

# Isolated Singularities of Polyharmonic Inequalities

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## Abstract

We study nonnegative classical solutions  $u$  of the polyharmonic inequality

$$-\Delta^m u \geq 0 \quad \text{in } B_1(0) - \{0\} \subset \mathbb{R}^n.$$

We give necessary and sufficient conditions on integers  $n \geq 2$  and  $m \geq 1$  such that these solutions  $u$  satisfy a pointwise a priori bound as  $x \rightarrow 0$ . In this case we show that the optimal bound for  $u$  is

$$u(x) = O(\Gamma(x)) \quad \text{as } x \rightarrow 0$$

where  $\Gamma$  is the fundamental solution of  $-\Delta$  in  $\mathbb{R}^n$ .

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## 1 Introduction

It is easy to show that there does not exist a pointwise a priori bound as  $x \rightarrow 0$  for  $C^2$  nonnegative solutions  $u(x)$  of

$$-\Delta u \geq 0 \quad \text{in } B_1(0) - \{0\} \subset \mathbb{R}^n, \quad n \geq 2. \quad (1.1)$$

That is, given any continuous function  $\psi: (0, 1) \rightarrow (0, \infty)$  there exists a  $C^2$  nonnegative solution  $u(x)$  of (1.1) such that

$$u(x) \neq O(\psi(|x|)) \quad \text{as } x \rightarrow 0.$$

The same is true if the inequality in (1.1) is reversed.

In this paper we study  $C^{2m}$  nonnegative solutions of the polyharmonic inequality

$$-\Delta^m u \geq 0 \quad \text{in } B_1(0) - \{0\} \subset \mathbb{R}^n \quad (1.2)$$

where  $n \geq 2$  and  $m \geq 1$  are integers. We obtain the following result.

**Theorem 1.1.** *A necessary and sufficient condition on integers  $n \geq 2$  and  $m \geq 1$  such that  $C^{2m}$  nonnegative solutions  $u(x)$  of (1.2) satisfy a pointwise a priori bound as  $x \rightarrow 0$  is that*

$$\text{either } m \text{ is even or } n < 2m. \quad (1.3)$$

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In this case, the optimal bound for  $u$  is

$$u(x) = O(\Gamma_0(x)) \quad \text{as } x \rightarrow 0, \quad (1.4)$$

where

$$\Gamma_0(x) = \begin{cases} |x|^{2-n} & \text{if } n \geq 3 \\ \log \frac{5}{|x|} & \text{if } n = 2. \end{cases} \quad (1.5)$$

The  $m$ -Kelvin transform of a function  $u(x)$ ,  $x \in \Omega \subset \mathbb{R}^n - \{0\}$ , is defined by

$$v(y) = |x|^{n-2m}u(x) \quad \text{where } x = y/|y|^2. \quad (1.6)$$

By direct computation,  $v(y)$  satisfies

$$\Delta^m v(y) = |x|^{n+2m} \Delta^m u(x). \quad (1.7)$$

See [15, p. 221] or [16, p. 660]. This fact and Theorem 1.1 immediately imply the following result.

**Theorem 1.2.** *A necessary and sufficient condition on integers  $n \geq 2$  and  $m \geq 1$  such that  $C^{2m}$  nonnegative solutions  $v(y)$  of*

$$-\Delta^m v \geq 0 \quad \text{in } \mathbb{R}^n - B_1(0)$$

*satisfy a pointwise a priori bound as  $|y| \rightarrow \infty$  is that (1.3) holds. In this case, the optimal bound for  $v$  is*

$$v(y) = O(\Gamma_\infty(y)) \quad \text{as } |y| \rightarrow \infty \quad (1.8)$$

where

$$\Gamma_\infty(y) = \begin{cases} |y|^{2m-2} & \text{if } n \geq 3 \\ |y|^{2m-2} \log(5|y|) & \text{if } n = 2. \end{cases} \quad (1.9)$$

The estimates (1.4) and (1.8) are optimal because  $\Delta^m \Gamma_0 = 0 = \Delta^m \Gamma_\infty$  in  $\mathbb{R}^n - \{0\}$ .

The sufficiency of condition (1.3) in Theorem 1.1 and the estimate (1.4) are an immediate consequence of the following theorem, which gives for  $C^{2m}$  nonnegative solutions  $u$  of (1.2) one sided estimates for  $\Delta^\sigma u$ ,  $\sigma = 0, 1, 2, \dots, m$ , and estimates for  $|D^\beta u|$  for certain multi-indices  $\beta$ .

**Theorem 1.3.** *Let  $u(x)$  be a  $C^{2m}$  nonnegative solution of*

$$-\Delta^m u \geq 0 \quad \text{in } B_2(0) - \{0\} \subset \mathbb{R}^n, \quad (1.10)$$

*where  $n \geq 2$  and  $m \geq 1$  are integers. Then for each nonnegative integer  $\sigma \leq m$  we have*

$$(-1)^{m+\sigma} \Delta^\sigma u(x) \leq C \left| \frac{d^{2\sigma}}{d|x|^{2\sigma}} \Gamma_0(|x|) \right| \quad \text{for } 0 < |x| < 1 \quad (1.11)$$

*where  $\Gamma_0$  is given by (1.5) and  $C$  is a positive constant independent of  $x$ .*

*Moreover, if  $n < 2m$  and  $\beta$  is a multi-index then*

$$|D^\beta u(x)| = O \left( \left| \frac{d^{|\beta|}}{d|x|^{|\beta|}} \Gamma_0(|x|) \right| \right) \quad \text{as } x \rightarrow 0 \quad (1.12)$$

*for*

$$|\beta| \leq \begin{cases} 2m - n & \text{if } n \text{ is odd} \\ 2m - n - 1 & \text{if } n \text{ is even.} \end{cases} \quad (1.13)$$

There is a similar result when the singularity is at infinity.

**Theorem 1.4.** *Let  $v(y)$  be a  $C^{2m}$  nonnegative solution of*

$$-\Delta^m v \geq 0 \quad \text{in } \mathbb{R}^n - B_{1/2}(0), \quad (1.14)$$

where  $n \geq 2$  and  $m \geq 1$  are integers. Then for each nonnegative integer  $\sigma \leq m$  we have

$$(-1)^{m+\sigma} \Delta^\sigma (|y|^{2\sigma-2m} v(y)) \leq C \begin{cases} |y|^{-2} \log 5|y| & \text{if } \sigma = 0 \text{ and } n = 2 \\ |y|^{-2} & \text{if } \sigma \geq 1 \text{ or } n \geq 3 \end{cases} \quad \text{for } |y| > 1 \quad (1.15)$$

where  $C$  is a positive constant independent of  $y$ .

Moreover, if  $n < 2m$  and  $\beta$  is a multi-index satisfying (1.13) then

$$|D^\beta v(y)| = O \left( \left| \frac{d^{|\beta|}}{d|y|^{|\beta|}} \Gamma_\infty(|y|) \right| \right) \quad \text{as } |y| \rightarrow \infty \quad (1.16)$$

where  $\Gamma_\infty$  is given by (1.9).

Note that in Theorems 1.3 and 1.4 we do not require that  $m$  and  $n$  satisfy (1.3).

