j_\lambda) \neq 0$. If $\mu \leq \lambda$, then $l(\lambda,\mu)$ denotes the length of a maximal chain $\mu < \mu_1 < \cdots < \lambda$ in Θ.

Lemma 6.20. $\mathrm{Ext}^1_{\mathfrak{g}}(V_\mu, V_\lambda) \neq 0$ *if and only if* $\mu < \lambda$*. If* $\mu < \lambda$*,* $\dim \mathrm{Ext}^1_{\mathfrak{g}}(V_\mu, V_\lambda) = 2^{\mathbb{Z}}$*.*

Proof. Assume that there is a non-trivial extension

$$0 \to V_\lambda \to X \to V_\mu \to 0. \tag{8}$$

We will show that $\mu < \lambda$. Let, on the contrary, $\mathrm{Hom}_{\mathfrak{g}_i}(V^i_\mu, V^j_\lambda) = 0$ for all $j > i$. Then $\mathrm{Hom}_{\mathfrak{g}_i}(V^i_\mu, V_\lambda) = 0$. Since $\dim \mathrm{Hom}_{\mathfrak{g}_i}(V^i_\mu, V_\mu) = 1$, we have $\dim \mathrm{Hom}_{\mathfrak{g}_i}(V^i_\mu, X) = 1$. Let $\varphi : V^i_\mu \to X$ be a non-zero homomorphism. Then $U(\mathfrak{g})\cdot\varphi(V^i_\mu) \simeq X$. Therefore φ extends to a homomorphism of $\mathfrak{g}$-modules $V_\mu \to X$, and this yields a splitting of the sequence in (8). Thus, $\mathrm{Ext}^1_{\mathfrak{g}}(V_\mu, V_\lambda) \neq 0$ implies $\mu < \lambda$.

Now let $\mu < \lambda$. Then there exists an infinite sequence $i_1, i_2, \dots$ such that $\mathrm{Hom}_{\mathfrak{g}_{i_j}}(V^{i_j}_\mu, V^{i_{j+1}}_\lambda) \neq 0$ for all j. Consider a sequence of non-zero homomorphisms $\varphi_j \in \mathrm{Hom}_{\mathfrak{g}_{i_j}}(V^{i_j}_\mu, V^{i_{j+1}}_\lambda)$ and set $Z_j := V^{i_j}_\mu \oplus V^{i_j}_\lambda$. Denote by e_j (respectively, f_j) the inclusion $V^{i_j}_\mu \to V^{i_{j+1}}_\mu$ (resp., $V^{i_j}_\lambda \to V^{i_{j+1}}_\lambda$). Define $\psi_j : Z_j \to Z_{j+1}$ by

$$\psi(x, y) = (e_j(x), \varphi_j(x) + f_j(y)),\ x \in V^{i_j}_\mu, y \in V^{i_j}_\lambda.$$

Consider $Z = \varinjlim Z_j$. It is an exercise to check that Z is an extension of V_μ by V_λ, and it does not split if infinitely many $\varphi_j \neq 0$. Hence the dimension of $\mathrm{Ext}^1_{\mathfrak{g}}(V_\mu, V_\lambda)$ is at least $2^{\mathbb{Z}}$. On the other hand, the dimension of $\mathrm{Ext}^1_{\mathfrak{g}}(V_\mu, V_\lambda)$ is bounded by the multiplicity of V_μ in $\mathrm{soc}^1(I_\lambda)/\mathrm{soc}(I_\lambda)$. The dimension of $I_\mu = ((V_\mu)_*)^*$ is $2^{\mathbb{Z}}$, hence the dimension of $\mathrm{Ext}^1_{\mathfrak{g}}(V_\mu, V_\lambda)$ is at most $2^{\mathbb{Z}}$.

To finish the proof just note that $\mathrm{Ext}^1_{\mathfrak{g}}(V_\lambda, V_\lambda) = 0$ by Lemma 6.6. □

Corollary 6.21. *The category* $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ *consists of a single block.*

Proof. According to Lemma 6.20, $\mathrm{Ext}^1_{\mathfrak{g}}(\mathbb{C}, V_\mu) \neq 0$ for any $\mu \in \Theta$. □

Proposition 6.22. *For* $k \in \mathbb{Z}_{>0}$*, set*

$$\Theta^k(\lambda) = \{\mu < \lambda | l(\lambda, \mu) \geq k + 1\}.$$

Then

$$\mathrm{soc}^k(I_\lambda)/\mathrm{soc}^{k-1}(I_\lambda) = \bigoplus_{\mu \in \Theta^k(\lambda)} X^\mu \otimes V_\mu,$$

where each X^μ *is a trivial* $\mathfrak{g}$*-module of dimension* $2^{\mathbb{Z}}$*.*

