NORM CONVERGENCE OF SECTORIAL OPERATORS ON VARYING HILBERT SPACES

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ABSTRACT. Convergence of operators acting on a given Hilbert space is an old and well studied topic in operator theory. The idea of introducing a related notion for operators acting on varying spaces is natural. Many previous contributions to this subject consider either concrete examples of perturbations, or an abstract setting where weak or strong convergence of the resolvents is used. However, it seems that the first results on *norm* resolvent convergence in this direction have been obtained only recently, to the best of our knowledge. Here we consider sectorial operators on Hilbert spaces that depend on a parameter. We define a notion of convergence that generalises convergence of the resolvents in operator norm to the case when the operators act on different spaces. In addition, we show that this kind of convergence is compatible with the functional calculus of the operator and moreover implies convergence of the spectrum. Finally, we present examples for which this convergence can be checked, including convergence of coefficients of parabolic problems. Convergence of a manifold (roughly speaking consisting of thin tubes) towards the manifold's skeleton graph plays a prominent role, being our main application.

1. INTRODUCTION

Convergence of operators in the resolvent sense is a classical issue in operator theory. Early results go back, at least implicitly, to Rayleigh and Schrödinger. The first systematic investigations are due to Trotter, Rellich and Kato.

If the operators under consideration arise from sesquilinear forms on a Hilbert space, there are powerful methods available to study convergence of the operators, in particular in the self-adjoint case. In Kato's classical monograph [18] one finds a detailed study of various kinds of convergence with focus on strong and norm convergence in the resolvent sense and the consequences of the respective convergence for the behaviour of the spectrum. Moreover, Kato gives criteria in terms of the forms that allow to check easily in many situations that a sequence of operators arising from uniformly sectorial forms converge either strongly or in norm. Those criteria are particularly easy to verify if the forms satisfy some monotonicity assumptions, i.e., they converge from above or from below.

2010 *Mathematics Subject Classification.* 34B45, 35P05, 47D06.

Key words and phrases. Diffusion equations on networks; Approximation schemes; Sectorial operators; Norm convergence of operators in different Hilbert spaces; Spectral convergence.

A similar, very successful approach has been developed by Mosco [27] in the context of symmetric Dirichlet forms, i.e., forms associated with sub-Markovian self-adjoint C_0-semigroups. He succeeds in obtaining *strong* resolvent convergence, spectral convergence and convergence of the generated semigroups from simple conditions on the forms, and in fact resolvent convergence can be easily characterised via the forms.

In the context of homogenisation problems, convergence results for operators acting in different spaces have been considered e.g. in [34, 29, 20, 24, 25, 33] on an abstract level and in concrete examples like $L^2(\Omega, \mu_\varepsilon)$ with ε-depending measures μ_ε converging weakly to a measure μ_0, or even with changing domains. In the case of homogenisation problems on varying domains, the identification operators often consists of restrictions and extensions of functions, and the latter operator is not always bounded on the form domains (see e.g. [23]). Note that these results imply strong or weak convergence of the resolvents, and imply convergence of the discrete spectrum. Moreover, the strong convergence of the corresponding semigroups follows, see [33] and references therein. On the other hand, these methods can also be extended to certain nonlinear settings, cf. [26].

On the other hand, a natural approach to infinite dimensional problems is based on approximation via finite dimensional spaces, see e.g. [15]. If in particular one considers diffusion-like processes, form methods are a mighty tool. Convergence schemes for Dirichlet forms on varying spaces of finite dimension have been considered by Mosco and others, particularly in the context of stochastic diffusion equations and diffusion on fractals, see e.g. [19, 11, 14, 5, 28]. There are similar convergence results for manifolds, metric measure spaces, Hilbert spaces, quadratic forms on different Hilbert spaces in [21, 16, 17]. Though, in these works only the *strong* convergence of the associated operators is considered.

Moreover, elliptic equations on varying domains with respect to several boundary conditions on several spaces have been widely studied. We refer to the work of Stollmann [35] on strong and norm resolvent convergence of Dirichlet Laplacians on varying domains (see also [38] for the strong resolvent convergence), as well as the works of Stolz and Weidmann about the approximation of singular Sturm-Liouville operators by regular ones; using again a domain change, see e.g. [36]. Finally, we refer to Daners' survey article [9] for more results on problems in the spirit of form methods and further references.

In this article, in contrast, we are interested in convergence properties *in operator norm* of operators associated with forms that act on varying Hilbert spaces, where the identification operators are not necessarily given in a canonical way. Although our setting is more restrictive than e.g. the strong convergence one used in homogenisation problems, there is still a wide class of examples in which the necessary assumptions are naturally fulfilled. We would like to stress that the convergence in operator norm of the resolvents in [35] uses the fact that all spaces are canonically embedded in a common space $L^2(\mathbb{R}^d)$, which is not necessarily true in our situation.

Let us now describe the results of this article in more details. We investigate convergence of m-sectorial operators A_ε, which are allowed to act on different Hilbert spaces H_ε, towards an m-sectorial operator A_0 acting on a Hilbert space H_0 by form methods. Our notion of form convergence resembles a sufficient condition for convergence of the resolvent in operator norm due to Kato and is designed in a way that allows to check the conditions easily in many applications. The notation is introduced in Section 2. Our main abstract results are contained in Section 3. More precisely, in Section 3.1 we show that if A_ε converges to A_0, then also $\varphi(A_\varepsilon)$ converges to $\varphi(A_0)$ in norm if φ is in a suitable class of bounded holomorphic functions (Theorem 3.7). We prove in Section 3.2 that the spectra of A_ε converge to the spectrum of A_0 (Corollary 3.14 and Theorem 3.17). Similar results for self-adjoint operators can be found in [31]. In [32], also convergence of certain non-self-adjoint operators in a specific situation is considered. In Section 3.3 we consider invariance of subsets of the Hilbert spaces and extrapolated semigroups. In particular, if we assume that the Hilbert spaces H_ε are L^2-spaces and the semigroups $(\mathrm{e}^{tA_\varepsilon})_{t\geq 0}$ generated by the A_ε are bounded on the corresponding L^∞-spaces, then we can prove that under suitable assumptions on the convergence scheme the semigroups $\mathrm{e}^{tA_\varepsilon}$ converge to e^{tA_0} also as operators on L^p for $p \in [2,\infty)$ (Theorem 3.23).

Section 4 describes several situations to which our results can be applied without much effort. In Section 4.1 we put the Fourier series expansion with respect to eigenvectors into our framework to exhibit the ideas at an elementary example. In Section 4.2 we apply our results in a situation where A_ε are elliptic operators on a domain whose coefficients converge to the coefficients of an elliptic operator A_0. More precisely, we consider generalised Wentzell-Robin boundary conditions, which are a natural candidate for our framework because the natural choice of inner products on the underlying Hilbert space depends on the coefficients even if the Hilbert spaces coincide as sets. In this setting we generalise results of Coclite et al. [7] and complement those of [8] (Theorem 4.4). In Section 4.3 we adopt a variational approach to elliptic operators whose coefficient may vanish at the boundary, as in [2] (Theorem 4.5). Observe that in this situation the limiting Hilbert space differs from the approximating ones — not only with respect to the inner product, but even as a set —, so that Kato's classical results cannot be applied directly.

Our main example, however, is the convergence of tube-like manifolds to the skeleton graph, which we investigate in Section 5. More precisely, we let $H_\varepsilon = L^2(X_\varepsilon)$ for $\varepsilon > 0$, where X_ε is a manifold consisting of $(m+1)$-dimensional objects resembling tubes (*edge neighbourhoods*) that are connected in $(m+1)$-dimensional junction regions (*vertex neighbourhoods*). If these tubes have a uniform thickness ε, then it is natural to expect that the behaviour of physical processes on X_ε which are described by an elliptic operator is close to the behaviour of an analogous process on the *skeleton graph* X_0, which is a 1-dimensional manifold with singularities at the vertices. We show that under some uniformity assumptions we indeed have resolvent convergence and convergence of finite parts of the spectrum (Theorem 5.9). Note that the convergence results for manifolds and metric measure spaces of Kasue et al. [21, 16, 17] cannot be used here, since our families of manifolds $(X_\varepsilon)_\varepsilon$ do not satisfy the necessary curvature bounds (see e.g. [16], p. 1224).

Robin boundary conditions are closely related to Neumann boundary conditions from the perspective of the quadratic (or, more generally, sesquilinear) form approach. In fact, the form domain is the same, while the forms differs only by a (possibly non-symmetric) boundary term. This allows us to rely on the results in [31] for treating the principal term, so that we only have to handle the boundary term.

One of our motivations for this example is given by the articles of Grieser [12] and Cacciapuoti–Finco [6]. Grieser considered general boundary conditions (Dirichlet, Robin or Neumann) on a manifold (if embedded, the embedding is "straight") shrinking to a metric graph. He showed that the limit behaviour depends on the scattering matrix at the threshold of the essential spectrum, so that, generically, the limit operator is decoupling. Cacciapuoti and Finco use a simple wave-guide model (in our terminology, a flat manifold converging to a graph consisting of two (half-infinite) edges and one vertex only). Using curved embedded edges with different scalings of the transversal and longitudinal curvature, they obtain non-trivial couplings starting with Robin boundary conditions. However, their notion of convergence differs significantly from ours since one can use separation of variables due to the simple product topology of the space. For the convergence of unitary groups in a similar setting we refer to the work of Teufel and Wachsmuth [37].

Grieser and Cacciapuoti-Finco use scale-invariant Robin boundary conditions of the form $\frac{\partial u}{\partial \nu} = \beta_\varepsilon u$ with $\beta_\varepsilon = \beta/\varepsilon$. This scaling leads to transversal eigenvalues of the order ε^{-2}. In particular, a rescaling of the limit operator is necessary in order to expect convergence, see Remark 5.2. Using Robin boundary conditions with coupling of order $\beta_\varepsilon = \mathrm{O}(1)$ near the vertices and $\beta_\varepsilon = \mathrm{O}(\varepsilon^{3/2})$ along the edge neighbourhoods, we are able to construct a family of manifolds with boundary, such that, in the limit, the corresponding Laplacians converge to a Laplacian on the underlying metric graph with generalised, possibly non-local δ-interactions in the vertices. Using the same idea as in [10], we can further approximate other couplings like the δ'-interaction.

Acknowledgements. This article has been written while the third author was visiting the University of Ulm. He would like to thank the University of Ulm for the hospitality and the financial support.

2. Notation

We consider m-sectorial operators (in the sense of Kato) on Hilbert spaces. For our approach, it is convenient to work with such an operator in terms of its associated form. We briefly sketch the correspondence of m-sectorial operators and sesquilinear forms. For these results and much more information we refer to [18], Chapter VI. We point out that there is a one-to-one correspondence between bounded, H-elliptic forms and m-sectorial operators, so there is no loss of generality in working with an m-sectorial operator only in terms of its form.

Let H be a Hilbert space and let V be a dense subspace of H that is a Hilbert space in its own right, and which is continuously embedded into H. We say that a sesquilinear form $a\colon V \times V \to \mathbb{C}$ is *bounded* if there exists $M \geq 0$ such that

$$|a(u,v)| \leq M\|u\|_V\|v\|_V \qquad \text{for all } u,v \in V, \tag{2.1}$$

and we call a *H-elliptic* or simply *elliptic* if there exist $\omega \in \mathbb{R}$ and $\alpha > 0$ such that

$$\operatorname{Re} a(u,u) + \omega\|u\|_H^2 \geq \alpha\|u\|_V^2 \qquad \text{for all } u \in V. \tag{2.2}$$

In this case

$$\|u\|_a := \sqrt{\operatorname{Re} a(u,u) + \omega\|u\|_H^2}$$

defines an equivalent norm on V. More precisely, since V is continuously embedded into H, there exists $c \geq 0$ such that

$$\|u\|_H \leq c_V\|u\|_V \qquad \text{for all } u \in V. \tag{2.3}$$

For any such constant c_V, we obtain

$$\alpha\|u\|_V^2 \leq \|u\|_a^2 \leq \big(M + c_V^2\omega\big)\|u\|_V^2 \qquad \text{for all } u \in V. \tag{2.4}$$

We define the *associated operator* A of a by

$$u \in D(A) \text{ and } Au = f \quad :\Longleftrightarrow \quad u \in V \text{ and } a(u,v) = \langle f|v\rangle_H \ \forall v \in V,$$

and we emphasise that since the form a is not assumed to be symmetric, the associated operator A is in general not self-adjoint.

Consider for a moment the form $b\colon V \times V \to \mathbb{C}$ given by

$$b(u,v) := a(u,v) + \omega\langle u|v\rangle_H,$$

which is associated with the operator $A + \omega$. Then by (2.1) and (2.4)

$$|\operatorname{Im} b(u,u)| = |\operatorname{Im} a(u,u)| \leq |a(u,u)| \leq M\|u\|_V^2 \leq \frac{M}{\alpha}\|u\|_a^2 = \frac{M}{\alpha} b(u,u).$$

The proof of Theorem 1.53 of [30] now shows that $\sigma(A+\omega) \subset \overline{\Sigma}_{\arctan \frac{M}{\alpha}}$, where

$$\Sigma_\theta := \big\{ z \in \mathbb{C}\setminus\{0\} : |\arg(z)| < \theta \big\}. \tag{2.5}$$

Moreover, denoting here and in the following

$$R(z,A) := (z-A)^{-1},$$

for every $\theta \in (\arctan \frac{M}{\alpha}, \pi]$ we have

$$\|zR(z,A+\omega)\|_{\mathscr{L}(H)} \leq D_\theta \quad \text{for all } z \notin \Sigma_\theta,$$

i.e., $\sigma(A) \subset -\omega + \Sigma_{\arctan \frac{M}{\alpha}}$ and

$$\|R(z,A)\|_{\mathscr{L}(H)} \leq \frac{D_\theta}{|z+\omega|} \quad \text{for all } z \notin \Sigma_\theta - \omega \tag{2.6}$$

with

$$D_\theta := \frac{1}{\sin(\theta - \arctan \frac{M}{\alpha})}.$$

Operators satisfying such a condition are frequently called *m-sectorial* (in the sense of Kato).

Definition 2.1. Let $(H_\varepsilon)_{\varepsilon\geq 0}$ be a family of Hilbert spaces. We say that $(a_\varepsilon)_{\varepsilon\geq 0}$ is an *equi-elliptic family of sesquilinear forms with form domains* $(V_\varepsilon)_{\varepsilon\geq 0}$, if there exist M, ω, α and c_V not depending on ε such that (2.1), (2.2) and (2.3) are satisfied for all $\varepsilon \geq 0$, i.e., all the constants are uniform with respect to ε. We call ω the associated *vertex* and $\arctan(M/\alpha)$ the associated *semi-angle*.

Remark 2.2. If $(a_\varepsilon)_{\varepsilon\geq 0}$ is an equi-elliptic family of sesquilinear forms, then by (2.4) the norms $\|\cdot\|_{V_\varepsilon}$ and $\|\cdot\|_{a_\varepsilon}$ are equivalent with a uniform constant. This allows us to use either of these two norm interchangeably in the following. For the theoretical part, the form norm is more convenient. But for applications, we usually prefer to equip V_ε with other norms that are easier to handle.

Now let $(a_\varepsilon)_{\varepsilon\geq 0}$ be a family of equi-elliptic family of sesquilinear forms on Hilbert spaces $(H_\varepsilon)_{\varepsilon\geq 0}$. We want to "measure" the distance between the associated operators $(A_\varepsilon)_{\varepsilon>0}$ and A_0. For this, we introduce *identification operators* $J^{\uparrow\varepsilon}\colon H_0 \longrightarrow H_\varepsilon$ and $J^{\downarrow\varepsilon}\colon H_\varepsilon \longrightarrow H_0$, $\varepsilon > 0$, which are considered to be "almost unitary", i.e., unitary up to some error. For technical reasons, it is also convenient to introduce identification operators $J_1^{\uparrow\varepsilon}\colon V_0 \longrightarrow V_\varepsilon$ and $J_1^{\downarrow\varepsilon}\colon V_\varepsilon \longrightarrow V_0$ for the form domains, which are considered to be "almost the restrictions" of $J^{\uparrow\varepsilon}$ and $J^{\downarrow\varepsilon}$ to V_0 and V_ε, respectively.

We make this more explicit and use the following terminology, inspired by the technique developed in Appendix A of [31] and [32].

Definition 2.3. Let $\varepsilon > 0$, and let a_0 and a_ε be bounded, elliptic, sesquilinear forms on Hilbert spaces H_0 and H_ε with form domains V_0 and V_ε. Denote the associated operators by A_0 and A_ε, respectively. For parameters $\delta_\varepsilon > 0$ and $\kappa \geq 1$ we say that a_0 and a_ε are *δ_ε-κ-quasi-unitarily equivalent* if there exist bounded operators $J^{\uparrow\varepsilon} \in \mathscr{L}(H_0, H_\varepsilon)$, $J^{\downarrow\varepsilon} \in \mathscr{L}(H_\varepsilon, H_0)$, $J_1^{\uparrow\varepsilon} \in \mathscr{L}(V_0, V_\varepsilon)$ and $J_1^{\downarrow\varepsilon} \in \mathscr{L}(V_\varepsilon, V_0)$ that satisfy the following conditions.

(2.7a) $$\|J^{\uparrow\varepsilon} - J_1^{\uparrow\varepsilon}\|_{\mathscr{L}(V_0,H_\varepsilon)} \leq \delta_\varepsilon \quad \text{and} \quad \|J^{\downarrow\varepsilon} - J_1^{\downarrow\varepsilon}\|_{\mathscr{L}(V_\varepsilon,H_0)} \leq \delta_\varepsilon;$$

(2.7b) $$\|J^{\downarrow\varepsilon} - (J^{\uparrow\varepsilon})^*\|_{\mathscr{L}(H_\varepsilon,H_0)} \leq \delta_\varepsilon \quad \text{and} \quad \|J^{\uparrow\varepsilon} - (J^{\downarrow\varepsilon})^*\|_{\mathscr{L}(H_0,H_\varepsilon)} \leq \delta_\varepsilon;$$

(2.7c) $$\|\mathrm{id} - J^{\downarrow\varepsilon}J^{\uparrow\varepsilon}\|_{\mathscr{L}(V_0,H_0)} \leq \delta_\varepsilon \quad \text{and} \quad \|\mathrm{id} - J^{\uparrow\varepsilon}J^{\downarrow\varepsilon}\|_{\mathscr{L}(V_\varepsilon,H_\varepsilon)} \leq \delta_\varepsilon;$$

(2.7d) $$\|J^{\uparrow\varepsilon}\|_{\mathscr{L}(H_0,H_\varepsilon)} \leq \kappa \quad \text{and} \quad \|J^{\downarrow\varepsilon}\|_{\mathscr{L}(H_\varepsilon,H_0)} \leq \kappa;$$

(2.7e) $$\left|a_0(f, J_1^{\downarrow\varepsilon}u) - a_\varepsilon(J_1^{\uparrow\varepsilon}f, u)\right| \leq \delta_\varepsilon \|f\|_{V_0}\|u\|_{V_\varepsilon} \quad \text{for all } f \in V_0 \text{ and } u \in V_\varepsilon.$$

If $(a_\varepsilon)_{\varepsilon\in\geq 0}$ is an equi-elliptic family of sesquilinear forms and if there exists $\kappa \geq 1$ and a family $(\delta_\varepsilon)_{\varepsilon>0}$ of positive real numbers with $\lim_{\varepsilon\to 0}\delta_\varepsilon \to 0$ such that a_ε is δ_ε-κ-quasi-unitarily equivalent to a_0, then we say that the family $(a_\varepsilon)_{\varepsilon>0}$ *converges to a_0 (in norm)* as $\varepsilon \to 0$.

Remark 2.4.

(i) For $\delta_\varepsilon = 0$ the associated operators A_0 and A_ε are unitarily equivalent. In fact, if $\delta_\varepsilon = 0$, conditions (2.7b) and (2.7c) states that $J^{\uparrow\varepsilon}$ is unitary with inverse $J^{\downarrow\varepsilon}$. Since by (2.7a) the operators $J_1^{\uparrow\varepsilon}$ and $J_1^{\downarrow\varepsilon}$ are the restrictions of $J^{\uparrow\varepsilon}$ and $J^{\downarrow\varepsilon}$, condition (2.7e) states that $J^{\uparrow\varepsilon}$ realises the unitary equivalence of A_0 and A_ε.

(ii) In the applications we have in mind, it is easy to check that $J_1^{\uparrow\varepsilon}\colon V_0 \longrightarrow V_\varepsilon$ and $J_1^{\downarrow\varepsilon}\colon V_\varepsilon \longrightarrow V_0$ are bounded: if $J_1^{\uparrow\varepsilon}$ is bounded as an operator into H_ε and takes values in V_ε, then it is bounded as an operator into V_ε by the closed graph theorem, and an analogous argument applies to $J_1^{\downarrow\varepsilon}$. Note that in the context of homogenisation problems, the boundedness of $J_1^{\downarrow\varepsilon}$ is not always assured (see e.g. [23]).

(iii) The two conditions in (2.7b) are equivalent to each other. In fact, they can be rephrased as

$$|\langle J^{\uparrow\varepsilon} f|u\rangle_{H_\varepsilon} - \langle f|J^{\downarrow\varepsilon} u\rangle_{H_0}| \le \delta_\varepsilon \|f\|_{H_0}\|u\|_{H_\varepsilon} \text{ for all } f \in H_0 \text{ and } u \in H_\varepsilon. \tag{2.7b'}$$

(iv) Condition (2.7c) does *not* imply that $J^{\downarrow\varepsilon}J^{\uparrow\varepsilon}$ or $J^{\uparrow\varepsilon}J^{\downarrow\varepsilon}$ are invertible operators. In fact, in most of our examples one of the two operators will have a large kernel, whereas the other has a small range.

(v) Only (2.7e) depends on the evolution processes acting on H_ε, while the first four conditions are solely related to the function spaces. So if we have verified (2.7a)–(2.7d) in one situation, those conditions are satisfied for a large class of examples. To be more specific, we can reuse the results obtained in [31] for Neumann boundary conditions and do not have to check these four conditions once again for the discussion of the Laplace operator with Robin boundary conditions in Section 5.

