

Meson mass spectrum using the Cayley-Dickson algebra

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Abstract

From an injective map between the mass of the meson 16-plet and the eigenvalue of the right multiplication in the Cayley-Dickson algebra, we obtain the mass formula as $2m_{D_s} = m_{\eta_c} + m_{\eta'}$, which is in excellent agreement with experiment.

1 Introduction

In the standard quark model, mesons are bound states of a quark and anti-quark. Due to the difference between quark masses, the $SU(N)$ flavor symmetry is broken. Concerning the meson mass formula, the Gell-Mann–Okubo relation [1, 2] is well known. For $N = 3$, the Gell-Mann–Okubo formula for the pseudoscalar mesons is given by $3m_\eta \simeq 4m_K - m_\pi$, assuming no singlet-octet mixing, which, however, cannot be neglected due to the $SU(3)$ symmetry breaking. It is difficult to estimate the singlet-octet mixing from the improvement of this formula (or other meson mass formula based on the dual resonance model [3] in connection with string theory) by taking account of the quark-quark interaction mediated by the gluon exchange [4]. It may remain difficult to obtain the singlet-octet mixing, even if the $SU(4)$ meson 16-plet and $SU(5)$ 25-plet are taken into account [5, 6]. For the meson 16-plet, for example, the model such that the $SU(4)$ symmetry is broken but the $SU(3)$ symmetry is exact, leads to a simple mass formula of the form $12m_D^2 = 5m_{c\bar{c}}^2 + 7m_0^2$ [5], where m_D represents the average mass of $c\bar{u}$, $c\bar{d}$, and $c\bar{s}$; and m_0 stands for the average of the meson octet masses; due to the assumption of the exact $SU(3)$ symmetry, the mass relation between the physical states η and η' cannot be obtained.

The aim of this paper is obtain a simple but exact mass formula by relating the meson n -plet (denoted by \mathfrak{n} for brevity) to a vector space where some appropriate algebra is given. The basic technique was developed in Ref. [7], where the meson octet can be identified with a certain algebra. Here by algebra \mathcal{A} , we mean a hypercomplex system, that is, a vector space V with a given multiplication $L_x : V \rightarrow V$ (with $x \in V$). Denote by m_i (for $i = 1, \dots, n$) the mass of

the meson n -plet, and λ_i (for $i = 1, \dots, n$) the eigenvalue of L_x with $x \in V/V_0$, where $V_0 \subset V$ represents the subspace of V such that $\dim(V/V_0) = n$ (so that $\dim V$ should be a multiple of n). The reason of dividing V by V_0 is that we want to obtain a bijective map between the meson mass and the eigenvalues of L_x (in an actual case, L_x is replaced by its relative). Suppose that there is a bijective map $\phi : M \rightarrow \Lambda$, where $M = \cup_i \{m_i\}$ and $\Lambda = \cup_i \{\lambda_i\}$. Then it is found that there is a bijective map $\tilde{\phi} : \mathfrak{n} \rightarrow V/V_0$, due to the existence of a bijective maps $f : \mathfrak{n} \rightarrow M$ and $g : \Lambda \rightarrow V/V_0$, that is, $\tilde{\phi} = g \circ \phi \circ f$ [see Eq. (1)]. Conversely, if the algebra \mathcal{A} is given (this means that the maps $\tilde{\phi}$ and g are given), then the eigenvalue of (the relative of) L_x is related to the meson mass through the relation of $\phi \circ f = g^{-1} \circ \tilde{\phi}$ (= given).

$$\begin{array}{ccc} \mathfrak{n} & \xrightarrow{\tilde{\phi}} & V/V_0 \\ f \downarrow & & \uparrow g \\ M & \xrightarrow{\phi} & \Lambda \end{array} \quad (1)$$

As an example, consider the meson octet $\mathfrak{8}$, which may be composed as

$$\begin{aligned} \mathfrak{8} &= \mathbf{1} \oplus \mathbf{1} \oplus \mathbf{2} \oplus \mathbf{4} \\ &= \eta \oplus \pi^0 \oplus (\pi^+, \pi^-) \oplus (K^+, K^-; K^0, \bar{K}^0). \end{aligned} \quad (2)$$

Although there may be many algebras such that the map $\tilde{\phi} : \mathfrak{8} \rightarrow V/V_0$ is injective, we have already chosen in Ref. [7] the Cayley-Dickson algebra with $V = \mathbb{R}^{64}$ and $V_0 = \mathbb{R}^8$.