Inequality (1.15) gives one sided estimates for  $\Delta^\sigma (|y|^{2\sigma-2m} v(y))$ . Sometimes one sided estimates for  $\Delta^\sigma v$  also hold. For example, in the important case  $m = 2$ ,  $n = 2$  or  $3$ , and the singularity is at the infinity, we have the following corollary of Theorem 1.4.

**Corollary 1.1.** *Let  $v(y)$  be a  $C^4$  nonnegative solution of*

$$-\Delta^2 v \geq 0 \quad \text{in } \mathbb{R}^n - B_{1/2}(0)$$

where  $n = 2$  or  $3$ . Then

$$v(y) = O(\Gamma_\infty(|y|)) \quad \text{and} \quad |\nabla v(y)| = O \left( \left| \frac{d}{d|y|} \Gamma_\infty(|y|) \right| \right) \quad \text{as } |y| \rightarrow \infty \quad (1.17)$$

and

$$-\Delta v(y) < C \left| \frac{d^2}{d|y|^2} \Gamma_\infty(|y|) \right| \quad \text{for } |y| > 1 \quad (1.18)$$

where  $\Gamma_\infty$  is given by (1.9) and  $C$  is a positive constant independent of  $y$ .

The proof of Theorem 1.3 relies heavily on a representation formula for  $C^{2m}$  nonnegative solutions  $u$  of (1.2), which we state and prove in Section 3. This formula, which is valid for all integers  $n \geq 2$  and  $m \geq 1$  and which when  $m = 1$  is essentially a result of Brezis and Lions [2], may also be useful for studying nonnegative solutions in a punctured neighborhood of the origin—or near  $x = \infty$  via the  $m$ -Kelvin transform—of problems of the form

$$-\Delta^m u = f(x, u) \quad \text{or} \quad 0 \leq -\Delta^m u \leq f(x, u) \quad (1.19)$$

when  $f$  is a nonnegative function and  $m$  and  $n$  may or may not satisfy (1.3). Examples of such problems can be found in [4, 5, 9, 11, 12, 15, 16] and elsewhere.

Pointwise estimates at  $x = \infty$  of solutions  $u$  of problems (1.19) can be crucial for proving existence results for entire solutions of (1.19) which in turn can be used to obtain, via scaling methods, existence and estimates of solutions of boundary value problems associated with (1.19), see e.g. [13, 14]. An excellent reference for polyharmonic boundary value problems is [8].

Lastly, weak solutions of  $\Delta^m u = \mu$ , where  $\mu$  is a measure on a subset of  $\mathbb{R}^n$ , have been studied in [3] and [6], and removable isolated singularities of  $\Delta^m u = 0$  have been studied in [11].

## 2 Preliminary results

In this section we state and prove four lemmas. Lemmas 2.1, 2.2, and 2.3 will only be used to prove Lemma 2.4, which in turn will be used in Section 3 to prove Theorem 3.1.

Lemmas 2.1 and 2.2 are well-known. We include their very short proofs for the convenience of the reader.

**Lemma 2.1.** *Let  $f: (0, r_2] \rightarrow [0, \infty)$  be a continuous function where  $r_2$  is a finite positive constant. Suppose  $n \geq 2$  is an integer and the equation*

$$v'' + \frac{n-1}{r}v' = -f(r) \quad 0 < r < r_2 \quad (2.1)$$

has a nonnegative solution  $v(r)$ . Then

$$\int_0^{r_2} r^{n-1}f(r) dr < \infty. \quad (2.2)$$

*Proof.* Let  $r_1 = r_2/2$ . Integrating (2.1) we obtain

$$r^{n-1}v'(r) = r_1^{n-1}v'(r_1) + \int_r^{r_1} \rho^{n-1}f(\rho) d\rho \quad \text{for } 0 < r < r_1. \quad (2.3)$$

Suppose for contradiction that

$$r_1^{n-1}v'(r_1) + \int_{r_0}^{r_1} \rho^{n-1}f(\rho) d\rho \geq 1 \quad \text{for some } r_0 \in (0, r_1).$$

Then for  $0 < r < r_0$  we have by (2.3) that

$$v(r_0) - v(r) \geq \int_r^{r_0} \rho^{1-n} d\rho \rightarrow \infty \quad \text{as } r \rightarrow 0^+$$

which contradicts the nonnegativity of  $v(r)$ . □

**Lemma 2.2.** *Suppose  $f: (0, R] \rightarrow \mathbb{R}$  is a continuous function,  $n \geq 2$  is an integer, and*

$$\int_0^R \rho^{n-1}|f(\rho)| d\rho < \infty. \quad (2.4)$$

Define  $u_0: (0, R] \rightarrow \mathbb{R}$  by

$$u_0(r) = \begin{cases} \frac{1}{n-2} \left[ \frac{1}{r^{n-2}} \int_0^r \rho^{n-1}f(\rho) d\rho + \int_r^R \rho f(\rho) d\rho \right] & \text{if } n \geq 3 \\ \left( \log \frac{2R}{r} \right) \int_0^r \rho f(\rho) d\rho + \int_r^R \rho \left( \log \frac{2R}{\rho} \right) f(\rho) d\rho & \text{if } n = 2. \end{cases}$$

Then  $u = u_0(r)$  is a  $C^2$  solution of

$$-(\Delta u)(r) := - \left( u''(r) + \frac{n-1}{r}u'(r) \right) = f(r) \quad \text{for } 0 < r \leq R. \quad (2.5)$$

Moreover, all solutions  $u(r)$  of (2.5) are such that

$$\int_0^r \rho^{n-1}|u(\rho)| d\rho = \begin{cases} O(r^2) & \text{as } r \rightarrow 0^+ \text{ if } n \geq 3 \\ O \left( r^2 \log \frac{1}{r} \right) & \text{as } r \rightarrow 0^+ \text{ if } n = 2. \end{cases} \quad (2.6)$$

*Proof.* By (2.4) the formula for  $u_0(r)$  makes sense and it is easy to check that  $u = u_0(r)$  is a solution of (2.5) and, as  $r \rightarrow 0^+$ ,

$$u_0(r) = \begin{cases} O(r^{2-n}) & \text{if } n \geq 3 \\ O\left(\log \frac{1}{r}\right) & \text{if } n = 2. \end{cases}$$

Thus, since all solutions of (2.5) are given by

$$u = u_0(r) + C_1 + C_2 \begin{cases} r^{2-n} & \text{if } n \geq 3 \\ \log \frac{1}{r} & \text{if } n = 2 \end{cases}$$

where  $C_1$  and  $C_2$  are arbitrary constants, we see that all solutions of (2.5) satisfy (2.6).  $\square$

**Lemma 2.3.** *Suppose  $f: (0, R] \rightarrow \mathbb{R}$  is a continuous function,  $n \geq 2$  is an integer, and*

$$\int_{x \in B_R(0) \subset \mathbb{R}^n} |f(|x|)| dx < \infty. \quad (2.7)$$

*If  $u = u(|x|)$  is a radial solution of*

$$-\Delta^m u = f \quad \text{for } 0 < |x| \leq R, \quad m \geq 1 \quad (2.8)$$

*then*

$$\int_{|x| < r} |u(x)| dx = \begin{cases} O(r^2) & \text{as } r \rightarrow 0^+ \text{ if } n \geq 3 \\ O\left(r^2 \log \frac{1}{r}\right) & \text{as } r \rightarrow 0^+ \text{ if } n = 2. \end{cases} \quad (2.9)$$