Example 2.5. Let a_0 and a_ε be forms on a single Hilbert space H with equal form domain V and let $J^{\uparrow\varepsilon}$, $J_1^{\uparrow\varepsilon}$, $J^{\downarrow\varepsilon}$ and $J_1^{\downarrow\varepsilon}$ be the identity on H resp. V. Then the conditions of Definition 2.3 are satisfied if and only if

$$|a_0(u,v) - a_\varepsilon(u,v)| \le \delta_\varepsilon \|u\|_V \|v\|_V \tag{2.8}$$

for all $u, v \in V$, i.e., $\|a_0 - a_\varepsilon\| \to 0$ in the operator norm on the space of sesquilinear forms on V.

In the setting of Example 2.5, if (2.8) is satisfied for a family $(\delta_\varepsilon)_{\varepsilon>0}$ satisfying $\lim_{\varepsilon\to 0}\delta_\varepsilon = 0$, then the resolvent of A_ε converges to the resolvent of A_0 in operator norm uniformly on compact subsets of $\rho(A_0)$. In fact, it would suffice if (2.8) is satisfied whenever $u = v$, see Theorem VI.3.6 of [18]. In this sense, our results are a generalisation of this classical result to the setting of varying spaces. We can also deduce similar consequences like in the classical situation, e.g. convergence of the spectra.

3. Abstract results

For the whole section, let $(a_\varepsilon)_{\varepsilon\ge 0}$ be an equi-elliptic family of sesquilinear forms for constants M, ω, α and c_V as in (2.1), (2.2) and (2.3), and let $(A_\varepsilon)_{\varepsilon\ge 0}$ denote the associated operators. We always let the operators $J^{\uparrow\varepsilon}$, $J^{\downarrow\varepsilon}$, $J_1^{\uparrow\varepsilon}$ and $J_1^{\downarrow\varepsilon}$ and the constant κ be as in Definition 2.3.

3.1. **Functional calculus.** In our situation, each operator $A_\varepsilon + \omega$ is invertible by the Lax-Milgram theorem due to (2.2). It is known that in this situation the operators $A_\varepsilon + \omega$ have bounded H^∞-calculus, see Sec. 11 of [22], Sec. 5.2 of [4] or Sec. 7.3 of [13]. We are going to show that if $(a_\varepsilon)_\varepsilon$ converges to a_0 in the sense of Definition 2.3, then the operators $\varphi(A_\varepsilon)$ converge to $\varphi(A_0)$ in a suitable sense as $\varepsilon \to 0$ for an admissible holomorphic function φ.

We start with some auxiliary estimates. For brevity, in the proofs we write

$$R_\varepsilon(z) := R(z, A_\varepsilon) = (z - A_\varepsilon)^{-1}.$$

Lemma 3.1. *Let $\theta \in (\arctan \frac{M}{\alpha}, \pi]$. There exists $C_\theta \geq 0$ such that for all $\varepsilon \geq 0$ and $z \notin \Sigma_\theta - \omega$*

$$\|R(z, A_\varepsilon)\|_{\mathscr{L}(H_\varepsilon, V_\varepsilon)} \leq \frac{C_\theta}{\sqrt{|z+\omega|}} \quad \text{and} \quad \|R(z, A_\varepsilon)^*\|_{\mathscr{L}(H_\varepsilon, V_\varepsilon)} \leq \frac{C_\theta}{\sqrt{|z+\omega|}}.$$

Proof. Let $u \in H_\varepsilon$ be fixed. Then by (2.4)

$$\begin{aligned}
\alpha \|R_\varepsilon(z)u\|_{V_\varepsilon}^2 \leq \|R_\varepsilon(z)u\|_{a_\varepsilon}^2 &= \operatorname{Re} a_\varepsilon(R_\varepsilon(z)u, R_\varepsilon(z)u) + \omega \|R_\varepsilon(z)u\|_{H_\varepsilon}^2 \\
&= \operatorname{Re}\langle(\omega + A_\varepsilon)R_\varepsilon(z)u | R_\varepsilon(z)u\rangle_{H_\varepsilon} \\
&= \operatorname{Re}\langle(\omega + z)R_\varepsilon(z)u | R_\varepsilon(z)u\rangle_{H_\varepsilon} - \operatorname{Re}\langle u | R_\varepsilon(z)u\rangle_{H_\varepsilon} \\
&\leq \big(|\omega + z| \cdot \|R_\varepsilon(z)\|_{\mathscr{L}(H_\varepsilon)} + 1\big) \|R_\varepsilon(z)\|_{\mathscr{L}(H_\varepsilon)} \|u\|_{H_\varepsilon}^2
\end{aligned}$$

Now (2.6) implies the first estimate for

$$C_\theta := \sqrt{\frac{(1 + D_\theta)D_\theta}{\alpha}}.$$

The second estimate can be proved like the first. In fact, $R(z, A_\varepsilon)^* = R(\overline{z}, A_\varepsilon^*)$ and A_ε^* is associated with the form a_ε^* given by

$$a_\varepsilon^*(u, v) := \overline{a_\varepsilon(v, u)}.$$

Thus it suffices to realise that a_ε^* satisfies (2.1) and (2.2) for the same constants as a_ε. ☐

Lemma 3.2. *Let $J^{\uparrow\varepsilon}$ and $J^{\downarrow\varepsilon}$ be as in Definition 2.3, and let $B_\varepsilon \in \mathscr{L}(H_\varepsilon, V_\varepsilon)$ and $B_0 \in \mathscr{L}(H_0, V_0)$. Then*

$$\|B_\varepsilon - J^{\uparrow\varepsilon} B_0 J^{\downarrow\varepsilon}\|_{\mathscr{L}(H_\varepsilon)} \leq \kappa \|J^{\downarrow\varepsilon} B_\varepsilon - B_0 J^{\downarrow\varepsilon}\|_{\mathscr{L}(H_\varepsilon, H_0)} + \delta_\varepsilon \|B_\varepsilon\|_{\mathscr{L}(H_\varepsilon, V_\varepsilon)}.$$

Proof. By (2.7c) and (2.7d) we have

$$\begin{aligned}
\|B_\varepsilon - J^{\uparrow\varepsilon} B_0 J^{\downarrow\varepsilon}\|_{\mathscr{L}(H_\varepsilon)} &\leq \|B_\varepsilon - J^{\uparrow\varepsilon} J^{\downarrow\varepsilon} B_\varepsilon\|_{\mathscr{L}(H_\varepsilon)} + \|J^{\uparrow\varepsilon} J^{\downarrow\varepsilon} B_\varepsilon - J^{\uparrow\varepsilon} B_0 J^{\downarrow\varepsilon}\|_{\mathscr{L}(H_\varepsilon)} \\
&\leq \delta_\varepsilon \|B_\varepsilon\|_{\mathscr{L}(H_\varepsilon, V_\varepsilon)} + \kappa \|J^{\downarrow\varepsilon} B_\varepsilon - B_0 J^{\downarrow\varepsilon}\|_{\mathscr{L}(H_\varepsilon, H_0)}.
\end{aligned}$$ ☐

Lemma 3.3. *Let $J^{\uparrow\varepsilon}$ and $J^{\downarrow\varepsilon}$ be as in Definition 2.3, and let $f \in V_0$ and $u \in H_\varepsilon$. Then*

$$\|J^{\downarrow\varepsilon} u - f\|_{H_0} \leq \kappa \|u - J^{\uparrow\varepsilon} f\|_{H_\varepsilon} + \delta_\varepsilon \|f\|_{V_0}.$$

Proof. Let $g \in H_0$. By (2.7c) and (2.7d)

$$\begin{aligned}\left|\langle J^{\downarrow\varepsilon} u - f | g\rangle_{H_0}\right| &\le \left|\langle J^{\downarrow\varepsilon}(u - J^{\uparrow\varepsilon} f) | g\rangle_{H_\varepsilon}\right| + \left|\langle J^{\downarrow\varepsilon} J^{\uparrow\varepsilon} f - f | g\rangle_{H_\varepsilon}\right| \\ &\le \kappa \|u - J^{\uparrow\varepsilon} f\|_{H_\varepsilon} \|g\|_{H_0} + \delta_\varepsilon \|f\|_{V_0} \|g\|_{H_0}.\end{aligned}$$

Since g is arbitrary, this proves the claim. □

Now we prove the key estimate of this section.

Proposition 3.4. *Let $(a_\varepsilon)_\varepsilon$ be an equi-elliptic family of sesquilinear forms with vertex ω and semi-angle $\theta_0 := \arctan(M/\alpha)$ as in Definition 2.1, and let a_ε and a_0 be δ_ε-κ-quasi-unitarily equivalent (see Definition 2.3). Moreover, let $r > 0$ and $\theta \in (\theta_0, \pi]$. Then there exist constants $C_{\theta,r,1} > 0$ and $C_{\theta,r,2} > 0$ such that*

$$\|R(z, A_\varepsilon) J^{\uparrow\varepsilon} - J^{\uparrow\varepsilon} R(z, A_0)\|_{\mathscr{L}(H_0, H_\varepsilon)} \le \frac{\delta_\varepsilon C_{\theta,r,1}}{\sqrt{|z+\omega|}} \tag{3.1}$$

and

$$\|R(z, A_\varepsilon) - J^{\uparrow\varepsilon} R(z, A_0) J^{\downarrow\varepsilon}\|_{\mathscr{L}(H_\varepsilon)} \le \frac{\delta_\varepsilon C_{\theta,r,2}}{\sqrt{|z+\omega|}} \tag{3.2}$$

for all $z \notin \Sigma_\theta - \omega$ satisfying $|z + \omega| \ge r$.

Proof. Let D_θ be as in (2.6) and let C_θ be as in Lemma 3.1. Let $f \in H_0$ and $u \in H_\varepsilon$ be arbitrary, and fix $z \notin \Sigma_\theta - \omega$. Then by (2.7b) (see also (2.7b')), (2.6), (2.7a) and Lemma 3.1

$$\begin{aligned}&\left|\left\langle \big(R_\varepsilon(z) J^{\uparrow\varepsilon} - J^{\uparrow\varepsilon} R_0(z)\big) f \middle| u\right\rangle_{H_\varepsilon}\right| \\ &\quad\le \left|\langle f | J^{\downarrow\varepsilon} R_\varepsilon(z)^* u\rangle_{H_0} - \langle J^{\uparrow\varepsilon} R_0(z) f | u\rangle_{H_\varepsilon}\right| + \frac{\delta_\varepsilon D_\theta}{|z+\omega|} \|u\|_{H_\varepsilon} \|f\|_{H_0} \\ &\quad\le \left|\langle (z - A_0) R_0(z) f | J_1^{\downarrow\varepsilon} R_\varepsilon(z)^* u\rangle_{H_0} - \langle J_1^{\uparrow\varepsilon} R_0(z) f | (z - A_\varepsilon)^* R_\varepsilon(z)^* u\rangle_{H_\varepsilon}\right| \\ &\qquad + \Big(\frac{2\delta_\varepsilon C_\theta}{|z+\omega|^{1/2}} + \frac{\delta_\varepsilon D_\theta}{|z+\omega|}\Big) \|u\|_{H_\varepsilon} \|f\|_{H_0} \\ &\quad\le \left|a_0(R_0(z) f, J_1^{\downarrow\varepsilon} R_\varepsilon(z)^* u) - a_\varepsilon(J_1^{\uparrow\varepsilon} R_0(z) f, R_\varepsilon(z)^* u)\right| \\ &\qquad + |z| \left|\langle R_0(z) f | J_1^{\downarrow\varepsilon} R_\varepsilon(z)^* u\rangle_{H_0} - \langle J_1^{\uparrow\varepsilon} R_0(z) f | R_\varepsilon(z)^* u\rangle_{H_\varepsilon}\right| \\ &\qquad + \Big(\frac{2\delta_\varepsilon C_\theta}{|z+\omega|^{1/2}} + \frac{\delta_\varepsilon D_\theta}{|z+\omega|}\Big) \|u\|_{H_\varepsilon} \|f\|_{H_0}.\end{aligned}$$

Using (2.7e) and once again (2.7b) we can further estimate

$$\begin{aligned}
&\big|\langle\big(R_\varepsilon(z)J^{\uparrow\varepsilon}-J^{\uparrow\varepsilon}R_0(z)\big)f|u\rangle_{H_\varepsilon}\big|\\
&\quad\le \delta_\varepsilon\|R_0(z)f\|_{V_0}\|R_\varepsilon(z)^*u\|_{V_\varepsilon}\\
&\qquad+|z|\big|\langle R_0(z)f|J^{\downarrow\varepsilon}R_\varepsilon(z)^*u\rangle_{H_0}-\langle J^{\uparrow\varepsilon}R_0(z)f|R_\varepsilon(z)^*u\rangle_{H_\varepsilon}\big|\\
&\qquad+\Big(\frac{\delta_\varepsilon|z|C_\theta D_\theta}{|z+\omega|^{3/2}}+\frac{2\delta_\varepsilon C_\theta}{|z+\omega|^{1/2}}+\frac{\delta_\varepsilon D_\theta}{|z+\omega|}\Big)\|u\|_{H_\varepsilon}\|f\|_{H_0}\\
&\quad\le\Big(\frac{\delta_\varepsilon(D_\theta+C_\theta^2)}{|z+\omega|}+\frac{\delta_\varepsilon|z|D_\theta^2}{|z+\omega|^2}+\frac{\delta_\varepsilon|z|C_\theta D_\theta}{|z+\omega|^{3/2}}+\frac{2\delta_\varepsilon C_\theta}{|z+\omega|^{1/2}}\Big)\|u\|_{H_\varepsilon}\|f\|_{H_0}.
\end{aligned}$$

For $|z+\omega|\ge r$, this implies (3.1) with

$$C_{\theta,r,1}:=(C_\theta D_\theta+2C_\theta)+\frac{D_\theta+C_\theta^2+D_\theta^2}{r^{1/2}}+\frac{|\omega|C_\theta D_\theta}{r}+\frac{|\omega|D_\theta^2}{r^{3/2}}.$$

Estimate (3.2) is a consequence of (3.1) since by Lemma 3.2 and Lemma 3.1 we have

$$\|R_\varepsilon(z)-J^{\uparrow\varepsilon}R_0(z)J^{\downarrow\varepsilon}\|_{\mathscr{L}(H_\varepsilon)}\le\kappa\|J^{\downarrow\varepsilon}R_\varepsilon(z)-R_0(z)J^{\downarrow\varepsilon}\|_{\mathscr{L}(H_\varepsilon,H_0)}+\frac{\delta_\varepsilon C_\theta}{\sqrt{|z+\omega|}},$$

so we can choose $C_{\theta,r,2}:=\kappa C_{\theta,r,1}+C_\theta$. □

Remark 3.5. Estimate (3.2) tells us that we can find a good approximation of the operator A_ε in terms of the (often simpler) operators A_0, $J^{\uparrow\varepsilon}$ and $J^{\downarrow\varepsilon}$, at least for small ε. This is interesting by itself. In fact, we even have a rather explicit error estimate; in the proof we have given concrete (though certainly not optimal) constants. However, since these expressions are quite cumbersome, we prefer to work with the general constants $C_{\theta,r,1}$ and $C_{\theta,r,2}$.

Define

$$H^\infty(\Sigma_\theta-\omega):=\big\{\,\varphi\colon\Sigma_\theta-\omega\to\mathbb{C}\,:\,\varphi\text{ is holomorphic and bounded}\big\}$$

and

$$H_{00}^\infty(\Sigma_\theta-\omega):=\big\{\,\psi\in H^\infty(\Sigma_\theta-\omega)\,:\,\exists\mu>\tfrac12\text{ such that }\psi(z)\in\mathrm{O}(|z|^{-\mu})\ (z\to\infty)\big\}$$

and equip these spaces with the supremum norm. Let $\theta\in(\arctan\frac{M}{\alpha},\pi]$. We define the primary functional calculus of A_ε for $\psi\in H_{00}^\infty(\Sigma_\theta-\omega)$ by

$$\psi(A_\varepsilon):=\frac{1}{2\pi\mathrm{i}}\int_{\partial(\Sigma_\sigma-\omega)}\psi(z)R(z,A_\varepsilon)\,\mathrm{d}z, \tag{3.3}$$

where $\sigma\in(\arctan\frac{M}{\alpha},\theta)$. By Cauchy's integral theorem, this definition is independent of the choice of σ and agrees with the usual definition of the functional calculus, compare also Sec. 2.5.1 of [13].

Remark 3.6. In our setting, the natural space for the primary functional calculus would be the larger space

$$H_0^\infty(\Sigma_\theta):=\big\{\,\psi\in H^\infty(\Sigma_\theta)\,:\,\exists\,\mu>0\text{ such that }\psi(z)\in\mathrm{O}(|z|^{-\mu})\ (z\to\infty)\big\}$$

since in fact (3.3) is defined even for $\psi \in H_0^\infty(\Sigma_\theta - \omega)$. However, using estimate (3.2) we can show convergence of $\psi(A_\varepsilon)$ to $\psi(A_0)$ only for $\psi \in H_{00}^\infty(\Sigma_\theta - \omega)$.

Since the operators A_ε are m-sectorial in the sense of Kato, this functional calculus has a natural extension to $\varphi \in H^\infty(\Sigma_\theta - \omega)$, and the operator $\varphi(A_\varepsilon)$ is bounded with norm

$$\|\varphi(A_\varepsilon)\|_{\mathscr{L}(H_\varepsilon)} \le \Big(2 + \frac{2}{\sqrt{3}}\Big)\|\varphi\|_\infty, \tag{3.4}$$

cf. Corollary 7.1.17 of [13]. It is important for us to have a bound on the norm of $\varphi(A_\varepsilon)$ that is uniform with respect to ε.

We are now able to show that $\varphi(A_\varepsilon)$ converges to $\varphi(A_0)$ is the following sense if $(a_\varepsilon)_{\varepsilon>0}$ converges to a_0 in the sense of Definition 2.3, which is our main result in the context of the functional calculus.

Theorem 3.7. *Let a_0 and $(a_\varepsilon)_\varepsilon$ be equi-elliptic sesquilinear forms with associated operators A_0 and $(A_\varepsilon)_\varepsilon$ as in Section 2. Assume in addition that a_ε is δ_ε-κ-quasi-unitarily equivalent to a_0. Let $\theta \in (\arctan \frac{M}{\alpha}, \pi]$. Then for all $\psi \in H_{00}^\infty(\Sigma_\theta - \omega)$ there exists $C_\psi \ge 0$ such that*

$$\|J^{\uparrow\varepsilon}\psi(A_0)J^{\downarrow\varepsilon} - \psi(A_\varepsilon)\|_{\mathscr{L}(H_\varepsilon)} \le C_\psi\delta_\varepsilon. \tag{3.5}$$

Moreover, for all $\psi \in H^\infty(\Sigma_\theta - \omega)$ there exists $C_\psi \ge 0$ such that

$$\|J^{\uparrow\varepsilon}\varphi(A_0)J^{\downarrow\varepsilon}u - \varphi(A_\varepsilon)u\|_{H_\varepsilon} \le C_\varphi\delta_\varepsilon\|(\omega+1+A_\varepsilon)u\|_{H_\varepsilon} \tag{3.6}$$

for all $u \in D(A_\varepsilon)$.

Proof. Fix $\psi \in H_{00}^\infty(\Sigma_\theta - \omega)$ and $\sigma \in (\arctan \frac{M}{\alpha}, \theta)$. Let $\nu \in (0, \alpha c_V^{-2})$ be such that

$$\theta' := \arctan\Big(\frac{M}{\alpha - \nu c_V^2}\Big) < \sigma,$$

where α, M and c_V are as in (2.1), (2.2) and (2.3). Since

$$a_\varepsilon(u,u) + (\omega - \nu)\|u\|_{H_\varepsilon}^2 \ge \alpha\|u\|_{V_\varepsilon}^2 - \nu\|u\|_{H_\varepsilon}^2 \ge (\alpha - \nu c_V^2)\|u\|_{V_\varepsilon}^2,$$

the operator A_ε is m-sectorial with vertex $-\omega + \nu$ and semi-angle θ', and the same is true for A_0 on H_0. Hence by Proposition 3.4

$$\|R(z,A_\varepsilon) - J^{\uparrow\varepsilon}R(z,A_0)J^{\downarrow\varepsilon}\|_{\mathscr{L}(H_\varepsilon)} \le \frac{\delta_\varepsilon C_{\theta',\nu/2,2}}{\sqrt{|z+\omega-\nu|}}$$

for all $z \notin \Sigma_{\theta'} - \omega + \nu$ such that $|z + \omega - \nu| \ge \frac{\nu}{2}$. If $r > 0$ is sufficiently small, then $B(-\omega, r) \cap (\partial\Sigma_\sigma - \omega)$ has distance at least $\frac{\nu}{2}$ to $\Sigma_{\theta'} - \omega + \nu$, and hence

$$\|R(z,A_\varepsilon) - J^{\uparrow\varepsilon}R(z,A_0)J^{\downarrow\varepsilon}\|_{\mathscr{L}(H_\varepsilon)} \le \delta_\varepsilon c_{\sigma,\nu} \tag{3.7}$$

for all $z \in \partial(\Sigma_\sigma - \omega)$ satisfying $|z + \omega| \le r$, where $c_{\sigma,\nu}$ and r are constants depending on σ and ν, so in principle only on M, α and c_V.