The reason of adopting the Cayley-Dickson algebra is as follows. Originally, the flavor $SU(N)$ symmetry breaking is responsible for the meson mass difference. Recall that $SU(N)$ is one of the compact simple Lie groups, which are categorized into two types: classical and exceptional ones. The classical type represents the isometry transformation in the vector space over the real number \mathbb{R} ($= \mathbb{A}_0$), the complex number \mathbb{C} ($= \mathbb{A}_1$), and the Hamilton number (quaternion) \mathbb{H} ($= \mathbb{A}_2$). The isometry over the Cayley number (octonion) \mathbb{O} ($= \mathbb{A}_3$) is not a simple Lie group of a classical type, due to the lack of the associativity. However, due to the alternativity of \mathbb{O} , \mathbb{O} is still related to the simple Lie group of an exceptional type as [8] $G_2 \cong \text{Aut}(\mathbb{O})$, $F_4 \cong \text{Isom}(\mathbb{O}\mathbb{P}^2)$, $E_6 \cong \text{Isom}((\mathbb{C} \otimes \mathbb{O})\mathbb{P}^2)$, $E_7 \cong \text{Isom}((\mathbb{H} \otimes \mathbb{O})\mathbb{P}^2)$, $E_8 \cong \text{Isom}((\mathbb{O} \otimes \mathbb{O})\mathbb{P}^2)$, where Aut , Isom , and $\mathbb{K}\mathbb{P}^2$ represent the automorphism, isometry, and the projective plane over a (skew) field \mathbb{K} , respectively. Recall also that \mathbb{A}_n is the 2^n -dimensional Cayley-Dickson algebra over \mathbb{R} . The above mathematical facts implies that if part of the symmetry under simple Lie group is not broken, a subgroup of the isometry group over \mathbb{A}_n (with $n \leq 3$) may still survive. Noticing that \mathbb{A}_n is a subalgebra of $\mathbb{A}_{n'}$ (for $n < n'$), we may naturally choose \mathbb{A}_n (with $n \geq 4$) as an algebra by which the Lie group symmetry breaking in the sense of Eq. (1) (actually, $\mathbb{A}_6 = \mathbb{R}^{64}$ is chosen for the meson octet $\mathfrak{8}$).

Furthermore, it should be mentioned why we deal with the meson n -plet, rather than the baryon n -plet. The reason is simple. Consider, for example, the

baryon octet, which is composed of $\Lambda, \Sigma^0, \Sigma^+, \Sigma^-, p, n, \Xi^0, \Xi^-$. Difference from the meson octet, the baryon octet cannot be identified with $\mathbb{A}_6/\mathbb{R}^8$ because the baryon octet cannot be decomposed into $\mathbb{1} \oplus \mathbb{1} \oplus 2 \oplus 4$, where the masses in 4 are doubly degenerate.

The outline of this paper is as follows. In Sec. 2, we briefly review the basic property of the Cayley-Dickson algebra, where the eigenvalues of (the relative of) L_x are given. In Sec. 3, we apply to the pseudoscalar meson 16-plet the analogous map $\tilde{\phi}$ that can be applied to the meson octet, to finally obtain the mass formula $2m_{D_s} = m_{\eta_c} + m_{\eta'}$, which is well verified by experiment. In Sec. 4, we make further application to the vector meson 16-plet and to the meson 25-plet. In Sec. 5, we give summary.

