# Distributed Source Coding of Correlated Gaussian Sources

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***Abstract*—We consider the distributed source coding system of $L$ correlated Gaussian sources $Y_l, l = 1,2,\cdots,L$ which are noisy observations of correlated Gaussian remote sources $X_k, k = 1,2,\cdots,K$. We assume that $Y^L = {}^{\rm t}(Y_1,Y_2,\cdots,Y_L)$ is an observation of the source vector $X^K = {}^{\rm t}(X_1,X_2,\cdots,X_K)$, having the form $Y^L = AX^K + N^L$, where $A$ is a $L \times K$ matrix and $N^L = {}^{\rm t}(N_1,N_2,\cdots,N_L)$ is a vector of $L$ independent Gaussian random variables also independent of $X^K$. In this system $L$ correlated Gaussian observations are separately compressed by $L$ encoders and sent to the information processing center. We study the remote source coding problem where the decoder at the center attempts to reconstruct the remote source $X^K$. We consider three distortion criteria based on the covariance matrix of the estimation error on $X^K$. For each of those three criteria we derive explicit inner and outer bounds of the rate distortion region. Next, in the case of $K = L$ and $A = I_L$, we study the multiterminal source coding problem where the decoder wishes to reconstruct the observation $Y^L = X^L + N^L$. To investigate this problem we shall establish a result which provides a strong connection between the remote source coding problem and the multiterminal source coding problem. Using this result, we drive several new partial solutions to the multiterminal source coding problem.**

***Index Terms*—Multiterminal source coding, rate distortion region, CEO problem.**

## I. INTRODUCTION

Distributed source coding systems of correlated information sources are a form of communication system which is significant from both theoretical and practical points of view in multi-user source networks. The first fundamental theory in those coding systems was established by Slepian and Wolf [1]. They considered a distributed source coding system of two correlated information sources. Those two sources are separately encoded and sent to a single destination, where the decoder wishes to decode the original sources. In the above distributed source coding systems we can consider a situation where the source outputs should be reconstructed with average distortions smaller than prescribed levels. This situation yields a kind of multiterminal rate distortion theory in the framework of distributed source coding. The rate distortion region is defined by the set of all rate vectors for which the source outputs are reconstructed with average distortions smaller than prescribed levels. The determination problem of the rate distortion region is often called the multiterminal source coding problem.

The multiterminal source coding problem was intensively studied by [2]-[12]. Wagner and Anantharam [10] gave a new method to evaluate an outer bound of the rate distortion region. Wagner *et al.* [11] gave a complete solution to this problem in the case of Gaussian information sources and quadratic distortion by proving that the sum rate part of the inner bound of Berger [4] and Tung [5] is optimal. Wang *et al.* [12] gave a new alternative proof of the sum rate part optimality. In spite of a recent progress made by those three works, the multiterminal source coding problem still largely remains open.

As a practical situation of the distributed source coding system, we can consider a case where the distributed encoders can not directly access the source outputs but can access their noisy observations. This situation was first studied by Yamamoto and Ito [13]. They call the investigated coding system the communication system with a remote source. Subsequently, a similar distributed source coding system was studied by Flynn and Gray [14].

In this paper we consider the distributed source coding system of $L$ correlated Gaussian sources $Y_l, l = 1,2,\cdots,L$ which are noisy observations of $X_k, k = 1,2,\cdots,K$. We assume that $Y^L = {}^{\rm t}(Y_1,Y_2,\cdots,Y_L)$ is an observation of the source vector $X^K = {}^{\rm t}(X_1,X_2,\cdots,X_K)$, having the form $Y^L = AX^K + N^L$, where $A$ is a $L \times K$ matrix and $N^L = {}^{\rm t}(N_1,N_2,\cdots,N_L)$ is a vector of $L$ independent Gaussian random variables also independent of $X^K$. In this system $L$ correlated Gaussian observations are separately compressed by $L$ encoders and sent to the information processing center. We study the remote source coding problem where the decoder at the center attempts to reconstruct the remote source $X^K$.

We consider three distortion criteria based on the covariance matrix of the average estimation error on $X^K$. The first criterion is called the distortion matrix criterion, where the estimation error must not exceed an arbitrary prescribed covariance matrix in the meaning of positive semi definite. The second criterion is called the vector distortion criterion, where for a fixed positive vector $D^K = (D_1,D_2,\cdots,D_K)$ and for each $k = 1,2,\cdots,K$, the diagonal $(k,k)$ element of the covariance matrix is upper bounded by $D_k$. The third criterion is called the sum distortion criterion, where the trace of the covariance matrix must not exceed a prescribed positive level $D$. For each distortion criterion the rate distortion region is defined by a set of all rates vectors for which the estimation error does not exceed an arbitrary prescribed distortion level.

For the first distortion criterion, i.e., the distortion matrix criterion we derive explicit inner and outer bounds of the rate distortion region. Those two bounds have a form of positive semi definite programming with respect to covariance matrices. Using this results, for each of the second and third distortion criteria we derive explicit inner and outer bounds of the rate distortion region. In the case of vector distortion

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criterion our outer bound includes that of Oohama [22] as a special case by letting $K = L$ and $A = I_L$. In the case of sum distortion criterion we derive more explicit outer bound of the rate distortion region having a form of water filling solution. In this case we further show that if the prescribed distortion level $D$ does not exceed a certain threshold, the inner and outer bounds match and derive two different thresholds. The first threshold improves the threshold obtained by Oohama [23],[24] in the case of $K = L, A = I_L$. The second threshold improves the first one for some cases but neither subsumes the other.

When $K = 1$, the distributed source coding system treated in this paper becomes the quadratic Gaussian CEO problem investigated by [12], [15]-[18]. The system in the case of $K = L$ and sum distortion criterion was studied by Pandya *et al.* [19]. They derived lower and upper bounds of the minimum sum rate in the rate distortion region. Several partial solutions in the case of $K = L$, $A = I_L$, and sum distortion criterion were obtained by [20]-[24]. The case of $K = L$, $A = I_L$, and vector distortion criterion was studied by [22].

Recently, Yang and Xiong [26] have studied the same problem. They have derived two outer bounds of the rate distortion region in the case of sum rate distortion criterion. When $K = L, A = I_L$, the first outer bound does not coincide with the outer bound obtained by Oohama [21]-[24]. When ${}^{\mathrm{t}}AA = I_K$, they have obtained the second outer bound tighter than the first one. This bound is the same as that of our result of this paper. When ${}^{\mathrm{t}}AA = I_K$, Yang *et al.* [27] have derived a threshold on the distortion level $D$ such that for $D$ below this threshold their second outer bound is tight. Their threshold also improves that of Oohama [23],[24] in the case of $K = L, A = I_L$. Comparing the formula of our first threshold with that of and Yang *et al.* [27], we can see that we have no obvious superiority of either to the other. On the other hand, our second threshold is better than their threshold for some nontrivial cases.

In this paper, in the case of $K = L$ and $A = I_L$, we study the multiterminal source coding problem where the decoder wishes to reconstruct the observation $Y^L = X^L + N^L$. Similarly to the case of remote source coding problem, we consider three types of distortion criteria based on the covariance matrix of the estimation error on $Y^L$. Based on the above three criteria, three rate distortion regions are defined.

The remote source coding problem is often referred to as the indirect distributed source coding problem. On the other hand, the multiterminal source coding problem in the frame work of distributed source coding is often called the direct distributed source coding problem. As shown in the paper of Wagner *et al.* [11] and in the recent work by Wang *et al.* [12], we have a strong connection between the direct and indirect distributed source coding problems. To investigate the determination problem of the three rate distortion regions for the multiterminal source coding problem we shall establish a result which provides a strong connection between the remote source coding problem and the multiterminal source coding problem. This result states that all results on the rate distortion region of the remote source coding problem can be converted into those on the rate distortion region of the multiterminal source coding problem. Using this relation and our results on the remote source coding problem, we drive new three outer bounds of the rate distortion regions for each of three distortion criteria.

In the case of vector distortion criterion, we can obtain a lower bound of the sum rate part of the rate distortion region by using the established outer bound in this case. This bound has a form of positive semidefinite programming. By some analytical computation we can show that this lower bound is equal to the lower bound obtained by Wang *et al.* [12] and tight when $L = 2$. Our method to derive this result essentially differs from the method of Wang *et al.* [12]. It is also quite different from that of Wagner *et al.* [11]. Hence in the case of two terminal Gaussian sources there exists three different proofs of the optimality of the sum rate part of the inner bound of Berger [4] and Tung [5].

In the case of sum distortion criterion we derive an explicit threshold such that for the distortion level $D$ below this threshold the outer bound coincides with the inner bound. An important feature of the multiterminal rate distortion problem is that the rate distortion region remains the same for any choice of covariance matrix $\Sigma_{X^L}$ and diagonal covariance matrix $\Sigma_{N^L}$ satisfying $\Sigma_{Y^L} = \Sigma_{X^L} + \Sigma_{N^L}$. Using this feature, we find a pair $(\Sigma_{X^L},\ \Sigma_{N^L})$ which maximizes the threshold subject to $\Sigma_{Y^L} = \Sigma_{X^L} + \Sigma_{N^L}$.

Let $\tau(Y^L) \triangleq (Y_2, Y_3, \cdots, Y_L, Y_1)$ be a cyclic shift of the source $Y^L = (Y_1, Y_2, Y_3,\ \cdots, Y_L)$. We say that the source $Y^L$ has the cyclic shift invariant property if the covariance matrix $\Sigma_{\tau(Y^L)}$ of $\tau(Y^L)$ is the same as the covariance matrix $\Sigma_{Y^L}$ of $Y^L$. When $Y^L$ has the cyclic shift invariant property, we investigate the sum rate part of the rate distortion region. We derive an explicit upper bound of the sum rate part from the inner bounds of the rate distortion region. On a lower bound of the sum rate part we derive a new explicit bound by making full use of the cyclic shift invariance property of $\Sigma_{Y^L}$. We further derive an explicit sufficient condition for the lower bound to coincide with the upper bound. We show that the lower and upper bounds match if the distortion does not exceed a threshold which is a function of $\Sigma_{Y^L}$ and find an explicit form of this threshold. As a corollary of this result, in the case of vector distortion criterion we obtain the optimal sum rate when $Y^L$ is cyclic shift invariant and $D^L$ has $L$ components with an identical value $D$ below a certain threshold depending only on $\Sigma_{Y^L}$

## II. Problem Statement and Previous Results

### A. *Formal Statement of Problem*

In this subsection we present a formal statement of problem. Throughout this paper all logarithms are taken to the base natural. Let $\Lambda_K \triangleq \{1, 2, \cdots, K\}$ and $\Lambda_L \triangleq \{1, 2, \cdots, L\}$. Let $X_k, k \in \Lambda_K$ be correlated zero mean Gaussian random variables. For each $k \in \Lambda_K$, $X_k$ takes values in the real line $\mathbb{R}$. We write a $K$ dimensional random vector as $X^K = {}^{\mathrm{t}}(X_1, X_2,\ \cdots, X_K)$. We denote the covariance matrix of $X^K$ by $\Sigma_{X^K}$. Let $Y^L \triangleq {}^{\mathrm{t}}(Y_1, Y_2,\ \cdots, Y_L)$ be an observation of the source vector $X^K$, having the form $Y^L = AX^K + N^L$, where $A$ is a $L \times K$ matrix and $N^L = {}^{\mathrm{t}}(N_1, N_2, \cdots, N_L)$

