

ON THE MORSE-SARD PROPERTY AND LEVEL SETS OF SOBOLEV AND BV FUNCTIONS

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Abstract

We establish Luzin N and Morse–Sard properties for BV_2 -functions defined on open domains in the plane. Using these results we prove that almost all level sets are finite disjoint unions of Lipschitz arcs whose tangent vectors are of bounded variation. In the case of $W^{2,1}$ -functions we strengthen the conclusion and show that almost all level sets are finite disjoint unions of C^1 -arcs whose tangent vectors are absolutely continuous.

Key words: BV^2 and $W^{2,1}$ -functions, Luzin N -property, Morse–Sard property, level sets.

Introduction

For C^2 -smooth functions $v: \Omega \rightarrow \mathbb{R}$, defined on an open subset Ω of \mathbb{R}^2 , the classical Morse–Sard theorem [21], [26] (see also [11] or [13]) guarantees that

$$\mathcal{L}^1(v(Z_v)) = 0, \tag{1}$$

where \mathcal{L}^1 is the 1-dimensional Lebesgue measure on \mathbb{R} and Z_v is the critical set of v , $Z_v = \{x \in \Omega : \nabla v(x) = 0\}$. Whitney demonstrated [27] that the C^2 -smoothness condition in the above assertion cannot be dropped. Namely, he constructed a C^1 -smooth function $v: (0, 1)^2 \rightarrow \mathbb{R}$ for which the set Z_v of critical points contains an arc on which v is not constant (subsequently called a Whitney arc).

However, some analogs of Sard’s theorem are valid for the functions lacking the required smoothness in the classical theorem. Although (1) may be no longer valid then, A. Ya. Dubovitskiĭ [10] obtained some results on the structure of level sets in the case of reduced smoothness (also see [4]).

Another Sard–type theorem was obtained by A.V. Pogorelov (see [24, Chapter 9, Section 4]): For a function $v \in C^1(\Omega)$ on a plane domain Ω , the equality (1) holds if for any

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linear map $L: \mathbb{R}^2 \rightarrow \mathbb{R}$ the sum $v(x) + L(x)$ satisfies the maximum principle (see also [15] for another proof of this result). In particular, the equality (1) holds if the gradient range $\nabla v(\Omega)$ has no interior points (see also [15, 17, 16]).

Another direction of the research was the generalization of Sard's theorem to functions in Hölder and Sobolev spaces (for example, see [4, 8, 12, 14, 22]). In particular, De Pascale (see also [12]) proved that (1) holds when $v \in W_{\text{loc}}^{2,p}(\Omega)$ for $p > 2$. Note that in this case v is C^1 -smooth by virtue of the Sobolev imbedding theorem, and so the critical set is defined as usual.

In the paper [6] it was proved that for functions $v \in W_{\text{loc}}^{2,p}(\mathbb{R}^2)$ with $p > 1$ there are no Whitney arcs.

Landis [19] proved that the equality (1) holds if $v: \Omega \rightarrow \mathbb{R}$ is a difference of two convex functions (sometimes called a d.c.-function), a result which answered a question raised previously by A.V. Pogorelov. D. Pavlica and L. Zajíček [23] presented the detailed and modern proof of the Landis result. Moreover, they proved in [23] that the equality (1) holds for Lipschitz functions $v \in BV_{2,\text{loc}}(\Omega)$, where $BV_{2,\text{loc}}(\Omega)$ is the space of functions $v \in L_{\text{loc}}^1(\Omega)$ such that all its partial (distributional) derivatives of the second order are \mathbb{R} -valued Radon measures on Ω .

In this paper we extend the last result to the case of any BV_2 -function defined on a planar domain (without the additional Lipschitz assumption, see Theorem 3.1). Moreover, as we understand the critical set in a wider sense than in [23], our result is also an improvement in the Lipschitz case. More precisely, in [23] the critical set is defined as the set of points x , where v is (Frechet-)differentiable with total (Frechet-)differential $v'(x) = 0$. But it is known [9] (see also Lemma 3.2 below) that in general a function $v \in BV_{2,\text{loc}}(\Omega)$ admits a continuous representative which is differentiable outside an at most \mathcal{H}^1 - σ -finite (rectifiable) set, and that has “half-space differentials” \mathcal{H}^1 -almost everywhere. We include in the critical set Z_v the points $x \in \Omega$ such that one of the “half-space differentials” is zero at x .

Our main result, contained in Theorem 2.1 and Corollary 2.2, is to establish the Luzin N -property with respect to \mathcal{H}^1 for BV^2 functions on plane domains. More precisely, we show that if v is BV^2 on the open domain $\Omega \subset \mathbb{R}^2$, then for any $\varepsilon > 0$ there exists $\delta > 0$ such that for all subsets $E \subset \Omega$ with $\mathcal{H}_{\infty}^1(E) < \delta$ we have $\mathcal{L}^1(v(E)) < \varepsilon$. In particular, it follows that $\mathcal{L}^1(v(E)) = 0$ whenever $\mathcal{H}^1(E) = 0$. So the image of the exceptional “bad” set, where neither the differential nor the half-space differentials are defined, has zero Lebesgue measure. This ties nicely in with our definition of the critical set and our version of the Morse–Sard result for BV^2 -functions on the plane.

Finally, using these results we prove that almost all level sets of BV_2 -functions defined on open domains in the plane, are finite disjoint unions of Lipschitz arcs whose tangent vectors have bounded variations (Theorem 5.1 and Corollary 5.2). In the $W^{2,1}$ -case we can strengthen the conclusions and show that almost all level sets are finite disjoint unions of C^1 -arcs whose tangent vectors are absolutely continuous functions (Theorem 4.1 and Corollary 4.2).

After this work was completed we learned that [1] have also recently established the Morse–Sard property for $W^{2,1}$ functions on the plane.

1 Preliminaries

Throughout this paper Ω denotes an open subset of \mathbb{R}^2 . By a *domain* we mean an open connected set. For a general subset $E \subset \mathbb{R}^2$, we let $\text{Cl } E$ stand for its closure, and ∂E for its boundary.

For a distribution T on Ω denote by $D_i T$, $i = 1, 2$, the distributional partial derivatives of T , and write $DT = (D_1 T, D_2 T)$. For \mathbb{R} -valued and \mathbb{R}^2 -valued Radon measures μ we denote by $\|\mu\|$ the total variation measure of μ . The space $\text{BV}(\Omega)$ is as usual defined as consisting of those functions $f \in L^1(\Omega)$ whose distributional partial derivatives $D_i f$ are Radon measures with $\|D_i f\|(\Omega) < \infty$ (for detailed definitions see [7]). As a consequence of Radon–Nikodym’s theorem we have for any $f \in \text{BV}(\Omega)$ the polar decomposition of the distributional derivative $Df(E) = \int_E \nu \, d\|Df\|$, where $\nu: \Omega \rightarrow \mathbb{S}^1$ is a Borel vector field valued in the unit sphere $\mathbb{S}^1 \subset \mathbb{R}^2$, and $\|Df\|$ is the total variation measure of Df .