2 Cayley-Dickson algebra

In this section, we briefly review the basic property of the Cayley-Dickson algebra. The Cayley-Dickson algebra \mathbb{A}_n over the real number \mathbb{R} represents the algebra structure on \mathbb{R}^{2^n} , which is given inductively. Let $x = (x_1, x_2), y = (y_1, y_2)$ be in $\mathbb{R}^{2^n} = \mathbb{R}^{2^{n-1}} \times \mathbb{R}^{2^{n-1}}$. Then the multiplication xy is given by

$$xy = (x_1y_1 - \bar{y}_2x_2, y_2x_1 + x_2\bar{y}_1), \quad \text{with } \bar{x} = (\bar{x}_1, -x_2).$$

For $n \leq 3$, \mathbb{A}_n corresponds to

$$\mathbb{A}_0 = \mathbb{R}, \mathbb{A}_1 = \mathbb{C}, \mathbb{A}_2 = \mathbb{H}, \mathbb{A}_3 = \mathbb{O},$$

which and only which are normed division algebras. The basic property of \mathbb{A}_n is summarized in Table 1. The Euclidean norm and inner product are given by $\|x\|^2 = x\bar{x} = \bar{x}x$ and $\langle x, y \rangle = \frac{1}{2}(x\bar{y} + y\bar{x})$, respectively. Due to the flexibility of \mathbb{A}_n , one obtains for all $x, y, z \in \mathbb{A}_n$ the identities [9]

$$\langle x, yz \rangle = \langle x\bar{z}, y \rangle = \langle \bar{y}x, z \rangle. \quad (3)$$

For further analysis of the algebra structure of \mathbb{A}_n , it is convenient to define the left and right multiplications $L_x, R_x : \mathbb{A}_n \rightarrow \mathbb{A}_n$ by

$$L_x(y) = xy, \quad R_x(y) = yx$$

Table 1: Basic property of \mathbb{A}_n [9], where the commutator and associator are given by $[x, y] = xy - yx$, $[x, y, z] = (xy)z - x(yz)$, respectively.

n	Property	Identity
0	Self conjugate	$x = \bar{x}$
0, 1	Commutative	$[x, y] = 0$
0, 1, 2	Associative	$[x, y, z] = 0$
0, 1, 2, 3	Alternative	$[x, x, y] = 0$
All	Flexible	$[x, y, x] = 0$

for $x \in \mathbb{A}_n$ fixed. If one tries to decompose the vector space \mathbb{R}^{2^n} into the eigenspaces of L_x , one encounters an obstacle; the eigenvalue of L_x is not necessarily given by a real number so that the eigenspace cannot be given by a real number. To remove the obstacle, we deal with the eigenvalue of $N_x := L_{\bar{x}}L_x$, instead of the eigenvalue of L_x itself. Since N_x is (real) symmetric, that is, $\langle y, N_x(z) \rangle = \langle y, \bar{x}(xz) \rangle = \langle xy, xz \rangle = \langle xz, xy \rangle = \langle z, N_x(y) \rangle$ by Eq. (3), all the eigenvalues of N_x turn out to be real numbers, so that the vector space \mathbb{R}^{2^n} can be decomposed into the eigenspaces of N_x .

Denote by S_n the set of the eigenvalues of $N_x - \|x\|^2$ for $x = (x_1, x_2) \in \mathbb{A}_n = \mathbb{A}_{n-1} \times \mathbb{A}_{n-1}$. For $n \leq 4$, it is relatively easy to calculate S_n as

$$S_i = \underbrace{\{0, \dots, 0\}}_{2^i \text{ times}} \quad (\text{for } i = 0, 1, 2, 3),$$

$$S_4 = \underbrace{\{0, \dots, 0\}}_{8 \text{ times}}; \underbrace{\{\pm\Delta^2, \dots, \pm\Delta^2\}}_{4 \text{ times}},$$

where $\Delta = 2|\mathbf{x}_1 \times \mathbf{x}_2| := 2\sqrt{\|\mathbf{x}_1\|^2\|\mathbf{x}_2\|^2 - \langle \mathbf{x}_1, \mathbf{x}_2 \rangle^2}$, with the bold face letter \mathbf{x} representing the imaginary part of x , that is, $\mathbf{x} = x - \text{Re}(x) = \frac{1}{2}(x - \bar{x})$. For $n \geq 5$, the calculation of S_n turns out to be so difficult that it may not be useful for a physical application. Before proceeding further, it should be noticed that S_n satisfies the following inclusion relation:

$$\begin{cases} S_{n-1} \subset S_n & (\text{for } n \leq 4), \\ S_{n-1} \not\subset S_n & (\text{for } n \geq 5). \end{cases}$$