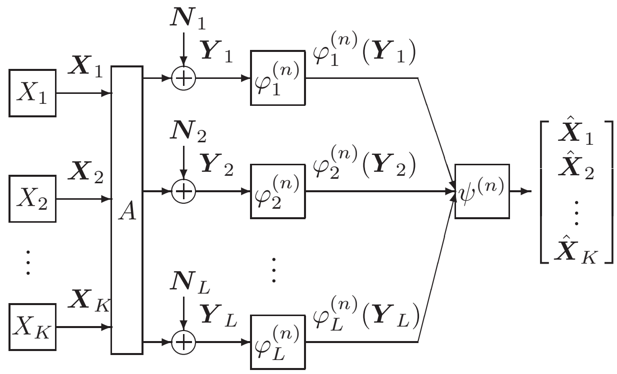


Fig. 1. Distributed source coding system for $L$ correlated Gaussian observations

is a vector of $L$ independent zero mean Gaussian random variables also independent of $X^K$. For $l \in \Lambda_L$, $\sigma^2_{N_l}$ stands for the variance of $N_l$. Let $\{(X_1(t),\ X_2(t),\cdots,X_K(t))\}_{t=1}^{\infty}$ be a stationary memoryless multiple Gaussian source. For each $t=1,2,\cdots,\ X^K(t) \stackrel{\triangle}{=} {}^{\rm t}(X_1(t),X_2(t),\cdots,X_k(t))$ has the same distribution as $X^K$. A random vector consisting of $n$ independent copies of the random variable $X_k$ is denoted by

$$\boldsymbol{X}_k \stackrel{\triangle}{=} (X_k(1),X_k(2),\cdots,X_k(n)).$$

For each $t=1,2,\cdots$, $Y^L(t) \stackrel{\triangle}{=} {}^{\rm t}(Y_1(t),\ Y_2(t),\cdots,Y_L(t))$ is a vector of $L$ correlated observations of $X^K(t)$, having the form $Y^L(t)=AX^K(t)+N^L(t)$, where $N^L(t), t=1,2,\cdots$, are independent identically distributed (i.i.d.) Gaussian random vector having the same distribution as $N^L$. We have no assumption on the number of observations $L$, which may be $L \geq K$ or $L<K$.

The distributed source coding system for $L$ correlated Gaussian observations treated in this paper is shown in Fig. 1. In this coding system the distributed encoder functions $\varphi_l, l\in\Lambda_L$ are defined by $\varphi_l^{(n)}:\mathbb{R}^n \mapsto \mathcal{M}_l \stackrel{\triangle}{=} \{1,2,\cdots,M_l\}$. For each $l \in \Lambda_L$, set $R_l^{(n)} \stackrel{\triangle}{=} \frac{1}{n}\log M_l$, which stands for the transmission rate of the encoder function $\varphi_l^{(n)}$. The joint decoder function $\psi^{(n)}=(\psi_1^{(n)},\ \psi_2^{(n)},\ \cdots,\psi_K^{(n)})$ is defined by

$$\psi^{(n)} \stackrel{\triangle}{=} (\psi_1^{(n)},\psi_2^{(n)},\cdots,\psi_K^{(n)}),$$
$$\psi_k^{(n)}:\mathcal{M}_1\times\cdots\times\mathcal{M}_L \mapsto \mathbb{R}^n, k\in\Lambda_K.$$

For $\boldsymbol{X}^K=(\boldsymbol{X}_1,\ \boldsymbol{X}_2,\ \cdots,\ \boldsymbol{X}_K)$, set

$$\varphi^{(n)}(\boldsymbol{Y}^L) \stackrel{\triangle}{=} (\varphi_1^{(n)}(\boldsymbol{Y}_1),\varphi_2^{(n)}(\boldsymbol{Y}_2),\cdots,\varphi_L^{(n)}(\boldsymbol{Y}_L)),$$
$$\hat{\boldsymbol{X}}^K=\begin{bmatrix}\hat{\boldsymbol{X}}_1\\ \hat{\boldsymbol{X}}_2\\ \vdots \\ \hat{\boldsymbol{X}}_K\end{bmatrix} \stackrel{\triangle}{=} \begin{bmatrix}\psi_1^{(n)}(\varphi^{(n)}(\boldsymbol{Y}^L))\\ \psi_2^{(n)}(\varphi^{(n)}(\boldsymbol{Y}^L))\\ \vdots \\ \psi_K^{(n)}(\varphi^{(n)}(\boldsymbol{Y}^L))\end{bmatrix},$$
$$d_{kk} \stackrel{\triangle}{=} {\rm E}||\boldsymbol{X}_k-\hat{\boldsymbol{X}}_k||^2, 1\leq k\leq K,$$
$$d_{kk'} \stackrel{\triangle}{=} {\rm E}\langle \boldsymbol{X}_k-\hat{\boldsymbol{X}}_k, \boldsymbol{X}_{k'}-\hat{\boldsymbol{X}}_{k'}\rangle, 1\leq k\neq k'\leq K,$$

where $||\boldsymbol{a}||$ stands for the Euclid norm of $n$ dimensional vector $\boldsymbol{a}$ and $\langle\boldsymbol{a},\boldsymbol{b}\rangle$ stands for the inner product between $\boldsymbol{a}$ and $\boldsymbol{b}$. Let $\Sigma_{\boldsymbol{X}^K-\hat{\boldsymbol{X}}^K}$ be a covariance matrix with $d_{kk'}$ in its $(k,k')$ element. Let $\Sigma_d$ be a given $K\times K$ covariance matrix which serves as a distortion criterion. We call this matrix a distortion matrix.

For a given distortion matrix $\Sigma_d$, the rate vector $(R_1, R_2,\cdots,R_L)$ is $\Sigma_d$*-admissible* if there exists a sequence $\{(\varphi_1^{(n)},\ \varphi_2^{(n)},\cdots,\ \varphi_L^{(n)},\ \psi^{(n)})\}_{n=1}^{\infty}$ such that

$$\limsup_{n\to\infty} R_l^{(n)} \leq R_l,\ \text{for } l\in\Lambda_L\,,$$
$$\limsup_{n\to\infty} \tfrac{1}{n}\Sigma_{\boldsymbol{X}^K-\hat{\boldsymbol{X}}^K} \preceq \Sigma_d\,,$$

where $A_1 \preceq A_2$ means that $A_2-A_1$ is a positive semidefinite matrix. Let $\mathcal{R}_L(\Sigma_d|\Sigma_{X^KY^L})$ denote the set of all $\Sigma_d$-admissible rate vectors. We often have a particular interest in the minimum sum rate part of the rate distortion region. To examine this quantity, we set

$$R_{{\rm sum},L}(\Sigma_d|\Sigma_{X^KY^L}) \stackrel{\triangle}{=} \min_{\substack{(R_1,R_2,\cdots,R_L)\\ \in\mathcal{R}_L(\Gamma,D^K|\Sigma_{X^KY^L})}} \left\{\sum_{l=1}^{L} R_l\right\}.$$

We consider two types of distortion criterion. For each distortion criterion we define the determination problem of the rate distortion region.

*Problem 1. Vector Distortion Criterion:* Fix $K\times K$ invertible matrix $\Gamma$ and positive vector $D^K=(D_1,\ D_2,\cdots,D_K)$. For given $\Gamma$ and $D^K$, the rate vector $(R_1,R_2,\cdots,R_L)$ is $(\Gamma,D^K)$*-admissible* if there exists a sequence $\{(\varphi_1^{(n)},\ \varphi_2^{(n)},\cdots,\ \varphi_L^{(n)},\ \psi^{(n)})\}_{n=1}^{\infty}$ such that

$$\limsup_{n\to\infty} R^{(n)} \leq R_l,\ \text{for } l\in\Lambda_L,$$
$$\limsup_{n\to\infty} \left[\Gamma\left(\tfrac{1}{n}\Sigma_{\boldsymbol{X}^K-\hat{\boldsymbol{X}}^K}\right){}^{\rm t}\Gamma\right]_{kk} \leq D_k\,,\ \text{for } k\in\Lambda_K,$$

where $[C]_{ij}$ stands for the $(i,j)$ element of the matrix $C$. Let $\mathcal{R}_L(\Gamma,D^K|\Sigma_{X^KY^L})$ denote the set of all $(\Gamma,D^K)$-admissible rate vectors. When $\Gamma$ is equal to the $K\times K$ identity matrix $I_K$, we omit $\Gamma$ in $\mathcal{R}_L(\Gamma,D|\Sigma_{X^KY^L})$ to simply write $\mathcal{R}_L(D|\Sigma_{X^KY^L})$. Similar notations are used for other sets or quantities. The sum rate part of $\mathcal{R}_L(\Gamma,D^K|\Sigma_{X^KY^L})$ is defined by

$$R_{{\rm sum},L}(\Gamma,D^K|\Sigma_{X^KY^L}) \stackrel{\triangle}{=} \min_{\substack{(R_1,R_2,\cdots,R_L)\\ \in\mathcal{R}_L(\Gamma,D^K|\Sigma_{X^KY^L})}} \left\{\sum_{l=1}^{L} R_l\right\}.$$

*Problem 2. Sum Distortion Criterion:* Fix $K\times K$ positive definite invertible matrix $\Gamma$ and positive $D$. For given $\Gamma$ and $D$, the rate vector $(R_1,R_2,\cdots,R_L)$ is $(\Gamma,D)$*-admissible* if there exists a sequence $\{(\varphi_1^{(n)},\ \varphi_2^{(n)},\cdots,\ \varphi_L^{(n)},\ \psi^{(n)})\}_{n=1}^{\infty}$ such that

$$\limsup_{n\to\infty} R^{(n)} \leq R_l,\ \text{for } l\in\Lambda_L,$$
$$\limsup_{n\to\infty} {\rm tr}\left[\Gamma\left(\tfrac{1}{n}\Sigma_{\boldsymbol{X}^K-\hat{\boldsymbol{X}}^K}\right){}^{\rm t}\Gamma\right] \leq D.$$

The sum rate part of $\mathcal{R}_L(\Gamma,D|\Sigma_{X^KY^L})$ is defined by

$$R_{{\rm sum},L}(\Gamma,D|\Sigma_{X^KY^L}) \stackrel{\triangle}{=} \min_{\substack{(R_1,R_2,\cdots,R_L)\\ \in\mathcal{R}_L(\Gamma,D|\Sigma_{X^KY^L})}} \left\{\sum_{l=1}^{L} R_l\right\}.$$

Let $\mathcal{S}_K(D^K)$ be a set of all $K\times K$ covariance matrices whose $(k,k)$ element do not exceed $D_k$ for $k\in\Lambda_K$. Then we have

$$\mathcal{R}_L(\Gamma,D^K|\Sigma_{X^KY^L}) = \bigcup_{\Gamma\Sigma_d{}^{\rm t}\Gamma\in\mathcal{S}_K(D^K)} \mathcal{R}_L(\Sigma_d|\Sigma_{X^KY^L}), \quad (1)$$

$$\mathcal{R}_L(\Gamma,D|\Sigma_{X^KY^L}) = \bigcup_{{\rm tr}[\Gamma\Sigma_d{}^{\rm t}\Gamma]\le D} \mathcal{R}_L(\Sigma_d|\Sigma_{X^KY^L}). \quad (2)$$

Furthermore, we have

$$\mathcal{R}_L(\Gamma,D|\Sigma_{X^KY^L}) = \bigcup_{\sum_{k=1}^K D_k\le D} \mathcal{R}_L(\Gamma,D^K|\Sigma_{X^KY^L}). \quad (3)$$

In this paper we establish explicit inner and outer bounds of $\mathcal{R}_L(\Sigma_d|\Sigma_{X^KY^L})$. Using the above bounds and equations (1) and (2), we give new outer bounds of $\mathcal{R}_L(\Gamma,D|\Sigma_{X^KY^L})$ and $\mathcal{R}_L(\Gamma,D^K|\Sigma_{X^KY^L})$.