A central role is played by $\text{BV}^2(\Omega)$ defined as the space of functions $v \in L^1(\Omega)$ such that $D_i v \in \text{BV}(\Omega)$, $i = 1, 2$. It is known (see [20]) that each function $v \in \text{BV}_2(\Omega)$ has a continuous representative, and subsequently we shall always select this representative when discussing BV^2 -functions. For $v \in \text{BV}_2(\Omega)$ denote by ∇v the gradient mapping $\nabla v = (D_1 v, D_2 v): \Omega \rightarrow \mathbb{R}^2$, well-defined as a $\text{BV}(\Omega, \mathbb{R}^2)$ mapping. Denote also

$$\begin{aligned} \|v\|_{\text{BV}_2(\Omega)} &= \|v\|_{L^1(\Omega)} + \|\nabla v\|_{L^1(\Omega)} + \|D^2 v\|(\Omega), \\ W^{1,1}(\Omega) &= \{f \in L^1(\Omega) : D_i f \in L^1(\Omega), i = 1, 2\}, \\ W^{2,1}(\Omega) &= \{v \in L^1(\Omega) : D_i f \in W^{1,1}(\Omega), i = 1, 2\}. \end{aligned}$$

We write $\|v\|_{\text{BV}}$ instead of $\|v\|_{\text{BV}(\mathbb{R}^2)}$.

For a Lebesgue measurable set $F \subset \mathbb{R}^2$ and a point $x \in \mathbb{R}^2$ we use the following notation:

$$\begin{aligned} \bar{D}(F, x) &= \limsup_{r \rightarrow 0^+} \frac{\mathcal{L}^2(F \cap B(x, r))}{\mathcal{L}^2(B(x, r))}, & \underline{D}(F, x) &= \liminf_{r \rightarrow 0^+} \frac{\mathcal{L}^2(F \cap B(x, r))}{\mathcal{L}^2(B(x, r))}, \\ \text{Int}_M F &= \{x : \underline{D}(F, x) = 1\}, & \text{Cl}_M F &= \{x : \bar{D}(F, x) > 0\}, \\ \partial^M F &= \text{Cl}_M F \setminus \text{Int}_M F. \end{aligned}$$

Here \mathcal{L}^2 is the Lebesgue measure on \mathbb{R}^2 . Denote by \mathcal{H}^1 , \mathcal{H}_∞^1 the 1-dimensional Hausdorff measure, Hausdorff content, respectively: for any $F \subset \mathbb{R}^2$, $\mathcal{H}^1(F) = \lim_{\alpha \rightarrow 0^+} \mathcal{H}_\alpha^1(F)$, where

$$\mathcal{H}_\alpha^1(F) = \inf \left\{ \sum_{i=1}^{\infty} \text{diam } F_i : \text{diam } F_i \leq \alpha, F \subset \bigcup_{i=1}^{\infty} F_i \right\}.$$

Recall that for any function $f \in \text{BV}(U)$, where U is an open set in \mathbb{R}^2 , the coarea formula

$$\|Df\|(U) = \int_{-\infty}^{+\infty} \mathcal{H}^1(U \cap \partial^M \{f \leq \lambda\}) \, d\lambda$$

holds (see [7]).

2 On images of sets of small capacities under BV_2 functions on the plane.

The main result of this section is the following Luzin N -property for BV^2 -functions:

Theorem 2.1. *Let $v \in BV_2(\mathbb{R}^2)$. Then for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any set $E \subset \mathbb{R}^2$ if $\mathcal{H}_\infty^1(E) < \delta$ then $\mathcal{H}^1(v(E)) < \varepsilon$.*

Corollary 2.2. *If $v \in BV_2(\mathbb{R}^2)$, $E \subset \mathbb{R}^2$, and $\mathcal{H}^1(E) = 0$, then $\mathcal{H}^1(v(E)) = 0$.*

Fix a function $v \in BV_2(\mathbb{R}^2)$. To prove the above results we need some preliminary lemmas.

Lemma 2.3. *For any $\varepsilon > 0$ there exists $\delta > 0$ such that for any set $E \subset \mathbb{R}^2$ if $\mathcal{H}_\infty^1(E) < \delta$ then $\|D^2v\|(E) < \varepsilon$.*

Proof. This is a consequence of the Coarea formula. □

Lemma 2.4. *For each $f \in BV(\mathbb{R}^2)$ and for any $\varepsilon_0 > 0$ there exists a pair of functions $f_0, f_1 \in BV(\mathbb{R}^2)$ such that*

$$f = f_0 + f_1, \tag{2}$$

$$\|f_0\|_{L^\infty} \leq K, \tag{3}$$

$$\|f_1\|_{BV} < \varepsilon_0, \tag{4}$$

where $K = K(\varepsilon_0, f)$.

Proof. The proof is similar to the proof of Theorem 3 in [7, §5.9].

Fix $K > 0$ and denote

$$f_0(x) = \begin{cases} f(x), & |f(x)| \leq K; \\ K, & f(x) > K, \\ -K, & f(x) < -K, \end{cases}$$

$$f_1(x) = f(x) - f_0(x).$$

Obviously $\|f_1\|_{L^1} < \frac{1}{2}\varepsilon_0$ for sufficiently large K . By construction we have inclusions $f_0, f_1 \in BV(\mathbb{R}^2)$ (see, for example, Theorem 4(iii) in [7, §4.2.2] for the Sobolev case). Then by the coarea formula

$$\|Df_1\|(\mathbb{R}^2) = \int_{|\lambda| > K} \mathcal{H}^1(\partial^M \{f \leq \lambda\}) \, d\lambda.$$

Consequently $\|f_1\|_{BV} < \frac{1}{2}\varepsilon_0$ for sufficiently large K . □

Corollary 2.5. *For any $\varepsilon_0 > 0$ there exists a pair of functions $f_0, f_1 \in BV(\mathbb{R}^2, \mathbb{R}^2)$ such that*

$$\forall x \in \mathbb{R}^2 \quad \nabla v(x) \equiv f_0(x) + f_1(x); \tag{5}$$

$$\|f_0\|_{L^\infty} \leq K; \tag{6}$$

$$\|f_1\|_{BV} < \varepsilon_0. \tag{7}$$

By *interval* we mean a square with the sides parallel to the coordinate axis.

Lemma 2.6 (see, for example, [20]). *Let $I \subset \Omega$ be an interval of the size $\ell(I)$. Then*

$$\mathcal{H}_1(v(I)) \leq C \left\{ \|D^2 v\|(I) + \frac{1}{\ell(I)} \int_I |\nabla v| \right\}, \quad (8)$$

where C does not depend on I, v .