The inclusion relation of $S_{n-1} \subset S_n$ implies that half of the eigenspaces of N_x for $x \in \mathbb{A}_n$ are given by the eigenspaces of N_x with x restricted to \mathbb{A}_{n-1} , which is the subspace of \mathbb{A}_n . In the present study, we extend to the map applied to the meson octet to the meson 16-plet, so that the inclusion relation of $S_{n-1} \subset S_n$ should be satisfied. Otherwise, the original map $\tilde{\phi} : \mathfrak{B} \rightarrow \mathbb{A}_6/\mathbb{R}^8$, in itself, would be violated when applied to the meson 16-plet.

Now we obtain the necessary and sufficient condition for $S_{n-1} \subset S_n$. Recall that for $n \geq 5$, \mathbb{A}_n is not given by a pair of alternative elements in \mathbb{A}_{n-1} (which is referred to as alternative entries, for short), where $S_{n-1} \subset S_n$ does not hold in general. Thus it is necessary for $S_{n-1} \subset S_n$ that an element in \mathbb{A}_n is given by alternative entries. Here, the alternative element is given by the following definition [9]:

Definition 1 *a in \mathbb{A}_n is an alternative element, if $[a, a, x] = 0$ holds for all x in \mathbb{A}_n .*

Recall that if all the elements in \mathbb{A}_n are alternative, then we simply call \mathbb{A}_n alternative. Fortunately, it is sufficient for $S_{n-1} \subset S_n$ that an element in \mathbb{A}_n is given by alternative entries:

Proposition 2 *If an element in \mathbb{A}_n is given by a pair of alternative elements in \mathbb{A}_{n-1} , then it follows that $S_{n-1} \subset S_n$.*

The proof, however, is somewhat complicated, and is referred to our previous work [7].

Once an element $x \in \mathbb{A}_n$ (for $n \geq 3$) is given by alternative entries, the eigenpolynomial for N_x turns out to be an even function with quadruple degeneracy. In this case, S_n can be written as

$$S_n = \bigcup_{i=1}^4 (\tilde{S}_n \cup (-\tilde{S}_n)) \quad (\text{for } n \geq 3)$$

where the element in the set \tilde{S}_n represents the quadruply degenerated non-negative eigenvalues of S_n . Under an appropriate parameterization, \tilde{S}_n is given by [7]

$$\begin{aligned} \tilde{S}_3 &= \{0\}, \\ \tilde{S}_4 \setminus \tilde{S}_3 &= \{\Delta^2\}, \\ \tilde{S}_5 \setminus \tilde{S}_4 &= \{\Delta_+^2, \Delta_-^2\}, \\ \tilde{S}_6 \setminus \tilde{S}_5 &= \{\Delta_{+-}^2, \Delta_{-+}^2; \Delta_{++}^2, \Delta_{--}^2\}, \\ \tilde{S}_7 \setminus \tilde{S}_6 &= \{\Delta_{+--}^2, \Delta_{-++}^2; \Delta_{+-+}^2, \Delta_{-+-}^2; \Delta_{+--}^2, \Delta_{-++}^2; \Delta_{+++}^2, \Delta_{---}^2\}, \end{aligned} \quad (4)$$

and so on, where $\Delta_{\underbrace{\pm \dots \pm}_{k \text{ times}}} = \Delta \cdot \cos(\pm\theta_k \pm \theta_{k-1} \dots \pm \theta_1)$. The parameters θ_i ($i = 1, 2, \dots, k$) are introduced by the requirement from the alternative entries. Thus for $n \leq 4$, there is no such parameter in S_n , because all the elements in \mathbb{A}_n are given by alternative entries.