### *B. Inner Bounds and Previous Results*

In this subsection we present inner bounds of $\mathcal{R}_L(\Sigma_d|\Sigma_{X^KY^L})$, $\mathcal{R}_L(\Gamma,D^L|\Sigma_{X^KY^L})$, and $\mathcal{R}_L(\Gamma,D|\Sigma_{X^KY^L})$. Those inner bounds can be obtained by a standard technique developed in the field of multiterminal source coding.

For $l\in\Lambda_L$, let $U_l$ be a random variable taking values in the real line $\mathbb{R}$. For any subset $S\subseteq\Lambda_L$, we introduce the notation $U_S=(U_l)_{l\in S}$. In particular $U_{\Lambda_L}=U^L=(U_1,U_2,\cdots,U_L)$. Define

$$\mathcal{G}(\Sigma_d) \triangleq \{U^L : U^L \text{ is a Gaussian random vector that satisfies } U_S\to Y_S\to X^K\to Y_{S^c}\to U_{S^c},\ U^L\to Y^L\to X^K \text{ for any } S\subseteq\Lambda_L \text{ and } \Sigma_{X^K-\psi(U^L)}\preceq\Sigma_d \text{ for some linear mapping } \psi:\mathbb{R}^L\to\mathbb{R}^K.\}$$

and set

$$\hat{\mathcal{R}}_L^{(\rm in)}(\Sigma_d|\Sigma_{X^KY^L}) \triangleq {\rm conv}\Big\{R^L : \text{There exists a random vector } U^L\in\mathcal{G}(\Sigma_d) \text{ such that } \sum_{l\in S}R_l\ge I(U_S;Y_S|U_{S^c}) \text{ for any } S\subseteq\Lambda_L.\Big\},$$

where ${\rm conv}\{A\}$ stands for the convex hull of the set $A$. Set

$$\hat{\mathcal{R}}_L^{(\rm in)}(\Gamma,D^K|\Sigma_{X^KY^L}) \triangleq {\rm conv}\left\{\bigcup_{\Gamma\Sigma_d{}^{\rm t}\Gamma\in\mathcal{S}_K(D^K)}\mathcal{R}_L(\Sigma_d|\Sigma_{X^KY^L})\right\},$$

$$\hat{\mathcal{R}}_L^{(\rm in)}(\Gamma,D|\Sigma_{X^KY^L}) \triangleq {\rm conv}\left\{\bigcup_{{\rm tr}[\Gamma\Sigma_d{}^{\rm t}\Gamma]\le D}\mathcal{R}_L(\Sigma_d|\Sigma_{X^KY^L})\right\}.$$

Define

$$\Sigma_{X^K|Y^L} \triangleq (\Sigma_{X^K}^{-1}+{}^{\rm t}A\Sigma_{N^L}^{-1}A)^{-1}$$

and set

$$d^K(\Gamma\Sigma_{X^K|Y^L}{}^{\rm t}\Gamma) \triangleq \big([\Gamma\Sigma_{X^K|Y^L}{}^{\rm t}\Gamma]_{11},[\Gamma\Sigma_{X^K|Y^L}{}^{\rm t}\Gamma]_{22}, \cdots,[\Gamma\Sigma_{X^K|Y^L}{}^{\rm t}\Gamma]_{KK}\big).$$

We can show that $\hat{\mathcal{R}}_L^{(\rm in)}(\Sigma_d|\Sigma_{X^KY^L})$, $\hat{\mathcal{R}}_L^{(\rm in)}(\Gamma,D^L|\Sigma_{X^KY^L})$, and $\hat{\mathcal{R}}_L^{(\rm in)}(\Gamma,D|\Sigma_{X^KY^L})$ satisfy the following property.

*Property 1:*

a) The set $\hat{\mathcal{R}}_L^{(\rm in)}(\Sigma_d|\Sigma_{X^KY^L})$ is not void if and only if $\Sigma_d\succ\Sigma_{X^K|Y^L}$.

b) The set $\hat{\mathcal{R}}_L^{(\rm in)}(\Gamma,D^K|\Sigma_{X^KY^L})$ is not void if and only if $D^K>d^K(\Gamma\ \Sigma_{X^K|Y^L}{}^{\rm t}\Gamma)$.

c) The set $\hat{\mathcal{R}}_L^{(\rm in)}(\Gamma,D|\Sigma_{X^KY^L})$ is not void if and only if $D>{\rm tr}[\Gamma\Sigma_{X^K|Y^L}{}^{\rm t}\Gamma]$.

On inner bounds of $\mathcal{R}_L(\Sigma_d|\Sigma_{X^KY^L})$, $\mathcal{R}_L(\Gamma,D^L|\Sigma_{X^KY^L})$, and $\hat{\mathcal{R}}_L(\Gamma,D|\Sigma_{X^KY^L})$, we have the following result.

*Theorem 1 (Berger [4] and Tung [5]):* For any $\Sigma_d\succ\Sigma_{X^K|Y^L}$, we have

$$\hat{\mathcal{R}}_L^{(\rm in)}(\Sigma_d|\Sigma_{X^KY^L})\subseteq\mathcal{R}_L(\Sigma_d|\Sigma_{X^KY^L}).$$

For any $\Gamma$ and any $D^K>d^K(\Gamma\Sigma_{X^K|Y^L}{}^{\rm t}\Gamma)$, we have

$$\hat{\mathcal{R}}_L^{(\rm in)}(\Gamma,D^K|\Sigma_{X^KY^L})\subseteq\mathcal{R}_L(\Gamma,D^K|\Sigma_{X^KY^L}).$$

For any $\Gamma$ and any $D>{\rm tr}[\Gamma\Sigma_{X^K|Y^L}{}^{\rm t}\Gamma]$, we have

$$\hat{\mathcal{R}}_L^{(\rm in)}(\Gamma,D|\Sigma_{X^KY^L})\subseteq\mathcal{R}_L(\Gamma,D|\Sigma_{X^KY^L}).$$

The above three inner bounds can be regarded as variants of the inner bound which is well known as that of Berger [4] and Tung [5].

When $K=1$ and $L\times 1$ column vector $A$ has the form $A={}^{\rm t}[11\cdots 1]$, the system considered here becomes the quadratic Gaussian CEO problem. This problem was first posed and investigated by Viswanathan and Berger [15]. They further assumed $\Sigma_{N^L}=\sigma^2 I_L$. Set $\sigma_X^2\triangleq\Sigma_X$ and

$$R_{\rm sum}(D|\sigma_X^2,\sigma^2)\triangleq\liminf_{L\to\infty}R_{{\rm sum},L}(D|\Sigma_{XY^L}).$$

Viswanathan and Berger [15] studied an asymptotic form of $R_{\rm sum}(D|\sigma_X^2,\sigma^2)$ for small $D$. Subsequently, Oohama [16] determined an exact form of $R_{\rm sum}(D|\sigma_X^2,\sigma^2)$. The region $\mathcal{R}_L(D|\Sigma_{XY^L})$ was determined independently by Oohama [17] and Prabhakaram *et al.* [18]. Wang *et al.*[12] obtained the same characterization of $R_{{\rm sum},L}(D|\Sigma_{XY^L})$ as that of Oohama [17] in a new alternative method. Their method is based on the order of the variances associated with the minimum mean square error (MMSE) estimation. Unlike the method of Oohama [17], the method of Wang *et al.* [12] is not directly applicable to the characterization of the entire rate distortion region $\mathcal{R}_L(D|\Sigma_{XY^L})$.

In the case where $K=L=2$ and $\Gamma=A=I_2$, Wagner *et al.* [11] determined $\mathcal{R}_2(D^2|\Sigma_{X^2Y^2})$. Their result is as follows.

*Theorem 2 (Wagner et al. [11]):* For any $D^2 > d^2([\Sigma_{X^2|Y^2}])$, we have

$$\mathcal{R}_2(D^2|\Sigma_{X^2Y^2}) = \hat{\mathcal{R}}_2^{(\rm in)}(D^2|\Sigma_{X^2Y^2}).$$

Their method for the proof depends heavily on the specific property of $L=2$. It is hard to generalize it to the case of $L \geq 3$.

In the case where $K = L$ and $\Gamma = A = I_L$, Oohama [20]-[24] derived inner and outer bounds of $\mathcal{R}_L(D|\Sigma_{X^LY^L})$. Oohama [21], [23], [24] also derived explicit sufficient conditions for inner and outer bounds to match. In [22], Oohama derived explicit outer bounds of $\mathcal{R}_L(\Sigma_d\ |\Sigma_{X^LY^L})$, $\mathcal{R}_L(D^L\ |\Sigma_{X^LY^L})$, and $\mathcal{R}_L(D\ |\Sigma_{X^LY^L})$.

The determination problem of $\mathcal{R}_L(D|\Sigma_{X^KY^L})$ in the case where $A$ is a general $K \times L$ matrix and $\Gamma = I_K$ was studied by Yang and Xiong [26] and Yang *et al.* [27]. Relations between their results and our results of the present paper will be discussed in the next section.