Lemma 2.7 (see also [5]). *Denote by \mathcal{C} the collection of all functions of the form*

$$\varphi = \frac{1}{\mathcal{H}^1(\partial\Omega)} 1_\Omega,$$

where 1_Ω is the indicator function of the set Ω and Ω is a bounded domain in \mathbb{R}^2 with a smooth boundary $\partial\Omega$. If $f \in \text{BV}(\mathbb{R}^2)$ and

$$\|f\|_{\text{BV}} \leq 1, \quad (9)$$

then there exists a sequence of functions $f_i: \mathbb{R}^2 \rightarrow \mathbb{R}$ such that $f_i \rightarrow f$ almost everywhere, and each f_i is a convex combination of functions from $\mathcal{C} \cup (-\mathcal{C})$.

Proof. We may assume without loss of generality that

$$f \geq 0, \quad \|\nabla f\|_{L^1} < 1 \quad (10)$$

(see the proof of Lemma 2.4). Since each function from $\text{BV}(\mathbb{R}^2)$ can be approximated by functions from $C_0^\infty(\mathbb{R}^2)$ (see [7, §5.2.2]), we may also assume without loss of generality that

$$f \in C_0^\infty(\mathbb{R}^2), \quad \text{supp } f \subset B(0, R), \quad f(\mathbb{R}^2) \subset [0, M]. \quad (11)$$

For a parameter $\delta < 1$ consider $f_\delta = f + g + c$, where c is a constant and $g: \mathbb{R}^2 \rightarrow \mathbb{R}$ is a linear function with small norm such that

- (i) $\|\nabla f_\delta\|_{L^1(B(0, R))} < 1$,
- (ii) $\sup_{x \in B(0, R)} |f(x) - f_\delta(x)| < \delta$,
- (iii) all the critical values of the function f_δ are irrational numbers and they are regular in the sense of Morse theory,
- (iv) for each rational $t > \delta$ we can decompose the preimage as

$$\{x \in B(0, R) : f_\delta(x) > t\} = \bigcup_{i=1}^{m_t} \Omega_i,$$

where Ω_i are bounded smooth domains, and

$$\Omega_i \cap \Omega_j = (\partial\Omega_i) \cap (\partial\Omega_j) = \emptyset \quad \text{for } i \neq j,$$

$$(\partial\Omega_i) \cap \partial B(0, R) = \emptyset \quad \text{for } i = 1, \dots, m_t.$$

Then the function $h: [\delta, M+1] \rightarrow \mathbb{R}$, defined by the formula

$$h(t) = \mathcal{H}^1\left(B(0, R) \cap \{f_\delta = t\}\right),$$

is continuous, and hence in particular integrable in the Riemann sense. By (i) and by the Coarea formula we get

$$\int_\delta^{M+1} h(t) dt < 1.$$

In view of the definition of the Riemann integral we have for sufficiently large $k \in \mathbb{N}$ that

$$\sum_{\mathbb{N} \ni j > k\delta} \frac{1}{k} h(t_j) < 1,$$

where $t_j = \frac{j}{k}$. Write $E_j = \{x \in B(0, R) : f_\delta(x) > \frac{j}{k}\}$ and $\tilde{f}_j = \frac{1}{k} 1_{E_j}$. By construction

$$\|f - \sum_{\mathbb{N} \ni j > k\delta} \tilde{f}_j\|_{L^\infty} < 3\delta + \frac{2}{k}. \quad (12)$$

Let $E_j = \bigcup_{i=1}^{m_j} \Omega_j^i$, where the Ω_j^i are defined in (iv). By construction

$$\sum_{\mathbb{N} \ni j > k\delta} \sum_{i=1}^{m_j} \frac{1}{k} \mathcal{H}^1(\partial\Omega_j^i) = \sum_{\mathbb{N} \ni j > k\delta} \frac{1}{k} h(t_j) < 1. \quad (13)$$

Finally

$$\sum_{\mathbb{N} \ni j > k\delta} \tilde{f}_j = \sum_{\mathbb{N} \ni j > k\delta} \sum_{i=1}^{m_j} \alpha_{ij} \frac{1_{\Omega_j^i}}{\mathcal{H}^1(\partial\Omega_j^i)}, \quad (14)$$

where

$$\alpha_{ij} = \frac{\mathcal{H}^1(\partial\Omega_j^i)}{k}, \quad (15)$$

and consequently by (13),

$$\sum_{\mathbb{N} \ni j > k\delta} \sum_{i=1}^{m_j} \alpha_{ij} < 1. \quad (16)$$

Formulas (12), (14) and (16) give the required assertion. \square

Definition 2.8. Let μ be a positive measure on \mathbb{R}^2 . We say that μ has property $(*)$ if μ is absolutely continuous with respect to Lebesgue measure (so $\mu(I) = \int_I g(x) dx$, where $g \in L^1(\mathbb{R}^2)$) and

$$\mu(I) \leq \ell(I) \tag{17}$$

for any interval $I \subset \mathbb{R}^2$.

Lemma 2.9. If $f \in \text{BV}(\mathbb{R}^2)$ and μ has property $(*)$, then

$$\left| \int f d\mu \right| \leq C \|f\|_{\text{BV}}, \tag{18}$$

where C does not depend on μ, f .

Proof. Because of Lemma 2.7 and the Fatou lemma, it is sufficient to bound $\int \varphi d\mu$ for the functions of the form

$$\varphi = \frac{1}{\mathcal{H}^1(\partial\Omega)} 1_\Omega,$$

where Ω is a bounded domain in \mathbb{R}^2 with a smooth boundary $\partial\Omega$. Obviously $\Omega \subset I$, where I is an interval of size $\ell(I) \sim \text{diam } \Omega \leq \mathcal{H}^1(\partial\Omega)$. Hence

$$\int \varphi d\mu \leq \frac{\mu(I)}{\mathcal{H}^1(\partial\Omega)} \lesssim \frac{\mu(I)}{\ell(I)} < C,$$

as required. □

Corollary 2.10. If $f \in \text{BV}_2(\mathbb{R}^2)$ and μ is a measure with property $(*)$, then

$$\int |\nabla f| d\mu \leq C \|f\|_{\text{BV}_2}, \tag{19}$$

where C does not depend on μ, f .

By a *dyadic interval* we understand a square of the form $[\frac{k}{2^m}, \frac{k+1}{2^m}] \times [\frac{l}{2^m}, \frac{l+1}{2^m}]$, where k, l, m are integers.

The following assertion is straightforward, and hence we omit its proof here.

Lemma 2.11. For any bounded set $F \subset \mathbb{R}^2$ there exist dyadic intervals I_1, \dots, I_4 such that $F \subset I_1 \cup \dots \cup I_4$ and $\ell(I_1) = \dots = \ell(I_4) \leq 2 \text{diam } F$.