There exist $\mu > \frac{1}{2}$ and $K \geq 0$ such that

$$|\psi(z)| \leq \frac{K}{|z+\omega|^{\mu}} \quad \text{and} \quad |\psi(z)| \leq K$$

for all $z \in \Sigma_\theta - \omega$. Thus, by (3.3), Proposition 3.4 and (3.7)

$$\begin{aligned}
&\|J^{\uparrow\varepsilon}\psi(A_0)J^{\downarrow\varepsilon} - \psi(A_\varepsilon)\|_{\mathscr{L}(H_\varepsilon)} \\
&\quad \leq \frac{1}{2\pi}\int_{\partial(\Sigma_\sigma-\omega)} |\psi(z)|\, \|J^{\uparrow\varepsilon}R_0(z)J^{\downarrow\varepsilon} - R_\varepsilon(z)\|_{\mathscr{L}(H_\varepsilon)}\,\mathrm{d}z \\
&\quad \leq \frac{1}{2\pi}\int_{\partial(\Sigma_\sigma-\omega)\setminus B(-\omega,r)} \frac{\delta_\varepsilon C_{\sigma,r,2}K}{|z+\omega|^{\mu+\frac{1}{2}}}\,\mathrm{d}z + \frac{1}{2\pi}\int_{\partial(\Sigma_\sigma-\omega)\cap B(-\omega,r)} \delta_\varepsilon c_{\sigma,\nu} K\,\mathrm{d}z.
\end{aligned}$$

Therefore, we have shown (3.5) with

$$C_\psi := \frac{C_{\sigma,r,2}K}{(\mu-\frac{1}{2})r^{\mu-\frac{1}{2}}\pi} + \frac{c_{\sigma,\nu}rK}{\pi}.$$

In particular, there exists a constant C_{ψ^*} belonging to the function $\psi^* \in H^\infty_{00}(\Sigma_\theta)$ defined by

$$\psi^*(z) := \frac{1}{\omega+1+z}$$

such that

$$\|J^{\uparrow\varepsilon}\psi^*(A_0)J^{\downarrow\varepsilon} - \psi^*(A_\varepsilon)\|_{\mathscr{L}(H_\varepsilon)} \leq C_{\psi^*}\delta_\varepsilon,$$

i.e.,

$$\|J^{\uparrow\varepsilon}(\omega+1+A_0)^{-1}J^{\downarrow\varepsilon} - (\omega+1+A_\varepsilon)^{-1}\|_{\mathscr{L}(H_\varepsilon)} \leq C_{\psi^*}\delta_\varepsilon. \tag{3.8}$$

Now let $\varphi \in H^\infty(\Sigma_\theta - \omega)$ be fixed and define $\psi \in H^\infty_{00}(\Sigma_\theta - \omega)$ by

$$\psi(z) := \frac{\varphi(z)}{\omega+1+z}$$

Then $\varphi(A_\varepsilon)(\omega+1+A_\varepsilon)^{-1} = \psi(A_\varepsilon)$ by the construction of the functional calculus (see Sec. 2.3.2 of [13]. Hence for all $u \in D(A_\varepsilon)$ we have

$$\begin{aligned}
&\|J^{\uparrow\varepsilon}\varphi(A_0)(\omega+1+A_0)^{-1}J^{\downarrow\varepsilon}(\omega+1+A_\varepsilon)u - \varphi(A_\varepsilon)u\|_{H_\varepsilon} \\
&\quad = \|J^{\uparrow\varepsilon}\psi(A_0)J^{\downarrow\varepsilon}(\omega+1+A_\varepsilon)u - \psi(A_\varepsilon)(\omega+1+A_\varepsilon)u\|_{H_\varepsilon} \\
&\quad \leq C_\psi\delta_\varepsilon\|(\omega+1+A_\varepsilon)u\|_{H_\varepsilon}.
\end{aligned}$$

Moreover, from (2.7d) and (3.4) we obtain that

$$\begin{aligned}
&\|J^{\uparrow\varepsilon}\varphi(A_0)J^{\downarrow\varepsilon}u - J^{\uparrow\varepsilon}\varphi(A_0)(\omega+1+A_0)^{-1}J^{\downarrow\varepsilon}(\omega+1+A_\varepsilon)u\|_{H_\varepsilon} \\
&\quad \leq \kappa\big(2+\frac{2}{\sqrt{3}}\big)\|\varphi\|_\infty\|J^{\downarrow\varepsilon}u - (\omega+1+A_0)^{-1}J^{\downarrow\varepsilon}(\omega+1+A_\varepsilon)u\|_{H_0}.
\end{aligned}$$

Finally, by Lemma 3.3, (3.8), Lemma 3.1 (for $z = -\omega - 1$) and (2.7d)

$$\begin{aligned}
&\|J^{\downarrow\varepsilon}u - (\omega+1+A_0)^{-1}J^{\downarrow\varepsilon}(\omega+1+A_\varepsilon)u\|_{H_0} \\
&\quad \le \kappa\|u - J^{\uparrow\varepsilon}(\omega+1+A_0)^{-1}J^{\downarrow\varepsilon}(\omega+1+A_\varepsilon)u\|_{H_\varepsilon} \\
&\qquad + \delta_\varepsilon\|(\omega+1+A_0)^{-1}J^{\downarrow\varepsilon}(1+\omega+A_\varepsilon)u\|_{V_0} \\
&\quad \le \kappa C_{\psi^*}\delta_\varepsilon\|(\omega+1+A_\varepsilon)u\|_{H_\varepsilon} + \delta_\varepsilon C_\theta\kappa\|(\omega+1+A_\varepsilon)u\|_{H_0}.
\end{aligned}$$

Combining the previous three estimates, we have proved (3.6) with

$$C_\varphi := C_\psi + \kappa^2\Big(2+\frac{2}{\sqrt{3}}\Big)\|\varphi\|_\infty\big(C_{\psi^*}+C_\theta\big). \qquad \square$$

Corollary 3.8. *If $(a_\varepsilon)_{\varepsilon>0}$ converges to a_0 in the sense of Definition 2.3, then the family $(J^{\downarrow\varepsilon}\varphi(A_\varepsilon)J^{\uparrow\varepsilon})_{\varepsilon>0}$ converges in operator norm to $\varphi(A_0)$ (regarded as operators on H_0) for every $\varphi \in H^\infty_{00}(\Sigma_\theta - \omega)$, $\theta \in (\arctan\frac{M}{\alpha}, \pi]$. If merely $\varphi \in H^\infty(\Sigma_\theta - \omega)$, then we have at least convergence in the strong operator topology.*

Proof. Since all conditions in Definition 2.3 are symmetric with respect to a_0 and a_ε, interchanging the roles of the two forms we obtain as in Theorem 3.7 that

$$\|\psi(A_0) - J^{\downarrow\varepsilon}\psi(A_\varepsilon)J^{\uparrow\varepsilon}\|_{\mathscr{L}(H_0)} \le C_\psi\delta_\varepsilon$$

for $\psi \in H^\infty_{00}(\Sigma_\theta - \omega)$, which proves the first claim. Similarly,

$$\|\varphi(A_0)f - J^{\downarrow\varepsilon}\varphi(A_\varepsilon)J^{\uparrow\varepsilon}f\|_{\mathscr{L}(H_0)} \le C_\varphi\delta_\varepsilon\|(\omega+1+A_0)f\|_{H_0}$$

for all $f \in D(A_0)$ if $\varphi \in H^\infty(\Sigma_\theta - \omega)$, implying that $(J^{\downarrow\varepsilon}\varphi(A_\varepsilon)J^{\uparrow\varepsilon})_{\varepsilon>0}$ converges to $\varphi(A_0)$ on a dense subspace of H_0. Since the operators are uniformly bounded by (2.7d) and (3.4), this implies strong convergence. $\square$

Example 3.9. The function $z \mapsto e^{-tz}$ is in $H^\infty_{00}(\Sigma_\theta - \omega)$ for $\theta \in (0, \frac{\pi}{2} - \sigma)$ and for all $t \in \Sigma_\sigma$, $\sigma \in (0, \frac{\pi}{2})$. Hence the semigroups $(e^{-tA_\varepsilon})_{\varepsilon>0}$ converge to e^{-tA_0} in operator norm (in the sense of Theorem 3.7 and Corollary 3.8) for t in the common sector of holomorphy of the semigroups, i.e., for every fixed $t \in \Sigma_\sigma$, where $\sigma := \frac{\pi}{2} - \arctan\frac{M}{\alpha}$.

Note, however, that we cannot expect this for $t = 0$ since typically $J^{\uparrow\varepsilon}J^{\downarrow\varepsilon}$ does not tend to the identity in operator norm even if $H_\varepsilon = H_0$ for all $\varepsilon \ge 0$. Thus we cannot expect uniform convergence near $t = 0$. However, the explicit constant C_ψ in the proof of Theorem 3.7 shows that the convergence is uniform on compact subsets of Σ_σ.

3.2. **Spectral convergence.** It has been proven in Sec. A.5 of [31] that if a family of non-negative forms $(a_\varepsilon)_{\varepsilon>0}$ converges to a_0 in the sense of Definition 2.3, then the spectra of the associated operators $\sigma(A_\varepsilon)$ converge to $\sigma(A_0)$. In [32], a similar result for certain non-self-adjoint operators arising in the treatment of resonances via complex scaling are considered. Here we prove that a similar result is true in the general (m-sectorial) case, where the spectra need not to be real. This is a part of the justification why we regard our notion of convergence as a reasonable generalisation of the classical resolvent convergence (see also Example 2.5).

We consider the following notion of spectral convergence, which is quite natural. It is often called "upper semi-continuity" of the spectrum. This type of convergence is precisely what we obtain if in a fixed Hilbert space we have a family of operators whose resolvents converge in operator norm, see Theorem IV.3.1 of [18].

Definition 3.10. We say that *the spectra* $\sigma(A_\varepsilon)$ *of the family* $(A_\varepsilon)_{\varepsilon>0}$ *converge to the spectrum* $\sigma(A_0)$ *of* A_0 *as* $\varepsilon \to 0$ if for each compact set $K \subset \rho(A_0)$ there exists $\varepsilon_1 > 0$ such that $K \subset \rho(A_\varepsilon)$ for all $\varepsilon \in (0, \varepsilon_1)$.

Ideally, we could hope that the spectra $\sigma(A_\varepsilon)$ converge to $\sigma(A_0)$ if $(a_\varepsilon)_{\varepsilon>0}$ converges to a_0. In fact, this is true if in addition $\rho(A_0)$ is connected, see Corollary 3.14.

We start with an auxiliary lemma, allowing us to estimate the resolvent of A_ε if we have a priori information about the resolvent of A_0. For the whole section, the operators $(A_\varepsilon)_{\varepsilon>0}$ are assumed to satisfy the conditions in Section 2.

Lemma 3.11. *For every* $\ell > 0$ *and* $r > 0$ *there exist* $\delta_0 = \delta_0(\ell, r) > 0$ *and* $L = L(\ell, r) > 0$ *with the following property: if* a_ε *is* δ_ε*-*κ*-quasi-unitarily equivalent to* a_0 *for some* $\delta_\varepsilon \in (0, \delta_0]$*, if* $z \in \rho(A_0) \cap \rho(A_\varepsilon) \cap B(0, r)$*, and if* $\|R(z, A_0)\|_{\mathscr{L}(H_0)} \le \ell$*, then* $\|R(z, A_\varepsilon)\|_{\mathscr{L}(H_\varepsilon)} \le L$*.*

Proof. For $z \in \rho(A_0) \cap \rho(A_\varepsilon)$ we define

$$V(z) := J^{\downarrow\varepsilon} R_\varepsilon(z) - R_0(z) J^{\downarrow\varepsilon}.$$

Let z and z_0 be in $\rho(A_0) \cap \rho(A_\varepsilon)$. Then by the resolvent identity we have

$$\big(R_0(z_0) - R_0(z)\big) J^{\downarrow\varepsilon} R_\varepsilon(z) R_\varepsilon(z_0) = R_0(z) R_0(z_0) J^{\downarrow\varepsilon} \big(R_\varepsilon(z_0) - R_\varepsilon(z)\big)$$

and thus

$$R_0(z_0) V(z) R_\varepsilon(z_0) = R_0(z) V(z_0) R_\varepsilon(z).$$

Hence

$$\begin{aligned} V(z) &= (z_0 - A_0) R_0(z) V(z_0) R_\varepsilon(z)(z_0 - A_\varepsilon) \\ &= (\mathrm{id} + (z_0 - z) R_0(z)) V(z_0)(\mathrm{id} + (z_0 - z) R_\varepsilon(z)) \end{aligned}$$

on $D(A_\varepsilon)$ and thus on H_ε by density. Setting $z_0 := -\omega - 1$ and using the dual version of (3.1), which follows by exchanging the roles of A_ε and A_0 in Proposition 3.4, to estimate $V(z_0)$ we deduce that

$$\|V(z)\|_{\mathscr{L}(H_\varepsilon, H_0)} \le \delta_\varepsilon C_{\theta,1,1} \big(1 + \ell|\omega + 1 + z|\big)\big(1 + |\omega + 1 + z|\, \|R_\varepsilon(z)\|_{\mathscr{L}(H_\varepsilon)}\big). \tag{3.9}$$

Next, we note that for all $u \in H_\varepsilon$

$$\begin{aligned} \|R_\varepsilon(z) u\|_{a_\varepsilon}^2 &= \langle (\omega + A_\varepsilon) R_\varepsilon(z) u | R_\varepsilon(z) u \rangle_{H_\varepsilon} \\ &\le \big(\|u\|_{H_\varepsilon} + |\omega + z|\, \|R_\varepsilon(z) u\|_{H_\varepsilon}\big) \|R_\varepsilon(z) u\|_{H_\varepsilon}, \end{aligned}$$

proving by (2.4) that

$$\begin{aligned} \|R_\varepsilon(z)\|_{\mathscr{L}(H_\varepsilon, V_\varepsilon)}^2 &\le \frac{1}{\alpha} \big(1 + |\omega + z|\, \|R_\varepsilon(z)\|_{\mathscr{L}(H_\varepsilon)}\big) \|R_\varepsilon(z)\|_{\mathscr{L}(H_\varepsilon)} \\ &\le \frac{1}{\alpha} \big(1 + \beta \|R_\varepsilon(z)\|_{\mathscr{L}(H_\varepsilon)}\big)^2 \end{aligned} \tag{3.10}$$

with $\beta := \max\{1, |\omega + z|\}$.

Now write

$$R_\varepsilon(z) = \big(\mathrm{id} - J^{\uparrow\varepsilon}J^{\downarrow\varepsilon}\big)R_\varepsilon(z) + J^{\uparrow\varepsilon}\big(J^{\downarrow\varepsilon}R_\varepsilon(z) - R_0(z)J^{\downarrow\varepsilon}\big) + J^{\uparrow\varepsilon}R_0(z)J^{\downarrow\varepsilon}.$$

This representation, combined with (3.9) and (3.10), shows that

$$\begin{aligned}
\|R_\varepsilon(z)\|_{\mathscr{L}(H_\varepsilon)} &\le \delta_\varepsilon \|R_\varepsilon(z)\|_{\mathscr{L}(H_\varepsilon, V_\varepsilon)} + \kappa \|V(z)\|_{\mathscr{L}(H_\varepsilon)} + \kappa^2 \ell \\
&\le \Big(\frac{\delta_\varepsilon}{\sqrt{\alpha}} + \kappa\delta_\varepsilon C_{\theta,1,1}\big(1 + \ell|\omega + 1 + z|\big) + \kappa^2\ell\Big) \\
&\qquad + \delta_\varepsilon\Big(\frac{\beta}{\sqrt{\alpha}} + C_{\theta,1,1}\big(1 + \ell|\omega + 1 + z|\big)\,|\omega + 1 + z|\Big)\|R_\varepsilon(z)\|_{\mathscr{L}(H_\varepsilon)} \\
&=: \ell_1 + \delta_\varepsilon c\|R_\varepsilon(z)\|_{\mathscr{L}(H_\varepsilon)}.
\end{aligned}$$

Thus, if $\delta_\varepsilon \in (0, \delta_0]$ with $\delta_0 := \frac{1}{2c}$, then

$$\frac{1}{2}\|R_\varepsilon(z)\|_{\mathscr{L}(H_\varepsilon)} \le (1 - \delta_\varepsilon c)\|R_\varepsilon(z)\|_{\mathscr{L}(H_\varepsilon)} \le \ell_1,$$

i.e., we have proved the claim with $L := 2\ell_1$. □

Now we come to our main theorem regarding convergence of the spectrum.

Theorem 3.12. *Let A_0 be an m-sectorial operator with vertex ω, semi-angle θ and associated form a_0. Let $K \subset \rho(A_0)$ be compact and connected. Then there exist constants $\delta_0 > 0$ and $C_{\theta,K}, D_{\theta,K} \ge 0$ with the following property: if a_ε is δ_ε-κ-quasi-unitarily equivalent to a_0 for $\delta_\varepsilon \in (0, \delta_0]$, if $(a_\varepsilon)_\varepsilon$ is equi-elliptic, and if in addition $K \cap \rho(A_\varepsilon) \ne \emptyset$, then $K \subset \rho(A_\varepsilon)$,*

$$\|J^{\downarrow\varepsilon}R(z, A_\varepsilon) - R(z, A_0)J^{\downarrow\varepsilon}\|_{\mathscr{L}(H_\varepsilon, H_0)} \le C_{\theta,K}\delta_\varepsilon \tag{3.11}$$

and

$$\|J^{\uparrow\varepsilon}R(z, A_0)J^{\downarrow\varepsilon} - R(z, A_\varepsilon)\|_{\mathscr{L}(H_\varepsilon)} \le D_{\theta,K}\delta_\varepsilon \tag{3.12}$$

for all $z \in K$.

Note that $C_{\theta,K}, D_{\theta,K} \ge 0$ also depend on A_0 and, as usual, on the constants of equi-ellipticity (see Definition 2.1).

Proof. Since K is compact, $K \subset B(0, r)$ for some $r > 0$ and

$$\ell := \sup_{z\in K}\|R_0(z)\|_{\mathscr{L}(H_\varepsilon)} < \infty.$$

Choose $\delta_0 = \delta_0(\ell, r, \omega)$ as in Lemma 3.11. Let $\delta_\varepsilon \in (0, \delta_0)$ and let a_ε be δ_ε-κ-quasi-unitarily equivalent to a_0. Let $K_0 := \rho(A_\varepsilon) \cap K$, which is non-empty by assumption. Since $\rho(A_\varepsilon)$ is open, the set K_0 is relatively open in K.

Let (z_n) be a sequence in K_0 converging to $z \in K$. Then from Lemma 3.11 we know that $\|R_\varepsilon(z_n)\|_{\mathscr{L}(H_\varepsilon)}$ is bounded, hence $z \in \rho(A_\varepsilon)$. We have shown that K_0 is closed in K. Since K is connected, $K_0 = K$, i.e., $K \subset \rho(A_\varepsilon)$.

Since by Lemma 3.11 we have $\|R_\varepsilon(z)\|_{\mathscr{L}(H_\varepsilon)} \le L$ for some $L > 0$, it follows from (3.9) that

$$\|J^{\downarrow\varepsilon} R_\varepsilon(z) - R_0(z) J^{\downarrow\varepsilon}\|_{\mathscr{L}(H_\varepsilon, H_0)} \le \delta_\varepsilon C_{\theta,1,1}\big(1+\ell(|\omega|+1+r)\big)\big(1+L(|\omega|+1+r)\big).$$

for all $z \in K$. This is (3.11) for

$$C_{\theta,K} := C_{\theta,1,1}\big(1+\ell(|\omega|+1+r)\big)\big(1+L(|\omega|+1+r)\big)$$

Now (3.12) follows from Lemma 3.2 and estimate (3.10) with

$$D_{\theta,K} := \kappa C_{\theta,K} + \frac{1}{\sqrt{\alpha}}\big(1+\beta L\big). \qquad \square$$

Remark 3.13. It can be difficult to check the condition $K \cap \rho(A_\varepsilon) \neq \emptyset$ of the previous theorem. On the other hand, in the classical situation, i.e., if $H_\varepsilon = H_0$ for all $\varepsilon \ge 0$ and $R(z, A_\varepsilon)$ converges to $R(z, A_0)$ in operator norm, this is automatically fulfilled by the fact that the set of invertible operators is open in $\mathscr{L}(H_0)$:

Let $\lambda \in \rho(A_0) \cap K$ and $\mu < -\omega$ such that $\lambda \neq \mu$. Then $R_\varepsilon(\mu)$ converges in operator norm to $R_0(\mu)$, since μ is outside the sector Σ_θ. Moreover, $\lambda \in \rho(A_\varepsilon)$ is equivalent with the invertibility of $\frac{1}{\mu-\lambda} - R_\varepsilon(\mu)$ by the spectral mapping theorem. For the same reason, $\frac{1}{\mu-\lambda} - R_0(\mu)$ is invertible. Since the set of invertible operators is open, $\lambda \in \rho(A_\varepsilon)$ for sufficiently small ε.

If the resolvent set is connected, a given compact set $K \subset \rho(A_0)$ can be enlarged to a connected compact set $K' \subset \rho(A_0)$ in such a way that we can guarantee $K' \cap \rho(A_\varepsilon) \neq \emptyset$, so that the theorem applies. We make this explicit in the following corollary. Note that in particular if the spectrum is real or discrete, the resolvent set is connected. Hence for self-adjoint operators and operators with compact resolvent we obtain spectral convergence.

Corollary 3.14. *Assume that $\rho(A_0)$ is connected and that $(a_\varepsilon)_{\varepsilon>0}$ converges to a_0 in the sense of Definition* 2.3. *Then $\sigma(A_\varepsilon)$ converges to $\sigma(A_0)$ in the sense of Definition* 3.10.

Proof. Let $K \subset \rho(A_0)$ be compact. Since $\rho(A_0)$ is connected, there exists a connected compact set $K' \subset \rho(A_0)$ such that $K \subset K'$ and $-\omega - 1 \in K'$. In fact, let $R > |\omega| + 1$ be such that $K \subset B(0,R)$, and let $(O_{\rho,\mu})_\mu$ denote the family of (open) connected components of the open set

$$O_\rho := \big\{\, z \in \rho(A_0) \cap B(0,R) \,:\, \operatorname{dist}(z, \sigma(A_0)) < \rho \,\big\}.$$

Then

$$K \cup \{-\omega - 1\} \subset \bigcup\nolimits_{\rho,\mu} O_{\rho,\mu},$$

and hence by compactness there exist a finite subcover (O_{ρ_i,μ_i}). Let K'' the be the union of the compact, connected sets $\overline{O}_{\rho_i,\mu_i} \subset \rho(A_0)$. Now K'' has only finitely many connected components. Since $\rho(A_0)$ is arcwise connected, we can join these connected components by finitely many paths (γ_k) in $\rho(A_0)$. Then $K' := K'' \cup \bigcup_k \gamma_k$ is a connected, compact subset of $\rho(A_0)$ that contains K and $-\omega - 1$.