3 Mass formula

In this section, we obtain a simple mass formula as $2m_{D_s} = m_{\eta_c} + m_{\eta'}$ by extending the injective map $\tilde{\phi}$ to the meson 16-plet. Comparing Eq. (2) and \tilde{S}_6 in Eq. (4), we readily find that there is an injective map $\phi \circ f : \mathfrak{B} \rightarrow \tilde{S}_6$, so that there is a one-to-one correspondence between the meson octet \mathfrak{B} and \mathbb{A}_6 as

$$\tilde{\phi} : \mathfrak{B} \rightarrow \mathbb{A}_6 / \mathbb{R}^8. \quad (5)$$

The reason of dividing \mathbb{A}_6 by \mathbb{R}^8 is due to the inclusion map i

$$i : \tilde{S}_n \hookrightarrow S_n.$$

To make the correspondence concrete, let $\theta : \mathfrak{n} \rightarrow \mathbb{R}$ be defined as $\theta = j \circ \phi \circ f$ with $j : \mathbb{R} \rightarrow \mathbb{R}$ by $j(x) = \arccos \sqrt{x/\Delta^2}$. Then the correspondence between the meson octet \mathfrak{B} and the element in \tilde{S}_6 is summarized as in the left-hand column of Table 2, from which it is found that the parameter θ_1 represents the difference between u and d quarks, and the parameter θ_2 the difference between s and u (or d) quarks. At the present stage, we have no mass relation unless the map

$\phi : M \rightarrow \tilde{S}_6$ is specified.

If we take account of the meson 16-plet, which is constructed from the SU(4) flavor system, it is expected that an analogous map to Eq. (5) should hold: $\tilde{\phi} : \mathbb{16} \rightarrow \mathbb{A}_7/\mathbb{R}^8$. If so, we have an injective map $\phi \circ f : \mathbb{16} \setminus \mathfrak{B} \rightarrow \tilde{S}_7 \setminus \tilde{S}_6$, where $\mathbb{16} \setminus \mathfrak{B}$ is given by (see Fig. 1)

$$\mathbb{16} \setminus \mathfrak{B} = (D^0, \bar{D}^0) \oplus (D^+, D^-) \oplus (D_s^+, D_s^-) \oplus \eta_c \oplus \eta'. \quad (6)$$

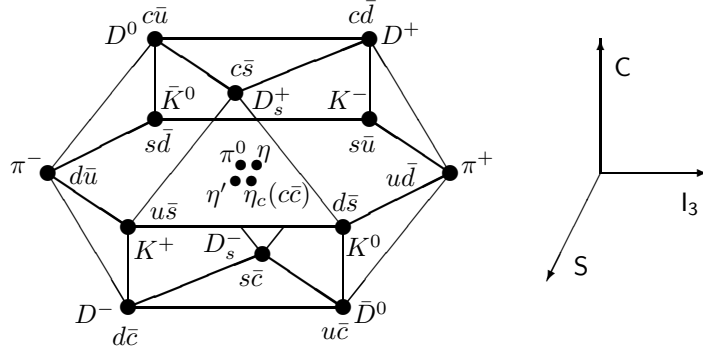


Figure 1: Meson 16-plet, where S represents the strangeness and I_3 is the z-component isospin I.

At first glance, there seems to be no such injective map $\phi \circ f$. This is because while the elements in $\tilde{S}_7 \setminus \tilde{S}_6$ are doubly degenerate, the masses in $\mathbb{16} \setminus \mathfrak{B}$ are not necessarily so; the mass of η_c is not equal to that of η' . However, a slight modification of the basis of the meson field of η_c and η' will lead to a desirable result. Notice that among the mesons in $\mathbb{16} \setminus \mathfrak{B}$, η_c and η' are the (only) mesons that have the same quantum numbers: zero electric charge, zero isospin, zero strangeness, and zero charm. Thus, it is physically possible to consider a superposed state of η_c and η' . To make the two (orthogonal) superposed mesons have the same mass, $\eta_c \oplus \eta'$ in Eq. (6) should be transformed to a “doublet” as

$$\eta_c \oplus \eta' \longrightarrow (\eta_+, \eta_-),$$

Table 2: Correspondence between the meson 16-plet and the elements in \tilde{S}_7 , where θ is a parameter representing an element in \tilde{S}_n such that $x = \Delta^2 \cos^2 \theta$ for $x \in \tilde{S}_n$.