Proof of Theorem 2.1. Fix $\varepsilon_0 > 0$ and take a decomposition $\nabla v = f_0 + f_1$ from Lemma 2.5. If δ from the conditions of Theorem 2.1 is sufficiently small, we may write

$$E \subset \bigcup I_\alpha,$$

where $\{I_\alpha\}$ is a collection of dyadic intervals satisfying

$$\sum_{\alpha} \ell(I_\alpha) < 16\delta < \frac{1}{K+1}\varepsilon_0 \quad (20)$$

(see Lemma 2.11). Define

$$\mathcal{F} = \left\{ J : J \subset \mathbb{R}^2 \text{ dyadic interval; } \sum_{I_\alpha \subset J} \ell(I_\alpha) \geq \ell(J) \right\}.$$

Thus $I_\alpha \in \mathcal{F}$ for each α . Denote by $\mathcal{F}^* = \{J_\beta\}$ the collection of maximal elements of \mathcal{F} . Clearly

$$E \subset \bigcup_{\alpha} I_\alpha \subset \bigcup_{\beta} J_\beta, \quad (21)$$

and since dyadic intervals are either disjoint or contained in one another, the $\{J_\beta\}$ are mutually disjoint. It follows that

$$\sum_{\beta} \ell(J_\beta) \leq \sum_{\beta} \sum_{I_\alpha \subset J_\beta} \ell(I_\alpha) \leq \sum_{\alpha} \ell(I_\alpha) \stackrel{(20)}{<} 16\delta < \frac{1}{K+1}\varepsilon_0. \quad (22)$$

Observe also that for any dyadic interval $Q \subset \mathbb{R}^2$,

$$\sum_{J_\beta \subset Q} \ell(J_\beta) \leq \sum_{I_\alpha \subset Q} \ell(I_\alpha) \leq 2\ell(Q). \quad (23)$$

We used here that if $J_\beta \subset Q$ for some β , then either $J_\beta = Q$ or $Q \notin \mathcal{F}$ (because J_β is maximal); and in both cases (23) holds. Define the measure μ by

$$\mu = \left(\sum_{\beta} \frac{1}{\ell(J_\beta)} 1_{J_\beta} \right) \mathcal{L}^2. \quad (24)$$

Claim. $\frac{1}{48}\mu$ has property (*).

Indeed, write for a dyadic interval Q ,

$$\mu(Q) = \sum_{J_\beta \subset Q} \ell(J_\beta) + \sum_{Q \subset J_\beta} \frac{\ell(Q)^2}{\ell(J_\beta)} \leq 3\ell(Q),$$

where we invoked (23) and the fact that $Q \subset J_\beta$ for at most one β . Then for any interval I we have the estimate $\mu(I) \leq 48\ell(I)$ (see Lemma 2.11). This proves the claim.

Now return to $\mathcal{H}^1(v(E))$. From (21) we get

$$v(E) \subset \bigcup_{\beta} v(J_\beta).$$

Given $\varepsilon_0 > 0$ it follows from the conditions of Theorem 2.1 and using Lemma 2.3 and inequality (22) that if $\delta > 0$ is sufficiently small, then we may assume

$$\sum_{\beta} \|D^2 v\|(J_{\beta}) < \varepsilon_0, \quad (25)$$

By Lemma 2.6 and properties (5)–(7), (19)

$$\begin{aligned} \sum_{\beta} \mathcal{H}^1(v(J_{\beta})) &\leq C \sum_{\beta} \|D^2 v\|(J_{\beta}) + C \sum_{\beta} \frac{1}{l(J_{\beta})} \int_{J_{\beta}} |\nabla v| \\ &\leq C \varepsilon_0 + C \frac{K}{K+1} \varepsilon_0 + C \sum_{\beta} \frac{1}{l(J_{\beta})} \int_{J_{\beta}} |f_1| \\ &= C' \varepsilon_0 + C \int |f_1| d\mu \leq C'' \varepsilon_0. \end{aligned}$$

Since ε_0 may be taken arbitrary small, it follows that Theorem 2.1 is proved. \square

3 Sard–type theorem

Before stating the main result of this section we shall define our notion of critical set for $v \in \text{BV}_{\text{loc}}^2(\Omega)$, where $\Omega \subset \mathbb{R}^2$ is open. First we let for $\varepsilon > 0$,

$$E_{\varepsilon} = \{x \in \Omega : |\nabla v(x)| \leq \varepsilon\},$$

and note that $\text{Cl}_M E_{\varepsilon}$ does not depend on the particular representative we use for ∇v when defining E_{ε} . Define

$$Z_{0v} = \bigcap_{\varepsilon > 0} \text{Cl}_M E_{\varepsilon},$$

and

$$Z_{1v} = \{x \in \Omega : v \text{ is differentiable at } x \text{ and } v'(x) = 0\},$$

where we refer to the continuous representative of v alluded to in the introduction (see also Lemma 3.2 below). The critical set for v is the union $Z_v = Z_{0v} \cup Z_{1v}$.

Theorem 3.1. *Suppose $v \in \text{BV}_{\text{loc}}^2(\Omega)$, where Ω is a domain in \mathbb{R}^2 . Then $\mathcal{H}^1(v(Z_v)) = 0$.*

The proof of Theorem 3.1 splits into a number of lemmas. Further we may assume, without loss of generality, that $\Omega = B(0, 1) \subset \mathbb{R}^2$ and $v \in \text{BV}^2(\Omega)$.

We require the following known result about differentiability properties of BV^2 -functions.

Lemma 3.2 (see [9], Theorems B and 1). *We can choose the Borel representative of ∇v such that there exist a decomposition $\mathbb{R}^2 = K_v \cup G_v \cup A_v$ and mappings $\lambda: \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $\mu: \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $\nu: K_v \rightarrow \mathbb{S}^1$ with the following properties:*

- (i) $\mathcal{H}^1(A_v) = 0$.
(ii) $K_v = \bigcup_i K_i$, each K_i is a compact subset of some C^1 -curve L_i ; moreover, $\nu(x)$ is perpendicular to L_i if $x \in K_i$.
(iii) for all $x \in G_v$, $\nabla v(x) = \lambda(x) = \mu(x)$ and

$$\lim_{r \searrow 0} \int_{B(x,r)} |\nabla v(z) - \nabla v(x)|^2 dz = 0,$$

$$\sup_{y \in B(x,r)} r^{-1} |v(y) - v(x) - y \cdot \nabla v(x)| \rightarrow 0 \quad \text{as } r \searrow 0$$

(i.e., v is differentiable at x);

- (iv) for all $x \in K_v$,

$$\lim_{r \searrow 0} \int_{B_+(x,r)} |\nabla v(z) - \lambda(x)|^2 dz = 0,$$

$$\lim_{r \searrow 0} \int_{B_-(x,r)} |\nabla v(z) - \mu(x)|^2 dz = 0,$$

$$\sup_{y \in B_+(x,r)} r^{-1} |v(y) - v(x) - y \cdot \lambda(x)| \rightarrow 0 \quad \text{as } r \searrow 0,$$

$$\sup_{y \in B_-(x,r)} r^{-1} |v(y) - v(x) - y \cdot \mu(x)| \rightarrow 0 \quad \text{as } r \searrow 0,$$

where

$$B_+(x, r) = \{y \in B(x, r) : (y - x) \cdot \nu(x) > 0\},$$

$$B_-(x, r) = \{y \in B(x, r) : (y - x) \cdot \nu(x) < 0\}.$$

Observe that by our definitions the inclusion

$$Z_v \supset \{x \in G_v : \nabla v(x) = 0\} \cup \{x \in K_v : \mu(x) = 0 \text{ or } \lambda(x) = 0\}$$

holds.