## III. Main Results

### A. Inner and Outer Bounds of the Rate Distortion Region

In this subsection we state our result on the characterizations of $\mathcal{R}_L(\Sigma_d\ |\Sigma_{X^KY^L})$, $\mathcal{R}_L(\Gamma, D^K\ |\Sigma_{X^KY^L})$, and $\mathcal{R}_L(\Gamma, D\ |\Sigma_{X^KY^L})$. To describe those results we define several functions and sets. For each $l \in \Lambda_L$ and for $r_l \geq 0$, let $N_l(r_l)$ be a Gaussian random variable with mean 0 and variance $\sigma^2_{N_l}/(1-{\rm e}^{-2r_l})$. We assume that $N_l(r_l), l \in \Lambda_L$ are independent. When $r_l = 0$, we formally think that the inverse value $\sigma^{-1}_{N(0)}$ of the variance of $N_l(0)$ is zero. Let $\Sigma_{N^L(r^L)}$ be a covariance matrix of the random vector

$$N^L(r^L) = N_{\Lambda_L}(r_{\Lambda_L}) = \{N_l(r_l)\}_{l\in\Lambda}.$$

When $r_S = \mathbf{0}$, we formally define

$$\Sigma^{-1}_{N_{S^{\rm c}}(r_{S^{\rm c}})} \stackrel{\triangle}{=} \Sigma^{-1}_{N^L(r^L)}\Big|_{r_S=\mathbf{0}}.$$

Fix nonnegative vector $r^L$. For $\theta > 0$ and for $S \subseteq \Lambda_L$, define

$$\underline{J}_S(\theta, r_S|r_{S^{\rm c}}) \stackrel{\triangle}{=} \frac{1}{2}\log^+\left[\frac{\prod_{l\in S}{\rm e}^{2r_l}}{\theta\left|\Sigma^{-1}_{X^K} + {}^{\rm t}A\Sigma^{-1}_{N_{S^{\rm c}}(r_{S^{\rm c}})}A\right|}\right],$$

$$J_S\left(r_S|r_{S^{\rm c}}\right) \stackrel{\triangle}{=} \frac{1}{2}\log\left[\frac{\left|\Sigma^{-1}_{X^K} + {}^{\rm t}A\Sigma^{-1}_{N^L(r^L)}A\right|\prod_{l\in S}{\rm e}^{2r_l}}{\left|\Sigma^{-1}_{X^K} + {}^{\rm t}A\Sigma^{-1}_{N_{S^{\rm c}}(r_{S^{\rm c}})}A\right|}\right],$$

where $S^{\rm c} = \Lambda_L - S$ and $\log^+[x] \stackrel{\triangle}{=} \max\{\log x, 0\}$. Set

$$\mathcal{A}_L(\Sigma_d) \stackrel{\triangle}{=} \left\{r^L \geq 0 : \left[\Sigma^{-1}_{X^K} + {}^{\rm t}A\Sigma^{-1}_{N^L(r^L)}A\right]^{-1} \preceq \Sigma_d\right\}.$$

We can show that for $S \subseteq \Lambda_L$, $\underline{J}_S(|\Sigma_d|,\ r_S|r_{S^{\rm c}})$ and $J_S(r_S|r_{S^{\rm c}})$ satisfy the following two properties.

*Property 2:*

a) If $r^L \in \mathcal{A}_L(\Sigma_d)$, then for any $S \subseteq \Lambda_L$,

$$\underline{J}_S(|\Sigma_d|, r_S|r_{S^{\rm c}}) \leq J_S(r_S|r_{S^{\rm c}}).$$

b) Suppose that $r^L \in \mathcal{A}_L(\Sigma_d)$. If $r^L\big|_{r_S=\mathbf{0}}$ still belongs to $\mathcal{A}_L(\Sigma_d)$, then

$$\underline{J}_S(|\Sigma_d|, r_S|r_{S^{\rm c}})|_{r_S=\mathbf{0}} = J_S(r_S|r_{S^{\rm c}})|_{r_S=\mathbf{0}} = 0.$$

*Property 3:* Fix $r^L \in \mathcal{A}_L(\Sigma_d)$. For $S \subseteq \Lambda_L$, set

$$f_S = f_S(r_S|r_{S^{\rm c}}) \stackrel{\triangle}{=} \underline{J}_S(|\Sigma_d|, r_S|r_{S^{\rm c}}).$$

By definition, it is obvious that $f_S, S \subseteq \Lambda_L$ are nonnegative. We can show that $f \stackrel{\triangle}{=} \{f_S\}_{S\subseteq\Lambda_L}$ satisfies the followings:

a) $f_\emptyset = 0$.
b) $f_A \leq f_B$ for $A \subseteq B \subseteq \Lambda_L$.
c) $f_A + f_B \leq f_{A\cap B} + f_{A\cup B}$.

In general $(\Lambda_L, f)$ is called a *co-polymatroid* if the nonnegative function $\rho$ on $2^{\Lambda_L}$ satisfies the above three properties. Similarly, we set

$$\tilde{f}_S = \tilde{f}_S(r_S|r_{S^{\rm c}}) \stackrel{\triangle}{=} J_S(r_S|r_{S^{\rm c}}), \quad \tilde{f} = \left\{\tilde{f}_S\right\}_{S\subseteq\Lambda_L}.$$

Then $(\Lambda_L, \tilde{f})$ also has the same three properties as those of $(\Lambda_L, f)$ and becomes a co-polymatroid.

To describe our result on $\mathcal{R}_L(\Sigma_d|\Sigma_{X^KY^L})$, set

$$\begin{aligned}
&\mathcal{R}_L^{(\rm out)}(\theta, r^L|\Sigma_{X^KY^L}) \\
&\stackrel{\triangle}{=} \Big\{R^L : \sum_{i\in S} R_l \geq \underline{J}_S\left(\theta, r_S|r_{S^{\rm c}}\right) \\
&\qquad \text{for any } S \subseteq \Lambda_L. \Big\}, \\
&\mathcal{R}_L^{(\rm out)}(\Sigma_d|\Sigma_{X^KY^L}) \\
&\stackrel{\triangle}{=} \bigcup_{r^L\in\mathcal{A}_L(\Sigma_d)} \mathcal{R}_L^{(\rm out)}(|\Sigma_d|, r^L|\Sigma_{X^KY^L}), \\
&\mathcal{R}_L^{(\rm in)}(r^L) \\
&\stackrel{\triangle}{=} \Big\{R^L : \sum_{l\in S} R_l \geq J_S\left(r_S|r_{S^{\rm c}}\right) \\
&\qquad \text{for any } S \subseteq \Lambda_L. \Big\}, \\
&\mathcal{R}_L^{(\rm in)}(\Sigma_d|\Sigma_{X^KY^L}) \\
&\stackrel{\triangle}{=} {\rm conv}\left\{\bigcup_{r^L\in\mathcal{A}_L(\Sigma_d)} \mathcal{R}_L^{(\rm in)}(r^L|\Sigma_{X^KY^L})\right\}.
\end{aligned}$$

We can show that $\mathcal{R}_L^{(\rm in)}(\Sigma_d|\Sigma_{X^KY^L})$ and $\mathcal{R}_L^{(\rm out)}(\Sigma_d|\Sigma_{X^KY^L})$ satisfy the following property.

*Property 4:* The sets $\mathcal{R}_L^{(\rm in)}(\Sigma_d|\Sigma_{X^KY^L})$ and $\mathcal{R}_L^{(\rm out)}(\Sigma_d\ |\Sigma_{X^KY^L})$ are not void if and only if $\Sigma_d \succ \Sigma_{X^K|Y^L}$.

Our result on inner and outer bounds of $\mathcal{R}_L(\Sigma_d|\Sigma_{X^KY^L})$ is as follows.

*Theorem 3:* For any $\Sigma_d \succ \Sigma_{X^K|Y^L}$, we have

$$\begin{aligned}
&\mathcal{R}_L^{(\rm in)}(\Sigma_d|\Sigma_{X^KY^L}) = \hat{\mathcal{R}}_L^{(\rm in)}(\Sigma_d|\Sigma_{X^KY^L}) \\
&\subseteq \mathcal{R}_L(\Sigma_d|\Sigma_{X^KY^L}) \subseteq \mathcal{R}_L^{(\rm out)}(\Sigma_d|\Sigma_{X^KY^L}).
\end{aligned}$$

Proof of this theorem is given in Section V. This result includes the result of Oohama [22] as a special case by letting $K = L$ and $\Gamma = A = I_L$. From this theorem we can derive outer and inner bounds of $\mathcal{R}_L(\Gamma, D^K|\ \Sigma_{X^K Y^L})$ and $\mathcal{R}_L(\Gamma,D|\Sigma_{X^K Y^L})$. To describe those bounds, set

$$\begin{aligned}
&\mathcal{R}_L^{(\mathrm{out})}(\Gamma, D^K|\Sigma_{X^K Y^L})\\
&\triangleq \bigcup_{\Gamma\Sigma_d{}^{\mathrm{t}}\Gamma\in\mathcal{S}_K(D^K)} \mathcal{R}_L^{(\mathrm{out})}(\Sigma_d|\Sigma_{X^K Y^L}),\\
&\mathcal{R}_L^{(\mathrm{in})}(\Gamma, D^K|\Sigma_{X^K Y^L})\\
&\triangleq \mathrm{conv}\left\{\bigcup_{\Gamma\Sigma_d{}^{\mathrm{t}}\Gamma\in\mathcal{S}_K(D^K)} \mathcal{R}_L^{(\mathrm{in})}(\Sigma_d|\Sigma_{X^K Y^L})\right\},\\
&\mathcal{R}_L^{(\mathrm{out})}(\Gamma, D|\Sigma_{X^K Y^L})\\
&\triangleq \bigcup_{\mathrm{tr}[\Gamma\Sigma_d{}^{\mathrm{t}}\Gamma]\le D} \mathcal{R}_L^{(\mathrm{out})}(\Sigma_d|\Sigma_{X^K Y^L}),\\
&\mathcal{R}_L^{(\mathrm{in})}(\Gamma, D|\Sigma_{X^K Y^L})\\
&\triangleq \mathrm{conv}\left\{\bigcup_{\mathrm{tr}[\Gamma\Sigma_d{}^{\mathrm{t}}\Gamma]\le D} \mathcal{R}_L^{(\mathrm{in})}(\Sigma_d|\Sigma_{X^K Y^L})\right\}.
\end{aligned}$$

Set

$$\begin{aligned}
\mathcal{A}(r^L) &\triangleq \left\{\Sigma_d : \Sigma_d \succeq (\Sigma_{X^K}^{-1} + {}^{\mathrm{t}}A\Sigma_{N^L(r^L)}^{-1}A)^{-1}\right\},\\
\theta(\Gamma, D^K, r^L) &\triangleq \max_{\substack{\Sigma_d:\Sigma_d\in\mathcal{A}_L(r^L),\\ \Gamma\Sigma_d{}^{\mathrm{t}}\Gamma\in\mathcal{S}_K(D^K)}} |\Sigma_d|,\\
\theta(\Gamma, D, r^L) &\triangleq \max_{\substack{\Sigma_d:\Sigma_d\in\mathcal{A}_L(r^L),\\ \mathrm{tr}[\Gamma\Sigma_d{}^{\mathrm{t}}\Gamma]\le D}} |\Sigma_d|.
\end{aligned}$$

Furthermore, set

$$\begin{aligned}
&\mathcal{B}_L(\Gamma, D^K)\\
&\triangleq \left\{r^L \ge 0 : \Gamma(\Sigma_{X^K}^{-1} + {}^{\mathrm{t}}A\Sigma_{N^L(r^L)}^{-1}A)^{-1}{}^{\mathrm{t}}\Gamma \in \mathcal{S}_K(D^K)\right\},\\
&\mathcal{B}_L(\Gamma, D)\\
&\triangleq \left\{r^L \ge 0 : \mathrm{tr}[\Gamma(\Sigma_{X^K}^{-1} + {}^{\mathrm{t}}A\Sigma_{N^L(r^L)}^{-1}A)^{-1}{}^{\mathrm{t}}\Gamma] \le D\right\}.
\end{aligned}$$

It can easily be verified that $\mathcal{R}_L^{(\mathrm{out})}(\Gamma,\ D^K|\Sigma_{X^K Y^L})$, $\mathcal{R}_L^{(\mathrm{in})}(\Gamma,\ D^K|\Sigma_{X^K Y^L})$, $\mathcal{R}_L^{(\mathrm{out})}(\Gamma,\ D|\Sigma_{X^K Y^L})$, and $\mathcal{R}_L^{(\mathrm{in})}(\Gamma,\ D|\Sigma_{X^K Y^L})$ satisfies the following property.