\mathfrak{B}	θ	$\mathbb{16} \setminus \mathfrak{B}$	θ
π^0	0	D^0, \bar{D}^0	$\pm(\theta_3 - \theta_2 - \theta_1)$
π^\pm	$\pm\theta_1$	D^\pm	$\pm(\theta_3 - \theta_2 + \theta_1)$
K^\pm	$\pm(\theta_2 - \theta_1)$	D_s^\pm	$\pm(\theta_3 + \theta_2 - \theta_1)$
K^0, \bar{K}^0	$\pm(\theta_2 + \theta_1)$	$\frac{1}{\sqrt{2}}(\eta_c \pm \eta')$	$\pm(\theta_3 + \theta_2 + \theta_1)$

where $\eta_{\pm} = \frac{1}{\sqrt{2}}(\eta_c \pm \eta')$. In this case, the θ -assignment for $\mathbb{16} \setminus \mathfrak{8}$ is summarized in the right-hand column of Table 2.

To obtain a mass relation, it should be recalled that the parameter θ_1 represents the mass difference between the u and d quarks. Considering the empirical relation of $m_u \approx m_d$, we find that $\theta_1 \approx 0$ as long as the map $\theta : \mathfrak{n} \rightarrow \mathbb{R}$ is continuous. In this case, we obtain $m_{D_s} \approx m_{\eta_{\pm}}$ from $\theta(D_s) \approx \theta(\eta_{\pm})$, that is

$$2m_{D_s} \approx m_{\eta_c} + m_{\eta'}. \quad (7)$$

The recent experimental value of the meson mass (see Table 3) leads to

$$\frac{m_{\eta_c} + m_{\eta'}}{2m_{D_s}} = 1.00033 \pm 0.00035,$$

which indicates the validity of the relation Eq. (7). In evaluating the standard deviation, we have assumed that m_{η_c} , $m_{\eta'}$, and m_{D_s} are independent variables.

Table 3: Recent experimental value of the meson mass [10].

Meson	Mass (MeV)
η_c	2980.5 ± 1.2
η'	957.78 ± 0.06
D_s	1968.49 ± 0.34

At the end of this section, we discuss the usefulness of Eq. (7) as an η' -including mass formula. In the standard quark model, however, it is not so easy a task to relate $m_{\eta'}$ to other meson masses, due to several reasons. One is that η' is not a pure SU(3) singlet η_1 , but a mixture with one of the SU(3) octet, η_8 , through the relation

$$\begin{pmatrix} \eta' \\ \eta \end{pmatrix} = \begin{pmatrix} \cos \vartheta & \sin \vartheta \\ -\sin \vartheta & \cos \vartheta \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_8 \end{pmatrix}, \quad (8)$$

where $\eta_1 = \frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s})$ and $\eta_8 = \frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s})$. To eliminate the mixing angle ϑ from the theory, one of the orthodox methods is to introduce the pion and kaon masses, with the result known as the Schwinger relation [11]. However, the Schwinger relation, which holds under the assumption of the ‘‘ideal mixing’’ $\eta' = s\bar{s}$ and $\eta = \frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d})$, is not satisfactory for the pseudoscalar mesons, due to the large deviation from the ideal mixing. For the pseudoscalar mesons, the interaction between the different flavor (such as $u\bar{u} \leftrightarrow s\bar{s}$) cannot be neglected, so that the mass caused by this interaction is comparable to the constituent quark mass [12]. Taking these things into account, we find it somewhat marvelous that $m_{\eta'}$ satisfies so simple a relation like Eq. (7).

4 Further application

In this section, we apply an analogous map $\tilde{\phi}$ to the vector meson 16-plet and to the meson 25-plet. However, it is found that further application to the vector 16-plet causes a delicate problem and that the application to the 25-plet is not viable as follows.