Since $-\omega-1 \in \rho(A_\varepsilon)$ for all $\varepsilon \geq 0$ we obtain from Theorem 3.12 that $K' \subset \rho(A_\varepsilon)$ if δ_ε is sufficiently small. Hence $K \subset \rho(A_\varepsilon)$ for small ε, which implies the claim. □

In the rest of this section, we show that the discrete spectra of $(A_\varepsilon)_{\varepsilon>0}$ converge to the discrete spectrum of A_0 as the forms $(a_\varepsilon)_{\varepsilon>0}$ converge to a_0. In fact, we show that for an eigenvalue λ of A_0 of finite algebraic multiplicity $m_0(\lambda)$ and for sufficiently small δ_ε, there exist exactly $m_0(\lambda)$ eigenvalues of A_ε near λ, where we count the eigenvalues according to their algebraic multiplicity.

Recall that the *algebraic multiplicity* $m_0(\lambda)$ of an isolated point $\lambda \in \sigma(A_0)$ is the rank $\operatorname{rk} P_0 := \dim \operatorname{Rg} P_0$ of the spectral projection

$$P_0 := \frac{1}{2\pi \mathrm{i}} \int_{\partial B(\lambda,r)} R(z,A_0)\,\mathrm{d}z,$$

where $r > 0$ is such that $\overline{B(\lambda,r)} \cap \sigma(A_0) = \{\lambda\}$, compare Sec. 1.3 of [3]. By Cauchy's integral theorem, this definition does not depend on the particular choice of r, and in fact we could replace the circle $\partial B(\lambda,r)$ by any positively oriented, smooth curve that surrounds λ, but no other point of $\sigma(A_0)$.

Remark 3.15. Since $R(z,A_0)$ is locally bounded as a $\mathscr{L}(H_0,V_0)$-valued function, see for example estimate (3.10), the spectral projection P_0 is a bounded operator from H_0 to V_0.

Lemma 3.16. *There exist* $\delta_0 > 0$ *such that* $\|J^{\uparrow\varepsilon} f\|_{H_\varepsilon} \geq \frac{1}{2}\|f\|_{H_0}$ *for all* $f \in \operatorname{Rg} P_0$ *if* $\delta_\varepsilon \in (0,\delta_0]$.

Proof. For all $f \in V_0$ we have by (2.7b'), (2.7c) and (2.7d) that

$$\begin{aligned}\|f\|_{H_\varepsilon}^2 - \|J^{\uparrow\varepsilon} f\|_{H_0}^2 &= \langle f|f\rangle_{H_0} - \langle J^{\uparrow\varepsilon} f|J^{\uparrow\varepsilon} f\rangle_{H_\varepsilon} \\ &= \langle f - J^{\downarrow\varepsilon} J^{\uparrow\varepsilon} f|f\rangle_{H_\varepsilon} + \delta_\varepsilon \|J^{\uparrow\varepsilon} f\|_{H_\varepsilon}\|f\|_{H_0} \\ &\leq \delta_\varepsilon \|f\|_{V_0}\|f\|_{H_0} + \delta_\varepsilon \kappa \|f\|_{H_0}^2.\end{aligned}$$

Now if $f \in \operatorname{Rg} P_0$, i.e., $f = P_0 f$, and $f \neq 0$, we obtain that

$$\begin{aligned}\|f\|_{H_\varepsilon} - \|J^{\uparrow\varepsilon} f\|_{H_0} &\leq \delta_\varepsilon \frac{\|P_0\|_{\mathscr{L}(H_0,V_0)} + \kappa}{\|J^{\uparrow\varepsilon} f\|_{H_\varepsilon} + \|f\|_{H_0}} \|f\|_{H_0}^2 \\ &\leq \delta_0 \big(\|P_0\|_{\mathscr{L}(H_0,V_0)} + \kappa\big)\|f\|_{H_0} = \frac{1}{2}\|f\|_{H_0}\end{aligned}$$

for

$$\delta_0 := \frac{1}{2}\big(\|P_0\|_{\mathscr{L}(H_0,V_0)} + \kappa\big)^{-1}.$$ □

We now prove our main theorem about continuous dependence of the discrete spectrum. For simplicity we assume that $\rho(A_0)$ is connected, even though it would suffice that $\rho(A_\varepsilon) \cap B(\lambda,r) \neq \emptyset$ for small ε and all $r > 0$.

Theorem 3.17. *Let* $\rho(A_0)$ *be connected, let* λ *be an isolated point of* $\sigma(A_0)$ *with finite algebraic multiplicity* $m_0(\lambda) \in \mathbb{N}$*, and let* D *be a bounded, open set such that* $\overline{D} \cap \sigma(A_0) = \{\lambda\}$*. Then there exists* $\delta_0 > 0$ *such that if* a_ε *is* δ_ε-κ-*quasi-unitarily* *equivalent to* a_0 *for* $\delta_\varepsilon \in (0, \delta_0]$ *and if* $(a_\varepsilon)_\varepsilon$ *is equi-elliptic, then there exist eigenvalues* $(\lambda_{\varepsilon,i})_{i=1}^{m_0(\lambda)}$ *of* A_ε *such that*

$$\sigma(A_\varepsilon) \cap D = \{\lambda_{\varepsilon,1}, \dots, \lambda_{\varepsilon,m_0(\lambda)}\}.$$

Here, the values $(\lambda_{\varepsilon,i})$ *are not necessarily pairwise different, but rather each value is repeated according to its algebraic multiplicity with respect to* A_ε.

Proof. We may assume that D has smooth boundary. In fact, otherwise we can replace D by an open set $D_1 \subset D$ with smooth boundary still containing λ. Since $(\overline{D} \setminus D_1) \cap \sigma(A_\varepsilon) = \emptyset$ for small δ_ε by Corollary 3.14, the result carries over from D_1 to D.

By Corollary 3.14, the integral

$$P_\varepsilon := \frac{1}{2\pi \mathrm{i}} \int_{\partial D} R_\varepsilon(z)\,\mathrm{d}z$$

is defined for sufficiently small δ_ε, and using Theorem 3.12 we see that there exist $\delta_1 > 0$ and $C_1 \geq 0$ such that

$$\|J^{\downarrow\varepsilon} P_\varepsilon - P_0 J^{\downarrow\varepsilon}\|_{\mathscr{L}(H_\varepsilon)} < C_1 \delta_\varepsilon$$

if $\delta_\varepsilon \in (0, \delta_1]$.

Now let $u \in \mathrm{Rg}(P_\varepsilon)$, i.e., $P_\varepsilon u = u$. Then by Lemma 3.16 there exists $\delta_2 \in (0, \delta_1)$ such that

$$\|P_0 J^{\downarrow\varepsilon} u\|_{H_0} \geq \|J^{\downarrow\varepsilon} P_\varepsilon u\|_{H_0} - \|(J^{\downarrow\varepsilon} P_\varepsilon - P_0 J^{\downarrow\varepsilon})u\|_{H_0} \geq \frac{1}{2}\|u\|_{H_\varepsilon} - C_1 \delta_\varepsilon \|u\|_{H_\varepsilon} > 0$$

whenever $\delta_\varepsilon \in (0, \delta_2]$. This proves that $P_0 J^{\downarrow\varepsilon}$ is injective on $\mathrm{Rg}(P_\varepsilon)$, showing that $\mathrm{rk}\, P_0 \geq \mathrm{rk}\, P_\varepsilon$ whenever $\delta_\varepsilon \in (0, \delta_2]$.

For the converse inequality, we interchange the roles of P_0 and P_ε. In fact, the estimate $\|J^{\downarrow\varepsilon} u\|_{H_0} \geq \frac{1}{2}\|u\|_{H_\varepsilon}$ for $u \in \mathrm{Rg}(P_\varepsilon)$ can be obtained as in Lemma 3.16 by exploiting the fact that Lemma 3.11 and (3.10) provide a uniform bound for $\|P_\varepsilon\|_{\mathscr{L}(H_\varepsilon, V_\varepsilon)}$, compare also Remark 3.15. Now it readily follows that $\mathrm{rk}\, P_\varepsilon \geq \mathrm{rk}\, P_0$ for sufficiently small δ_ε.

Thus there exists $\delta_0 > 0$ such that $m_0(\lambda) = \mathrm{rk}\, P_\varepsilon$ whenever $\delta_\varepsilon \in (0, \delta_0]$. This implies that $\sigma(A_\varepsilon) \cap D$ consists of finitely many eigenvalues whose algebraic multiplicities add up to $m_0(\lambda)$, compare Theorem 1.32 of [3]. □

Remark 3.18. Theorem 3.17 says that near an isolated eigenvalue λ of A_0 any sufficiently close operator A_ε also possess only isolated eigenvalues, whose multiplicities add up to the multiplicity of λ. This is a version of Corollary A.15 of [31] for non-self-adjoint operators, see also [32].

The following corollary is a trivial consequence of Theorem 3.17 and Corollary 3.14.

Corollary 3.19. *Let* $\rho(A_0)$ *be connected, let* λ *be an isolated point of* $\sigma(A_0)$ *with finite algebraic multiplicity* $m_0(\lambda) \in \mathbb{N}$,*, and let* $(a_\varepsilon)_{\varepsilon>0}$ *converge to* a_0 *in the sense of Definition* 2.3. *Then the eigenvalues* $\lambda_{\varepsilon,i}$ *in Theorem* 3.17 *converge to* λ, *i.e.,* $\lim_{\varepsilon\to 0} \lambda_{\varepsilon,i} \to \lambda$ *for every* $i = 1, \dots, m_0(\lambda)$.

3.3. **Invariance and extrapolation.** Assume that $(a_\varepsilon)_{\varepsilon>0}$ converges to a_0 in the sense of Definition 2.3. We have already shown that the generated semigroups also converge in an appropriate sense, see Example 3.9. It is now natural to ask whether certain properties of the semigroups $(\mathrm{e}^{-tA_\varepsilon})_{\varepsilon>0}$ are inherited by e^{-tA_0} under appropriate assumptions on the operators $J^{\uparrow\varepsilon}$ and $J^{\downarrow\varepsilon}$.

In this short section, we formulate a simple result of this kind and apply it to obtain convergence of the semigroups in extrapolation spaces under natural assumptions.

Theorem 3.20. *Let $(a_\varepsilon)_{\varepsilon>0}$ converge to a_0 as $\varepsilon \to 0$ in the sense of Definition 2.3, let $\theta \in (\arctan \frac{M}{\alpha}, \pi]$, and let $\varphi \in H^\infty(\Sigma_\theta - \omega)$. For every $\varepsilon \geq 0$, let C_ε be a closed subset of H_ε such that*

$$J^{\uparrow\varepsilon} C_0 \subset C_\varepsilon \quad \text{and} \quad J^{\downarrow\varepsilon} C_\varepsilon \subset C_0. \tag{3.13}$$

If $\varphi(A_\varepsilon)C_\varepsilon \subset C_\varepsilon$ for all $\varepsilon > 0$, then $\varphi(A_0)C_0 \subset C_0$.

Proof. By the assumptions,

$$(J^{\downarrow\varepsilon} \varphi(A_\varepsilon) J^{\uparrow\varepsilon}) C_0 \subset C_0 \tag{3.14}$$

for all $\varepsilon > 0$. Thus the result follows from Corollary 3.8 because the invariance of a closed set is stable under strong convergence. □

Remark 3.21. In some applications, for example in Section 5, Condition (3.13) is only satisfied up to a rescaling of the identification operators, i.e., we can write the identification operators as

$$J^{\uparrow\varepsilon} = c_\varepsilon \widetilde{J^{\uparrow\varepsilon}}, \qquad J^{\downarrow\varepsilon} = c_\varepsilon^{-1} \widetilde{J^{\downarrow\varepsilon}}$$

for operators $\widetilde{J^{\uparrow\varepsilon}}$ and $\widetilde{J^{\downarrow\varepsilon}}$ that do satisfy (3.13). It is clear that also in this more general situation the inclusion (3.14) is satisfied and hence the assertion of Theorem 3.20 remains valid.

It is well-known how invariance of closed convex subsets under the action of a semigroup on a Hilbert space H generated by an operator associated with a sesquilinear form can be efficiently characterised by a Beurling-Deny-type criterion due to Ouhabaz, see Thm. 2.2 of [30]. Assuming that $H = L^2(\Omega, \mu)$ with a measure space Ω, typical applications of this criterion involve positivity and L^∞-contractivity (i.e., invariance of the subset of those L^2-functions taking a.e. values in the interval $[0,\infty)$ or $[-1,1]$).

A typical application of Theorem 3.20 is the following.

Corollary 3.22. *Let $p \in [1,\infty]$. Assume that $H_\varepsilon = L^2(\Omega_\varepsilon)$ for measure spaces Ω_ε, $\varepsilon \geq 0$. Assume that $(a_\varepsilon)_{\varepsilon>0}$ converges to a_0, where the operators $J^{\uparrow\varepsilon}$ and $J^{\downarrow\varepsilon}$ in the definition of convergence are positive (L^p-contractive). Assume that the semigroup $(\mathrm{e}^{-tA_\varepsilon})_{t\geq 0}$ is positive (L^p-contractive) on H_ε for every $\varepsilon > 0$. Then $(\mathrm{e}^{-tA_0})_{t\geq 0}$ is positive (L^p-contractive) on H_0.*

Proof. Apply Theorem 3.20 to the closed (and convex) sets

$$C_\varepsilon := \{u \in L^2(\Omega_\varepsilon) : u \geq 0 \text{ a.e.}\}$$

and

$$C_\varepsilon := \{u \in L^2(\Omega_\varepsilon) \cap L^p(\Omega_\varepsilon) : \|u\|_{L^p(\Omega_\varepsilon)} \le 1\}, \qquad p \in [1,\infty],$$

respectively. □

If we are in the situation that the semigroups on $H_\varepsilon = L^2(\Omega_\varepsilon)$ are L^∞-contractive, we can even establish convergence in $\mathscr{L}(L^p(\Omega_\varepsilon))$. We could also state the result in a more general version for arbitrary interpolation spaces. But this would involve several technical assumption that we prefer to avoid. It is clear that the analogous result for $1 < p < 2$ holds if we assume the semigroups to be L^1-contractive.

Theorem 3.23. *Let $(a_\varepsilon)_{\varepsilon>0}$ converge to a_0 in the sense of Definition 2.3 as $\varepsilon \to 0$, assume that $H_\varepsilon = L^2(\Omega_\varepsilon)$ for $\varepsilon \ge 0$ with measure spaces (Ω_ε), let $\theta \in (\arctan \frac{M}{\alpha}, \pi]$, let $\varphi \in H^\infty(\Sigma_\theta - \omega)$, and let $p \in [2,\infty)$. Assume that there exists a family $(c_\varepsilon)_{\varepsilon>0}$ of positive real numbers such that $c_\varepsilon J^{\uparrow\varepsilon}$, $c_\varepsilon^{-1} J^{\downarrow\varepsilon}$ and $\varphi(A_\varepsilon)$ are L^∞-contractive for all $\varepsilon > 0$.*

Then $J^{\downarrow\varepsilon}\varphi(A_\varepsilon)J^{\uparrow\varepsilon} \to \varphi(A_0)$ strongly as operators on $L^p(\Omega_0)$. If $\varphi \in H^\infty_{00}(\Sigma_\theta - \omega)$, the operators convergence even in operator norm, and in this case we have

$$\|J^{\uparrow\varepsilon}\varphi(A_0)J^{\downarrow\varepsilon} - \varphi(A_\varepsilon)\|_{\mathscr{L}(L^p(\Omega_\varepsilon))} \to 0.$$

Proof. By Corollary 3.22 and Remark 3.21 also $\varphi(A_0)$ is L^∞-contractive. Moreover, by Corollary 3.8,

$$\|\varphi(A_0)f - J^{\downarrow\varepsilon}\varphi(A_\varepsilon)J^{\uparrow\varepsilon}f\|_{H_0} \to 0$$

for all $f \in H_0$. Now by the interpolation inequality

$$\begin{aligned}
&\|\varphi(A_0)f - J^{\downarrow\varepsilon}\varphi(A_\varepsilon)J^{\uparrow\varepsilon}f\|_{L^p(\Omega_0)} \\
&\quad\le \|\varphi(A_0)f - J^{\downarrow\varepsilon}\varphi(A_\varepsilon)J^{\uparrow\varepsilon}f\|_{L^\infty(\Omega_0)}^{(p-2)/p}\ \|\varphi(A_0)f - J^{\downarrow\varepsilon}\varphi(A_\varepsilon)J^{\uparrow\varepsilon}f\|_{L^2(\Omega_0)}^{2/p} \\
&\quad\le \big(2\|f\|_{L^\infty(\Omega_0)}\big)^{(p-2)/p}\ \|\varphi(A_0)f - J^{\downarrow\varepsilon}\varphi(A_\varepsilon)J^{\uparrow\varepsilon}f\|_{L^2(\Omega_0)}^{2/p} \to 0
\end{aligned}$$

for all f in the dense subspace $L^2(\Omega_0) \cap L^\infty(\Omega_0)$ of $L^p(\Omega_0)$. Since in addition

$$\|J^{\downarrow\varepsilon}\varphi(A_\varepsilon)J^{\uparrow\varepsilon}\|_{\mathscr{L}(L^p(\Omega_0))} \le 1$$

by the Riesz-Thorin interpolation theorem, this proves strong convergence.

Now if $\psi \in H^\infty_{00}(\Sigma_\theta)$, then as in the proof of Corollary 3.8 there exists $C_\psi \ge 0$ such that

$$\|\psi(A_0) - J^{\downarrow\varepsilon}\psi(A_\varepsilon)J^{\uparrow\varepsilon}\|_{\mathscr{L}(H_0)} \le C_\psi \delta_\varepsilon.$$

Moreover, by assumption and Corollary 3.22, see also Remark 3.21,

$$\|\psi(A_0)f - J^{\downarrow\varepsilon}\psi(A_\varepsilon)J^{\uparrow\varepsilon}f\|_{L^\infty(\Omega_0)} \le 2\|f\|_{L^\infty(\Omega_0)}.$$

Thus by the Riesz-Thorin interpolation theorem

$$\|\psi(A_0) - J^{\downarrow\varepsilon}\psi(A_\varepsilon)J^{\uparrow\varepsilon}\|_{\mathscr{L}(L^p(\Omega_0))} \le 2^{(p-2)/p} C_\psi^{p/2} \delta_\varepsilon^{p/2} \to 0.$$

Employing Theorem 3.7 instead of Corollary 3.8, the same reasoning shows that

$$\|J^{\uparrow\varepsilon}\varphi(A_0)J^{\downarrow\varepsilon} - \varphi(A_\varepsilon)\|_{\mathscr{L}(L^p(\Omega))} \to 0.$$

□

4. SIMPLE EXAMPLES

In this section, we collect some examples to which the theory of the previous section can be applied without much effort. On the other hand, our main application, which involves some delicate calculations, is contained in a section by its own.

4.1. **Fourier series.** We start with an almost trivial example. Let a_0 be sectorial form with form domain V_0 on a Hilbert space H_0 as introduced in Section 2, and let A_0 be the associated operator. Assume that V_0 is compactly embedded into H_0, i.e., that A_0 has compact resolvent. For simplicity we also assume that A_0 is self-adjoint.

It is classical that in this situation there exists an orthonormal basis $(e_k)_{k\in\mathbb{N}}$ of H_0 consisting of eigenvectors of A_0 to eigenvalues $(\lambda_k)_{k\in\mathbb{N}}$, and $\lambda_k \to \infty$. We can assume that $\lambda_k \le \lambda_{k+1}$ for all $k \in \mathbb{N}$, and to make the notation simpler we assume that $\lambda_1 > 0$. Passing to an equivalent norm,

$$V_0 = \Big\{ f \in H_0 \,:\, \|f\|_{V_0}^2 = \sum_{k=1}^{\infty} \lambda_k \big|\langle f|e_k\rangle_{H_0}\big|^2 < \infty \Big\}.$$

We explain how this situation can be embedded into our framework. To this aim, it is convenient to index the Hilbert spaces and operators by $n \in \mathbb{N}$ instead of ε. Let P_n denote the orthogonal projection onto $H_n := V_n := \operatorname{span}(e_k)_{k=1}^n$, $J^{\uparrow n} := J_1^{\uparrow n} := P_n$, and $J^{\downarrow n} := J_1^{\downarrow n} := \mathrm{id}$, where H_n and V_n carry the norms induced by H_0 and V_0, respectively, and let a_n be the restriction of a_0 to V_n, so that A_n is the restriction of A_0 to $H_n = H_n \cap D(A_0)$.

Now (2.7a), (2.7b) and (2.7d) are trivial; in fact, these conditions hold with $\delta_\varepsilon = 0$ and $\kappa = 1$. Moreover,

$$\|f - P_n f\|_{H_0}^2 = \sum_{k=n+1}^{\infty} \big|\langle f|e_k\rangle_{H_0}\big| \le \frac{1}{\lambda_{n+1}} \|f\|_{V_0}^2,$$

which implies both conditions in (2.7c). Finally, (2.7e) follows from the fact that

$$a_0(f,u) - a_n(P_n f, u) = \sum_{k=1}^{\infty} \lambda_k \langle f - P_n f|e_k\rangle_{H_0} \langle e_k|u\rangle_{H_0} = 0$$

for all $f \in V_0$ and $u \in V_n$. Hence the forms a_n converge to a_0 in the sense of Definition 2.3.

The results in Section 3 now tell us that

$$\|P_n R(z, A_0) - R(z, A_n)\|_{\mathscr{L}(H_0)} \to 0,$$

as well as that other functions of these operators like the generated semigroup converge in this sense. Convergence of the spectrum as in Corollary 3.14 and Theorem 3.17 is of course built into this approximation.

4.2. **Varying coefficients.** Studying the convergence of elliptic operators with varying coefficients is a very classical topic. In fact, the underlying spaces typically do not change, so that the theory in Kato's book applies. However, sometimes it is convenient to incorporate the coefficients into the measure of the underlying L^2-space. Although such problems are still accessible by classical methods if all the norms are uniformly equivalent, it is quite natural to work with varying Hilbert spaces instead.

The following example is taken from [7], where the authors proved strong convergence in $\mathrm{C}(\overline{\Omega})$ as well as in L^p for every $p \in [1,\infty)$ for a class of elliptic operators with Wentzell boundary conditions. Applying our results, on the other hand, we obtain convergence in operator norm for all $p \in (1,\infty)$, see Theorem 3.23. Tracing the constants in the proofs, we in addition have explicit error estimates, and in particular we know the order of convergence, which answers the open question that closes [7]. We also mention the later article [8], where these results are refined by obtaining a detailed estimate on the order of convergence in operator norm in H^1.