*Property 5:*

a) The sets $\mathcal{R}_L^{(\mathrm{in})}(\Gamma, D^K|\Sigma_{X^K Y^L})$ and $\mathcal{R}_L^{(\mathrm{out})}(\Gamma, D^K|\Sigma_{X^K Y^L})$ are not void if and only if $D^K > d^K(\Gamma\Sigma_{X^K|Y^L}{}^{\mathrm{t}}\Gamma)$.

b) The sets $\mathcal{R}_L^{(\mathrm{in})}(\Gamma, D|\Sigma_{X^K Y^L})$ and $\mathcal{R}_L^{(\mathrm{out})}(\Gamma, D|\Sigma_{X^K Y^L})$ are not void if and only if $D > \mathrm{tr}[\Gamma\Sigma_{X^K|Y^L}{}^{\mathrm{t}}\Gamma]$.

c)

$$\begin{aligned}
&\mathcal{R}_L^{(\mathrm{out})}(\Gamma, D^K|\Sigma_{X^K Y^L})\\
&= \bigcup_{r^L\in\mathcal{B}_L(\Gamma,D^K)} \mathcal{R}_L^{(\mathrm{out})}(\theta(\Gamma, D^K, r^L), r^L|\Sigma_{X^K Y^L}),\\
&\mathcal{R}_L^{(\mathrm{in})}(\Gamma, D^K|\Sigma_{X^K Y^L})\\
&= \mathrm{conv}\left\{\bigcup_{r^L\in\mathcal{B}_L(\Gamma,D^K)} \mathcal{R}_L^{(\mathrm{in})}(r^L|\Sigma_{X^K Y^L})\right\},\\
&\mathcal{R}_L^{(\mathrm{out})}(\Gamma, D|\Sigma_{X^K Y^L})\\
&= \bigcup_{r^L\in\mathcal{B}_L(\Gamma,D)} \mathcal{R}_L^{(\mathrm{out})}(\theta(\Gamma, D, r^L), r^L|\Sigma_{X^K Y^L}),\\
&\mathcal{R}_L^{(\mathrm{in})}(\Gamma, D|\Sigma_{X^K Y^L})\\
&= \mathrm{conv}\left\{\bigcup_{r^L\in\mathcal{B}_L(\Gamma,D)} \mathcal{R}_L^{(\mathrm{in})}(r^L)\right\}.
\end{aligned}$$

The following result is obtained as a simple corollary from Theorem 3.

*Corollary 1:* For any $\Gamma$ and any $D^K > d^K(\Gamma\Sigma_{X^K|Y^L}{}^{\mathrm{t}}\Gamma)$, we have

$$\begin{aligned}
&\mathcal{R}_L^{(\mathrm{in})}(\Gamma, D^K|\Sigma_{X^K Y^L}) = \hat{\mathcal{R}}_L^{(\mathrm{in})}(\Gamma, D^K|\Sigma_{X^K Y^L})\\
&\subseteq \mathcal{R}_L(\Gamma, D^K|\Sigma_{X^K Y^L}) \subseteq \mathcal{R}_L^{(\mathrm{out})}(\Gamma, D^K|\Sigma_{X^K Y^L}).
\end{aligned}$$

For any $\Gamma$ and any $D > \mathrm{tr}[\Gamma\Sigma_{X^K|Y^L}{}^{\mathrm{t}}\Gamma]$, we have

$$\begin{aligned}
&\mathcal{R}_L^{(\mathrm{in})}(\Gamma, D|\Sigma_{X^K Y^L}) = \hat{\mathcal{R}}_L^{(\mathrm{in})}(\Gamma, D|\Sigma_{X^K Y^L})\\
&\subseteq \mathcal{R}_L(\Gamma, D|\Sigma_{X^K Y^L}) \subseteq \mathcal{R}_L^{(\mathrm{out})}(\Gamma, D|\Sigma_{X^K Y^L}).
\end{aligned}$$

Those result includes the result of Oohama [22] as a special case by letting $K = L$ and $\Gamma = A = I_L$. Next we compute $\theta(\Gamma, D, r^L)$ to derive a more explicit expression of $\mathcal{R}_L^{(\mathrm{out})}(\Gamma, D|\Sigma_{X^K Y^L})$. This expression will be quite useful for finding a sufficient condition for the outer bound $\mathcal{R}_L^{(\mathrm{out})}(\Gamma, D|\Sigma_{X^K Y^L})$ to be tight. Let $\alpha_k = \alpha_k(r^L), k \in \Lambda_K$ be $K$ eigenvalues of the matrix

$$\Gamma^{-1}\left(\Sigma_{X^K}^{-1} + {}^{\mathrm{t}}A\Sigma_{N^L(r^L)}^{-1}A\right){}^{\mathrm{t}}\Gamma^{-1}.$$

Let $\xi$ be a nonnegative number that satisfy

$$\sum_{k=1}^{K}\left\{[\xi - \alpha_k^{-1}]^+ + \alpha_k^{-1}\right\} = D.$$

Define

$$\omega(\Gamma, D, r^L) \triangleq |\Gamma|^{-2}\prod_{k=1}^{K}\left\{[\xi - \alpha_k^{-1}]^+ + \alpha_k^{-1}\right\}.$$

The function $\omega(\Gamma, D, r^L)$ has an expression of the so-called water filling solution to the following optimization problem:

$$\omega(\Gamma, D, r^L) = |\Gamma|^{-2}\max_{\substack{\xi_k\alpha_k\ge 1, k\in\Lambda_K,\\ \sum_{k=1}^K \xi_k \le D}} \prod_{k=1}^{K}\xi_k. \quad (4)$$

Then we have the following theorem.

*Theorem 4:* For any $\Gamma$ and any positive $D$, we have

$$\theta(\Gamma, D, r^L) = \omega(\Gamma, D, r^L).$$

A more explicit expression of $\mathcal{R}_L^{(\mathrm{out})}(\Gamma, D|\Sigma_{X^K Y^L})$ using $\omega(\Gamma, D, r^L)$ is given by

$$\begin{aligned}
&\mathcal{R}_L^{(\mathrm{out})}(\Gamma, D|\Sigma_{X^K Y^L})\\
&\triangleq \bigcup_{r^L\in\mathcal{B}_L(\Gamma,D)} \mathcal{R}_L^{(\mathrm{out})}(\omega(\Gamma, D, r^L), r^L|\Sigma_{X^K Y^L}).
\end{aligned}$$

Proof of this theorem will be given in Section V. The above expression of the outer bound includes the result of Oohama [22] as a special case by letting $K = L$ and $\Gamma = A = I_L$. In the next subsection we derive a matching condition for $\mathcal{R}_L^{(\mathrm{out})}(\Gamma, D|\Sigma_{X^K Y^L})$ to coincide with $\mathcal{R}_L(\Gamma, D|\Sigma_{X^K Y^L})$.

Two other outer bounds of $\mathcal{R}_L(D|\Sigma_{X^K Y^L})$ were obtained by Yang and Xiong [26]. They derived the first outer bound for general $L\times K$ matrix $A$. This outer bound denoted by $\check{\mathcal{R}}_L^{(\mathrm{out})}(D|\Sigma_{X^K Y^L})$ does not coincide with $\mathcal{R}_L^{(\mathrm{out})}(D|\Sigma_{X^K Y^L})$ when $K=L$ and $A=I_L$. When $A$ is semi orthogonal, i.e., ${}^{\mathrm{t}}AA=I_K$, Yang and Xiong [26] derived the second outer bound $\tilde{\mathcal{R}}_L^{(\mathrm{out})}(D|\Sigma_{X^K Y^L})$ tighter than $\check{\mathcal{R}}_L^{(\mathrm{out})}(D|\Sigma_{X^K Y^L})$. The outer bound $\tilde{\mathcal{R}}_L^{(\mathrm{out})}(D|\Sigma_{X^K Y^L})$ is the same as our outer bound $\mathcal{R}_L^{(\mathrm{out})}(D|\Sigma_{X^K Y^L})$ although it has a form different from that of our outer bound. They further derived a matching condition for $\tilde{\mathcal{R}}_L^{(\mathrm{out})}(D|\Sigma_{X^K Y^L})$ to coincide with $\mathcal{R}_L(D|\Sigma_{X^K Y^L})$. Their matching condition and its relation to our matching condition will be presented in the next subsection.

### B. Matching Condition Analysis

For $L\geq 3$, we present a sufficient condition for $\mathcal{R}_L^{(\mathrm{out})}(\Gamma, D|\Sigma_{X^K Y^L})\subseteq\mathcal{R}_L^{(\mathrm{in})}(D|\Sigma_{X^K Y^L})$. We consider the following condition on $\theta(\Gamma,D,r^L)$.

*Condition:* For any $l\in\Lambda_L$, $\mathrm{e}^{-2r_l}\theta(\Gamma,D,r^L)$ is a monotone decreasing function of $r_l\geq 0$.

We call this condition the MD condition. The following is a key lemma to derive the matching condition. This lemma is due to Oohama [21], [23].

*Lemma 1 (Oohama [21],[23]):* If $\theta(\Gamma,D,r^L)$ satisfies the MD condition on $\mathcal{B}_L(\Gamma,D)$, then

$$\begin{aligned}\mathcal{R}_L^{(\mathrm{in})}(\Gamma,D|\Sigma_{X^K Y^L})&=\mathcal{R}_L(\Gamma,D|\Sigma_{X^K Y^L})\\&=\mathcal{R}_L^{(\mathrm{out})}(\Gamma,D|\Sigma_{X^K Y^L}).\end{aligned}$$

Based on Lemma 1, we derive a sufficient condition for $\theta(\Gamma,D,r^L)$ to satisfy the MD condition.