*Proof.* The lemma is true for  $m = 1$  by Lemma 2.2. Assume, inductively, that the lemma is true for  $m - 1$  where  $m \geq 2$ . Let  $u$  be a radial solution of (2.8). Then

$$-\Delta(\Delta^{m-1}u) = -\Delta^m u = f \quad \text{for } 0 < |x| \leq R.$$

Hence by (2.7) and Lemma 2.2,

$$g := -\Delta^{m-1}u \in L^1(B_R(0)).$$

So by the inductive assumption, (2.9) holds.  $\square$

**Lemma 2.4.** *Suppose  $f: \overline{B_R(0)} - \{0\} \rightarrow \mathbb{R}$  is a nonnegative continuous function and  $u$  is a  $C^{2m}$  solution of*

$$\left. \begin{array}{l} -\Delta^m u = f \\ u \geq 0 \end{array} \right\} \quad \text{in } \overline{B_R(0)} - \{0\} \subset \mathbb{R}^n, \quad n \geq 2, \quad m \geq 1. \quad (2.10)$$

*Then*

$$\int_{|x| < r} u(x) dx = \begin{cases} O(r^2) & \text{as } r \rightarrow 0^+ \text{ if } n \geq 3 \\ O\left(r^2 \log \frac{1}{r}\right) & \text{as } r \rightarrow 0^+ \text{ if } n = 2 \end{cases} \quad (2.11)$$

*and*

$$\int_{|x| < R} |x|^{2m-2} f(x) dx < \infty. \quad (2.12)$$

*Proof.* By averaging (2.10) we can assume  $f = f(|x|)$  and  $u = u(|x|)$  are radial functions. The lemma is true for  $m = 1$  by Lemmas 2.1 and 2.2. Assume inductively that the lemma is true for  $m - 1$ , where  $m \geq 2$ . Let  $u = u(|x|)$  be a radial solution of (2.10). Let  $v = \Delta^{m-1}u$ . Then  $-\Delta v = -\Delta^m u = f$  and integrating this equation we obtain as in the proof of Lemma 2.1 that

$$r^{n-1}v'(r) = r_2^{n-1}v'(r_2) + \int_r^{r_2} \rho^{n-1} f(\rho) d\rho \quad \text{for all } 0 < r < r_2 \leq R. \quad (2.13)$$

We can assume

$$\int_0^R \rho^{n-1} f(\rho) d\rho = \infty \quad (2.14)$$

for otherwise  $\int_{|x| < R} f(x) dx < \infty$  and hence (2.12) obviously holds and (2.11) holds by Lemma 2.3.

By (2.13) and (2.14) we have for some  $r_1 \in (0, R)$  that

$$v'(r_1) \geq 1. \quad (2.15)$$

Replacing  $r_2$  with  $r_1$  in (2.13) we get

$$v'(\rho) = \frac{r_1^{n-1}v'(r_1)}{\rho^{n-1}} + \frac{1}{\rho^{n-1}} \int_\rho^{r_1} s^{n-1} f(s) ds \quad \text{for } 0 < \rho \leq r_1$$

and integrating this equation from  $r$  to  $r_1$  we obtain for  $0 < r \leq r_1$  that

$$-v(r) = -v(r_1) + r_1^{n-1}v'(r_1) \int_r^{r_1} \frac{1}{\rho^{n-1}} d\rho + \int_r^{r_1} \frac{1}{\rho^{n-1}} \int_\rho^{r_1} s^{n-1} f(s) ds d\rho$$

and hence by (2.15) for some  $r_0 \in (0, r_1)$  we have

$$-\Delta^{m-1}u(r) = -v(r) > \int_r^{r_0} \frac{1}{\rho^{n-1}} \int_\rho^{r_0} s^{n-1} f(s) ds d\rho \geq 0 \quad \text{for } 0 < r \leq r_0.$$

So by the inductive assumption,  $u$  satisfies (2.11) and

$$\begin{aligned} \infty &> \frac{1}{n\omega_n} \int_{|x| < r_0} |x|^{2m-4} (-v(|x|)) dx \\ &= \int_0^{r_0} r^{2m+n-5} (-v(r)) dr \\ &\geq \int_0^{r_0} r^{2m+n-5} \left( \int_r^{r_0} \frac{1}{\rho^{n-1}} \int_\rho^{r_0} s^{n-1} f(s) ds d\rho \right) dr \\ &= C \int_0^{r_0} s^{2m-2} f(s) s^{n-1} ds \\ &= C \int_{|x| < r_0} |x|^{2m-2} f(x) dx \end{aligned}$$

where in the above calculation we have interchanged the order of integration and  $C$  is a positive constant which depends only on  $m$  and  $n$ . This completes the inductive proof.  $\square$

### 3 Representation formula

A fundamental solution of  $\Delta^m$  in  $\mathbb{R}^n$ , where  $n \geq 2$  and  $m \geq 1$  are integers, is given by

$$\Phi(x) := a \begin{cases} (-1)^m |x|^{2m-n}, & \text{if } 2 \leq 2m < n \\ (-1)^{\frac{n-1}{2}} |x|^{2m-n}, & \text{if } 3 \leq n < 2m \text{ and } n \text{ is odd} \\ (-1)^{\frac{n}{2}} |x|^{2m-n} \log \frac{5}{|x|}, & \text{if } 2 \leq n \leq 2m \text{ and } n \text{ is even} \end{cases} \quad (3.1)$$

where  $a = a(m, n)$  is a *positive* constant. In the sense of distributions,  $\Delta^m \Phi = \delta$ , where  $\delta$  is the Dirac mass at the origin in  $\mathbb{R}^n$ . For  $x \neq 0$  and  $y \neq x$ , let

$$\Psi(x, y) = \Phi(x - y) - \sum_{|\alpha| \leq 2m-3} \frac{(-y)^\alpha}{\alpha!} D^\alpha \Phi(x) \quad (3.4)$$

be the error in approximating  $\Phi(x - y)$  with the partial sum of degree  $2m - 3$  of the Taylor series of  $\Phi$  at  $x$ .

The following theorem gives representation formula (3.6) for nonnegative solutions of inequality (3.5).

**Theorem 3.1.** *Let  $u(x)$  be a  $C^{2m}$  nonnegative solution of*

$$-\Delta^m u \geq 0 \quad \text{in } B_2(0) - \{0\} \subset \mathbb{R}^n, \quad (3.5)$$

where  $n \geq 2$  and  $m \geq 1$  are integers. Then

$$u = N + h + \sum_{|\alpha| \leq 2m-2} a_\alpha D^\alpha \Phi \quad \text{in } B_1(0) - \{0\} \quad (3.6)$$

where  $a_\alpha, |\alpha| \leq 2m - 2$ , are constants,  $h \in C^\infty(B_1(0))$  is a solution of

$$\Delta^m h = 0 \quad \text{in } B_1(0),$$

and

$$N(x) = \int_{|y| \leq 1} \Psi(x, y) \Delta^m u(y) dy \quad \text{for } x \neq 0. \quad (3.7)$$

When  $m = 1$ , equation (3.6) becomes

$$u = N + h + a_0 \Phi_1 \quad \text{in } B_1(0) - \{0\},$$

where

$$N(x) = \int_{|y| < 1} \Phi_1(x - y) \Delta u(y) dy$$

and  $\Phi_1$  is the fundamental solution of the Laplacian in  $\mathbb{R}^n$ . Thus, when  $m = 1$ , Theorem 3.1 is essentially a result of Brezis and Lions [2].