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Then $\Gamma := \partial\Omega$ becomes an oriented compact Riemannian manifold with Riemannian metric g in a natural way, where the charts are Lipschitz regular, and the metric is bounded and measurable. As in the smooth case, there exists a volume measure σ on Γ, which coincides with the $(n-1)$-dimensional Hausdorff measure. Let $H^1(\Gamma)$ be the completion of Lipschitz-continuous functions u on Γ with respect to the norm defined by

$$\|u\|^2_{H^1(\Gamma)} := \int_\Gamma \big(|u|^2 + |\mathrm{d}u|^2_g\big)\,\mathrm{d}\sigma,$$

where

$$|\mathrm{d}u|^2_g = \sum_{i,j=1}^{n-1} g^{ij}\partial_i u \partial_j \overline{u} \tag{4.1}$$

in a chart $U \subset \Gamma$ with coordinates $x_i \colon U \longrightarrow \mathbb{R}$, $i = 1,\dots,n-1$, and tangential vectors $\partial_i = \partial/\partial x_i$. Moreover, (g^{ij}) is the inverse of $(g_{ij}) = (g(\partial_i,\partial_j))$. For an ad hoc definition of Lipschitz-regular manifolds, we refer to [1].

Now let the families $(\mathscr{A}_\varepsilon)_{\varepsilon\geq 0} \subset L^\infty(\Omega;\mathscr{L}(\mathbb{C}^n))$, $(\beta_\varepsilon)_{\varepsilon\geq 0} \subset L^\infty(\Gamma)$, $(\gamma_\varepsilon)_{\varepsilon\geq 0} \subset L^\infty(\Gamma)$ and $(q_\varepsilon)_{\varepsilon\geq 0} \subset \mathbb{R}$ be bounded in the respective spaces, and assume that there exist $\alpha > 0$ and $b > 0$ such that for all $\varepsilon \geq 0$ we have $q_\varepsilon \geq \alpha$,

$$\mathrm{Re}\langle \mathscr{A}_\varepsilon \xi | \xi \rangle_{\mathbb{C}^n} \geq \alpha |\xi|^2$$

on Ω for all $\xi \in \mathbb{C}^n$ and $\beta_\varepsilon \geq b$ on Γ. For $\varepsilon \geq 0$, define

$$H_\varepsilon := L^2(\Omega) \times L^2\Big(\Gamma; \frac{\mathrm{d}\sigma}{\beta_\varepsilon}\Big)$$

and

$$V_\varepsilon := \big\{(u,f) \in H^1(\Omega) \times H^1(\Gamma) : u|_\Gamma = f\big\} \subset H_\varepsilon,$$

and equip these spaces with the natural scalar products. Note that the space V_ε and its norm do in fact not depend on ε.

Proposition 4.1. *The family $(a_\varepsilon)_{\varepsilon\geq 0}$ of sesquilinear forms with form domains V_ε which is defined by*

$$a_\varepsilon\big((u,u|_\Gamma),(v,v|_\Gamma)\big) := \int_\Omega \langle \mathscr{A}_\varepsilon \nabla u | \nabla v \rangle_{\mathbb{C}^n} + \int_\Gamma \gamma_\varepsilon u \overline{v} \frac{\mathrm{d}\sigma}{\beta_\varepsilon} + q_\varepsilon \int_\Gamma \langle \mathrm{d}u | \mathrm{d}v \rangle_g \,\mathrm{d}\sigma$$

is equi-elliptic.

Proof. By the uniformity conditions on the coefficients,

$$\|u\|_{H_\varepsilon}^2 = \|u\|_{L^2(\Omega)}^2 + \|u\|_{L^2(\Gamma;\frac{d\sigma}{\beta_\varepsilon})}^2 \le \|u\|_{H^1(\Omega)}^2 + \tfrac{1}{b}\|u\|_{H^1(\Gamma)}^2 \le \bigl(1+\tfrac{1}{b}\bigr)\|u\|_{V_\varepsilon}^2,$$

which shows that the embedding of V_ε into H_ε has a uniform constant. Moreover,

$$\begin{aligned}|a_\varepsilon\bigl((u,u|_\Gamma),(v,v|_\Gamma)\bigr)| &\le \|\mathscr{A}_\varepsilon\|_{L^\infty(\Omega,\mathscr{L}(\mathbb{C}^n))}\|\nabla u\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)}\\ &\quad + \|\gamma_\varepsilon\|_\infty \|u\|_{L^2(\Gamma;\frac{d\sigma}{\beta_\varepsilon})}\|v\|_{L^2(\Gamma;\frac{d\sigma}{\beta_\varepsilon})} + q_\varepsilon \|u\|_{H^1(\Gamma)}\|v\|_{H^1(\Gamma)},\end{aligned}$$

which shows that the forms are uniformly bounded, and

$$\begin{aligned}&\operatorname{Re} a_\varepsilon\bigl((u,u|_\Gamma),(u,u|_\Gamma)\bigr)\\ &\quad\ge \alpha\|\nabla u\|_{L^2(\Omega)}^2 + \alpha\|du\|_{L^2(\Gamma)}^2 - \|\gamma_\varepsilon\|_\infty\|u\|_{L^2(\Gamma;\frac{d\sigma}{\beta_\varepsilon}))}^2\\ &\quad\ge \alpha\Bigl(\|u\|_{H^1(\Omega)}^2 + \|u\|_{H^1(\Gamma)}^2\Bigr) - \alpha\|u\|_{L^2(\Omega)}^2 - \bigl(\|\gamma_\varepsilon\|_\infty + \alpha\|\beta_\varepsilon\|_\infty\bigr)\|u\|_{L^2(\Gamma;\frac{d\sigma}{\beta_\varepsilon})}^2,\end{aligned}$$

which shows that the ellipticity constants are uniform with respect to $\varepsilon \ge 0$. □

Remark 4.2. Integration by parts shows that (at least formally) the operator A_ε on H_ε associated with a_ε acts as

$$A_\varepsilon\bigl((u,u|_\Gamma)\bigr) = \Bigl(-\operatorname{div}(\mathscr{A}_\varepsilon\nabla u), \beta_\varepsilon\left((\mathscr{A}_\varepsilon\nabla u)\cdot\nu\right) + \gamma_\varepsilon u|_\Gamma - q_\varepsilon\beta_\varepsilon\Delta_\Gamma u|_\Gamma\Bigr),$$

where ν denotes the outer unit normal of Ω, and Δ_Γ is the Laplace-Beltrami operator on Γ, i.e., A_ε is the operator considered in [7].

Proposition 4.3. *There exists a constant K depending only on b and $\|\beta_0\|_\infty$ and $\|\gamma_0\|_\infty$ such that the forms a_ε are δ_ε-κ-quasi-unitarily equivalent to a_0 for $\kappa = 1$ and*

$$\delta_\varepsilon = \mathrm{O}\Bigl(\|\mathscr{A}_\varepsilon - \mathscr{A}_0\|_\infty + \|\beta_\varepsilon - \beta_0\|_\infty + \|\gamma_\varepsilon - \gamma_0\|_\infty + |q_\varepsilon - q_0|\Bigr).$$

Moreover, this equivalence can be realised by taking the identification operators to be the identity operators.

Proof. Let $J^{\uparrow\varepsilon}$, $J_1^{\uparrow\varepsilon}$, $J^{\downarrow\varepsilon}$ and $J_1^{\downarrow\varepsilon}$ be the identity operators between the respective spaces. Then (2.7a), (2.7c) and (2.7d) hold trivially with $\delta_\varepsilon = 0$ and $\kappa = 1$.

To check (2.7b), fix $(u,f) \in H_0$ and $(v,g) \in H_\varepsilon$. Then

$$\begin{aligned}&\bigl|\langle J^{\uparrow\varepsilon}(u,f)|(v,g)\rangle_{H_\varepsilon} - \langle (u,f)|J^{\downarrow\varepsilon}(v,g)\rangle_{H_0}\bigr|\\ &\quad\le \int_\Gamma \bigl|\tfrac{1}{\beta_\varepsilon} - \tfrac{1}{\beta_0}\bigr|\,|f||g|\,d\sigma \le \|\beta_\varepsilon - \beta_0\|_\infty \int_\Gamma \frac{|fg|}{b\sqrt{\beta_0\beta_\varepsilon}}\,d\sigma\\ &\quad\le \frac{\|\beta_\varepsilon - \beta_0\|_\infty}{b}\|(u,f)\|_{H_0}\|(v,g)\|_{H_\varepsilon},\end{aligned}$$

i.e., (2.7b) holds with $\delta_\varepsilon = b^{-1}\|\beta_\varepsilon - \beta_0\|_\infty$, compare Remark 2.4.

Finally, to check (2.7e), fix $(u,u|_\Gamma)\in V_0$ and $(v,v|_\Gamma)\in V_\varepsilon$. Then

$$\begin{aligned}&\big|a_0\big((u,u|_\Gamma),J_1^{\downarrow\varepsilon}(v,v|_\Gamma)\big)-a_\varepsilon\big(J_1^{\uparrow\varepsilon}(u,u|_\Gamma),(v,v|_\Gamma)\big)\big|\\&\quad\le\|\mathscr{A}_\varepsilon-\mathscr{A}_0\|_\infty\|\nabla u\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)}\\&\qquad+\Big\|\frac{\gamma_0}{\beta_0}-\frac{\gamma_\varepsilon}{\beta_\varepsilon}\Big\|_\infty\|u\|_{L^2(\Gamma)}\|v\|_{L^2(\Gamma)}+|q_0-q_\varepsilon|\|u\|_{H^1(\Gamma)}\|v\|_{H^1(\Gamma)}.\end{aligned}$$

Finally, note that

$$\Big\|\frac{\gamma_0}{\beta_0}-\frac{\gamma_\varepsilon}{\beta_\varepsilon}\Big\|_\infty\le\frac{\|\beta_\varepsilon\gamma_0-\beta_0\gamma_\varepsilon\|_\infty}{b^2}\le\frac{\|\gamma_0\|_\infty}{b^2}\|\beta_\varepsilon-\beta_0\|_\infty+\frac{\|\beta_0\|_\infty}{b^2}\|\gamma_0-\gamma_\varepsilon\|_\infty,$$

which concludes the proof. □

It is easy to check using the Beurling-Deny criteria that the semigroups $(\mathrm{e}^{-tA_\varepsilon})$ are positive and quasi-contractive in the norm of

$$L^\infty(\Omega)\times L^\infty\Big(\Gamma;\frac{\mathrm{d}\sigma}{\beta_\varepsilon}\Big),$$

i.e.,

$$\|\mathrm{e}^{-tA_\varepsilon}\|_{L^\infty(\Omega)\times L^\infty(\Gamma;\frac{\mathrm{d}\sigma}{\beta_\varepsilon})}\le\mathrm{e}^{rt},$$

where $r\in\mathbb{R}$ depends only on a lower bound of $(\gamma_\varepsilon)_{\varepsilon\ge0}$. By duality, we obtain uniform quasi-contractivity also in the space

$$L^1(\Omega)\times L^1\Big(\Gamma;\frac{\mathrm{d}\sigma}{\beta_\varepsilon}\Big).$$

In fact, the adjoint operator satisfies the same conditions as A_ε itself.

Thus, by Theorem 3.23 (and its dual version for $p<2$) we obtain the following result. In fact, the proof of Theorem 3.23 provides an explicit error estimate.

Theorem 4.4. *Assume that $\mathscr{A}_\varepsilon\to\mathscr{A}_0$, $\beta_\varepsilon\to\beta_0$, $\gamma_\varepsilon\to\gamma_0$ and $q_\varepsilon\to q_0$ uniformly on Ω or Γ, respectively. Then for every $t\ge0$ and $p\in(1,\infty)$ the operators $\mathrm{e}^{-tA_\varepsilon}$ converge to e^{-tA_0} as $\varepsilon\to0$ in the operator norm of $L^p(\Omega)\times L^p(\Gamma;\mathrm{d}\sigma)$ and the convergence rate can be estimated by $\delta_\varepsilon^{p/2}$.*

4.3. **Degenerate equations in non-divergence form.** Now we show that our machinery also applies to the approximation of degenerate elliptic operators in non-divergence form. More precisely, we study the operator $m\Delta$ with Dirichlet boundary conditions on a bounded domain $\Omega\subset\mathbb{R}^N$ for a possibly degenerate function m. This operator has been studied for example by Arendt and Chovanec [2], who investigated under which conditions its part in $\mathrm{C}_0(\Omega)$ generates a C_0-semigroup.

Let $m_0\colon\Omega\to(0,\infty)$ be a bounded, measurable function and assume that $\frac{1}{m_0}\in L^q(\Omega)$, where $q=1$ if $N=1$, $q>1$ if $N=2$, and $q=\frac{N}{2}$ if $N\ge3$. Define $m_\varepsilon:=\max\{m_0,\varepsilon\}$ Our goal is to show that the forms associated to the uniformly elliptic operators $m_\varepsilon\Delta$

converge to the (possibly degenerate) form associated to the operator $m_0\Delta$ in the sense of our abstract framework as $\varepsilon \to 0$, where

$$\operatorname{dom}(m_\varepsilon\Delta) := \big\{\, u \in H_0^1(\Omega) \cap L^2(\Omega; \tfrac{\mathrm{d}x}{m_\varepsilon(x)}) : \\ \exists f \in L^2(\Omega; \tfrac{\mathrm{d}x}{m_\varepsilon(x)}) \text{ such that } \Delta u = \tfrac{f}{m_\varepsilon} \,\big\}, \qquad (m_\varepsilon\Delta)u := f.$$

Here, in the definition of $D(m_\varepsilon\Delta)$, the expression Δu has to be understood as a distribution.

We start by introducing the forms that give rise to these operators. Define

$$H_\varepsilon := L^2(\Omega; \tfrac{\mathrm{d}x}{m_\varepsilon(x)}) \quad \text{and} \quad V_\varepsilon := H_0^1(\Omega).$$

Note that $V_\varepsilon \subset H_\varepsilon$ even for $\varepsilon = 0$ by the Sobolev embedding theorem and Hölder's inequality due to the integrability assumption $\frac{1}{m_0} \in L^q(\Omega)$. Thus the natural inner product

$$\langle u|v\rangle_{V_\varepsilon} := \int_\Omega \nabla u \overline{\nabla v} + \langle u|v\rangle_{H_\varepsilon}, \tag{4.2}$$

turns $V_\varepsilon \cap H_\varepsilon = V_\varepsilon$ into a Hilbert space. Here we used the equivalent norm $u \mapsto \|\nabla u\|$ on $H_0^1(\Omega)$. For the norm associated to (4.2), the embedding constant of V_ε into H_ε is at most 1. We emphasise that in general the Hilbert spaces H_ε do not agree with H_0, not even as sets. We define the form $a_\varepsilon\colon V_\varepsilon \times V_\varepsilon \to \mathbb{C}$ by

$$a_\varepsilon(f,g) := \int_\Omega \nabla u \overline{\nabla v}.$$

Then a_ε is bounded with constant $M = 1$, and a_ε is elliptic with constants $\omega = 1$ and $\alpha = 1$. Hence the family $(a_\varepsilon)_{\varepsilon\geq 0}$ is equi-elliptic in the sense of Definition 2.1. The form a_ε is associated with the operator $-m_\varepsilon\Delta$ as defined above, compare [2].

Theorem 4.5. *The forms associated with the operators $m_\varepsilon\Delta$ and $m_0\Delta$ are δ_ε-κ-quasi unitarily equivalent for $\kappa = 1$ and a family $(\delta_\varepsilon)_{\varepsilon>0}$ of real numbers such that $\delta_\varepsilon \to 0$ as $\varepsilon \to 0$.*

Proof. For simplicity, we assume that $N \geq 3$. Define

$$J^{\downarrow\varepsilon} u := \sqrt{\frac{m_0}{m_\varepsilon}}\, u \quad \text{and} \quad J^{\uparrow\varepsilon} f := \sqrt{\frac{m_\varepsilon}{m_0}}\, f.$$

Then $J^{\uparrow\varepsilon}\colon H_\varepsilon \to H_0$ and $J^{\downarrow\varepsilon}\colon H_0 \to H_\varepsilon$ are isometric isomorphisms, hence unitary. Moreover, $J^{\uparrow\varepsilon}$ and $J^{\downarrow\varepsilon}$ are inverse to each other, so (2.7b), (2.7c) and (2.7d) are satisfied with $\delta_\varepsilon = 0$ and $\kappa = 1$.

We take $J_1^{\uparrow\varepsilon}$ and $J_1^{\downarrow\varepsilon}$ to be the identity. Then (2.7e) is fulfilled with $\delta_\varepsilon = 0$. Moreover, by Hölder's inequality

$$\begin{aligned} \|J^{\uparrow\varepsilon}u - J_1^{\uparrow\varepsilon}u\|_{H_\varepsilon}^2 &\leq \int_\Omega \Big|\sqrt{\frac{m_\varepsilon}{m_0}} - 1\Big|^2 \frac{|u|^2}{m_\varepsilon} \leq \|\tfrac{1}{\sqrt{m_0}} - \tfrac{1}{\sqrt{m_\varepsilon}}\|_{L^N(\Omega)}^2 \|u\|_{L^{\frac{2N}{N-2}}(\Omega)}^2 \\ &\leq c^2 \delta_\varepsilon^{2/N} \|u\|_{V_0}^2 \end{aligned}$$

for all $u \in V_0$ with the embedding constant c of $H_0^1(\Omega)$ into $L^{\frac{2N}{N-2}}(\Omega)$, and for

$$\delta_\varepsilon := \int_\Omega \Big| \frac{1}{\sqrt{m_0}} - \frac{1}{\sqrt{m_\varepsilon}} \Big|^N.$$

Since

$$\Big| \frac{1}{\sqrt{m_0}} - \frac{1}{\sqrt{m_\varepsilon}} \Big|^N \leq \frac{1}{m_0^{N/2}} \in L^1(\Omega)$$

by assumption and in addition $m_\varepsilon(x) \to m_0(x)$ for all $x \in \Omega$, we obtain that $\delta_\varepsilon \to 0$ by the dominated convergence theorem. The other inequality in (2.7b) is proved in a similar way; in fact, the calculations are symmetric in m_0 and m_ε. □

5. SHRINKING TUBES WITH ROBIN BOUNDARY CONDITIONS

In this section we present our main example of convergence of Laplacians acting in different Hilbert spaces. We consider a family of manifolds X_ε with boundary together with the corresponding Laplacian A_ε with (in general) non-local boundary conditions.

We use Robin-type boundary condition of a certain scaling. In Remark 5.2 we compare our approach with the ones used in [12] and [6].

5.1. **The metric graph model.** As an example of our approximation scheme, we consider a diffusive process on a family of $(m+1)$-dimensional manifolds (X, g_ε) converging to a limit space given by a metric graph X_0. We will now present the construction in detail. We consider compact spaces only. For the non-compact case, see Remark 5.10.

Let $(\mathsf{V}, \mathsf{E}, \partial)$ be a directed graph where V and E are finite sets, the set of *vertices* and *edges*. Furthermore, $\partial \colon \mathsf{E} \longrightarrow \mathsf{V} \times \mathsf{V}$ encodes the graph structure and orientation by associating to an edge $\mathsf{e} \in \mathsf{E}$ the pair $(\partial_-\mathsf{e}, \partial_+\mathsf{e})$ of its *initial* and *terminal* vertex. The orientation is only introduced for convenience. The definition of A_0 below does not depend on the choice of orientation. We denote by

$$\mathsf{E}_\mathsf{v}^\pm := \{ \mathsf{e} \in \mathsf{E} : \partial_\pm \mathsf{e} = \mathsf{v} \} \qquad \text{and} \qquad \mathsf{E}_\mathsf{v} := \mathsf{E}_\mathsf{v}^- \,\dot\cup\, \mathsf{E}_\mathsf{v}^+$$

the set of edges *terminating in* v $(+)$, *starting in* v $(-)$ resp. *adjacent with* v. We denote by $\deg \mathsf{v} := |\mathsf{E}_\mathsf{v}|$ the *degree* of a vertex v, i.e., the number of edges terminating and starting in v.

Let X_0 be the topological graph associated to $(\mathsf{V}, \mathsf{E}, \partial)$, i.e., the edges are 1-dimensional intervals meeting in the vertices according to the graph structure. The *metric* structure of X_0 is defined by a function $\ell \colon \mathsf{E} \longrightarrow (0, \infty)$ associating to each edge e a length ℓ_e. We parametrise each edge with a coordinate $s = s_\mathsf{e}$, i.e., we identify the directed edge e with the associated *metric edge* $I_\mathsf{e} := [0, \ell_\mathsf{e}]$ in such a way that $\partial_-\mathsf{e}$ corresponds to $s = 0$ and $\partial_+\mathsf{e}$ corresponds to $s = \ell_\mathsf{e}$. Introducing the obvious distance function now turns the topological graph X_0 into a metric space, the *metric graph*. Similarly, we have a natural measure on X_0 given by the Lebesque measure on each edge $\mathrm{d}s = \mathrm{d}s_\mathsf{e}$.

The basic Hilbert space is

$$H_0 := L^2(X_0) := \bigoplus_{\mathsf{e} \in \mathsf{E}} L^2(I_\mathsf{e}),$$

with norm[1] $\|f\|^2_{X_0} = \sum_{\mathsf{e}\in\mathsf{E}} \|f_\mathsf{e}\|^2_{I_\mathsf{e}}$, where $L^2(I_\mathsf{e})$ is the usual L^2-space with norm given by $\|f_\mathsf{e}\|^2_{I_\mathsf{e}} = \int_0^{\ell_\mathsf{e}} |f_\mathsf{e}|^2 \,\mathrm{d}s$.

The Hilbert space, which will serve as domain of the sesquilinear form defined below, is given by

$$V_0 := H^1(X_0) := C(X_0) \cap \bigoplus_{\mathsf{e}\in\mathsf{E}} H^1(I_\mathsf{e})$$

with norm defined by

$$\|f\|^2_{V_0} := \sum_{\mathsf{e}\in\mathsf{E}} \bigl(\|f'\|^2_{I_\mathsf{e}} + \|f\|^2_{I_\mathsf{e}}\bigr),$$

i.e., a function in V_0 is of class H^1 along the edges and also continuous at each vertex. Trivially, $\|f\|_{H_0} \le \|f\|_{V_0}$, i.e., we can choose $C_V = 1$ in (2.3).