Let $a_{lk}$ be the $(l,k)$ element of $A$. Set $\boldsymbol{a}_l\stackrel{\triangle}{=}[a_{l1}a_{l2}\cdots a_{lK}]$ and $\hat{\boldsymbol{a}}_l\stackrel{\triangle}{=}\boldsymbol{a}_l\Gamma^{-1}$. Let $\mathcal{O}_K$ be the set of all $K\times K$ orthogonal matrices. For $(l,k)\in\Lambda_L\times\Lambda_K$, let $\mathcal{O}_K(\hat{\boldsymbol{a}}_l,k)$ be a set of all $T\in\mathcal{O}_K$ that satisfy

$$[\hat{\boldsymbol{a}}_l T]_j=\begin{cases}||\hat{\boldsymbol{a}}_l||, & \text{if } j=k,\\ 0, & \text{if } j\neq k.\end{cases}$$

For $T\in\mathcal{O}_K(\hat{\boldsymbol{a}}_l,k)$, we consider the following matrix:

$$\begin{aligned}&C(\Gamma^{-1}T,r^L)\stackrel{\triangle}{=}{}^{\mathrm{t}}T{}^{\mathrm{t}}\Gamma^{-1}(\Sigma_{X^K}^{-1}+{}^{\mathrm{t}}A\Sigma_{N^L(r^L)}^{-1}A)\Gamma^{-1}T\\&={}^{\mathrm{t}}T{}^{\mathrm{t}}\Gamma^{-1}\Sigma_{X^K}^{-1}\Gamma^{-1}T+\sum_{l=1}^{L}\tfrac{1}{\sigma_{N_l}^2}(1-\mathrm{e}^{-2r_l}){}^{\mathrm{t}}(\hat{\boldsymbol{a}}_l T)(\hat{\boldsymbol{a}}_l T).\end{aligned}$$

Let $r_{[l]}^L\stackrel{\triangle}{=}r_1\cdots r_{l-1}r_{l+1}\cdots r_L$ and set

$$\begin{aligned}&\eta_k(\Gamma^{-1}T,r_{[l]}^L)\\&\stackrel{\triangle}{=}\left[{}^{\mathrm{t}}T{}^{\mathrm{t}}\Gamma^{-1}\Sigma_{X^K}^{-1}\Gamma^{-1}T\right]_{kk}\\&\qquad+\sum_{i\neq l}\tfrac{1}{\sigma_{N_i}^2}(1-\mathrm{e}^{-2r_i})\left[{}^{\mathrm{t}}(\hat{\boldsymbol{a}}_i T)(\hat{\boldsymbol{a}}_i T)\right]_{kk},\\&\chi_{lk}(\Gamma^{-1}T,r_{[l]}^L)\stackrel{\triangle}{=}||\hat{\boldsymbol{a}}_l||^2\tfrac{1}{\sigma_{N_l}^2}+\eta_k(\Gamma^{-1}T,r_{[l]}^L).\end{aligned}$$

Then we have

$$\begin{aligned}[C(\Gamma^{-1}T,r^L)]_{kk}&=||\hat{\boldsymbol{a}}_l||^2\tfrac{1}{\sigma_{N_l}^2}(1-\mathrm{e}^{-2r_l})+\eta_i(\Gamma^{-1}T,r_{[l]}^L)\\&=\chi_{lk}(\Gamma^{-1}T,r_{[l]}^L)-||\hat{\boldsymbol{a}}_l||^2\tfrac{1}{\sigma_{N_l}^2}\mathrm{e}^{-2r_l}.\end{aligned}\quad (5)$$

If $(i',i'')\neq(k,k)$, then the value of

$$\begin{aligned}&[C(\Gamma^{-1}T,r^L)]_{i'i''}\\&=[{}^{\mathrm{t}}T{}^{\mathrm{t}}\Gamma^{-1}\Sigma_{X^K}^{-1}\Gamma^{-1}T]_{i'i''}\\&\qquad+\sum_{j=1}^{L}\tfrac{1}{\sigma_{N_j}^2}(1-\mathrm{e}^{-2r_j})\left[{}^{\mathrm{t}}(\hat{\boldsymbol{a}}_j T)(\hat{\boldsymbol{a}}_j T)\right]_{i'i''}\end{aligned}$$

does not depend on $r_l$. Note that the matrix $C(\Gamma^{-1}T,r^L)$ has the same eigenvalue set as that of

$$C(\Gamma^{-1},r^L)={}^{\mathrm{t}}\Gamma^{-1}(\Sigma_{X^K}^{-1}+{}^{\mathrm{t}}A\Sigma_{N^L(r^L)}^{-1}A)\Gamma^{-1}.$$

We recall here that $\alpha_k=\alpha_k(r^L),k\in\Lambda_K$ are $K$ eigenvalues of the above two matrices. Let $\alpha_{\min}=\alpha_{\min}(r^L)$ and $\alpha_{\max}=\alpha_{\max}(r^L)$ be the minimum and maximum eigenvalues among $\alpha_k,k\in\Lambda_K$. The matrix $C(\Gamma^{-1}T,r^L)$ for $T\in\mathcal{O}_K(\hat{\boldsymbol{a}}_l,k)$, has a structure that the $(k,k)$ element of this matrix is only one element which depends on $r_l$ and this element is a monotone increasing function of $r_l\geq 0$. Properties on eigenvalues of matrices having the above structure were studied in detail by Oohama [21],[23]. The following lemma is a variant of his result.

*Lemma 2 (Oohama [21],[23]):* For each $(l,k)\in\Lambda_L\times\Lambda_K$ and each $T\in\mathcal{O}_K(\hat{\boldsymbol{a}}_l,k)$, we have the followings.

$$\begin{aligned}&\alpha_{\min}(r^L)\\&\leq||\hat{\boldsymbol{a}}_l||^2\tfrac{1}{\sigma_{N_l}^2}(1-\mathrm{e}^{-2r_l})+\eta_{lk}(\Gamma^{-1}T,r_{[l]}^L)\leq\alpha_{\max}(r^L),\end{aligned}$$

$$\frac{\partial\alpha_j}{\partial r_l}\geq 0,\ \text{for } j\in\Lambda_K,\quad\sum_{j=1}^{K}\frac{\partial\alpha_j}{\partial r_l}=\frac{2||\hat{\boldsymbol{a}}_l||^2}{\mathrm{e}^{2r_l}\sigma_{N_l}^2}.$$

The following is a key lemma to derive a sufficient condition for the MD condition to hold.

*Lemma 3:* If $\alpha_{\min}(r^L)$ and $\alpha_{\max}(r^L)$ satisfy

$$\left(\frac{1}{\alpha_{\min}(r^L)}-\frac{1}{\alpha_{\max}(r^L)}\right)\cdot\frac{\alpha_{\max}(r^L)}{\alpha_{\min}(r^L)}\leq\frac{\mathrm{e}^{2r_l}\sigma_{N_l}^2}{||\hat{\boldsymbol{a}}_l||^2}\quad (6)$$
$$\text{for } l\in\Lambda_L,$$

on $\mathcal{B}_L(\Gamma,D)$, then $\theta(\Gamma,D,r^L)$ satisfies the MD condition on $\mathcal{B}_L(\Gamma,D)$.

Proof of Lemma 3 will be stated in Section V. Set

$$\begin{aligned}C^*(\Gamma^{-1}T,r_l)&\stackrel{\triangle}{=}\lim_{r_{[l]}^L\to\infty}C(\Gamma^{-1}T,r^L),\\\chi_k^*(\Gamma^{-1}T)&\stackrel{\triangle}{=}\lim_{r_{[l]}^L\to\infty}\chi_{lk}(\Gamma^{-1}T,r_{[l]}^L)\\&=[{}^{\mathrm{t}}T{}^{\mathrm{t}}\Gamma^{-1}\left(\Sigma_{X^K}^{-1}+{}^{\mathrm{t}}A\Sigma_{N^L}^{-1}A\right)\Gamma^{-1}T]_{kk}.\end{aligned}$$

For $k\in\Lambda_K$, we denote the $(k,k)$ element of $C^*(\Gamma^{-1}T,r_l)$ by $c_{kk}^*=c_{kk}^*(\Gamma^{-1}T,r_l)$. When $(j,j')\in\Lambda_K^2$ and $(j,j')\neq(k,k)$, the $(j,j')$ element of $C^*(\Gamma^{-1}T,r_l)$ does not depend on $r_l$. We denote it by $c_{jj'}^*=c_{jj'}^*(\Gamma^{-1}T)$. Furthermore, set

$$\boldsymbol{c}_{k[k]}^*=\boldsymbol{c}_{k[k]}^*(\Gamma^{-1}T)\stackrel{\triangle}{=}[c_{k1}^*\cdots c_{kk-1}^*c_{kk+1}^*\cdots c_{kK}^*].$$

By definition we have

$$c^*_{kk}(\Gamma^{-1}T, r_l) = \chi^*_k(\Gamma^{-1}T) - \frac{||\hat{\boldsymbol{a}}_l||^2}{\mathrm{e}^{2r_l}\sigma^2_{N_l}}.$$

Define

$$\alpha^*_{\max} \triangleq \lim_{r^L\to\infty} \alpha_{\max}(r^L), \alpha^*_{\min} \triangleq \lim_{r^L\to\infty} \alpha_{\min}(r^L),$$
$$\alpha^*_{\max}(r_i) \triangleq \lim_{r^L_{[l]}\to\infty} \alpha_{\max}(r^L) \text{ for } l \in \Lambda_L.$$

By definition, $\alpha^*_{\max}$ and $\alpha^*_{\min}$ are the maximum and minimum eigenvalues of ${}^{\mathrm{t}}\Gamma^{-1}(\Sigma^{-1}_{X^K} + {}^{\mathrm{t}}A\Sigma^{-1}_{N^L}A\ )\Gamma^{-1}$, respectively. By Lemma 2, we have

$$\alpha_{\min}(r^L) \le \alpha^*_{\min}(r_l) \le \alpha^*_{\min}, \text{ for } l \in \Lambda_L, \quad (7)$$
$$\chi_{lk}(\Gamma^{-1}T, r^L_{[l]}) \le \chi^*_k(\Gamma^{-1}T) \le \alpha^*_{\max}, \text{ for } l \in \Lambda_L. \quad (8)$$

The following lemma provides an effective lower bound of $\mathrm{e}^{2r_l}\sigma^2_{N_l}/||\hat{\boldsymbol{a}}_l||^2$.