Lemma 3.3 ([3]). *For any Lebesgue measurable set $F \subset \mathbb{R}^2$ with $\mathcal{H}^1(\partial^M F) < \infty$ there is a finite or countable family $\{F_i\}_{i \in I}$ and a set $T \subset \mathbb{R}^2$ with the following properties:*

- (i) F_i are measurable sets, $\mathcal{L}^2(F_i) > 0$, $\mathcal{H}^1(\partial^M F_i) < \infty$;
(ii) $F_i \cap F_j = \emptyset$ for $i \neq j$;
(iii) $(\partial^M F_i) \cap (\partial^M F_j) = \emptyset \pmod{\mathcal{H}^1}$ for $i \neq j$.
(iv) $\partial^M F = \bigcup_{i \in I} \partial^M F_i \pmod{\mathcal{H}^1}$, so in particular,

$$\mathcal{H}^1(\partial^M F) = \sum_{i \in I} \mathcal{H}^1(\partial^M F_i).$$

$$(v) \mathcal{H}^1\left(\text{Int}_M F \setminus \left(\bigcup_{i \in I} \text{Int}_M F_i\right)\right) = 0.$$

$$(vi) \mathcal{H}^1(T) = 0.$$

(vii) For any set L with $\mathcal{H}^1(L) = 0$ and for any $x, y \in \text{Int}_M F_i \setminus (T \cup L)$ and $\delta > 0$ there exists a rectifiable curve $\Gamma \subset (\text{Int}_M F_i) \setminus (T \cup L)$ joining x to y so that

$$\mathcal{H}^1(\Gamma) \leq |x - y| + \mathcal{H}^1(\partial^M F_i) + \delta.$$

Proof. See Proposition 3, Theorems 1 and 8 (together with the subsequent remark) from [3]. \square

Lemma 3.4. *If the set F in Lemma 3.3 is bounded, then we can reformulate the property (vii) in the following way:*

(vii') *for any set L with $\mathcal{H}^1(L) = 0$ and for any $x, y \in (\text{Int}_M F_i) \setminus (T \cup L)$ and $\delta > 0$ there exists a rectifiable curve $\Gamma \subset (\text{Int}_M F_i) \setminus (T \cup L)$ joining x to y so that*

$$\mathcal{H}^1(\Gamma) \leq 2\mathcal{H}^1(\partial^M F_i) + \delta.$$

Proof. See [23, Lemma 4.2]. \square

Lemma 3.5. *Suppose $\mathcal{H}^1(\partial^M E_\varepsilon) < \infty$. Let E_ε^i be the sets from Lemmas 3.3-3.4 applying to $F = E_\varepsilon$. Then $\text{diam}(v(\text{Cl}_M E_\varepsilon)) \leq 2\varepsilon\mathcal{H}^1(\partial^M E_\varepsilon)$.*

Proof. In property (vii') of Lemma 3.4 put $L = A_v$, where A_v is defined in Lemma 3.2. Then the restriction $v|_\Gamma$ is ε -Lipschitz. \square

Lemma 3.6. *For any $\varepsilon > 0$ the inequality $\mathcal{H}^1(v(\text{Cl}_M E_\varepsilon)) \leq 2\varepsilon\mathcal{H}^1(\partial^M E_\varepsilon)$ holds.*

Proof. Suppose $\mathcal{H}^1(\partial^M E_\varepsilon) < \infty$. From properties (iv)-(v) of Lemma 3.3 we have $\text{Cl}_M E_\varepsilon = \bigcup_{i \in I} \text{Cl}_M E_\varepsilon^i \pmod{\mathcal{H}^1}$. So from Corollary 2.2 we obtain

$$\mathcal{H}^1(v(\text{Cl}_M E_\varepsilon)) \leq \sum_{i \in I} \mathcal{H}^1(v(\text{Cl}_M E_\varepsilon^i)) \leq 2\varepsilon \sum_{i \in I} \mathcal{H}^1(\partial^M E_\varepsilon^i) = 2\varepsilon\mathcal{H}^1(\partial^M E_\varepsilon),$$

where the last equality follows from property (iv) of Lemma 3.3. \square

Corollary 3.7. *For any $\varepsilon > 0$ the estimate*

$$\mathcal{H}^1(v(\text{Cl}_M E_\varepsilon)) \leq 2\varepsilon[\mathcal{H}^1(\Omega \cap \partial^M E_\varepsilon) + \mathcal{H}^1(\partial\Omega)] \quad (26)$$

holds.

Corollary 3.8. *The convergence*

$$\mathcal{H}^1(v(\text{Cl}_M E_\varepsilon)) \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0+ \quad (27)$$

holds.

Proof. It follows from Lemma 3.7 and the Coarea formula (see also the proof of Proposition 4.3 in [23]). \square

Obviously the last corollary, together with Lemma 3.2 and Corollary 2.2, imply the statement of Theorem 3.1.

4 Application to the level sets of $W^{2,1}$ functions

By a *cycle* we mean a set which is homeomorphic to the unit circle $\mathbb{S}^1 \subset \mathbb{R}^2$. Now the purpose of the section is to prove the following result.

Theorem 4.1. *Suppose $v \in W^{2,1}(\mathbb{R}^2)$. Then for almost all $y \in \mathbb{R}$ the preimage $v^{-1}(y)$ is a finite disjoint family of C^1 -cycles S_j , $j = 1, \dots, N(y)$. Moreover, the tangent vector to each S_j is an absolutely continuous function.*

Invoking extension theorems for Sobolev spaces (see, for example, [20]), we obtain the following:

Corollary 4.2. *Suppose $\Omega \subset \mathbb{R}^2$ is a bounded domain with a Lipschitz boundary and $v \in W^{2,1}(\Omega)$. Then for almost all $y \in \mathbb{R}$ the preimage $v^{-1}(y)$ is a finite disjoint family of C^1 -curves Γ_j , $j = 1, \dots, N(y)$. Each Γ_j is a cycle or it is a simple arc with endpoints on $\partial\Omega$ (in case of the latter, Γ_j is transversal to $\partial\Omega$). Moreover, the tangent vector to each Γ_j is an absolutely continuous function.*

Fix a function $v \in W^{2,1}(\mathbb{R}^2)$.