We define the operator governing an evolution process via the sesquilinear form

$$a_0(f,g) := a_{0,\mathsf{V}}(f,g) + a_{0,\mathsf{E}}(f,g) \tag{5.1}$$

for functions $f, g \in V_0$, where

$$a_{0,\mathsf{V}}(f,g) := \sum_{\mathsf{v}\in\mathsf{V}} \sum_{\mathsf{w}\in\mathsf{V}} \gamma_{\mathsf{vw}} f(\mathsf{w}) \overline{g(\mathsf{v})} \deg \mathsf{v} \quad \text{and}$$

$$a_{0,\mathsf{E}}(f,g) := \langle f' | g' \rangle_{X_0} = \sum_{\mathsf{e}\in\mathsf{E}} \int_0^{\ell_\mathsf{e}} f'_\mathsf{e} \overline{g}'_\mathsf{e} \,\mathrm{d}s$$

for a given coefficient matrix $(\gamma_{\mathsf{vw}})_{\mathsf{v},\mathsf{w}\in\mathsf{V}}$.

The following estimate follows easily from a standard Sobolev estimate on an interval. In particular, we have

$$\|\underline{f}\|^2_\mathsf{V} := \sum_{\mathsf{v}\in\mathsf{V}} |f(\mathsf{v})|^2 \deg \mathsf{v} \le 4b\|f'\|^2_{X_0} + \frac{8}{b}\|f\|^2_{X_0} \tag{5.2}$$

for $f \in H^1(X_0)$, where $\underline{f} = (f(\mathsf{v}))_{\mathsf{v}\in\mathsf{V}}$ and $0 < b \le \min_\mathsf{e} \ell_\mathsf{e}$, see e.g. [10]. The next proposition is an easy consequence of (5.2).

Proposition 5.1. *The sesquilinear form a_0 is well-defined on $V_0 = H^1(X_0)$. Moreover, given $\alpha \in (0,1)$, there exists $\omega \ge 0$ such that*

$$\operatorname{Re} a_0(f,f) + \omega\|f\|^2_{H_0} \ge \alpha\|f\|^2_{V_0}$$

for all $f \in V_0 = H^1(X_0)$. In particular, a_0 is H_0-elliptic.

It is easily seen that the corresponding operator A_0 acts as $(A_0 f)_\mathsf{e} = -f''_\mathsf{e}$ on each edge for $f \in \operatorname{dom} A_0$, where $f \in \operatorname{dom} A_0$ iff $f \in C(X_0) \cap \bigoplus_{\mathsf{e}\in\mathsf{E}} H^2(I_\mathsf{e})$ and

$$\frac{1}{\deg \mathsf{v}} \sum_{\mathsf{e}\in\mathsf{E}_\mathsf{v}} f'_\mathsf{e}(\mathsf{v}) + \sum_{\mathsf{w}\in\mathsf{V}} \gamma_{\mathsf{vw}} f(\mathsf{w}) = 0,$$

where $f'_\mathsf{e}(\mathsf{v}) = -f'_\mathsf{e}(0)$ if $\mathsf{v} = \partial_-\mathsf{e}$ and $f'_\mathsf{e}(\mathsf{v}) = f'_\mathsf{e}(\ell_\mathsf{e})$ if $\mathsf{v} = \partial_+\mathsf{e}$. Observe that for a non-diagonal matrix γ the vertex conditions defined above turn out to be non-local.

[1] Here and in the sequel, we use the notation $\|f\|_M$ for the L^2-norm of a measurable function $u\colon M \longrightarrow \mathbb{C}$ on a measure space M.

5.2. **The manifold model.** In the sequel, we will construct a manifold X according to the graph $(\mathsf{V},\mathsf{E},\partial)$ together with a family of metrics g_ε such that (X,g_ε) shrinks to the metric graph X_0 in a suitable sense (see Figure 1).

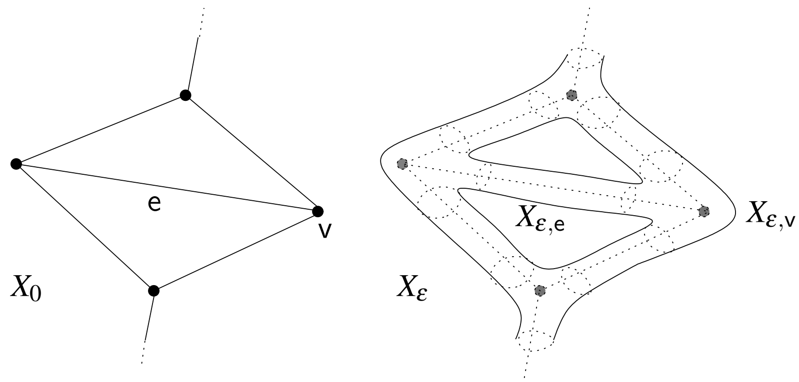

FIGURE 1. The metric graph X_0 and the family of manifolds (X,g_ε) shrinking to the metric graph. Here, (X,g_ε) can be considered as a subset of $\mathbb{R}^3$, i.e., as a full cylinder with boundary consisting of the surface of the pipeline network.

Let X be an $(m+1)$-dimensional connected manifold with boundary ∂X. We assume that X decomposes as

$$X = \dot{\bigcup_{\mathsf{v}\in\mathsf{V}}} X_\mathsf{v} \cup \dot{\bigcup_{\mathsf{e}\in\mathsf{E}}} X_\mathsf{e}, \tag{5.3}$$

where the *vertex* and *edge manifolds*, X_v and X_e, are compact connected subsets with non-empty interior. Moreover, we assume that $\{X_\mathsf{v}\}_{\mathsf{v}\in\mathsf{V}}$ and $\{X_\mathsf{e}\}_{\mathsf{e}\in\mathsf{E}}$ are families of pairwise disjoint sets, respectively (indicated by $\dot\cup$), and that

$$X_\mathsf{e} \cong I_\mathsf{e} \times Y_\mathsf{e},$$

where Y_e is a compact, connected, m-dimensional manifold, the *transversal* or *cross-section manifold* at the edge e. Note that Y_e has a boundary (as far as $\partial X \cap X_\mathsf{e}$ is non-empty). In the sequel, we will identify X_e with the product $I_\mathsf{e} \times Y_\mathsf{e}$. Finally, we assume that

$$Y_{\mathsf{v},\mathsf{e}} := X_\mathsf{v} \cap X_\mathsf{e} \cong \begin{cases} Y_\mathsf{e}, & \text{if } \mathsf{v} \in \partial\mathsf{e}, \\ \emptyset & \text{otherwise.} \end{cases}$$

Let g be a smooth Riemannian metric on X having product structure on X_e, i.e.,

$$g_\mathsf{e} = \mathrm{d}s_\mathsf{e}^2 + h_\mathsf{e}$$

on X_e, where h_e is a Riemannian metric on Y_e. Here, and in the sequel, we use the subscripts $(\cdot)_\mathsf{v}$ and $(\cdot)_\mathsf{e}$ to indicate the restriction to X_v and X_e for objects on the manifold.

By assumption, X_{v} is a manifold with boundary in which the disjoint union of transversal manifolds

$$Y_{\mathsf{v}} = \dot{\bigcup_{\mathsf{e}\in \mathsf{E}_{\mathsf{v}}}} Y_{\mathsf{v},\mathsf{e}}$$

is embedded. In addition, the embedding is isometric. We can think of X as being constructed from the graph $(\mathsf{V},\mathsf{E},\partial)$ and the family of transversal manifolds $\{Y_{\mathsf{e}}\}_{\mathsf{e}\in\mathsf{E}}$ and vertex manifolds $\{X_{\mathsf{v}}\}_{\mathsf{v}\in\mathsf{V}}$ according to the graph.

Let us now define the family of ε-depending metrics on X via

$$g_{\varepsilon,\mathsf{v}} := \varepsilon^2 g_{\mathsf{v}} \qquad \text{and} \qquad g_{\varepsilon,\mathsf{e}} := \mathrm{d}s_{\mathsf{e}}^2 + \varepsilon^2 h_{\mathsf{e}},$$

i.e., (X, g_ε) is obtained from the manifold (X, g) by ε-homothetically shrinking of the vertex manifold X_{v} and the transversal manifold Y_{e} of the edge manifold X_{e} (thin tube). Note that (X, g_ε) defines a smooth Riemannian manifold. The smoothness of the metric along the passage from X_{v} to X_{e} is assured since the original metric $g = g_1$ is assumed to be smooth on X.

If the metric graph X_0 is embedded in $\mathbb{R}^{m+1}$, then one can choose a closed neighbourhood X_ε of X_0 in $\mathbb{R}^{m+1}$ with smooth boundary and thickness of order ε. The smoothness is assumed only for simplicity; a Lipschitz boundary would be enough. Note that a decomposition as in (5.3) does not give an *isometric* decomposition, since the edge neighbourhood $X_{\varepsilon,\mathsf{e}}$ is slightly shorter than ℓ_{e} due to the presence of the vertex neighbourhoods. Nevertheless, this example can be treated in the same way after a longitudinal rescaling of the edge variable. The error made is only of order ε. For details, we refer to Lem. 2.7 of [10] or [32, Prop. 5.3.10].

The decomposition (5.3) induces a decomposition of the boundary $\Gamma = \partial X$, an m-dimensional Riemannian manifold,

$$\Gamma = \dot{\bigcup_{\mathsf{v}\in\mathsf{V}}} \Gamma_{\mathsf{v}} \cup \dot{\bigcup_{\mathsf{e}\in\mathsf{E}}} \Gamma_{\mathsf{e}},$$

where $\Gamma_{\mathsf{v}} \subset \partial X_{\mathsf{v}}$ and $\Gamma_{\mathsf{e}} = I_{\mathsf{e}} \times \partial Y_{\mathsf{e}} \subset \partial X_{\mathsf{e}}$ are pairwise disjoint (or intersect only in sets of m-dimensional measure 0). Moreover, we have

$$\partial X_{\mathsf{v}} = \Gamma_{\mathsf{v}} \cup \bigcup_{\mathsf{e}\in\mathsf{E}_{\mathsf{v}}} Y_{\mathsf{v},\mathsf{e}} \qquad \text{and} \qquad \partial X_{\mathsf{e}} = \Gamma_{\mathsf{e}} \cup \bigcup_{\mathsf{v}\in\partial\mathsf{e}} Y_{\mathsf{v},\mathsf{e}}.$$

The Riemannian measure associated with a Riemannian manifold (M, g_ε) is denoted by $\mathrm{d}M_\varepsilon$. In particular, we have

(5.4a) $\qquad \mathrm{d}X_{\varepsilon,\mathsf{v}} = \varepsilon^{m+1}\,\mathrm{d}X_{\mathsf{v}}, \qquad \mathrm{d}\Gamma_{\varepsilon,\mathsf{v}} = \varepsilon^m\,\mathrm{d}\Gamma_{\mathsf{v}},$

(5.4b) $\qquad \mathrm{d}X_{\varepsilon,\mathsf{e}} = \varepsilon^m\,\mathrm{d}X_{\mathsf{e}} = \varepsilon^m\,\mathrm{d}s_{\mathsf{e}}\,\mathrm{d}Y_{\mathsf{e}}, \qquad \mathrm{d}\Gamma_{\varepsilon,\mathsf{e}} = \varepsilon^{m-1}\,\mathrm{d}\Gamma_{\mathsf{e}} = \varepsilon^{m-1}\,\mathrm{d}s_{\mathsf{e}}\,\mathrm{d}\partial Y_{\mathsf{e}}.$

We will use the abbreviation X_ε, $X_{\varepsilon,\mathsf{v}}$ etc. for the measure spaces $(X, \mathrm{d}X_\varepsilon)$, $(X_{\mathsf{v}}, \mathrm{d}X_{\varepsilon,\mathsf{v}})$, etc.

Here and in the sequel, we use the notation

$$X_{\mathsf{E}} := \bigcup_{\mathsf{e}\in\mathsf{E}} X_{\mathsf{e}}, \qquad X_{\mathsf{V}} := \bigcup_{\mathsf{v}\in\mathsf{V}} X_{\mathsf{v}}, \qquad \Gamma_{\mathsf{E}} := \bigcup_{\mathsf{e}\in\mathsf{E}} \Gamma_{\mathsf{e}}$$

etc. for the (disjoint) union of the corresponding manifolds. Similarly, $X_{\varepsilon,\mathsf{E}}$, $X_{\varepsilon,\mathsf{V}}$, $\Gamma_{\varepsilon,\mathsf{E}}$ etc. denote the corresponding Riemannian manifolds with the ε-depending metric.

The basic Hilbert space we are working in is $H_\varepsilon := L^2(X_\varepsilon)$. We often write $\|u\|_{X_\varepsilon}$ instead of $\|u\|_{L^2(X_\varepsilon)}$ for the corresponding norm. The ε-dependence of the norms for the scaled spaces can easily calculated using (5.4); e.g. for $X_{\varepsilon,\mathsf{v}}$ and $X_{\varepsilon,\mathsf{e}}$ we have

$$\|u\|^2_{X_{\varepsilon,\mathsf{v}}} = \varepsilon^{m+1} \int_{X_\mathsf{v}} |u|^2 \, \mathrm{d}X_\mathsf{v} \qquad \text{and} \qquad \|u\|^2_{X_{\varepsilon,\mathsf{e}}} = \varepsilon^{m} \int_{I_\mathsf{e} \times Y_\mathsf{e}} |u|^2 \, \mathrm{d}Y_\mathsf{e} \, \mathrm{d}x_\mathsf{e}.$$

Let $V_\varepsilon := H^1(X_\varepsilon)$ be the Sobolev space of first order defined as the completion of smooth functions on X_ε with respect to the norm defined by

$$\|u\|^2_{H^1(X_\varepsilon)} = \|u\|^2_{L^2(X_\varepsilon)} + \|\mathrm{d}u\|^2_{L^2(X_\varepsilon)},$$

where $\|\mathrm{d}u\|^2_{L^2(X_\varepsilon)} = \int_X |\mathrm{d}u|^2_{g_\varepsilon} \, \mathrm{d}X_\varepsilon$, and $|\mathrm{d}u|^2_{g_\varepsilon}$ is given in (4.1). Trivially, $\|u\|_{H_\varepsilon} \le \|u\|_{V_\varepsilon}$, i.e., we can choose $C_V = 1$ in (2.3).

We define a sesquilinear form by

$$a_\varepsilon(u,v) = \int_X \langle \mathrm{d}u | \mathrm{d}v \rangle_{g_\varepsilon} \, \mathrm{d}X_\varepsilon + \int_\Gamma \beta_\varepsilon u \overline{v} \, \mathrm{d}\Gamma_\varepsilon + \int_{\Gamma \times \Gamma} \gamma_\varepsilon (u \otimes \overline{v}) \, \mathrm{d}\Gamma_\varepsilon \otimes \mathrm{d}\Gamma_\varepsilon \tag{5.5}$$

for functions $u \in V_\varepsilon = H^1(X_\varepsilon)$. Here $\langle \cdot | \cdot \rangle_{g_\varepsilon}$ is the (pointwise) inner product on T^*X defined via the Riemannian metric g_ε. Moreover, $(u \otimes \overline{v})(x_1, x_2) := u(x_1)\overline{v(x_2)}$. We assume that $\beta_\varepsilon \in L_\infty(\Gamma)$ and $\gamma_\varepsilon \in L^2(\Gamma \times \Gamma)$. In this case, a_ε is indeed well-defined for all $u \in H^1(X_\varepsilon)$ (see Proposition 5.7).

The associated operator is given by $A_\varepsilon u = -\Delta u = \mathrm{d}^*\mathrm{d}u$ for functions $u \in \operatorname{dom} A_\varepsilon$. Moreover, $u \in \operatorname{dom} A_\varepsilon$ iff $u \in H^2(X_\varepsilon)$ and

$$\frac{1}{\varepsilon} \partial_\mathrm{n} u + \beta_\varepsilon u + \int_\Gamma \gamma_\varepsilon u \, \mathrm{d}\Gamma_\varepsilon = 0,$$

where the integral is taken with respect to the first variable of $\gamma_\varepsilon \colon \Gamma \times \Gamma \longrightarrow \mathbb{C}$, and where ∂_n is the normal derivative on X.

Remark 5.2. Let us illustrate the effect of scaling the underlying space for Robin boundary conditions in a simple example: Assume that the transversal manifold $Y_{\varepsilon,\mathsf{e}}$ is isometric to the interval $[0,\varepsilon]$ and that there is no non-local contribution, i.e., $\gamma_\varepsilon = 0$. We consider the Laplacian with Neumann boundary conditions at 0 and Robin boundary conditions at ε, i.e.,

$$v'(\varepsilon) + \beta_\varepsilon v(\varepsilon) = 0,$$

where $\beta_\varepsilon \in \mathbb{R} \setminus \{0\}$ is the coupling constant. An eigenfunction for the Laplacian $\Delta v = -v''$ with Neumann boundary conditions at 0 is of the form

$$v(s) = v(0) \cos(\omega s) \qquad \text{and} \qquad v(s) = v(0) \cosh(\omega s),$$

with eigenvalue ω^2 and $-\omega^2$ if $\beta_\varepsilon > 0$ and $\beta_\varepsilon < 0$, respectively. The lowest eigenvalue $\mu(\varepsilon)$ is then of the same order as $\varepsilon^{-1}\beta_\varepsilon$ for $\varepsilon \to 0$.

If one chooses *scale-invariant* Robin boundary conditions, i.e., $\beta_\varepsilon = \varepsilon^{-1}\beta$ for some $\beta \neq 0$, as e.g. in [12] and [6], then the lowest eigenvalue $\mu(\varepsilon)$ is of order ε^{-2} and one has to substract the divergent term $\mu(\varepsilon)$ in order to expect convergence to a limit.

Here, we use a different approach. We assume that the coupling constant β_ε is of order $\varepsilon^{3/2}$ on the edge neighbourhoods, see (5.13) (actually, $\varepsilon^{1+\eta}$ would be enough for some $\eta > 0$). In this case, the lowest (transversal) eigenvalue $\mu(\varepsilon)$ is of order $\varepsilon^{1/2}$ (resp. ε^{η}), and converges to 0. We are then in the situation, where the Robin Laplacian is close to the Neumann Laplacian. This is the reason why we are in the same setting as in the (simpler) Neumann boundary condition case treated already in [31].

5.3. **Some estimates on the manifold.** Let us collect some estimates needed later on. Basically, we need a Sobolev trace estimate. Let M be a compact Riemannian manifold of dimension n with metric g and boundary ∂M, and B a compact $(n-1)$-dimensional submanifold of ∂M carrying the induced metric. It follows that there is a constant $C^{\mathrm{tr}}_{B,M} > 0$ such that

$$\|u\|_B^2 \le C^{\mathrm{tr}}_{B,M}\big(\|\mathrm{d}u\|_M^2 + \|u\|_M^2\big) \tag{5.6a}$$

for all $u \in H^1(M)$. The constant $C^{\mathrm{tr}}_{B,M}$ geometrically depends on the shape of B embedded in M.

If we scale the metric by a factor, $g_\varepsilon = \varepsilon^2 g$, then the estimate changes to

$$\|u\|_{B_\varepsilon}^2 \le C^{\mathrm{tr}}_{B,M}\Big(\varepsilon\|\mathrm{d}u\|_{M_\varepsilon}^2 + \frac{1}{\varepsilon}\|u\|_{M_\varepsilon}^2\Big), \tag{5.6b}$$

using $\mathrm{d}M_\varepsilon = \varepsilon^n\,\mathrm{d}M$, $\mathrm{d}B_\varepsilon = \varepsilon^{n-1}\,\mathrm{d}B$ and the fact that $|\mathrm{d}u|^2_{g_\varepsilon} = \varepsilon^{-2}|\mathrm{d}u|^2_g$. Here, B_ε and M_ε denote the corresponding Riemannian manifolds with the ε-depending metric.

We will apply this trace estimate basically in the situations $(\Gamma_{\mathsf{v}}, X_{\mathsf{v}})$, $(Y_{\mathsf{v}}, X_{\mathsf{v}})$ and $(\partial Y_{\mathsf{e}}, Y_{\mathsf{e}})$. Let us first prove the following lemma, which shows that the trace estimate for $(\partial Y_{\mathsf{e}}, Y_{\mathsf{e}})$ gives a trace estimate for the product $(\Gamma_{\mathsf{e}}, X_{\mathsf{e}}) = (I_{\mathsf{e}} \times \partial Y_{\mathsf{e}}, I_{\mathsf{e}} \times Y_{\mathsf{e}})$:

Lemma 5.3. *We have*

$$\|u\|^2_{\Gamma_{\varepsilon,\mathsf{e}}} \le C^{\mathrm{tr}}_{\partial Y_{\mathsf{e}},Y_{\mathsf{e}}}\Big(\varepsilon\|\mathrm{d}_{Y_{\mathsf{e}}}u\|^2_{X_{\varepsilon,\mathsf{e}}} + \frac{1}{\varepsilon}\|u\|^2_{X_{\varepsilon,\mathsf{e}}}\Big)$$

for all $u \in H^1(X_{\varepsilon,\mathsf{e}})$, where $\mathrm{d}_{Y_{\mathsf{e}}}u$ denotes the exterior derivative with respect to the second variable of $I_{\mathsf{e}} \times Y_{\mathsf{e}}$.