*Lemma 4:* For any $(l,k) \in \Lambda_L \times \Lambda_K$ and $T \in \mathcal{O}_Y(\hat{\boldsymbol{a}}_l, k)$, we have

$$\begin{aligned} c^*_{kk}(\Gamma^{-1}T, r_l) &= \chi^*_k(\Gamma^{-1}T) - \frac{||\hat{\boldsymbol{a}}_l||^2}{\mathrm{e}^{2r_l}\sigma^2_{N_l}} \\ &\ge \alpha^*_{\min}(r_l) + \frac{||\boldsymbol{c}^*_{k[k]}(\Gamma^{-1}T)||^2}{\alpha^*_{\max}(r_l) - \alpha^*_{\min}(r_l)} \\ &\ge \alpha_{\min}(r^L) + \frac{||\boldsymbol{c}^*_{k[k]}(\Gamma^{-1}T)||^2}{\alpha^*_{\max} - \alpha_{\min}(r^L)}. \end{aligned}$$

Proof of this lemma will be given in Section V. Set

$$\Upsilon_l(\Gamma^{-1}) \triangleq \max_{\substack{k\in\Lambda_K \\ T\in\mathcal{O}_K(\boldsymbol{a}_l\Gamma^{-1},k)}} \frac{1 + \frac{||\boldsymbol{c}^*_{k[k]}(\Gamma^{-1}T)||^2}{(\alpha^*_{\max})^2}}{\chi^*_k(\Gamma^{-1}T) - \frac{||\boldsymbol{c}^*_{k[k]}(\Gamma^{-1}T)||^2}{\alpha^*_{\max}}}.$$

When $\Gamma = I_K$, we simply write $\Upsilon_l(I_K) = \Upsilon_l$. From Lemmas 1-4 and an elementary computation we obtain the following.

*Theorem 5:* If we have

$$\mathrm{tr}[\Gamma\Sigma_{X^K|Y^L}{}^{\mathrm{t}}\Gamma] < D \le \frac{K}{\alpha^*_{\max}} + \min_{l\in\Lambda_L} \Upsilon_l(\Gamma^{-1}) \quad (9)$$

then

$$\begin{aligned} &\mathcal{R}^{(\mathrm{in})}_L(\Gamma, D|\Sigma_{X^KY^L}) = \hat{\mathcal{R}}^{(\mathrm{in})}_L(\Gamma, D|\Sigma_{X^KY^L}) \\ &= \mathcal{R}_L(\Gamma, D|\Sigma_{X^KY^L}) = \mathcal{R}^{(\mathrm{out})}_L(\Gamma, D|\Sigma_{X^KY^L}). \end{aligned}$$

Using (8), we obtain $\Upsilon_l(\Gamma^{-1}) \ge 1/\alpha^*_{\max}$. Hence we have the following matching condition simpler than (9):

$$\mathrm{tr}[\Gamma\Sigma_{X^K|Y^L}{}^{\mathrm{t}}\Gamma] < D \le \frac{K+1}{\alpha^*_{\max}}. \quad (10)$$

Proof of Theorem 5 will be stated in Section V. When $K = L, A = I_L$, the matching condition (10) is the same as that of Oohama [23],[24]. It is obvious that in the case of $K = L, A = I_L$, the matching condition (9) improves that of Oohama [23],[24]. Yang *et al.* [27] have obtained a matching condition on $\mathcal{R}_L(D|\Sigma_{X^KY^L})$ by an argument quite similar to that of Oohama [23]. The matching condition by Yang *et al.* [27] is as follows:

$$\mathrm{tr}[\Sigma_{X^K|Y^L}] < D \le \frac{K}{\alpha^*_{\max}} + \min_{l\in\Lambda_L} \tilde{\Upsilon}_l, \quad (11)$$

where

$$\tilde{\Upsilon}_l \triangleq \max_{T\in\mathcal{O}_K} \max_{k\in\Lambda_K} \left\{ \frac{1}{\chi^*_k(T)} \frac{[\boldsymbol{a}_lT]^2_k}{||\boldsymbol{a}_lT||^2} \right\}.$$

The matching condition (11) by Yang *et al.* [27] also improves that of Oohama [23],[24] in the case of $K = L, A = I_L$. When $\Gamma = I_K$, for $l \in \Lambda_L$, we have

$$\begin{aligned} \Upsilon_l &= \max_{\substack{k\in\Lambda_K \\ T\in\mathcal{O}_K(\boldsymbol{a}_l,k)}} \frac{1 + \frac{||\boldsymbol{c}^*_{k[k]}(T)||^2}{(\alpha^*_{\max})^2}}{\chi^*_k(T) - \frac{||\boldsymbol{c}^*_{k[k]}(T)||^2}{\alpha^*_{\max}}} \\ &\ge \max_{\substack{k\in\Lambda_K \\ T\in\mathcal{O}_K(\boldsymbol{a}_l,k)}} \frac{1}{\chi^*_k(T)} \triangleq \underline{\Upsilon}_l. \end{aligned} \quad (12)$$

On the other hand, for $i \in \Lambda_L$, we have

$$\begin{aligned} \tilde{\Upsilon}_l &= \max_{T\in\mathcal{O}_K} \max_{k\in\Lambda_K} \left\{ \frac{1}{\chi^*_k(T)} \frac{[\boldsymbol{a}_lT]^2_k}{||\boldsymbol{a}_lT||^2} \right\} \\ &= \max_{k\in\Lambda_K} \max_{T\in\mathcal{O}_K} \left\{ \frac{1}{\chi^*_k(T)} \frac{[\boldsymbol{a}_lT]^2_k}{||\boldsymbol{a}_lT||^2} \right\} \\ &\ge \max_{k\in\Lambda_K} \max_{T\in\mathcal{O}_K(\boldsymbol{a}_l,k)} \left\{ \frac{1}{\chi^*_k(T)} \frac{[\boldsymbol{a}_lT]^2_k}{||\boldsymbol{a}_lT||^2} \right\} \\ &= \max_{k\in\Lambda_K} \max_{T\in\mathcal{O}_K(\boldsymbol{a}_l,k)} \frac{1}{\chi^*_k(T)} = \underline{\Upsilon}_l. \end{aligned} \quad (13)$$

Thus, we have $\Upsilon_l \ge \underline{\Upsilon}_l$ and $\tilde{\Upsilon}_l \ge \underline{\Upsilon}_l$. Comparing the two inequalities (12) and (13), we can see that the improvement of $\Upsilon_l$ from $\underline{\Upsilon}_l$ is quite differnt from that of $\tilde{\Upsilon}_l$ from $\underline{\Upsilon}_l$. Hence we have no obvious superiority of $\Upsilon_l$ or $\tilde{\Upsilon}_l$ to the other.

Next we derive another matching condition, which is better than the second matching condition (10) in Theorem 5 and the matching condition (11) of Yang *et al.* [27] for some nontrivial cases. Set

$$\tau_l \triangleq \frac{\sigma^2_{N_l}}{||\hat{\boldsymbol{a}}_l||^2}, \tau^* \triangleq \min_{l\in\Lambda_L} \tau_l.$$

From Lemmas 1-3 and an elementary computation we obtain the following.

*Theorem 6:* If we have

$$\begin{aligned} &\mathrm{tr}[\Gamma\Sigma_{X^K|Y^L}{}^{\mathrm{t}}\Gamma] \\ &< D \le \frac{K}{\alpha^*_{\max}} + \frac{1}{2\alpha^*_{\max}} \left\{ \sqrt{1 + 4\alpha^*_{\max}\tau^*} - 1 \right\}, \end{aligned} \quad (14)$$

then

$$\begin{aligned} &\mathcal{R}^{(\mathrm{in})}_L(\Gamma, D|\Sigma_{X^KY^L}) = \hat{\mathcal{R}}^{(\mathrm{in})}_L(\Gamma, D|\Sigma_{X^KY^L}) \\ &= \mathcal{R}_L(\Gamma, D|\Sigma_{X^KY^L}) = \mathcal{R}^{(\mathrm{out})}_L(\Gamma, D|\Sigma_{X^KY^L}). \end{aligned}$$

Proof of Theorem 6 will be stated in Section V. When $\tau^*$ becomes large, $\alpha^*_{\max}$ and $\alpha^*_{\min}$ approach to the maximum and minimum eigenvalues of $\Sigma^{-1}_{X^L}$, respectively. Hence we have

$$\lim_{\tau^*\to+\infty} \frac{1}{2\alpha^*_{\max}} \left\{ \sqrt{1 + 4\alpha^*_{\max}\tau^*} - 1 \right\} = +\infty, \quad (15)$$

which implies that there exists a sufficiently large $\tau^*$ such that

$$\frac{1}{\alpha^*_{\max}} \le \frac{1}{\alpha^*_{\min}} < \frac{1}{2\alpha^*_{\max}} \left\{ \sqrt{1 + 4\alpha^*_{\max}\tau^*} - 1 \right\}. \quad (16)$$

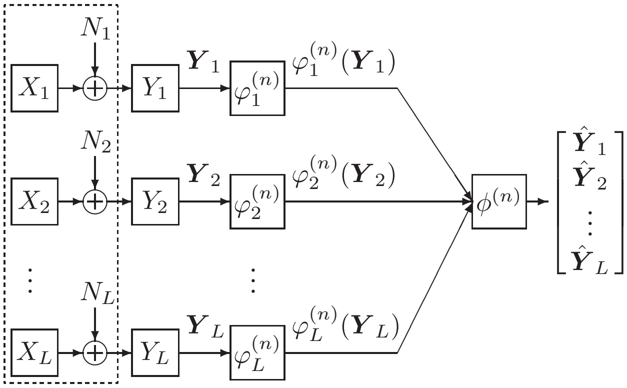


Fig. 2. Distributed source coding system for $L$ correlated Gaussian sources

On the other hand, it follows from the definition of $\tilde{\Upsilon}_l$ that we have for $l \in \Lambda_L$,

$$\tilde{\Upsilon}_l \le \max_{T \in \mathcal{O}_K} \max_{k \in \Lambda_K} \frac{1}{\chi_k^*(T)} \le \frac{1}{\alpha_{\min}^*}. \tag{17}$$

Thus we can see from (16) and (17) that for sufficiently large $\tau^*$, the matching condition (14) in Theorem 6 is better than the second matching condition (10) in Theorem 5 and the matching condition (11) of Yang *et al.* [27].

## IV. Application to the Multiterminal Source Coding Problem

In this section we consider the case where $K = L$ and $A = I_L$. In this case we have $Y^L = X^L + N^L$; Gaussian random variables $Y_l$, $l \in \Lambda_L$ are $L$-noisy components of the Gaussian random vector $X^L$. We study the multiterminal source coding problem for the Gaussian observations $Y_l, l \in \Lambda$. The random vector $X^L$ can be regarded as a "hidden" information source of $Y^L$. Note that $(X^L, Y^L)$ satisfies $Y_S \to X^L \to Y_{S^c}$ for any $S \subseteq \Lambda_L$.