Lemma 4.3. *For any $\alpha \in (0, 1)$, a ball $B(x, r) \subset \mathbb{R}^2$ and for any Lebesgue measurable set $E \subset B(x, r)$ satisfying $\frac{\mathcal{L}^2(E)}{\mathcal{L}^2(B(x, r))} \geq \alpha$ the estimate*

$$\sup_{y \in B(x, r)} \left| v(y) - v(x) - y \cdot \int_E \nabla v(z) dz \right| \leq c_\alpha \|D^2 v\|(B(x, r)) \quad (28)$$

holds, where c_α depends on α only.

Proof. Because of coordinate invariance it is sufficient to prove the estimate for the case $\Omega = B(0, 1) = B(x, r)$. By results of [20] for any $u \in W^{2,1}(\Omega)$ the estimate

$$\sup_{y \in \Omega} |u(y)| \leq c(p)(p(u) + \|D^2 u\|(\Omega)), \quad (29)$$

holds, where $p(\cdot)$ is a continuous seminorm in $W^{2,1}(\Omega)$ such that $p(g) = 0 \Leftrightarrow g = 0$ for all first-order polynomials g . Take $p_\alpha(u) = |u(0)| + \inf_{E \subset \Omega, \mathcal{L}^2(E) \geq \alpha} \left| \int_E \nabla u(z) dz \right|$. It is easy to check that p_α satisfies the above conditions. Fix a measurable set $E \subset \Omega$ with $\mathcal{L}^2(E) \geq \alpha$ and take $u(y) = v(y) - v(0) - y \cdot \int_E \nabla v(z) dz$. Then $p_\alpha(u) = 0$ and the inequality (29) turns to the estimate (28). \square

For functions $v \in W^{2,1}(\mathbb{R}^2)$ the set K_v from Lemma 3.2 is empty (see the proofs in [9]), so we have the following result.

Lemma 4.4 (see also Theorem 1 in [7], §4.8). *We can choose the representative of ∇v such that there exists a set $A_v \subset \mathbb{R}^2$ with the following properties:*

- (i) $\mathcal{H}^1(A_v) = 0$;
- (ii) for all $x \in \mathbb{R}^2 \setminus A_v$

$$\lim_{r \searrow 0} \int_{B(x,r)} |\nabla v(z) - \nabla v(x)|^2 dz = 0,$$

$$\sup_{y \in B(x,r)} r^{-1} |v(y) - v(x) - y \cdot \nabla v(x)| \rightarrow 0 \quad \text{as } r \searrow 0$$

(i.e., v is differentiable at x);

(iii) for any $\varepsilon > 0$ there exists an open set $U \subset \mathbb{R}^2$ such that $\text{Cap}_1(U) < \varepsilon$, $A_v \subset U$, and ∇v is continuous on $\mathbb{R}^2 \setminus U$.

Further we fix the above representative of ∇v . Here (see, for example, [7], §4.8) Cap_1 denotes the 1-capacity defined for any $E \subset \mathbb{R}^2$ as

$$\text{Cap}_1(E) = \inf \left\{ \|\nabla f\|_{L^1} : f \in L^2(\mathbb{R}^2), Df \in L^1(\mathbb{R}^2), f \geq 1 \text{ in an open neighborhood of } E \right\}.$$

The 1-capacity has the following simple description.

Lemma 4.5 (see the proof of Theorem 3 in [7], §5.6.3). *There is a constant $C_0 > 0$ such that for any set $E \subset \mathbb{R}^2$ the inequalities*

$$\frac{1}{C_0} \mathcal{H}_\infty^1(E) \leq \text{Cap}_1(E) \leq C_0 \mathcal{H}_\infty^1(E)$$

hold.

Lemma 4.6. *For any $\varepsilon > 0$ there exists an open set $U \subset \mathbb{R}^2$ and a function $g \in C^1(\mathbb{R}^2)$ such that $\text{Cap}_1(U) < \varepsilon$, $A_v \subset U$ and $v|_{\mathbb{R}^2 \setminus U} = g|_{\mathbb{R}^2 \setminus U}$, $\nabla v|_{\mathbb{R}^2 \setminus U} = \nabla g|_{\mathbb{R}^2 \setminus U}$.*

Proof. Denote

$$A_{\delta,\rho} = \{x \in \mathbb{R}^2 : \exists r \in (0, \rho] \text{ so } \frac{1}{r} \|D^2 v\|(B(x, r)) \geq \delta\}.$$

Using Vitali's covering theorem (see [7]) and that $\|D^2 v\|$ is absolutely continuous with respect to \mathcal{L}^2 (recall that v is $W^{2,1}$) it is easy to prove that for each fixed $\delta > 0$,

$$\text{Cap}_1(A_{\delta,\rho}) \rightarrow 0 \text{ as } \rho \searrow 0. \tag{30}$$

So we can choose a sequence $\rho_j > 0$ such that

$$\text{Cap}_1(A_{\frac{1}{j}, \rho_j}) \leq \frac{1}{2^j} \tag{31}$$

holds. Denoting

$$A_k = \bigcup_{j \geq k} A_{\frac{1}{j}, \rho_j},$$

we have

$$\text{Cap}_1(A_k) \leq \frac{1}{2^{k-1}}; \quad (32)$$

$$\forall k \in \mathbb{N} \forall \alpha > 0 \exists r_{k,\alpha} > 0 \forall x \in \mathbb{R}^2 \setminus A_k \forall r \in (0, r_{k,\alpha}) \quad \frac{1}{r} \|D^2 v\|(B(x, r)) < \alpha. \quad (33)$$

It follows from the proof of Theorem 1 in [7, §4.8] that there exists a sequence of mappings $f_i \in C_0^\infty(\mathbb{R}^2, \mathbb{R}^2)$ such that for the sets

$$B_i = \left\{ x \in \mathbb{R}^n : \exists r > 0 \int_{B(x,r)} |\nabla v(y) - f_i(y)| dy > \frac{1}{2^i} \right\}, \quad (34)$$

$$F_k = A_v \cap \left(\bigcup_{j=k}^{\infty} B_j \right)$$

we have

$$\text{Cap}_1 F_k \rightarrow 0 \quad \text{as} \quad k \rightarrow \infty,$$

and

$$\forall x \in \mathbb{R}^2 \setminus F_k \forall i \geq k \quad |f_i(x) - \nabla v(x)| \leq \frac{1}{2^i}. \quad (35)$$

Take a sequence of open sets $U_k \supset F_k \cup A_k$ such that

$$\text{Cap}_1 U_k \rightarrow 0 \quad \text{as} \quad k \rightarrow \infty. \quad (36)$$

Then from above formulas (33)–(35) and Lemma 4.3 we obtain that there exist a function $\omega: (0, +\infty) \rightarrow (0, +\infty)$ such that $\omega(\delta) \rightarrow 0$ as $\delta \searrow 0$ and for all $k \in \mathbb{N}$ and for any pair $x, y \in \mathbb{R}^2 \setminus U_k$ the estimates

$$\begin{aligned} |v(x) - v(y)| &\leq \omega(|x - y|), \\ |\nabla v(x) - \nabla v(y)| &\leq \omega(|x - y|), \\ |v(y) - v(x) - (y - x) \cdot \nabla v(x)| &\leq \omega(|x - y|)|x - y| \end{aligned}$$

hold. Then the assertion of Lemma 4.6 follows from the last estimates, the convergence (36), and from the classical Whitney extension theorem (see, for example, [7, Theorem 1 of §6.5]). \square

Using Theorems 2.1, 3.1 and Lemma 4.5 we can reformulate the last lemma in the following way.