Proof. Let $u \in H^1(X_\varepsilon)$ be smooth, then

$$\begin{aligned}\|u\|^2_{\Gamma_{\varepsilon,\mathsf{e}}} = \int_0^{\ell_{\mathsf{e}}} \|u(s)\|^2_{\partial Y_{\varepsilon,\mathsf{e}}}\,\mathrm{d}s &\le \int_0^{\ell_{\mathsf{e}}} C^{\mathrm{tr}}_{\partial Y_{\mathsf{e}},Y_{\mathsf{e}}}\Big(\varepsilon\|\mathrm{d}_{Y_{\mathsf{e}}}u(s)\|^2_{Y_{\varepsilon,\mathsf{e}}} + \frac{1}{\varepsilon}\|u(s)\|^2_{Y_{\varepsilon,\mathsf{e}}}\Big)\,\mathrm{d}s\\ &= C^{\mathrm{tr}}_{\partial Y_{\mathsf{e}},Y_{\mathsf{e}}}\Big(\varepsilon\|\mathrm{d}_{Y_{\mathsf{e}}}u\|^2_{X_{\varepsilon,\mathsf{e}}} + \frac{1}{\varepsilon}\|u\|^2_{X_{\varepsilon,\mathsf{e}}}\Big)\end{aligned}$$

using the Sobolev trace estimate (5.6b) for $\partial Y_{\mathsf{e}} \subset Y_{\mathsf{e}}$ pointwise. Since smooth functions are dense in $H^1(X_{\mathsf{e}})$ and since the operators $H^1(X_{\mathsf{e}}) \to L^2(\Gamma_{\mathsf{e}})$, $u \mapsto u\restriction_{\Gamma_{\mathsf{e}}}$ and $H^1(X_{\mathsf{e}}) \to L^2(X_{\mathsf{e}}, T^*X_{\mathsf{e}})$, $u \mapsto \mathrm{d}_{Y_{\mathsf{e}}}u$ are bounded, the estimate also holds for $u \in H^1(X_{\mathsf{e}})$. $\square$

It follows from these trace estimates that the global trace operator $u \mapsto u\restriction_\Gamma$ is bounded, either as operator $H^1(X) \to L^2(\Gamma)$ or $H^1(X_\varepsilon) \to L^2(\Gamma_\varepsilon)$ (see also Proposition 5.7 below).

In the following, we need several averaging operators. Let

$$f_{\mathsf{v}} u := \fint_{X_{\mathsf{v}}} u \, \mathrm{d}X_{\mathsf{v}}. \tag{5.7}$$

denote the average value of u on X_{v} (and also the corresponding constant function on X_{v}). Here, $\fint_M := (\operatorname{vol} M)^{-1} \int$ is the normalised volume integral. Denote by $\lambda_2(X_{\mathsf{v}})$ the second (first non-vanishing) eigenvalue of the Neumann problem on X_{v}. Let us now compare the average of u on Γ_{v} with the average of u on X_{v}:

Lemma 5.4. *For all $u \in H^1(X_{\mathsf{v}})$, we have*

$$\varepsilon^m \|u - \fint_{\mathsf{v}} u\|^2_{\Gamma_{\mathsf{v}}} \le \varepsilon C^{\mathrm{tr}}_{\Gamma_{\mathsf{v}}, X_{\mathsf{v}}} \Big(\frac{1}{\lambda_2^{\mathrm{N}}(X_{\mathsf{v}})} + 1 \Big) \|\mathrm{d}u\|^2_{X_{\varepsilon,\mathsf{v}}}.$$

Proof. Interpreting $\widetilde{u} := u - \fint_{\mathsf{v}} u$ as a function on Γ_{v} we have

$$\varepsilon^m \|\widetilde{u}\|^2_{\Gamma_{\mathsf{v}}} \le \varepsilon^m C^{\mathrm{tr}}_{\Gamma_{\mathsf{v}}, X_{\mathsf{v}}} \big(\|\widetilde{u}\|^2_{X_{\mathsf{v}}} + \|\mathrm{d}u\|^2_{X_{\mathsf{v}}} \big) \le \varepsilon^m C^{\mathrm{tr}}_{\Gamma_{\mathsf{v}}, X_{\mathsf{v}}} \Big(\frac{1}{\lambda_2^{\mathrm{N}}(X_{\mathsf{v}})} + 1 \Big) \|\mathrm{d}u\|^2_{X_{\mathsf{v}}}$$

using Cauchy-Schwarz, the Sobolev trace estimate (5.6a) for $\Gamma_{\mathsf{v}} \subset X_{\mathsf{v}}$ and the min-max principle

$$\lambda_2^{\mathrm{N}}(X_{\mathsf{v}}) \|\widetilde{u}\|^2_{X_{\mathsf{v}}} \le \|\mathrm{d}\widetilde{u}\|^2_{X_{\mathsf{v}}} = \|\mathrm{d}u\|^2_{X_{\mathsf{v}}},$$

since $\widetilde{u}$ is orthogonal to the first (constant) eigenfunction of the Neumann Laplacian on X_{v}. The scaling property $\varepsilon^{m-1} \|\mathrm{d}u\|^2_{X_{\mathsf{v}}} = \|\mathrm{d}u\|^2_{X_{\varepsilon,\mathsf{v}}}$ now gives the result. ☐

Next, we compare the average of u on $Y_{\mathsf{v},\mathsf{e}}$ with the average of u on X_{v}. To do so, we introduce a partial averaging operator also needed later on for the identification operators. For simplicity, we assume that

$$\operatorname{vol}_m Y_{\mathsf{e}} = 1 \tag{5.8}$$

for all $\mathsf{e} \in \mathsf{E}$. We set

$$(\fint_{\mathsf{e}} u)(s) := \fint_{Y_{\mathsf{e}}} u(s,y) \, \mathrm{d}Y_{\mathsf{e}}(y), \tag{5.9}$$

Note that the integral exists for almost every s and $\fint_{\mathsf{e}} u$ defines a function in $L^2(I_{\mathsf{e}})$. If $s = 0$ or $s = \ell_{\mathsf{e}}$ denotes the vertex $\mathsf{v} = \partial_- \mathsf{e}$ or $\mathsf{v} = \partial_+ \mathsf{e}$, respectively, we also write $(\fint_{\mathsf{e}} u)(\mathsf{v})$.

The proof of the following lemma is similar to the proof of Lemma 5.4 (see also Lem. 2.8 of [10]):

Lemma 5.5. *We have*

$$\varepsilon^m \sum_{\mathsf{e} \in \mathsf{E}_{\mathsf{v}}} |\fint_{\mathsf{v}} u - \fint_{\mathsf{e}} u(\mathsf{v})|^2 \le \varepsilon C^{\mathrm{tr}}_{Y_{\mathsf{v}}, X_{\mathsf{v}}} \Big(\frac{1}{\lambda_2^{\mathrm{N}}(X_{\mathsf{v}})} + 1 \Big) \|\mathrm{d}u\|^2_{X_{\varepsilon,\mathsf{v}}}$$

for all $u \in H^1(X_{\mathsf{v}})$.

We finally need an estimate over the vertex neighbourhoods. It will assure that in the limit $\varepsilon \to 0$, no family of normalised eigenfunctions $(u_\varepsilon)_\varepsilon$ with uniformly bounded eigenvalues can concentrate on $X_{\varepsilon,\mathsf{v}}$, i.e., $\|u\|_{X_{\varepsilon,\mathsf{v}}}/\|u\|_{X_\varepsilon} \to 0$. A proof of the following estimate was given e.g. in [10, Lem. 2.9]:

Lemma 5.6. *We have*

$$\|u\|^2_{X_{\varepsilon,\mathsf{v}}} \le \varepsilon^2 C_{\mathsf{v}} \|\mathrm{d}u\|^2_{X_{\varepsilon,\mathsf{v}}} + 8\varepsilon c_{\mathrm{vol},\mathsf{v}} \Big[b\|u'\|^2_{X_{\varepsilon,\mathsf{E}_\mathsf{v}}} + \frac{2}{b}\|u\|^2_{X_{\varepsilon,\mathsf{E}_\mathsf{v}}}\Big]$$

for $0 < b \le \min_\mathsf{e} \ell_\mathsf{e}$*, where*

$$C_\mathsf{v} := 4\Big[\frac{1}{\lambda_2(X_\mathsf{v})} + c_{\mathrm{vol},\mathsf{v}} C^{\mathrm{tr}}_{Y_\mathsf{v},X_\mathsf{v}}\big(\frac{1}{\lambda_2^{\mathrm{N}}(X_\mathsf{v})} + 1\big)\Big] \qquad \text{and} \qquad c_{\mathrm{vol},\mathsf{v}} := \frac{\mathrm{vol}_{m+1} X_\mathsf{v}}{\deg \mathsf{v}}. \tag{5.10}$$

Moreover, u' *denotes the derivative with respect to the longitudinal variable* $s \in I_\mathsf{e}$ *on each component* $X_\mathsf{e} = I_\mathsf{e} \times Y_\mathsf{e}$ *of* $X_{\mathsf{E}_\mathsf{v}}$.

5.4. **Equi-ellipticity.** Let us now show that the family of sesquilinear forms $(a_\varepsilon)_\varepsilon$ is equi-elliptic. To do so, we need assumptions on β_ε and γ_ε. We assume that

$$\gamma_\varepsilon \in \big(L^2(\Gamma_{\varepsilon,\mathsf{V}}) \otimes L^2(\Gamma_{\varepsilon,\mathsf{V}})\big) \oplus \big(L^2(\Gamma_{\varepsilon,\mathsf{E}}) \otimes L^2(\Gamma_{\varepsilon,\mathsf{E}})\big) \subset L^2(\Gamma_\varepsilon) \otimes L^2(\Gamma_\varepsilon), \tag{5.11a}$$

$$\|\beta_{\varepsilon,\mathsf{V}}\|_\infty + \|\gamma_\varepsilon\|_{\Gamma_{\varepsilon,\mathsf{V}}\times\Gamma_{\varepsilon,\mathsf{V}}} \le C_{\beta,\gamma,\mathsf{V}}, \quad \|\beta_{\varepsilon,\mathsf{E}}\|_\infty + \|\gamma_\varepsilon\|_{\Gamma_{\varepsilon,\mathsf{E}}\times\Gamma_{\varepsilon,\mathsf{E}}} \le \varepsilon C_{\beta,\gamma,\mathsf{E}} \tag{5.11b}$$

for all $\varepsilon > 0$ small enough, where $\beta_{\varepsilon,\mathsf{V}}$ is the restriction of β_ε to Γ_V etc. Note that we assumed for simplicity that γ_ε only couples edge neighbourhoods with edge neighbourhoods and vertex neighbourhoods with vertex neighbourhoods.

Proposition 5.7. *Assume that* (5.11a)–(5.11b) *are fulfilled. Then,* $a_\varepsilon(u,u)$ *is well-defined for* $u \in V_\varepsilon = H^1(X_\varepsilon)$*. Moreover, given* $\alpha \in (0,1)$*, there exists* $\omega \ge 0$ *and* $\varepsilon_0 = \varepsilon_0(\alpha) > 0$ *such that*

$$\mathrm{Re}\, a_\varepsilon(u,u) + \omega\|u\|^2_{H_\varepsilon} \ge \alpha\|u\|^2_{V_\varepsilon}$$

for all $u \in V_\varepsilon$ *and all* $\varepsilon \in (0,\varepsilon_0]$*. In particular,* $(a_\varepsilon)_{\varepsilon\in(0,\varepsilon_0]}$ *is an* $(H_\varepsilon)_{\varepsilon\in(0,\varepsilon_0]}$*-equi-elliptic family and* $H_\varepsilon = L^2(X_\varepsilon)$.

Proof. Let us show that (2.2) holds with uniform constants ω and α. Estimate (2.1) can be seen similarly; and (2.3) is fulfilled with $c_V = 1$.

We start estimating the difference $a_\varepsilon(u,u) - \|\mathrm{d}u\|^2_{X_\varepsilon}$. We have

$$\big|a_\varepsilon(u,u) - \|\mathrm{d}u\|^2_{X_\varepsilon}\big| \le C_{\beta,\gamma,\mathsf{V}}\|u\|^2_{\Gamma_{\varepsilon,\mathsf{V}}} + \varepsilon C_{\beta,\gamma,\mathsf{E}}\|u\|^2_{\Gamma_{\varepsilon,\mathsf{E}}}$$

by Cauchy-Schwarz, Fubini, (5.11a)–(5.11b). It follows from the Sobolev trace estimate (5.6b) and Lemma 5.6 that

$$\begin{aligned}\|u\|^2_{\Gamma_{\varepsilon,\mathsf{V}}} &\le \max_\mathsf{v} C^{\mathrm{tr}}_{\Gamma_\mathsf{v},X_\mathsf{v}}\Big(\varepsilon\|\mathrm{d}u\|^2_{X_{\varepsilon,\mathsf{V}}} + \frac{1}{\varepsilon}\|u\|^2_{X_{\varepsilon,\mathsf{V}}}\Big)\\ &\le \max_\mathsf{v}(C^{\mathrm{tr}}_{\Gamma_\mathsf{v},X_\mathsf{v}} + C_\mathsf{v})\varepsilon\|\mathrm{d}u\|^2_{X_{\varepsilon,\mathsf{V}}} + 16\max_\mathsf{v} C^{\mathrm{tr}}_{\Gamma_\mathsf{v},X_\mathsf{v}} c_{\mathrm{vol},\mathsf{v}}\Big[b\|u'\|^2_{X_{\varepsilon,\mathsf{E}}} + \frac{2}{b}\|u\|^2_{X_{\varepsilon,\mathsf{E}}}\Big]\end{aligned}$$

for $0 < b \le \min_{\mathsf{e}} \ell_{\mathsf{e}}$. For $\|u\|^2_{\Gamma_{\varepsilon,\mathsf{E}}}$, we have the estimate

$$\|u\|^2_{\Gamma_{\varepsilon,\mathsf{E}}} \le \max_{\mathsf{e}} C^{\mathrm{tr}}_{\partial Y_{\mathsf{e}},Y_{\mathsf{e}}}\Big(\varepsilon\|\mathrm{d}u\|^2_{X_{\varepsilon,\mathsf{E}}} + \frac{1}{\varepsilon}\|u\|^2_{X_{\varepsilon,\mathsf{E}}}\Big)$$

by Lemma 5.3. It follows that

$$\big|a_\varepsilon(u,u) - \|\mathrm{d}u\|^2_{X_\varepsilon}\big| \le C(\varepsilon,b)\|\mathrm{d}u\|^2_{X_\varepsilon} + \omega(a)\|u\|^2_{X_\varepsilon},$$

where

$$C(\varepsilon,b) := \max_{\mathsf{v},\mathsf{e}}\big\{\varepsilon C_{\beta,\gamma,\mathsf{V}}(C^{\mathrm{tr}}_{\Gamma_{\mathsf{v}},X_{\mathsf{v}}} + C_{\mathsf{v}}),\ 16bC_{\beta,\gamma,\mathsf{V}}C^{\mathrm{tr}}_{\Gamma_{\mathsf{v}},X_{\mathsf{v}}}c_{\mathrm{vol},\mathsf{v}},\ \varepsilon^2 C_{\beta,\gamma,\mathsf{E}}C^{\mathrm{tr}}_{\Gamma_{\mathsf{e}},X_{\mathsf{e}}}\big\},$$
$$\omega(b) := \max_{\mathsf{v},\mathsf{e}}\big\{16b^{-1}C_{\beta,\gamma,\mathsf{V}}C^{\mathrm{tr}}_{\Gamma_{\mathsf{v}},X_{\mathsf{v}}}c_{\mathrm{vol},\mathsf{v}},\ C_{\beta,\gamma,\mathsf{E}}C^{\mathrm{tr}}_{\Gamma_{\mathsf{e}},X_{\mathsf{e}}}\big\}.$$

For $\alpha \in (0,1)$ we choose

$$\varepsilon_0 := \min_{\mathsf{v},\mathsf{e}}\Big\{1,\ \frac{1-\alpha}{C_{\beta,\gamma,\mathsf{v}}(C^{\mathrm{tr}}_{\Gamma_{\mathsf{v}},X_{\mathsf{v}}} + C_{\mathsf{v}})},\ \frac{1-\alpha}{C_{\beta,\gamma,\mathsf{e}}C^{\mathrm{tr}}_{\Gamma_{\mathsf{e}},X_{\mathsf{e}}}}\Big\}$$

and

$$b := \min_{\mathsf{v},\mathsf{e}}\Big\{\ell_{\mathsf{e}},\ \frac{1-\alpha}{16C_{\beta,\gamma,\mathsf{V}}C^{\mathrm{tr}}_{\Gamma_{\mathsf{v}},X_{\mathsf{v}}}c_{\mathrm{vol},\mathsf{v}}}\Big\}.$$

Then $C(\varepsilon,b) \le C(\varepsilon_0,b) \le 1-\alpha$ and we have

$$\begin{aligned}\operatorname{Re} a_\varepsilon(u) &\ge \|\mathrm{d}u\|^2_{X_\varepsilon} - \big|a_\varepsilon(u) - \|\mathrm{d}u\|^2_{X_\varepsilon}\big|\\ &\ge \big(1 - C(\varepsilon_0,b)\big)\|\mathrm{d}u\|^2_{X_\varepsilon} - \omega(b)\|u\|^2_{X_\varepsilon} \ge \alpha\|\mathrm{d}u\|^2_{X_\varepsilon} - \omega\|u\|^2_{X_\varepsilon}\end{aligned}$$

for all $\varepsilon \in (0,\varepsilon_0]$ with $\omega := \omega(b)$. In particular, the family $(a_\varepsilon)_\varepsilon$ is equi-elliptic. □

5.5. **The identification operators.** We now fix the identification operators $J^{\uparrow\varepsilon}$ and $J^{\downarrow\varepsilon}$ similar as in [31] (see also Remark 5.2). In particular, we set

$$J^{\uparrow\varepsilon}\colon L^2(X) \longrightarrow L^2(X_\varepsilon), \qquad (J^{\uparrow\varepsilon}f)_{\mathsf{v}} := 0, \quad (J^{\uparrow\varepsilon}f)_{\mathsf{e}} := f_{\mathsf{e}} \otimes \mathbb{1}_{\varepsilon,\mathsf{e}}, \tag{5.12}$$

where we use the decomposition of $u = J^{\uparrow\varepsilon}f$ with respect to (5.3). Here $\mathbb{1}_{\varepsilon,\mathsf{e}}(y) := \varepsilon^{-m/2}$ for all $y \in Y_{\mathsf{e}}$, thus

$$(J^{\uparrow\varepsilon}f)_{\mathsf{e}}(s,y) = \varepsilon^{-m/2}f_{\mathsf{e}}(s).$$

Note that $\|J^{\uparrow\varepsilon}f\|_{H_\varepsilon} \le \|f\|_{H_0}$. For $J^{\downarrow\varepsilon}$, we just choose the adjoint, i.e., $J^{\downarrow\varepsilon} := (J^{\uparrow\varepsilon})^*$. In particular, (2.7b) is fulfilled and we have

$$(J^{\downarrow\varepsilon}u)_{\mathsf{e}} = \varepsilon^{m/2}\fint_{\mathsf{e}} u.$$

Moreover, we need the corresponding identification operators on the level of quadratic form domains. As in [31], we define

$$(J_1^{\uparrow\varepsilon}f)_{\mathsf{e}} := (J^{\uparrow\varepsilon}f)_{\mathsf{e}}, \qquad (J_1^{\uparrow\varepsilon}f)_{\mathsf{v}} := \varepsilon^{-m/2}f(\mathsf{v})$$

(see (5.9) for the notation). Note that $f(\mathsf{v})$ is well defined for $f \in V_0$, and that $J_1^{\uparrow\varepsilon} f \in V_\varepsilon$. For the operator in the opposite direction, we choose

$$\begin{aligned}(J_1^{\downarrow\varepsilon} u)_{\mathsf{e}}(s) :=& (J^{\downarrow\varepsilon} u)_{\mathsf{e}}(s) + \varepsilon^{m/2} \sum_{\mathsf{v}\in\partial\mathsf{e}} \chi_{\mathsf{v},\mathsf{e}}(s)\big(⨍_{\mathsf{v}} u - ⨍_{\mathsf{e}} u(\mathsf{v})\big)\\ =& \varepsilon^{m/2}\Big(⨍_{\mathsf{e}} u(s) + \sum_{\mathsf{v}\in\partial\mathsf{e}} \chi_{\mathsf{v},\mathsf{e}}(s)\big(⨍_{\mathsf{v}} u - ⨍_{\mathsf{e}} u(\mathsf{v})\big)\Big)\end{aligned}$$

where $\chi_{\mathsf{v},\mathsf{e}}$ is the continuous function on the metric edge I_{e} with $\chi_{\mathsf{v},\mathsf{e}}(\mathsf{v}) = 1$, $\chi_{\mathsf{v},\mathsf{e}}$ being affine linear on $I_{\mathsf{v},\mathsf{e}} := \{\, s \in I_{\mathsf{e}} : d(s,v) \le \ell_0 \,\}$ and $\chi_{\mathsf{v},\mathsf{e}}(s) = 0$ for $s \in I_{\mathsf{v},\mathsf{e}}$. Recall the definition of $⨍_{\mathsf{v}} u$ in (5.7). In particular, it is easy to see that $(J_1^{\downarrow\varepsilon} u)_{\mathsf{e}}(\mathsf{v}) = \varepsilon^{m/2} ⨍_{\mathsf{v}} u$, independently of the edge $\mathsf{e} \in \mathsf{E}_{\mathsf{v}}$, i.e., $J_1^{\downarrow\varepsilon} u \in \mathscr{V}$.

Let γ be the matrix of Section 5.1. We additionally need that

$$\|\widetilde{\gamma}_\varepsilon - \gamma\|_{\mathscr{L}(\ell^2(\mathsf{V}))} \le \varepsilon^{1/2} C'_{\beta,\gamma,\mathsf{V}} \quad\text{and}\quad \|\beta_{\varepsilon,\mathsf{E}}\|_\infty + \|\gamma_\varepsilon\|_{\Gamma_{\varepsilon,\mathsf{E}}\times\Gamma_{\varepsilon,\mathsf{E}}} \le \varepsilon^{3/2} C'_{\beta,\gamma,\mathsf{E}}, \tag{5.13}$$

where $\tilde{\gamma}_\varepsilon$ is the $|\mathsf{V}| \times |\mathsf{V}|$-matrix defined by

$$\widetilde{\gamma}_{\varepsilon,\mathsf{vw}} := \frac{1}{\deg \mathsf{v}}\Big(\delta_{\mathsf{vw}} \int_{\Gamma_{\mathsf{v}}} \beta_\varepsilon \, \mathrm{d}\Gamma_{\mathsf{v}} + \varepsilon^m \int_{\Gamma_{\mathsf{v}}\times\Gamma_{\mathsf{w}}} \gamma_\varepsilon \, \mathrm{d}\Gamma_{\mathsf{v}} \otimes \mathrm{d}\Gamma_{\mathsf{w}}\Big).$$

Moreover, $(\widetilde{\gamma}_\varepsilon \varphi)(\mathsf{v}) := \sum_{\mathsf{w}\in\mathsf{V}} \widetilde{\gamma}_{\varepsilon,\mathsf{vw}} \varphi(\mathsf{w})$ denotes the corresponding operator in the Hilbert space $\ell^2(\mathsf{V})$ with weighted norm $\|\varphi\|_{\mathsf{V}}^2 := \sum_{\mathsf{v}} |\varphi(\mathsf{v})|^2 \deg \mathsf{v}$. Note that the second condition in (5.13) is stronger than the second condition in (5.11b).