Futamara, Kishi, and Mizuta [6, Theorem 1] and [7, Corollary 5.1] obtained a result very similar to our Theorem 3.1, but using their result we would have to let the index of summation  $\alpha$  in (3.4) range over the larger set  $|\alpha| \leq 2m - 2$ . This would not suffice for our proof of Theorem 1.1. We have however used their idea of using the remainder term  $\Psi(x, y)$  instead of  $\Phi(x - y)$  in (3.7). This is done so that the integral in (3.7) is finite. See also the book [10, p. 137].

*Proof of Theorem 3.1.* By (3.5),

$$f := -\Delta^m u \geq 0 \quad \text{in } B_2(0) - \{0\}. \quad (3.8)$$

Thus by Lemma 2.4,

$$\int_{|x|<1} |x|^{2m-2} f(x) dx < \infty \quad (3.9)$$

and

$$\int_{|x|<r} u(x) dx = O\left(r^2 \log \frac{1}{r}\right) \quad \text{as } r \rightarrow 0^+. \quad (3.10)$$

If  $|\alpha| = 2m - 2$  we claim

$$D^\alpha \Phi(x) = O(\Gamma_0(x)) \quad \text{as } x \rightarrow 0 \quad (3.11)$$

where  $\Gamma_0(x)$  is given by (1.5). This is clearly true if  $\Phi$  is given by (3.1) or (3.2) because then  $n \geq 3$  and  $\Gamma_0(x) = |x|^{2-n}$ . The estimate (3.11) is also true when  $\Phi$  is given by (3.3) because then  $|x|^{2m-n}$  is a *polynomial* of degree  $2m - n \leq 2m - 2 = |\alpha|$  with equality if and only if  $n = 2$ , and hence  $D^\alpha \Phi$  has a term with  $\log \frac{5}{|x|}$  as a factor if and only if  $n = 2$ . This proves (3.11).

By Taylor's theorem and (3.11) we have

$$\begin{aligned} |\Psi(x, y)| &\leq C|y|^{2m-2} \Gamma_0(x) \\ &\leq C|y|^{2m-2} |x|^{2-n} \log \frac{5}{|x|} \quad \text{for } |y| < \frac{|x|}{2} < 1. \end{aligned} \quad (3.12)$$

Differentiating (3.4) with respect to  $x$  we get

$$D_x^\beta (\Psi(x, y)) = (D^\beta \Phi)(x - y) - \sum_{|\alpha| \leq 2m-3} \frac{(-y)^\alpha}{\alpha!} (D^{\alpha+\beta} \Phi)(x) \quad \text{for } x \neq 0 \quad \text{and } y \neq x \quad (3.13)$$

and so by Taylor's theorem applied to  $D^\beta \Phi$  we have

$$|D_x^\beta \Psi(x, y)| \leq C|y|^{2m-2} |x|^{2-n-|\beta|} \log \frac{5}{|x|} \quad \text{for } |y| < \frac{|x|}{2} < 1. \quad (3.14)$$

Also,

$$\Delta_x^m \Psi(x, y) = 0 = \Delta_y^m \Psi(x, y) \quad \text{for } x \neq 0 \quad \text{and } y \neq x \quad (3.15)$$

(see also [10, Lemma 4.1, p. 137]) and

$$\begin{aligned} \int_{|x|<r} |\Phi(x - y)| dx &\leq Cr^{2m} \log \frac{5}{r} \\ &\leq C|y|^{2m-2} r^2 \log \frac{5}{r} \quad \text{for } 0 < r \leq 2|y| < 2. \end{aligned} \quad (3.16)$$

Before continuing with the proof of Theorem 3.1, we state and prove the following lemma.

**Lemma 3.1.** *For  $|y| < 1$  and  $0 < r < 1$  we have*

$$\int_{|x|<r} |\Psi(x, y)| dx \leq C|y|^{2m-2} r^2 \log \frac{5}{r}. \quad (3.17)$$



*Proof.* Since  $\Psi(x, 0) \equiv 0$  for  $x \neq 0$ , we can assume  $y \neq 0$ .

**Case I.** Suppose  $0 < r \leq |y| < 1$ . Then by (3.16)

$$\begin{aligned} \int_{0 < |x| < r} |\Psi(x, y)| dx &\leq \int_{0 < |x| < r} |\Phi(x - y)| dx + \sum_{|\alpha| \leq 2m-3} |y|^{|\alpha|} \int_{0 < |x| < r} |D^\alpha \Phi(x)| dx \\ &\leq C \left[ |y|^{2m-2} r^2 \log \frac{5}{r} + \sum_{|\alpha| < 2m-3} |y|^{|\alpha|} r^{2m-|\alpha|} \log \frac{5}{r} \right] \\ &\leq C |y|^{2m-2} r^2 \log \frac{5}{r}. \end{aligned}$$

**Case II.** Suppose  $0 < |y| < r < 1$ . Then by (3.16), with  $r = 2|y|$ , and (3.12) we have

$$\begin{aligned} \int_{|x| < 2r} |\Psi(x, y)| dx &= \int_{2|y| < |x| < 2r} |\Psi(x, y)| dx + \int_{|x| < 2|y|} |\Psi(x, y)| dx \\ &\leq C \left[ \int_{2|y| < |x| < 2r} |y|^{2m-2} |x|^{2-n} \log \frac{5}{|x|} dx + |y|^{2m} \log \frac{5}{|y|} \right. \\ &\quad \left. + \sum_{|\alpha| \leq 2m-3} |y|^{|\alpha|} \int_{|x| < 2|y|} |D^\alpha \Phi(x)| dx \right] \\ &\leq C \left[ |y|^{2m-2} r^2 \log \frac{5}{r} + |y|^{2m-2} |y|^2 \log \frac{5}{|y|} \right] \\ &\leq C |y|^{2m-2} r^2 \log \frac{5}{r} \end{aligned}$$

which proves the lemma.  $\square$

Continuing with the proof of Theorem 3.1, let  $N$  be defined by (3.7) and let  $2r \in (0, 1)$  be fixed. Then for  $2r < |x| < 1$  we have

$$\begin{aligned} N(x) &= \int_{r < |y| < 1} \left[ \Phi(y - x) - \sum_{|\alpha| \leq 2m-3} \frac{(-y)^\alpha}{\alpha!} D^\alpha \Phi(x) \right] \Delta^m u(y) dy \\ &\quad - \int_{0 < |y| < r} \Psi(x, y) f(y) dy. \end{aligned}$$

By (3.9) and (3.14), we can move differentiation of the second integral with respect to  $x$  under the integral. Hence by (3.15),

$$\Delta^m N = \Delta^m u \tag{3.18}$$

for  $2r < |x| < 1$  and since  $2r \in (0, 1)$  was arbitrary, (3.18) holds for  $0 < |x| < 1$ .