Corollary 4.7. *For any $\varepsilon > 0$ there exist an open set $V \subset \mathbb{R}$ and a function $g \in C^1(\mathbb{R}^2)$ such that $\mathcal{H}^1(V) < \varepsilon$, $v(A_v) \subset V$ and $v|_{v^{-1}(\mathbb{R} \setminus V)} = g|_{v^{-1}(\mathbb{R} \setminus V)}$, $\nabla v|_{v^{-1}(\mathbb{R} \setminus V)} = \nabla g|_{v^{-1}(\mathbb{R} \setminus V)} \neq 0$.*

The last corollary and Lemma 4.4 easily imply the statement of Theorem 3.1.

5 Application to the level sets of BV^2 functions

The main goal of this section is to prove the following result.

Theorem 5.1. *Suppose $v \in BV_2(\mathbb{R}^2)$. Then for almost all $y \in \mathbb{R}$ the preimage $v^{-1}(y) \cap \Omega$ is a finite disjoint family of cycles S_j , $j = 1, \dots, N(y)$. Moreover, the variation of the tangent vector to each S_j (i.e., the integral curvature of Γ_j) is finite.*

Corollary 5.2. *Suppose $\Omega \subset \mathbb{R}^2$ is a bounded domain with a Lipschitz boundary and $v \in BV^2(\Omega)$. Then for almost all $y \in \mathbb{R}$ the preimage $v^{-1}(y)$ is a finite disjoint family of Lipschitz curves Γ_j , $j = 1, \dots, N(y)$. Each Γ_j is a cycle or it is a simple arc with endpoints on $\partial\Omega$ (in the last case Γ_j is transversal to $\partial\Omega$). Moreover, the variation of the tangent vector to Γ_j (i.e., the integral curvature of Γ_j) is finite.*

Curves of this kind are called *curves of finite turn* and they have been systematically studied in [2] and [25].

Fix a function $v \in BV^2(\mathbb{R}^2)$. Let A_v , K_v , $\mu(x)$, $\lambda(x)$, $\nu(x)$ be objects defined in Lemma 3.2.

Lemma 5.3. *For almost all $y \in v(\mathbb{R}^2)$ the following assertions are true:*

- (i) $v^{-1}(y) \cap A_v = \emptyset$;
- (ii) for all $x \in v^{-1}(y)$ $\lambda(x) \neq 0 \neq \mu(x)$;
- (iii) for all $x \in v^{-1}(y) \cap K_v$ both vectors $\lambda(x)$, $\mu(x)$ are not parallel to $\nu(x)$;
- (iv) the intersection $v^{-1}(y) \cap K_v$ is at most countable;
- (v) $\mathcal{H}^1(v^{-1}(y)) < \infty$.

Proof. (i) follows from Theorem 2.2.

(ii) follows from Theorem 3.1.

(iii) follows from the classical one dimension version of the Sard theorem applied to the restriction $v|_{L_i}$ (see the assertion (ii), (iv) of Lemma 3.2);

(iv) follows from (iii);

(v) follows from the Coarea formula. □

By *connectedness* (without additional terms) we mean connectedness in the sense of general topology.

Lemma 5.4 (see, for example, Lemma 2.2 in [16]). *Let $\Omega \subset \mathbb{R}^2$ be a domain that is homeomorphic to the unit disc and let $G \subset \Omega$ be a subdomain of Ω . Then for each connected component Ω_i of the open set $\Omega \setminus \text{Cl } G$ the intersection $\Omega \cap \partial\Omega_i$ is connected.*

Lemma 5.5 (see, for example, [3]). *Suppose K is a compact connected set in \mathbb{R}^2 and $\mathcal{H}^1(K) < \infty$. Then K is arcwise connected.*

By *arc* we mean a set which is homeomorphic to an interval of the straight line.

Lemma 5.6. *For any $y \in \mathbb{R}$ satisfying (i)–(v) of Lemma 5.3, for any $x \in v^{-1}(y)$, and for all sufficiently small $r > 0$ the connected component $K \ni x$ of the set $B(x, r) \cap v^{-1}(y)$ contains an arc $J \ni x$ with endpoints on $\partial B(x, r)$. Moreover, the set $J \setminus \{x\}$ intersects two connected components of the set $B(x, r) \cap v^{-1}(y) \setminus \{x\}$.*

Proof. We may assume without loss of generality that $x = 0$, $v(x) = 0$ and the vector $\nu(x)$ (from Lemmas 3.2, 5.3) is vertical: $\nu(x) = (0, 1)$. Let L be the intersection of the open ball $B(0, r)$ with the horizontal axis: $L = \{(t, 0) : t \in (-r, r)\}$. Denote by A, C the endpoints of the segment L : $A = (r, 0)$, $C = (-r, 0)$. If $r > 0$ is sufficiently small, then by the differentiability properties recorded in Lemmas 3.2, 5.3 we infer that the function v is strictly monotone on L . For definiteness assume that $v(t, 0) > 0$ for $t \in (0, r]$ and $v(t, 0) < 0$ for $t \in [-r, 0)$. In particular, $v(A) > 0 > v(C)$. Denote $\Omega_+ = \{(t, s) \in B(0, r) : s > 0\}$, $\Omega_- = \{(t, s) \in B(0, r) : s < 0\}$. Denote by G the connected component of the open set $\{z \in \Omega_+ : v(z) > 0\}$ such that $A \in \partial G$. Denote by Ω_1 the connected component of the open set $\Omega_+ \setminus \text{Cl} G$ such that $C \in \partial \Omega_1$. Put $K_+ = \text{Cl}(\Omega_+ \cap \partial \Omega_1)$. Obviously $0 \in K_+$, $v \equiv 0$ on K_+ , and $K_+ \cap (\partial \Omega_+) \setminus \text{Cl} \Omega_- \neq \emptyset$. Let $D_+ \in K_+ \cap (\partial \Omega_+) \setminus \text{Cl} \Omega_-$. By Lemma 5.4 K_+ is a compact connected set, and by (v) of Lemma 5.3 $\mathcal{H}^1(K_+) < \infty$. Then by Lemma 5.5 there exists an arc $J_+ \subset K_+$ joining 0 to D_+ . Because $L \cap v^{-1}(0) = \{0\}$ we have equality $J_+ \cap \text{Cl} \Omega_- = \{0\}$. Analogously, there exists a point $D_- \in (\partial \Omega_-) \setminus \text{Cl} \Omega_+$ and an arc $J_- \subset \text{Cl}(\Omega_- \cap v^{-1}(0))$ joining 0 to D_- so that $J_- \cap \text{Cl} \Omega_+ = \{0\}$. Now $J = J_+ \cup J_-$ is the required arc. \square