First, we deal with the case of vector meson 16-plet, where $\rho, K^*, \phi, \omega, J/\psi, D^*, D_s^*$ take the place of $\pi, K, \eta, \eta', \eta_c, D, D_s$, respectively. Suppose that the map $\tilde{\phi}$ can be applied to the vector meson 16-plet, the relation of Eq. (7) might be replaced by $2m_{D_s^*} \approx m_{J/\psi} + m_\omega$, which, however, is not satisfactory compared to Eq. (7). The failure of the relation of $2m_{D_s^*} \approx m_{J/\psi} + m_\omega$ can be interpreted as follows. In deriving Eq. (7), it should be recalled that we assume that the injective map $\theta : \mathfrak{n} \rightarrow \mathbb{R}$ is continuous. This implies that $\cos \theta$ should decrease monotonically with respect to the meson mass, that is, $\cos \theta(\varphi) \geq \cos \theta(\varphi') \iff m_\varphi \leq m_{\varphi'}$ for $\varphi, \varphi' \in \mathfrak{n}$ (see Table 2, where m_{π^0} is the smallest). Thus the condition of $m_{\rho^0} < m_{\rho^\pm}$ is necessary for the relation of $2m_{D_s^*} \approx m_{J/\psi} + m_\omega$. Different from the pseudoscalar meson, it is quite a delicate problem to determine the sign of $\Delta m_\rho (\equiv m_{\rho^0} - m_{\rho^\pm})$. The Particle Data Group gives the values of $\Delta m_\rho = -0.7 \pm 0.8 \text{ MeV}$ [10], while some theoretical considerations indicate that $-0.4 \text{ MeV} < \Delta m_\rho < 0.7 \text{ MeV}$ [13], $\Delta m_\rho = -0.02 \pm 0.02 \text{ MeV}$ [14], $\Delta m_\rho = 0.62 \text{ MeV}$ [15], and $\Delta m_\rho \sim 1 \text{ MeV}$ [16]. As long as Δm_ρ is positive, it is not necessary to hold a relation of $2m_{D_s^*} \approx m_{J/\psi} + m_\omega$ in the sense above. Conversely, the failure of the relation of $2m_{D_s^*} \approx m_{J/\psi} + m_\omega$ suggests the relation of $m_{\rho^0} > m_{\rho^\pm}$.

Finally, we deal with the meson 25-plet, where b quark is taken into account. In this case, we cannot single out an enlarged algebra $\mathbb{A}_8/\mathbb{R}^8$, due to $\dim(\mathbb{A}_8/\mathbb{R}^8) = 32 (> 25)$. Thus, we can only make a decomposition as $25 = 1\mathbb{6} \oplus \mathfrak{8}_b \oplus 1_b$, where $\mathfrak{8}_b$ and $1_b (= b\bar{b})$ represent the b -quark related octet and singlet, respectively. Although the $\mathfrak{8}_b$ can be identified with $(\mathbb{A}_7 \setminus \mathbb{A}_6)/\mathbb{R}^8$ due to the injective map $\phi \circ f : \mathfrak{8}_b \rightarrow \tilde{\mathfrak{S}}_7 \setminus \tilde{\mathfrak{S}}_6$ [this should be contrasted with the injective map $\phi \circ f : \mathfrak{8} \rightarrow \tilde{\mathfrak{S}}_6$ for the original meson octet $\mathfrak{8}$ in Eq. (2)], no new mass relation is obtainable as in the case of the original $\mathfrak{8}$.

5 Summary

So far, we have obtained the mass formula Eq. (7) by identifying the pseudoscalar meson 16-plet with $\mathbb{A}_7/\mathbb{R}^8$ through the injective map $\tilde{\phi} : 1\mathbb{6} \rightarrow \mathbb{A}_7/\mathbb{R}^8$. The point is that η_c and η' can be mixed to form a “doublet” (η_+, η_-) so as to have the same mass. This mixture is possible because η_c and η' have the same quantum numbers as charge, isospin, strangeness, and charm. The resultant mass formula is well verified by experiment. The application of an analogous map $\tilde{\phi}$ to the vector meson 16-plet brings about quite a delicate problem in connection with the sign of $m_{\rho^0} - m_{\rho^\pm}$. Further application to the meson 25-plet is not viable due to the lack of an enlarged algebra.

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