By (3.7), (3.8), and Lemma 3.1, for  $0 < r < 1$  we have

$$\begin{aligned} \int_{|x|<r} |N(x)| dx &\leq \int_{|y|<1} \left( \int_{|x|<r} |\Psi(x,y)| dx \right) f(y) dy \\ &\leq Cr^2 \log \frac{5}{r} \int_{|y|<1} |y|^{2m-2} f(y) dy \\ &= O\left(r^2 \log \frac{1}{r}\right) \quad \text{as } r \rightarrow 0^+ \end{aligned}$$

by (3.9). Thus by (3.10)

$$v := u - N \in L^1_{\text{loc}}(B_1(0)) \subset \mathcal{D}'(B_1(0)) \quad (3.19)$$

and

$$\int_{|x|<r} |v(x)| dx = O\left(r^2 \log \frac{1}{r}\right) \quad \text{as } r \rightarrow 0^+. \quad (3.20)$$

By (3.18),

$$\Delta^m v(x) = 0 \quad \text{for } 0 < |x| < 1.$$

Thus  $\Delta^m v$  is a distribution in  $\mathcal{D}'(B_1(0))$  whose support is a subset of  $\{0\}$ . Hence

$$\Delta^m v = \sum_{|\alpha| \leq k} a_\alpha D^\alpha \delta$$

is a finite linear combination of the delta function and its derivatives.

We now use a method of Brezis and Lions [2] to show  $a_\alpha = 0$  for  $|\alpha| \geq 2m - 1$ . Choose  $\varphi \in C_0^\infty(B_1(0))$  such that

$$(-1)^{|\alpha|} (D^\alpha \varphi)(0) = a_\alpha \quad \text{for } |\alpha| \leq k.$$

Let  $\varphi_\varepsilon(x) = \varphi\left(\frac{x}{\varepsilon}\right)$ . Then, for  $0 < \varepsilon < 1$ ,  $\varphi_\varepsilon \in C_0^\infty(B_1(0))$  and

$$\begin{aligned} \int v \Delta^m \varphi_\varepsilon &= (\Delta^m v)(\varphi_\varepsilon) = \sum_{|\alpha| \leq k} a_\alpha (D^\alpha \delta) \varphi_\varepsilon \\ &= \sum_{|\alpha| \leq k} a_\alpha (-1)^{|\alpha|} \delta(D^\alpha \varphi_\varepsilon) = \sum_{|\alpha| \leq k} a_\alpha (-1)^{|\alpha|} (D^\alpha \varphi_\varepsilon)(0) \\ &= \sum_{|\alpha| \leq k} a_\alpha (-1)^{|\alpha|} \frac{1}{\varepsilon^{|\alpha|}} (D^\alpha \varphi)(0) = \sum_{|\alpha| \leq k} a_\alpha^2 \frac{1}{\varepsilon^{|\alpha|}}. \end{aligned}$$

On the other hand,

$$\begin{aligned} \int v \Delta^m \varphi_\varepsilon &= \int v(x) \frac{1}{\varepsilon^{2m}} (\Delta^m \varphi)\left(\frac{x}{\varepsilon}\right) dx \\ &\leq \frac{C}{\varepsilon^{2m}} \int_{|x|<\varepsilon} |v(x)| dx = O\left(\frac{1}{\varepsilon^{2m-2}} \log \frac{1}{\varepsilon}\right) \quad \text{as } \varepsilon \rightarrow 0^+ \end{aligned}$$

by (3.20). Hence  $a_\alpha = 0$  for  $|\alpha| \geq 2m - 1$  and consequently

$$\Delta^m v = \sum_{|\alpha| \leq 2m-2} a_\alpha D^\alpha \delta = \sum_{|\alpha| \leq 2m-2} a_\alpha D^\alpha \Delta^m \Phi.$$

That is

$$\Delta^m \left( v - \sum_{|\alpha| \leq 2m-2} a_\alpha D^\alpha \Phi \right) = 0 \quad \text{in } \mathcal{D}'(B_1(0)).$$

Thus for some  $C^\infty$  solution of  $\Delta^m h = 0$  in  $B_1(0)$  we have

$$v = \sum_{|\alpha| \leq 2m-2} a_\alpha D^\alpha \Phi + h \quad \text{in } B_1(0) - \{0\}.$$

Hence Theorem 3.1 follows from (3.19).  $\square$

## 4 Proofs of Theorems 1.3 and 1.4 and Corollary 1.1

In this section we prove Theorems 1.3 and 1.4 and Corollary 1.1.

*Proof of Theorem 1.3.* This proof is a continuation of the proof of Theorem 3.1. If  $m = 1$  then Theorem 1.3 is trivially true. Hence we can assume  $m \geq 2$ . Also, if  $\sigma = m$  then (1.11) follows trivially from (1.10). Hence we can assume  $\sigma \leq m - 1$  in (1.11).

If  $\alpha$  and  $\beta$  are multi-indices and  $|\alpha| = 2m - 2$  then it follows from (3.1)–(3.3) that

$$D^{\alpha+\beta} \Phi(x) = O \left( \left| \frac{d^{|\beta|}}{d|x|^{|\beta|}} \Gamma_0(|x|) \right| \right) \quad \text{as } x \rightarrow 0. \quad (4.1)$$

(This is clearly true if  $n = 2$ . If  $n \geq 3$  then  $|\alpha + \beta| = 2m - 2 + |\beta| > 2m - n$  and thus

$$D^{\alpha+\beta} \Phi(x) = O(|x|^{2m-n-(2m-2+|\beta|)}) = O \left( \left| \frac{d^{|\beta|}}{d|x|^{|\beta|}} \Gamma_0(|x|) \right| \right).$$

Let  $L^b$  be any linear partial differential operator of the form  $\sum_{|\beta|=b} c_\beta D^\beta$ , where  $b$  is a nonnegative integer and  $c_\beta \in \mathbb{R}$ . Then applying Taylor's theorem to (3.13) and using (4.1) we obtain

$$|L_x^b \Psi(x, y)| \leq C |y|^{2m-2} \left| \frac{d^b}{d|x|^b} \Gamma_0(|x|) \right| \quad \text{for } |y| < \frac{|x|}{2} < 1. \quad (4.2)$$

Here and later  $C$  is a positive constant, independent of  $x$  and  $y$ , whose value may change from line to line. For  $0 \leq b \leq 2m - 1$  we have

$$L^b N(x) = \int_{|y| < 1} -L_x^b \Psi(x, y) f(y) dy \quad \text{for } 0 < |x| < 1.$$

Hence by (4.1), (4.2), (3.6) and (3.9) we have

$$L^b u(x) \leq C \left| \frac{d^b}{d|x|^b} \Gamma_0(|x|) \right| \quad \text{for } 0 < |x| < 1 \quad (4.3)$$

provided  $0 \leq b \leq 2m - 1$  and

$$-L_x^b \Psi(x, y) \leq C |y|^{2m-2} \left| \frac{d^b}{d|x|^b} \Gamma_0(|x|) \right| \quad \text{for } 0 < \frac{|x|}{2} < |y| < 1. \quad (4.4)$$