Proposition 5.8. *Assume that* (5.11a)–(5.11b) *and* (5.13) *hold, then we have*

$$\big|a_\varepsilon\big(J_1^{\uparrow\varepsilon} f, u\big) - a_0\big(f, J_1^{\downarrow\varepsilon} u\big)\big| \le \delta_\varepsilon \|f\|_{V_0} \|u\|_{V_\varepsilon}$$

for all $f \in V_0 = H^1(X_0)$, $u \in V_\varepsilon = H^1(X_\varepsilon)$ and $\varepsilon \in (0, \varepsilon_0]$, where $\delta_\varepsilon = \mathrm{O}(\varepsilon^{1/2})$ depends only on the geometry of the unscaled manifold X and the metric graph.

Proof. In order to verify the estimate, we will split the estimate in its vertex and edge part. For the edge contribution, we have

$$\begin{aligned}&\big|a_{\varepsilon,\mathsf{E}}\big(J_1^{\uparrow\varepsilon} f, u\big) - a_{0,\mathsf{E}}\big(f, J_1^{\downarrow\varepsilon} u\big)\big|\\ &\qquad= \varepsilon^{m/2}\Big|\varepsilon^{-1}\Big(\int_{\Gamma_{\mathsf{E}}} \beta_\varepsilon f \overline{u} \, \mathrm{d}\Gamma + \varepsilon^{m-1} \int_{\Gamma_{\mathsf{E}}\times\Gamma_{\mathsf{E}}} \gamma_\varepsilon (f \otimes \overline{u}) \, \mathrm{d}\Gamma \otimes \mathrm{d}\Gamma\Big)\\ &\qquad\qquad + \sum_{\mathsf{e}\in\mathsf{E}} \sum_{\mathsf{v}\in\partial\mathsf{e}} \int_{I_{\mathsf{e}}} \chi'_{\mathsf{v},\mathsf{e}} f'_{\mathsf{e}} \big(⨍_{\mathsf{v}} \overline{u} - ⨍_{\mathsf{e}} \overline{u}(\mathsf{v})\big) \, \mathrm{d}s\Big|.\end{aligned}$$

The first two integrals can be estimated by

$$\begin{aligned}&\varepsilon C'_{\beta,\gamma,\mathsf{E}} (c_{\mathrm{vol},\mathsf{E}})^{1/2} \|f\|_{X_0} \|u\|_{\Gamma_{\varepsilon,\mathsf{E}}}\\ &\qquad\le \varepsilon^{1/2} C'_{\beta,\gamma,\mathsf{E}} (c_{\mathrm{vol},\mathsf{E}})^{1/2} \|f\|_{X_0} \big(\max_{\mathsf{e}} C^{\mathrm{tr}}_{\partial Y_{\mathsf{e}},Y_{\mathsf{e}}} (\varepsilon^2 \|\mathrm{d}u\|^2_{X_{\varepsilon,\mathsf{E}}} + \|u\|^2_{X_{\varepsilon,\mathsf{E}}})\big)^{1/2}\\ &\qquad\qquad \le \varepsilon^{1/2} C'_{\beta,\gamma,\mathsf{E}} \big(c_{\mathrm{vol},\mathsf{E}} \max_{\mathsf{e}} C^{\mathrm{tr}}_{\partial Y_{\mathsf{e}},Y_{\mathsf{e}}}\big)^{1/2} \|f\|_{X_0} \|u\|_{H^1(X_\varepsilon)}\end{aligned}$$

using the assumption in (5.13), Cauchy-Schwarz, Lemma 5.3 and $\varepsilon \le 1$. Here, we have set $c_{\mathrm{vol},\mathsf{E}} := \max_{\mathsf{e}}(\mathrm{vol}_{m-1} \partial Y_{\mathsf{e}})$. The last term of the edge contribution is small since

$$\begin{aligned}
&\varepsilon^{m/2}\Big|\sum_{\mathsf{e}\in\mathsf{E}}\sum_{\mathsf{v}\in\partial\mathsf{e}}\int_{I_{\mathsf{e}}} \chi'_{\mathsf{v},\mathsf{e}} f'_{\mathsf{e}} (\textstyle\fint_{\mathsf{v}}\overline{u} - \fint_{\mathsf{e}}\overline{u}(\mathsf{v}))\,\mathrm{d}s\Big| \\
&\qquad\le 2\varepsilon^{m/2}\ell_0^{-1/2}\|f'\|_{X_0}\Big(\sum_{\mathsf{v}\in\mathsf{V}}\sum_{\mathsf{e}\in\mathsf{E}_{\mathsf{v}}}\big|\textstyle\fint_{\mathsf{v}} u - \fint_{\mathsf{e}} u(\mathsf{v})\big|^2\Big)^{1/2} \\
&\qquad\le 2\varepsilon^{1/2}\max_{\mathsf{v}}\Big(\frac{C^{\mathrm{tr}}_{Y_{\mathsf{v}},X_{\mathsf{v}}}}{\ell_0}\Big)^{1/2}\Big(\frac{1}{\lambda_2^{\mathrm{N}}(X_{\mathsf{v}})}+1\Big)^{1/2}\|f'\|_{X_0}\|\mathrm{d}u\|_{X_{\varepsilon,\mathsf{V}}}
\end{aligned}$$

using Cauchy-Schwarz again, the fact that $\|\chi'_{\mathsf{v},\mathsf{e}}\|^2_{I_{\mathsf{e}}} = 1/\ell_0$, where $\ell_0 = \min_{\mathsf{e}}\{\ell_{\mathsf{e}}, 1\}$, and Lemma 5.5.

For the vertex contribution, we have

$$\begin{aligned}
(5.14)\quad &\big|a_{\varepsilon,\mathsf{V}}\big(J_1^{\uparrow\varepsilon} f, u\big) - a_{0,\mathsf{V}}\big(f, J_1^{\downarrow\varepsilon} u\big)\big| = \varepsilon^{m/2}\Big|\sum_{\mathsf{v}\in\mathsf{V}} f(\mathsf{v})\int_{\Gamma_{\mathsf{v}}}\beta_\varepsilon\overline{u}\,\mathrm{d}\Gamma_{\mathsf{v}} \\
&\quad + \sum_{\mathsf{v},\mathsf{w}\in\mathsf{V}} f(\mathsf{w})\Big(\varepsilon^m\int_{\Gamma_{\mathsf{v}}\times\Gamma_{\mathsf{w}}}\gamma_\varepsilon(\mathbb{1}\otimes\overline{u})\,\mathrm{d}\Gamma_{\mathsf{v}}\otimes\mathrm{d}\Gamma_{\mathsf{w}} - \gamma_{\mathsf{vw}}(\deg\mathsf{v})\textstyle\fint_{\mathsf{v}}\overline{u}\Big)\Big| \\
&\quad \le \varepsilon^{m/2}\Big|\sum_{\mathsf{v},\mathsf{w}\in\mathsf{V}} f(\mathsf{w})(\widetilde{\gamma}_{\varepsilon,\mathsf{vw}} - \gamma_{\mathsf{vw}})\textstyle\fint_{\mathsf{v}}\overline{u})\deg\mathsf{v}\Big| \\
&\quad\quad + \varepsilon^{m/2}\sum_{\mathsf{v}\in\mathsf{V}}\Big(|f(\mathsf{v})|\|\beta_\varepsilon\|_{\Gamma_{\mathsf{v}}} + \sum_{\mathsf{w}\in\mathsf{V}}|f(\mathsf{w})|\|\gamma_\varepsilon\|_{\Gamma_{\varepsilon,\mathsf{v}}\times\Gamma_{\varepsilon,\mathsf{w}}}\|\mathbb{1}\|_{\Gamma_{\mathsf{v}}}\Big)\|u - \textstyle\fint_{\mathsf{v}} u\|_{\Gamma_{\mathsf{v}}}
\end{aligned}$$

since the derivative vanishes as $J_1^{\uparrow\varepsilon} f$ is constant on X_{v}, and where we replaced u by $\fint_{\mathsf{v}} u + (u - \fint_{\mathsf{v}} u)$ in the first two integrals. The first sum of the last estimate can be estimated by

$$\varepsilon^{m/2}\big|\langle \underline{f}|(\widetilde{\gamma}_\varepsilon - \gamma)\underline{u}\rangle_{\mathsf{V}}\big| \le \|\widetilde{\gamma}_\varepsilon - \gamma\|_{\mathscr{L}(\ell^2(\mathsf{V}))}\|\underline{f}\|_{\mathsf{V}}(\varepsilon^{m/2}\|\underline{u}\|_{\mathsf{V}}),$$

where $\underline{f} = (f(\mathsf{v}))_{\mathsf{v}\in\mathsf{V}}$ and $\underline{u} = (\fint_{\mathsf{v}} u)_{\mathsf{v}\in\mathsf{V}}$. Now,

$$\begin{aligned}
\varepsilon^m\|\underline{u}\|^2_{\mathsf{V}} &\le \varepsilon^m\sum_{\mathsf{v}\in\mathsf{V}}\frac{\deg\mathsf{v}}{\mathrm{vol}_{m+1} X_{\mathsf{v}}}\|u\|^2_{X_{\mathsf{v}}} \\
&\qquad\le c'_{\mathrm{vol},\mathsf{V}}\cdot\Big(\varepsilon\max_{\mathsf{v}} C_{\mathsf{v}}\|\mathrm{d}u\|^2_{X_{\varepsilon,\mathsf{V}}} + \frac{16\max_{\mathsf{v}} c_{\mathrm{vol},\mathsf{v}}}{\ell_0}\big(\|u'\|^2_{X_{\varepsilon,\mathsf{E}}} + \|u\|^2_{X_{\varepsilon,\mathsf{E}}}\big)\Big)
\end{aligned}$$

by Lemma 5.6, where $c'_{\mathrm{vol},\mathsf{V}} := \max_{\mathsf{v}}(\deg\mathsf{v})/(\mathrm{vol}_{m+1} X_{\mathsf{v}})$. In particular, the first sum equals $\varepsilon^{m/2}\big|\langle \underline{f}|(\widetilde{\gamma}_\varepsilon - \gamma)\underline{u}\rangle_{\mathsf{V}}\big|$ and can be estimated from above by

$$\varepsilon^{1/2}C'_{\beta,\gamma,\mathsf{V}}\Big(\frac{8c'_{\mathrm{vol},\mathsf{V}}}{\ell_0}\max_{\mathsf{v}}\Big\{C_{\mathsf{v}},\ \frac{16c_{\mathrm{vol},\mathsf{v}}}{\ell_0}\Big\}\Big)^{1/2}\|f\|_{H^1(X_0)}\|u\|_{H^1(X_\varepsilon)}$$

by (5.2) and since $\varepsilon \le 1$. The second summand of the right hand side of (5.14) can be estimated by

$$\varepsilon^{m/2}\big(\|\beta_\varepsilon\|_\infty + \|\gamma_\varepsilon\|_{\Gamma_{\varepsilon,\mathsf{V}}\times\Gamma_{\varepsilon,\mathsf{V}}}\big)\|\underline{f}\|_{\mathsf{V}}\Big(\sum_{\mathsf{v}\in\mathsf{V}}\frac{\operatorname{vol}\Gamma_{\mathsf{v}}}{\deg \mathsf{v}}\|u - ⨍_{\mathsf{v}} u\|^2_{\Gamma_{\mathsf{v}}}\Big)^{1/2}$$
$$\le \varepsilon^{1/2} C_{\beta,\gamma,\mathsf{V}} \max_{\mathsf{v},\mathsf{e}}\bigg(\frac{8c''_{\mathrm{vol},\mathsf{v}} C^{\mathrm{tr}}_{\Gamma_{\mathsf{v}},X_{\mathsf{v}}}}{\ell_{\mathsf{e}}}\Big(\frac{1}{\lambda_2^{\mathrm{N}}(X_{\mathsf{v}})}+1\Big)\bigg)^{1/2}\|f\|_{H^1(X_0)}\|\mathrm{d}u\|_{X_{\varepsilon,\mathsf{V}}}$$

using (5.11b), (5.2) and Lemma 5.4, where $c''_{\mathrm{vol},\mathsf{v}} := (\operatorname{vol}_m \Gamma_{\mathsf{v}})/(\deg \mathsf{v})$. □

Let us now formulate the main theorem of this section.

Theorem 5.9. *Assume that* (5.11a)–(5.11b) *and* (5.13) *are fulfilled. Then the sesquilinear forms* $(a_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ *form an equi-elliptic family for some* $\varepsilon_0 > 0$. *Moreover,* a_ε *is* δ_ε-κ*-quasi-unitarily equivalent to* a_0 *for* $\delta_\varepsilon = \mathrm{O}(\varepsilon^{1/2})$ *and* $\kappa = 1$.

In particular, the convergence results of Section 2 *apply, e.g., the spectra* $\sigma(A_\varepsilon)$ *of the associated operators* A_ε *converges to the spectrum* $\sigma(A_0)$ *of* A_0 *in the sense of Definition* 3.10.

Proof. Condition (2.7e) has been shown in Proposition 5.8. The other conditions have already been shown in [31] or [10, Prp. 3.2]. Note that the spectrum of A_ε and A_0 is purely discrete, since the underlying spaces are compact. □

Remark 5.10. If the graph X_0 and the corresponding manifold X_ε are not compact, the corresponding forms a_0 and a_ε are still (equi-)sectorial and fulfil Definition 2.3 provided we have a uniform control of the geometry of the graph and the manifold building blocks (see the constants in the proofs). For example, we need a positive lower bound on the edge length, i.e., $\inf_{\mathsf{e}} \ell_{\mathsf{e}} > 0$ and a uniform finite upper bound on the Sobolev trace constants like $\sup_{\mathsf{v}} C^{\mathrm{tr}}_{\Gamma_{\mathsf{v}},X_{\mathsf{v}}} < \infty$. The uniform control of the geometry is in particular fulfilled if there is a finite set of of manifolds $\mathscr{M}$ such that the building blocks X_{v} and Y_{e} of the manifold X, constructed according to the graph $(\mathsf{V},\mathsf{E},\partial)$, are isometric to a member in $\mathscr{M}$. Coverings of compact spaces provide such examples.

Remark 5.11. Under suitable conditions on the coefficients we can apply Theorem 3.23 in the context of the approximation results of this section. More precisely, assume e.g. that $\beta_\varepsilon \ge 0$ and $\gamma_\varepsilon = 0$. Then it can easily be verified that the forms a_ε satisfy the Beurling-Deny conditions for all $\varepsilon \ge 0$. Thus the associated semigroups are positive and L^∞-contractive. Thus for $\varphi(z) = \mathrm{e}^{-tz}$ the assumptions of Theorem 3.23 are satisfied with $c_\varepsilon = \varepsilon^{m/2}$. Hence

$$\|J^{\uparrow\varepsilon}\mathrm{e}^{-tA_0}J^{\downarrow\varepsilon} - \mathrm{e}^{tA_\varepsilon}\|_{\mathscr{L}(L^p(X_\varepsilon))} \to 0$$

as $\varepsilon \to 0$.

Bibliography.

REFERENCES

[1] W. Arendt, M. Biegert, and A. F. M. ter Elst, *Diffusion determines the manifold*, arXiv:0806.0437v1, 2008.

[2] W. Arendt and M. Chovanec, *Dirichlet regularity and degenerate diffusion*, Trans. Amer. Math. Soc. **362** (2010), 5861–5878.

[3] M. Ahues, A. Largillier, and B. V. Limaye, *Spectral computations for bounded operators*, Applied Mathematics (Boca Raton), vol. 18, Chapman & Hall/CRC, Boca Raton, FL, 2001.

[4] W. Arendt, *Semigroups and evolution equations: functional calculus, regularity and kernel estimates*, Evolutionary equations. Vol. I, Handb. Differ. Equ., North-Holland, Amsterdam, 2004, pp. 1–85.

[5] S. Andres and M. Von Renesse, *Particle approximation of the Wasserstein diffusion*, Journal of Functional Analysis **258** (2009), 3879–3905.

[6] C. Cacciapuoti and D. Finco, *Graph-like models for thin waveguides with Robin boundary conditions*, Asymptot. Anal. **70** (2010), 199–230.

[7] G. M. Coclite, A. Favini, G. R. Goldstein, J. A. Goldstein, and S. Romanelli, *Continuous dependence on the boundary conditions for the Wentzell Laplacian*, Semigroup Forum **77** (2008), 101–108.

[8] G. M. Coclite, G. R. Goldstein, and J. A. Goldstein, *Stability estimates for parabolic problems with Wentzell boundary conditions*, J. Differential Equations **245** (2008), 2595–2626.

[9] D. Daners, *Domain perturbation for linear and semi-linear boundary value problems*, Handbook of differential equations: stationary partial differential equations. Vol. VI, Handb. Differ. Equ., Elsevier/North-Holland, Amsterdam, 2008, pp. 1–81.

[10] P. Exner and O. Post, *Approximation of quantum graph vertex couplings by scaled Schrödinger operators on thin branched manifolds*, J. Phys. A **42** (2009), 415305, 22.

[11] U. R. Freiberg and M. R. Lancia, *Energy forms on conformal C^1-diffeomorphic images of the Sierpinski gasket*, Mathematische Nachrichten **281** (2008), 337–349.

[12] D. Grieser, *Spectra of graph neighborhoods and scattering*, Proc. Lond. Math. Soc. (3) **97** (2008), 718–752.

[13] M. Haase, *The functional calculus for sectorial operators*, Operator Theory: Advances and Applications, vol. 169, Birkhäuser Verlag, Basel, 2006.

[14] M. Hinz, *Approximation of jump processes on fractals*, Osaka J. Math **46** (2009), 141–171.

[15] K. Ito and F. Kappel, *Evolution equations and approximations*, Series on Advances in Mathematics for Applied Sciences, vol. 61, World Scientific Publishing Co. Inc., River Edge, NJ, 2002.

[16] A. Kasue, *Convergence of Riemannian manifolds and Laplace operators. I*, Ann. Inst. Fourier (Grenoble) **52** (2002), 1219–1257.

[17] A. Kasue, *Convergence of Riemannian manifolds and Laplace operators. II*, Potential Anal. **24** (2006), 137–194.

[18] T. Kato, *Perturbation theory for linear operators*, Classics in Mathematics, Springer-Verlag, Berlin, 1995, Reprint of the 1980 edition.

[19] A. Kolesnikov, *Mosco convergence of Dirichlet forms in infinite dimensions with changing reference measures*, Journal of Functional Analysis **230** (2006), 382–418.

[20] S. M. Kozlov, O. A. Oleĭnik, and V. V. Zhikov, *Homogenization of differential operators and integral functionals*, Springer-Verlag, Berlin, 1994, Translated from the Russian by G. A. Yosifian [G. A. Iosif′yan].

[21] K. Kuwae and T. Shioya, *Convergence of spectral structures: a functional analytic theory and its applications to spectral geometry*, Comm. Anal. Geom. **11** (2003), 599–673.

[22] P. C. Kunstmann and L. Weis, *Maximal L_p-regularity for parabolic equations, Fourier multiplier theorems and H^∞-functional calculus*, Functional analytic methods for evolution equations, Lecture Notes in Math., vol. 1855, Springer, Berlin, 2004, pp. 65–311.

[23] T. A. Mel′nyk, *Asymptotic analysis of a spectral problem in a periodic thick junction of type 3: 2: 1*, Math. Meth. Appl. Sci. **23** (2000), 321–346.

[24] T. A. Mel′nyk, *Hausdorff convergence and asymptotic estimates of the spectrum of a perturbed operator*, Z. Anal. Anwendungen **20** (2001), 941–957.

[25] T. A. Mel′nyk, *A scheme for studying the spectrum of a family of perturbed operators and its application to spectral problems in thick junctions*, Nelīnīĭnī Koliv. **6** (2003), 233–251.

[26] T. A. Mel′nyk, *Homogenization of a boundary-value problem with a nonlinear boundary condition in a thick junction of type 3: 2: 1*, Math. Meth. Appl. Sci. **31** (2008), 1005–1027.

[27] U. Mosco, *Approximation of the solutions of some variational inequalities*, Ann. Sc. Norm. Super. Pisa, Cl. Sci., III Serie **21** (1967), 373–394.

[28] U. Mosco and M. Vivaldi, *Fractal reinforcement of elastic membranes*, Archive for rational mechanics and analysis **194** (2009), 49–74.

[29] O. A. Oleĭnik, A. S. Shamaev, and G. A. Yosifian, *Mathematical problems in elasticity and homogenization*, Studies in Mathematics and its Applications, vol. 26, North-Holland Publishing Co., Amsterdam, 1992.

[30] E. M. Ouhabaz, *Analysis of heat equations on domains*, London Mathematical Society Monographs Series, vol. 31, Princeton University Press, Princeton, NJ, 2005.

[31] O. Post, *Spectral convergence of quasi-one-dimensional spaces*, Ann. Henri Poincaré **7** (2006), 933–973.

[32] O. Post, *Spectral analysis on graph-like spaces*, to appear in Lecture Notes in Mathematics, Springer-Verlag, Berlin, 2011.

[33] S. E. Pastukhova and V. V. Zhikov, *On the Trotter-Kato theorem in a variable space*, Funktsional. Anal. i Prilozhen. **41** (2007), 22–29, 96.

[34] E. Sánchez-Palencia, *Nonhomogeneous media and vibration theory*, Lecture Notes in Physics, vol. 127, Springer-Verlag, Berlin, 1980.

[35] P. Stollmann, *A convergence theorem for Dirichlet forms with applications to boundary value problems with varying domains*, Math. Z. **219** (1995), 275–287.

[36] G. Stolz and J. Weidmann, *Approximation of isolated eigenvalues of general singular ordinary differential operators*, Results Math. **28** (1995), 345–358.

[37] S. Teufel and J. Wachsmuth, *Effective hamiltonians for constrained quantum systems*, preprint arXiv:0907.0351, 2009.

[38] J. Weidmann, *Stetige Abhängigkeit der Eigenwerte und Eigenfunktionen elliptischer Differentialoperatoren vom Gebiet*, Math. Scand. **54** (1984), 51–69.

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