Lemma 5.7. *For any $y \in \mathbb{R}$ satisfying (i)–(v) of Lemma 5.3 and for any connected component C of $v^{-1}(y)$ there exists a cycle $S \subset C$. Moreover, if there is only one cycle $S \subset C$, then $S = C$.*

Proof. Let J_1 be a maximal *open* arc (i.e., J_1 is homeomorphic to an *open* interval of \mathbb{R}) in C . Such an arc exists by the previous Lemma 5.6. By (v) of Lemma 5.3 the inequality $\mathcal{H}^1(J_1) < \infty$ holds. So the arc J_1 has endpoints, denote them by x, y . If $x = y$, then there is nothing to prove. The same applies for the case $x \in J_1$. If $x \neq y$ and $x \notin J_1$ we can continue the arc J_1 through x by Lemma 5.6. This contradiction establishes the existence of a cycle $S \subset C$.

To prove the second statement suppose that $z \in C \setminus S$. Take a maximal arc J_2 in C containing z . By the above arguments this arc generates a cycle $S_2 \neq S$, $S_2 \subset C$. \square

Corollary 5.8. *There exists at most countable set $Z \subset \mathbb{R}$ such that for any $y \in \mathbb{R} \setminus Z$ satisfying (i)–(v) of Lemma 5.3 any connected component C of $v^{-1}(y)$ is a cycle.*

Proof. Suppose $y \in \mathbb{R}$ satisfies (i)–(v) of Lemma 5.3 and a connected component C of $v^{-1}(y)$ is not a cycle. Then by Lemma 5.7 the set $\mathbb{R}^2 \setminus C$ has more than two connected components. By results of [18] this is possible only for at most countable many values of y . \square

We need the following classical estimate and its corollary:

Lemma 5.9 (see, for example, Lemma 1 of §4.8 in [7]). *There exists the constant $C_5 > 0$ such that the estimate*

$$\text{Cap}_1(\{x \in \mathbb{R}^2 : \exists r > 0 \int_{B(x,r)} |\nabla v(y)| dy \geq \delta\}) \leq C_5 \frac{1}{\delta} \|D^2 v\|(\mathbb{R}^2)$$

holds.

Corollary 5.10. *The estimate*

$$\text{Cap}_1(\{x \in G_v : |\nabla v(x)| > \delta\}) \leq C_5 \frac{1}{\delta} \|D^2 v\|(\mathbb{R}^2)$$

holds.

Lemma 5.11. *For any $\varepsilon > 0$ there exists a compact set $F_\varepsilon \subset v(\mathbb{R}^2)$ and constants $\delta_1, \delta_2 > 0$ such that $\mathcal{H}^1(v(\mathbb{R}^2) \setminus F_\varepsilon) < \varepsilon$ and for all $y \in F_\varepsilon$ the preimage $v^{-1}(y)$ satisfies the properties (i)-(v) from the Lemma 5.3 and the following additional conditions:*

(vi) *for all $x \in v^{-1}(y) \cap G_v$ the estimates $\delta_1 > |\nabla v(x)| > \delta_2$ hold;*

(vii) *each connected component of the set $v^{-1}(y)$ is a cycle.*

Proof. (vi) follows from Theorem 2.1, Lemma 4.5 and Corollaries 3.8, 5.10. (vii) follows from Lemma 5.8. \square

Proof of Theorem 5.1. Fix an arbitrary $\varepsilon > 0$ and take the set F_ε from Lemma 5.11. From the above results we have that

$$\forall y \in F_\varepsilon \quad v^{-1}(y) = \bigcup_{j=1}^{N(y)} S_j(y),$$

where $S_j(y)$ are cycles and $N(y) \in \mathbb{N} \cup \{+\infty\}$.

Take a sequence of functions $v_i \in C^\infty(\mathbb{R}^2)$ that approximates v as usual. In particular,

$$\forall x \in G_v \quad \nabla v_i(x) \rightarrow \nabla v(x); \quad (37)$$

$$\|D^2 v_i\|(\mathbb{R}^2) = \int_{\mathbb{R}^2} |D^2 v_i(x)| dx \leq 2 \|D^2 v\|(\mathbb{R}^2) \quad (38)$$

By the coarea formula

$$\int_{v^{-1}(F_\varepsilon)} |\nabla v(x)| \cdot |D^2 v_i(x)| dx = \int_{F_\varepsilon} \sum_{j=1}^{N(y)} \int_{S_j(y)} |D^2 v_i(x)| d\mathcal{H}^1 dy \leq 2\delta_1 \|D^2 v\|(\mathbb{R}^2), \quad (39)$$

where the last estimate follows from condition (vi) of Lemma 5.11. Consequently there exists a constant C_7 such that

$$\int_{F_\varepsilon} \sum_{j=1}^{N(y)} \text{Var}(\nabla v_i, S_j(y)) dy \leq C_7, \quad (40)$$

where $\text{Var}(\nabla v_i, S_j(y))$ is the variation of ∇v_i on $S_j(y)$.

From (37) and the properties (i), (iv) of Lemma 5.3 it is easy to deduce that

$$\text{Var}(\nabla v, S_j(y)) \leq \liminf_{i \rightarrow \infty} \text{Var}(\nabla v_i, S_j(y)), \quad (41)$$

consequently,

$$\sum_{j=1}^{N(y)} \text{Var}(\nabla v, S_j(y)) \leq \liminf_{i \rightarrow \infty} \sum_{j=1}^{N(y)} \text{Var}(\nabla v_i, S_j(y)). \quad (42)$$

Then by Fatou's lemma

$$\int_{F_\varepsilon} \sum_{j=1}^{N(y)} \text{Var}(\nabla v, S_j(y)) \, dy \leq \liminf_{i \rightarrow \infty} \int_{F_\varepsilon} \sum_{j=1}^{N(y)} \text{Var}(\nabla v_i, S_j(y)) \, dy \leq C_7. \quad (43)$$

Let τ denote the tangent vector to $S_j(y)$. By straightforward geometric considerations we have

$$2\pi \leq \text{Var}(\tau, S_j(y)) \leq \frac{\delta_1}{(\delta_2)^2} \text{Var}(\nabla v, S_j(y)) \quad (44)$$

From the last two formulas we deduce that $N(y) < \infty$ and $\sum_{j=1}^{N(y)} \text{Var}(\tau, S_j(y)) < \infty$ for almost all $y \in F_\varepsilon$. \square

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