We will complete the proof of Theorem 1.3 by proving (4.4) for various choices for  $L^b$ . For the rest of the proof of Theorem 1.3 we will always assume

$$0 < \frac{|x|}{2} < |y| < 1 \quad (4.5)$$

which implies

$$|x - y| \leq |x| + |y| \leq 3|y|. \quad (4.6)$$

**Case I.** Suppose  $\Phi$  is given by (3.1) or (3.2). It follows from (3.13) and (4.5) that

$$\begin{aligned} |D_x^\beta \Psi(x, y) - D_x^\beta \Phi(x - y)| &\leq C \sum_{|\alpha| \leq 2m-3} |y|^{|\alpha|} |x|^{2m-n-|\alpha|-|\beta|} \\ &\leq C |y|^{2m-2} |x|^{2-n-|\beta|}. \end{aligned}$$

Thus (4.4), and hence (4.3), holds provided  $0 \leq b \leq 2m - 1$  and

$$-(L^b \Phi)(x - y) \leq C |y|^{2m-2} |x|^{2-n-b}. \quad (4.7)$$

**Case I(a).** Suppose  $\Phi$  is given by (3.1). Let  $\sigma \in [0, m - 1]$  be an integer,  $b = 2\sigma$ , and  $L^b = (-1)^{m+\sigma} \Delta^\sigma$ . Then  $0 \leq b \leq 2m - 2$  and

$$\operatorname{sgn}(-L^b \Phi) = (-1)^{1+m+\sigma} \operatorname{sgn} \Delta^\sigma \Phi = (-1)^{1+2m+\sigma} \operatorname{sgn} \Delta^\sigma |x|^{2m-n} = (-1)^{1+2m+2\sigma} = -1.$$

Thus (4.7), and hence (4.3) holds with  $L^b = (-1)^{m+\sigma} \Delta^\sigma$  and  $0 \leq \sigma \leq m - 1$ . This completes the proof of Theorem 1.3 when  $\Phi$  is given by (3.1).

**Case I(b).** Suppose  $\Phi$  is given by (3.2). Then  $n$  is odd. It follows from (4.5) and (4.6) that for  $0 \leq |\beta| \leq 2m - n$  we have

$$|(D^\beta \Phi)(x - y)| \leq C |x - y|^{2m-n-|\beta|} \leq C |y|^{2m-n-|\beta|} \leq C |y|^{2m-2} |x|^{2-n-|\beta|}.$$

So (4.7) holds with  $L^b = \pm D^\beta$  and  $|\beta| = b$ . Hence

$$|D^\beta u(x)| \leq C |x|^{2-n-|\beta|} \quad \text{for } 0 \leq |\beta| \leq 2m - n \quad \text{and} \quad 0 < |x| < 1.$$

In particular

$$|\Delta^\sigma u(x)| \leq C |x|^{2-n-2\sigma} \quad \text{for } 2\sigma \leq 2m - n \quad \text{and} \quad 0 < |x| < 1.$$

Also, if  $2m - n + 1 \leq 2\sigma \leq 2m - 2$ ,  $b = 2\sigma$ , and  $L^b = (-1)^{m+\sigma} \Delta^\sigma$ , then  $0 \leq \sigma \leq m - 1$  and

$$\begin{aligned} \operatorname{sgn}(-L^b \Phi) &= (-1)^{m+\sigma+1} \operatorname{sgn} \Delta^\sigma \Phi = (-1)^{m+\sigma+1+\frac{n-1}{2}} \operatorname{sgn} \Delta^\sigma |x|^{2m-n} \\ &= (-1)^{m+\sigma+1+\frac{n-1}{2}} \operatorname{sgn}(\Delta^{\frac{b-(2m-n+1)}{2}} \Delta^{\frac{2m-n+1}{2}} |x|^{2m-n}) \\ &= (-1)^{m+\sigma+1+\frac{n-1}{2}+\sigma-m+\frac{n-1}{2}} = -1 \end{aligned}$$

because  $\Delta^{\frac{2m-n+1}{2}} |x|^{2m-n} = C |x|^{-1}$  where  $C > 0$ .

So (4.7) holds with  $L^b = (-1)^{m+\sigma} \Delta^\sigma$ . Hence  $(-1)^{m+\sigma} \Delta^\sigma u(x) \leq C |x|^{2-n-2\sigma}$  for  $0 \leq \sigma \leq m - 1$  and  $0 < |x| < 1$ . This completes the proof Theorem 1.3 when  $\Phi$  is given by (3.2).

**Case II.** Suppose  $\Phi$  is given by (3.3). Then  $2 \leq n \leq 2m$  and  $n$  is even. To prove Theorem 1.3 in Case II, it suffices to prove the following three statements.

(i) Estimate (1.12) holds when  $n = 2$ ,  $\beta = 0$ , and  $m \geq 2$ .

(ii) Estimate (1.12) holds when  $|\beta| \leq 2m - n - 1$  and either  $n \geq 3$  or  $|\beta| \geq 1$ .

(iii) Estimate (1.11) holds for  $2m - n \leq 2\sigma \leq 2m - 2$ .

*Proof of (i).* Suppose  $n = 2$ ,  $\beta = 0$ , and  $m \geq 2$ . Then, since  $u$  is nonnegative, to prove (i) it suffices to prove

$$u(x) \leq C \log \frac{5}{|x|} \quad \text{for } 0 < |x| < 1$$

which holds if (4.4) holds with  $b = 0$  and  $L^b = D^0 = \text{id}$ . That is if

$$-\Psi(x, y) \leq C|y|^{2m-2} \log \frac{5}{|x|} \quad (4.8)$$

By (3.4), (4.5), and (4.6) we have

$$\begin{aligned} |\Psi(x, y) - \Phi(x - y)| &\leq \sum_{|\alpha| \leq 2m-3} |y|^{|\alpha|} |D^\alpha \Phi(x)| \\ &\leq C \sum_{|\alpha| \leq 2m-3} |y|^{|\alpha|} |x|^{2m-2-|\alpha|} \log \frac{5}{|x|} \leq C|y|^{2m-2} \log \frac{5}{|x|} \end{aligned}$$

and

$$\begin{aligned} |\Phi(x - y)| &= a|x - y|^{2m-2} \log \frac{5}{|x - y|} \\ &\leq C|y|^{2m-2} \log \frac{5}{|y|} \leq C|y|^{2m-2} \log \frac{5}{|x|} \end{aligned}$$

which imply (4.8). This completes the proof of (i).

*Proof of (ii).* Suppose  $|\beta| \leq 2m - n - 1$  and either  $n \geq 3$  or  $|\beta| \geq 1$ . Then  $n + |\beta| \geq 3$  and in order to prove (ii) it suffices to prove

$$|D_x^\beta \Psi(x, y)| \leq C|y|^{2m-2} \left| \frac{d^{|\beta|}}{d|x|^{|\beta|}} \Gamma_0(|x|) \right| \quad (4.9)$$

because then (4.4), and hence (4.3), holds with  $L^b = \pm D^\beta$ .