Proof. For $k = 1$ the statement follows from Lemma 6.20. Now we proceed by induction on k. Note first that if V_μ is a simple constituent of $\mathrm{soc}^k(I_\lambda)/\mathrm{soc}^{k-1}(I_\lambda)$, then, by Lemma 6.20, $\mu < \chi$ for some simple constituent V_χ of $\mathrm{soc}^{k-1}(I_\lambda)/\mathrm{soc}^{k-2}(I_\lambda)$. By the induction assumption, $\chi \in \Theta^{k-1}(\lambda)$. In addition, it is clear that V_μ is a simple constituent of $\mathrm{soc}^k(I_\lambda)/\mathrm{soc}^{k-1}(I_\lambda)$ if and only if there exists a non-zero homomorphism $\varphi : I_\lambda \to I_\mu$, such that $\varphi(\mathrm{soc}^{k-1}(I_\lambda)) = 0$. By Lemma 6.10, φ is surjective, so all simple constituents of $\mathrm{soc}^1(I_\mu)/\mathrm{soc}(I_\mu)$ are also simple constituents of $\mathrm{soc}^k(I_\lambda)/\mathrm{soc}^{k-1}(I_\lambda)$. This implies that V_μ is a simple constituent of $\mathrm{soc}^k(I_\lambda)/\mathrm{soc}^{k-1}(I_\lambda)$ if and only if there exists $\psi \in \Theta^{k-1}(\lambda)$ such that $\mu \in \Theta^1(\psi)$. Since $\mu \in \Theta^1(\psi)$ if and only if $\mu \in \Theta^k(\lambda)$, the statement follows. □

Let $\mathrm{Tens}_{\mathfrak{g}}$ be the full subcategory of $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ consisting of modules M whose cardinality $\mathrm{card}M$ is bounded by $\beth_n$ for some n depending on M.

Theorem 6.23. $\mathrm{Tens}_{\mathfrak{g}}$ *is the unique minimal abelian full subcategory of* $\mathrm{Int}_{\mathfrak{g}}$ *which does not consist of trivial modules only and which is closed under* $\otimes$ *and* *.

Proof. Let $\mathcal{C}$ be a minimal abelian full subcategory of $\mathrm{Int}_{\mathfrak{g}}$ which contains a non-trivial module M and is closed under $\otimes$ and *. We will show that $V \in \mathcal{C}$. Since $\mathrm{End}_{\mathbb{C}}M$ is a $\mathfrak{g}$-submodule of $(M^* \otimes M)^*$ (through the map $\varphi(\psi \otimes m) = \psi(\varphi(m))$ for $m \in M$, $\psi \in M^*$, $\varphi \in \mathrm{End}_{\mathbb{C}}M$), we have $\mathrm{End}_{\mathbb{C}}M \in \mathcal{C}$. Furthermore, the adjoint module $\mathfrak{g}$ is a submodule of $\mathrm{End}_{\mathbb{C}}M$. Hence $\mathfrak{g} \in \mathcal{C}$. Recall that $\mathfrak{g}$ is the socle of $V_* \otimes V$ for $sl(\infty)$, of $\Lambda^2(V)$ for $o(\infty)$, and of $S^2(V)$ for $sp(\infty)$. In all cases it is easy to see that $\mathfrak{g}^*$ contains a subquotient isomorphic to V. Therefore $V \in \mathcal{C}$. In addition, $V_* = \mathrm{soc}(V^*) \in \mathcal{C}$. Therefore $T^{p,q} \in \mathcal{C}$ for all p, q, and $V_\lambda \in \mathcal{C}$ for all $\lambda \in \Theta$. Finally, by Corollary 6.7 a), any $M \in \widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ is a submodule of $(\mathrm{soc}(M)_*)^*$, and the statement follows. □

We conclude this paper with the remark that the category $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$, for $\mathfrak{g} = sl(\infty), o(\infty), sp(\infty)$, is functorial with respect to any homomorphism of locally semisimple Lie algebras $\varphi : \mathfrak{g}' \to \mathfrak{g}$. By this we mean that any $M \in \widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ considered as a $\mathfrak{g}'$-module is an object of $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$.

To prove this, recall that the image of φ', being a locally semisimple subalgebra of $\mathfrak{g}$, is isomorphic to a direct sum of copies of $sl(\infty), o(\infty), sp(\infty)$ and of finite-dimensional simple Lie algebras, [DP2]. Furthermore, the result of [DP2] implies that as $\mathfrak{g}'$-modules both V and V_* have Loewy length at most 2 and that all non-trivial simple constituents of V and V_* are isomorphic to the natural and conatural representations $V_{\mathfrak{s}}$ and $(V_{\mathfrak{s}})_*$ for some simple direct summands $\mathfrak{s}$ of $\varphi(\mathfrak{g}')$ and that all non-trivial constituents occur with finite multiplicity. (The simple trivial representation may occur with up to countable multiplicity in both $\mathrm{soc}(V)$ and $V/\mathrm{soc}(V)$ (respectively, $\mathrm{soc}(V_*)$ and $V_*/\mathrm{soc}(V_*)$.) This allows us to conclude that any single simple object of $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ is an object of $\widetilde{\mathrm{Tens}}_{\varphi(\mathfrak{g}')}$. Hence, by Theorem 6.12, any $M \in \widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ is an object of $\widetilde{\mathrm{Tens}}_{\varphi(\mathfrak{g}')}$.

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