Since  $\Phi$  is given by (3.3) we have  $n \geq 2$  is even and

$$\Phi(x) = P(x) \log \frac{5}{|x|}$$

where  $P(x) = a(-1)^{\frac{n}{2}} |x|^{2m-n}$  is a *polynomial* of degree  $2m - n$ . Since  $D^\beta P$  is a polynomial of degree  $2m - n - |\beta| \leq 2m - 3$  we have

$$D_x^\beta P(x - y) = \sum_{|\alpha| \leq 2m-3} \frac{(-y)^\alpha}{\alpha!} D^{\alpha+\beta} P(x). \quad (4.10)$$

Since  $D_x^\beta \Psi(x, y) = A_1 + A_2 + A_3$ , where

$$A_1 = D_x^\beta \Psi(x, y) - D_x^\beta \Phi(x - y) + (D_x^\beta P(x - y)) \log \frac{5}{|x|}$$

$$A_2 = D_x^\beta \Phi(x - y) - (D_x^\beta P(x - y)) \log \frac{5}{|x - y|}$$

$$A_3 = (D_x^\beta P(x - y)) \log \frac{|x|}{|x - y|},$$

to prove (4.9) it suffices to prove for  $j = 1, 2, 3$  that

$$|A_j| \leq C|y|^{2m-2} \left| \frac{d^{|\beta|}}{d|x|^{|\beta|}} \Gamma_0(|x|) \right|. \quad (4.11)$$

Since

$$\begin{aligned} \left| D^{\alpha+\beta} \Phi(x) - (D^{\alpha+\beta} P(x)) \log \frac{5}{|x|} \right| &= \left| \sum_{\substack{\gamma \leq \alpha+\beta \\ |\alpha+\beta-\gamma| \geq 1}} \binom{\alpha+\beta}{\gamma} (D^\gamma P(x)) \left( D^{\alpha+\beta-\gamma} \log \frac{5}{|x|} \right) \right| \\ &\leq C|x|^{2m-n-|\alpha|-|\beta|} \end{aligned}$$

it follows from (3.13), (4.10), and (4.5) that

$$\begin{aligned} |A_1| = |-A_1| &= \left| \sum_{|\alpha| \leq 2m-3} \frac{(-y)^\alpha}{\alpha!} D^{\alpha+\beta} \Phi(x) - \sum_{|\alpha| \leq 2m-3} \frac{(-y)^\alpha}{\alpha!} (D^{\alpha+\beta} P(x)) \log \frac{5}{|x|} \right| \\ &\leq C \sum_{|\alpha| \leq 2m-3} |y|^{|\alpha|} |x|^{2m-n-|\alpha|-|\beta|} \leq C|y|^{2m-2} |x|^{2-n-|\beta|} \\ &= C|y|^{2m-2} \left| \frac{d^{|\beta|}}{d|x|^{|\beta|}} \Gamma_0(|x|) \right|. \end{aligned}$$

Thus (4.11) hold when  $j = 1$ .

Since  $A_2 = 0$  when  $\beta = 0$ , we can assume for the proof of (4.11) when  $j = 2$  that  $|\beta| \geq 1$ . Then by (4.6) and (4.5),

$$\begin{aligned} |A_2| &= \left| \sum_{\substack{\alpha \leq \beta \\ |\beta-\alpha| \geq 1}} \binom{\beta}{\alpha} (D_x^\alpha P(x-y)) \left( D_x^{\beta-\alpha} \log \frac{5}{|x-y|} \right) \right| \\ &\leq C|x-y|^{2m-n-|\beta|} \leq C|y|^{2m-n-|\beta|} \\ &\leq C|y|^{2m-2} |x|^{2-n-|\beta|} \\ &= C|y|^{2m-2} \left| \frac{d^{|\beta|}}{d|x|^{|\beta|}} \Gamma_0(|x|) \right|. \end{aligned}$$

Thus (4.11) holds when  $j = 2$ .

Finally we prove (4.11) when  $j = 3$ . Let  $d = 2m - n - |\beta|$ . Then  $1 \leq d \leq 2m - 3$ ,

$$|A_3| \leq C|x-y|^d \left| \log \frac{|x|}{|x-y|} \right|$$

and by (4.5) and (4.6) we have

$$\begin{aligned} |x-y|^d \left| \log \frac{|x|}{|x-y|} \right| &\leq \begin{cases} |x-y|^d \left( \frac{|x|}{|x-y|} \right)^d = |x|^d \leq C|y|^{2m-2} |x|^{2-n-|\beta|} & \text{if } |x-y| \leq |x| \\ |x-y|^d \left( \frac{|x-y|}{|x|} \right)^{2m-2-d} = |x-y|^{2m-2} |x|^{2-n-|\beta|} & \text{if } |x| \leq |x-y| \end{cases} \\ &\leq C|y|^{2m-2} |x|^{2-n-|\beta|} = C|y|^{2m-2} \left| \frac{d^{|\beta|}}{d|x|^{|\beta|}} \Gamma_0(|x|) \right|. \end{aligned}$$

Thus (4.11) holds when  $j = 3$ . This completes the proof of (4.9) and hence of (ii).

*Proof of (iii).* Suppose  $2m - n \leq 2\sigma \leq 2m - 2$ . In order to prove (iii) it suffices to prove

$$(-1)^{m+\sigma+1} \Delta_x^\sigma \Psi(x, y) \leq C |y|^{2m-2} \left| \frac{d^{2\sigma}}{d|x|^{2\sigma}} \Gamma_0(|x|) \right| \quad (4.12)$$

because then (4.4), and hence (4.3), holds with  $L^b = (-1)^{m+\sigma} \Delta^\sigma$  and  $b = 2\sigma$ .

If  $|\beta| = 2\sigma$  then (4.5) implies

$$\begin{aligned} \left| \sum_{1 \leq |\alpha| \leq 2m-3} \frac{(-y)^\alpha}{\alpha!} D^{\alpha+\beta} \Phi(x) \right| &\leq C \sum_{1 \leq |\alpha| \leq 2m-3} |y|^{|\alpha|} |x|^{2m-n-|\alpha|-|\beta|} \\ &\leq C |y|^{2m-2} |x|^{2-n-|\beta|}. \end{aligned}$$

Thus it follows from (3.13) that

$$|\Delta_x^\sigma \Psi(x, y) - \Delta_x^\sigma \Phi(x - y) + \Delta^\sigma \Phi(x)| \leq C |y|^{2m-2} |x|^{2-n-2\sigma}.$$

Hence to prove (4.12) it suffices to prove

$$(-1)^{m+\sigma+1} (\Delta_x^\sigma \Phi(x - y) - \Delta^\sigma \Phi(x)) \leq C |y|^{2m-2} |x|^{2-n-2\sigma}. \quad (4.13)$$

We divide the proof of (4.13) into cases.

**Case 1.** Suppose  $2 \leq 2m - n + 2 \leq 2\sigma \leq 2m - 2$ . Then by (4.5)

$$|\Delta^\sigma \Phi(x)| \leq C |x|^{2m-n-2\sigma} \leq C |y|^{2m-2} |x|^{2-n-2\sigma}$$

and since

$$\Delta^{\frac{2m-n}{2}} \left( |x|^{2m-n} \log \frac{5}{|x|} \right) = A \log \frac{5}{|x|} - B \quad (4.14)$$

where  $A > 0$  and  $B \geq 0$  are constants, we have

$$\operatorname{sgn}((-1)^{m+\sigma+1} \Delta^\sigma \Phi(z)) = (-1)^{m+\sigma+\frac{n}{2}+1} (-1)^{\sigma-\frac{2m-n}{2}} = -1 \quad \text{for } |z| > 0.$$

This proves (4.13) and hence (iii) in Case 1.

**Case 2.** Suppose  $2\sigma = 2m - n$ . Then by (4.14) and (4.6) we have

$$\begin{aligned} (-1)^{m+\sigma+1} (\Delta_x^\sigma \Phi(x - y) - \Delta^\sigma \Phi(x)) &= (-1)^{\frac{n}{2}+m+\sigma+1} A \log \frac{|x|}{|x-y|} \\ &= A \log \frac{|x-y|}{|x|} \leq A \log \frac{3|y|}{|x|} \leq A \left( \frac{3|y|}{|x|} \right)^{2m-2} \\ &= A 3^{2m-2} |y|^{2m-2} |x|^{2-n-2\sigma}. \end{aligned}$$

This proves (4.13) and hence (iii) in Case 2, and thereby completes the proof of Theorem 1.3.  $\square$

*Proof of Theorem 1.4.* Let  $u(x)$  be defined in terms of  $v(y)$  by (1.6). Then by (1.7) and (1.14),  $u(x)$  is a  $C^{2m}$  nonnegative solution of (1.10), and hence  $u(x)$  satisfies the conclusion of Theorem 1.3. It is a straight-forward exercise to show that (1.16) follows from (1.12) when  $n < 2m$  and  $\beta$  satisfies (1.13). So to complete the proof of Theorem 1.4 we will now prove (1.15).

Suppose  $\sigma \leq m$  is a nonnegative integer. Let  $v_\sigma(y)$  be the  $\sigma$ -Kelvin transform of  $u(x)$ . Then  $v_\sigma(y) = |y|^{2\sigma-2m}v(y)$  and thus by (1.11), we have for  $|y| > 1$  that

$$\begin{aligned} (-1)^{m+\sigma} \Delta^\sigma(|y|^{2\sigma-2m}v(y)) &= (-1)^{m+\sigma} \Delta^\sigma v_\sigma(y) \\ &= (-1)^{m+\sigma} |x|^{n+2\sigma} \Delta^\sigma u(x) \\ &\leq C|x|^{n+2\sigma} \left| \frac{d^{2\sigma}}{d|x|^{2\sigma}} \Gamma_0(|x|) \right| \\ &\leq C \begin{cases} |x|^2 \log \frac{5}{|x|} & \text{if } \sigma = 0 \text{ and } n = 2 \\ |x|^2 & \text{if } \sigma \geq 1 \text{ or } n \geq 3 \end{cases} \end{aligned}$$

which implies (1.15) after replacing  $|x|$  with  $1/|y|$ . □

*Proof of Corollary 1.1.* Theorem 1.4 implies (1.17) and

$$-\Delta(|y|^{-2}v(y)) \leq C|y|^{-2} \quad \text{for } |y| > 1$$

and thus for  $|y| > 1$  we have

$$\begin{aligned} -|y|^{-2} \Delta v(y) &= -\Delta(|y|^{-2}v(y)) + (\Delta|y|^{-2})v(y) + 2\nabla|y|^{-2} \cdot \nabla v(y) \\ &\leq -\Delta(|y|^{-2}v(y)) + C \left( |y|^{-4} \Gamma_\infty(|y|) + |y|^{-3} \frac{d}{d|y|} \Gamma_\infty(|y|) \right) \\ &\leq C \begin{cases} |y|^{-2} & \text{if } n = 3 \\ |y|^{-2} \log 5|y| & \text{if } n = 2 \end{cases} \\ &\leq C|y|^{-2} \left| \frac{d^2}{d|y|^2} \Gamma_\infty(|y|) \right| \end{aligned}$$

which implies (1.18). □

## 5 Proof of Theorem 1.1

As noted in the introduction, the sufficiency of condition (1.3) in Theorem 1.1 and the estimate (1.4) follow from Theorem 1.3, which we proved in the last section. Consequently, we can complete the proof of Theorem 1.1 by proving the following proposition.

**Proposition 5.1.** *Suppose  $n \geq 2$  and  $m \geq 1$  are integers such that (1.3) does not hold. Let  $\psi: (0, 1) \rightarrow (0, \infty)$  be a continuous function. Then there exists a  $C^\infty$  positive solution of*

$$-\Delta^m u \geq 0 \quad \text{in } B_1(0) - \{0\} \subset \mathbb{R}^n \tag{5.1}$$

such that

$$u(x) \neq O(\psi(|x|)) \quad \text{as } x \rightarrow 0. \tag{5.2}$$

*Proof.* Let  $\{x_j\}_{j=1}^\infty \subset \mathbb{R}^n - \{0\}$  be a sequence such that  $4|x_{j+1}| < |x_j| < 1$ . Choose  $\alpha_j > 0$  such that

$$\frac{\alpha_j}{\psi(x_j)} \rightarrow \infty \quad \text{as } j \rightarrow \infty. \tag{5.3}$$



Since (1.3) does not hold, it follows from (3.1)–(3.3) that  $\lim_{x \rightarrow 0} -\Phi(x) = \infty$  and  $-\Phi(x) > 0$  for  $0 < |x| < 5$ . Hence we can choose  $R_j \in (0, |x_j|/4)$  such that

$$\int_{|z| < R_j} -\Phi(z) dz > R_j^n 2^j \alpha_j, \quad \text{for } j = 1, 2, \dots \quad (5.4)$$

Let  $\varphi: \mathbb{R} \rightarrow [0, 1]$  be a  $C^\infty$  function such that  $\varphi(t) = 1$  for  $t \leq 1$  and  $\varphi(t) = 0$  for  $t \geq 2$ . Define  $f_j \in C_0^\infty(B_{\frac{|x_j|}{2}}(x_j))$  by

$$f_j(x) = \frac{1}{2^j R_j^n} \varphi\left(\frac{|x - x_j|}{R_j}\right).$$

Then the functions  $f_j$  have disjoint supports and

$$\int_{\mathbb{R}^n} f_j(x) dx = \int_{|x - x_j| < 2R_j} f_j(x) dx \leq \frac{C(n)}{2^j}.$$

Thus  $f := \sum_{j=1}^{\infty} f_j \in L^1(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n - \{0\})$  and hence the function  $u: B_1(0) - \{0\} \rightarrow \mathbb{R}$  defined by

$$u(x) := \int_{|y| < 1} -\Phi(x - y) f(y) dy$$

is a  $C^\infty$  positive solution of (5.1). Also

$$\begin{aligned} u(x_j) &\geq \int_{|y| < 1} -\Phi(x_j - y) f_j(y) dy \\ &\geq \frac{1}{2^j R_j^n} \int_{|x - x_j| < R_j} -\Phi(x_j - y) dy \\ &= \frac{1}{2^j R_j^n} \int_{|z| < R_j} -\Phi(z) dz > \alpha_j \end{aligned}$$

by (5.4). Hence (5.3) implies that  $u$  satisfies (5.2). □

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