### A. *Problem Formulation and Previous Results*

The distributed source coding system for $L$ correlated Gaussian source treated here is shown in Fig. 2. Definitions of encoder functions $\varphi_l, l \in \Lambda_L$ are the same as the previous definitions. The decoder function $\phi^{(n)}$ is defined by

$$\phi^{(n)} = (\phi_1^{(n)}, \phi_2^{(n)}, \cdots, \phi_L^{(n)})$$
$$\phi_l^{(n)} : \mathcal{M}_1 \times \cdots \times \mathcal{M}_L \mapsto \mathbb{R}^n, l \in \Lambda_L.$$

For $\boldsymbol{Y}^L = (\boldsymbol{Y}_1, \boldsymbol{Y}_2, \cdots, \boldsymbol{Y}_L)$, set

$$\hat{\boldsymbol{Y}}^L = \begin{bmatrix} \hat{\boldsymbol{Y}}_1 \\ \hat{\boldsymbol{Y}}_2 \\ \vdots \\ \hat{\boldsymbol{Y}}_L \end{bmatrix} \triangleq \begin{bmatrix} \phi_1^{(n)}(\varphi^{(n)}(\boldsymbol{Y}^L)) \\ \phi_2^{(n)}(\varphi^{(n)}(\boldsymbol{Y}^L)) \\ \vdots \\ \phi_L^{(n)}(\varphi^{(n)}(\boldsymbol{Y}^L)) \end{bmatrix},$$
$$\tilde{d}_{ll} \triangleq \mathrm{E}||\boldsymbol{Y}_l - \hat{\boldsymbol{Y}}_l||^2, 1 \le l \le L,$$
$$\tilde{d}_{ll'} \triangleq \mathrm{E}\langle \boldsymbol{Y}_l - \hat{\boldsymbol{Y}}_l, \boldsymbol{Y}_{l'} - \hat{\boldsymbol{Y}}_{l'} \rangle, 1 \le l \ne l' \le L.$$

Let $\Sigma_{\boldsymbol{Y}^L - \hat{\boldsymbol{Y}}^L}$ be a covariance matrix with $\tilde{d}_{ll'}$ in its $(l, l')$ element.

For a given $\Sigma_d$, the rate vector $(R_1, R_2, \cdots, R_L)$ is $\Sigma_d$-*admissible* if there exists a sequence $\{(\varphi_1^{(n)}, \varphi_2^{(n)}, \cdots, \varphi_L^{(n)}, \psi^{(n)})\}_{n=1}^\infty$ such that

$$\limsup_{n \to \infty} R_l^{(n)} \le R_l, \text{ for } l \in \Lambda_L,$$
$$\limsup_{n \to \infty} \tfrac{1}{n} \Sigma_{\boldsymbol{Y}^L - \hat{\boldsymbol{Y}}^L} \preceq \Sigma_d.$$

Let $\mathcal{R}_L(\Sigma_d|\Sigma_{Y^L})$ denote the set of all $\Sigma_d$-admissible rate vectors. We consider two types of distortion criterion. For each distortion criterion we define the determination problem of the rate distortion region.

*Problem 3. Vector Distortion Criterion:* For given $L \times L$ invertible matrix $\Gamma$ and $D^L > 0$, the rate vector $(R_1, R_2, \cdots, R_L)$ is $(\Gamma, D^L)$-*admissible* if there exists a sequence $\{(\varphi_1^{(n)}, \varphi_2^{(n)}, \cdots, \varphi_L^{(n)}, \phi^{(n)})\}_{n=1}^\infty$ such that

$$\limsup_{n \to \infty} R^{(n)} \le R_l, \text{ for } l \in \Lambda_L,$$
$$\limsup_{n \to \infty} \left[ \Gamma \left( \tfrac{1}{n} \Sigma_{\boldsymbol{Y}^L - \hat{\boldsymbol{Y}}^L} \right) {}^{\mathrm{t}}\Gamma \right]_{ll} \le D_l, \text{ for } l \in \Lambda_L.$$

Let $\mathcal{R}_L(\Gamma, D^L|\Sigma_{Y^L})$ denote the set of all $(\Gamma, D^L)$-admissible rate vectors. The sum rate part of the rate distortion region is defined by

$$R_{\mathrm{sum},L}(\Gamma, D^L|\Sigma_{Y^L}) \triangleq \min_{\substack{(R_1, R_2, \cdots, R_L) \\ \in \mathcal{R}_L(\Gamma, D^L|\Sigma_{Y^L})}} \left\{ \sum_{l=1}^L R_l \right\}.$$

*Problem 4. Sum Distortion Criterion:* For given $L \times L$ invertible matrix $\Gamma$ and $D > 0$, the rate vector $(R_1, R_2, \cdots, R_L)$ is $(\Gamma, D)$-*admissible* if there exists a sequence $\{(\varphi_1^{(n)}, \varphi_2^{(n)}, \cdots, \varphi_L^{(n)}, \phi^{(n)})\}_{n=1}^\infty$ such that

$$\limsup_{n \to \infty} R^{(n)} \le R_l, \text{ for } l \in \Lambda_L,$$
$$\limsup_{n \to \infty} \mathrm{tr} \left[ \Gamma \left( \tfrac{1}{n} \Sigma_{\boldsymbol{Y}^L - \hat{\boldsymbol{Y}}^L} \right) {}^{\mathrm{t}}\Gamma \right] \le D.$$

Let $\mathcal{R}_L(\Gamma, D|\Sigma_{Y^L})$ denote the set of all admissible rate vectors. The sum rate part of the rate distortion region is defined by

$$R_{\mathrm{sum},L}(\Gamma, D|\Sigma_{Y^L}) \triangleq \min_{\substack{(R_1, R_2, \cdots, R_L) \\ \in \mathcal{R}_L(\Gamma, D|\Sigma_{Y^L})}} \left\{ \sum_{l=1}^L R_l \right\}.$$

Relations between $\mathcal{R}_L(\Sigma_d|\Sigma_{Y^L})$, $\mathcal{R}_L(\Gamma, D^L|\Sigma_{Y^L})$, and $\mathcal{R}_L(\Gamma, D|\Sigma_{Y^L})$ are as follows.

$$\mathcal{R}_L(\Gamma, D^L|\Sigma_{Y^L}) = \bigcup_{\Gamma \Sigma_d {}^{\mathrm{t}}\Gamma \in \mathcal{S}_L(D^L)} \mathcal{R}_L(\Sigma_d|\Sigma_{Y^L}), \tag{18}$$

$$\mathcal{R}_L(\Gamma, D|\Sigma_{Y^L}) = \bigcup_{\mathrm{tr}[\Gamma \Sigma_d {}^{\mathrm{t}}\Gamma] \le D} \mathcal{R}_L(\Sigma_d|\Sigma_{Y^L}). \tag{19}$$

Furthermore, we have

$$\mathcal{R}_L(\Gamma, D|\Sigma_{Y^L}) = \bigcup_{\sum_{l=1}^L D_l \le D} \mathcal{R}_L(\Gamma, D^L|\Sigma_{Y^L}). \tag{20}$$

We first present inner bounds of $\mathcal{R}_L(\Sigma_d\,|\Sigma_{Y^L})$, $\mathcal{R}_L(\Gamma, D^L\,|\Sigma_{Y^L})$, and $\mathcal{R}_L(\Gamma,\ D|\Sigma_{Y^L})$. Those inner bounds can be obtained by a standard technique of multiterminal source coding. Define

$$\tilde{\mathcal{G}}(\Sigma_d) \stackrel{\triangle}{=} \{U^L : U^L \text{ is a Gaussian random vector that satisfies } U_S \to Y_S \to X^L \to Y_{S^c} \to U_{S^c},\ U^L \to Y^L \to X^L \text{ for any } S \subset \Lambda_L \text{ and } \Sigma_{Y^L - \phi(U^L)} \preceq \Sigma_d \text{ for some linear mapping } \phi : \mathbb{R}^L \to \mathbb{R}^L. \}$$

and set

$$\hat{\mathcal{R}}_L^{(\rm in)}(\Sigma_d|\Sigma_{Y^L}) \stackrel{\triangle}{=} {\rm conv}\Big\{R^L : \text{There exists a random vector } U^L \in \tilde{\mathcal{G}}(\Sigma_d) \text{ such that } \sum_{i\in S} R_l \geq I(U_S;Y_S|U_{S^c}) \text{ for any } S \subseteq \Lambda_L. \Big\},$$

$$\hat{\mathcal{R}}_L^{(\rm in)}(\Gamma, D^L|\Sigma_{Y^L}) \stackrel{\triangle}{=} {\rm conv}\left\{ \bigcup_{\Gamma\Sigma_d{}^{\rm t}\Gamma \in \mathcal{S}_L(D^L)} \hat{\mathcal{R}}_L^{(\rm in)}(\Sigma_d|\Sigma_{Y^L}) \right\},$$

$$\hat{\mathcal{R}}_L^{(\rm in)}(\Gamma, D|\Sigma_{Y^L}) \stackrel{\triangle}{=} {\rm conv}\left\{ \bigcup_{{\rm tr}[\Gamma\Sigma_d{}^{\rm t}\Gamma] \leq D} \hat{\mathcal{R}}_L^{(\rm in)}(\Sigma_d|\Sigma_{Y^L}) \right\}.$$

Then we have the following result.

*Theorem 7 (Berger [4] and Tung [5]):* For any positive definite $\Sigma_d$, we have

$$\hat{\mathcal{R}}_L^{(\rm in)}(\Sigma_d|\Sigma_{Y^L}) \subseteq \mathcal{R}_L(\Sigma_d|\Sigma_{Y^L}).$$

For any invertible $\Gamma$ and any $D^L > 0$, we have

$$\hat{\mathcal{R}}_L^{(\rm in)}(\Gamma, D^L|\Sigma_{Y^L}) \subseteq \mathcal{R}_L(\Gamma, D^L|\Sigma_{Y^L}).$$

For any invertible $\Gamma$ and any $D > 0$, we have

$$\hat{\mathcal{R}}_L^{(\rm in)}(\Gamma, D|\Sigma_{Y^L}) \subseteq \mathcal{R}_L(\Gamma, D|\Sigma_{Y^L}).$$

The inner bound $\hat{\mathcal{R}}_L^{(\rm in)}(D^L|\Sigma_{Y^L})$ for $\Gamma = I_L$ is well known as the inner bound of Berger [4] and Tung [5]. The above three inner bounds are variants of this inner bound.

Optimality of $\hat{\mathcal{R}}_2^{(\rm in)}(D^2|\Sigma_{Y^2})$ was first studied by Oohama [9]. Let

$$\Sigma_{Y^2} = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}, \quad \rho \in [0,1).$$

For $l = 1, 2$, set

$$\mathcal{R}_{l,2}(D_l|\Sigma_{Y^2}) \stackrel{\triangle}{=} \bigcup_{D_{3-l}>0} \mathcal{R}_2(D^2|\Sigma_{Y^2}).$$

Oohama [9] obtained the following result.

*Theorem 8 (Oohama [9]):* For $l = 1, 2$, we have

$$\mathcal{R}_{l,2}(D_l|\Sigma_{Y^2}) = \mathcal{R}_{l,2}^*(D_l|\Sigma_{Y^2}),$$

where

$$\mathcal{R}_{l,2}^*(D_l|\Sigma_{Y^2}) \stackrel{\triangle}{=} \Big\{(R_1, R_2) : R_l \geq \tfrac{1}{2}\log^+\left[(1-\rho^2)\tfrac{\sigma_l^2}{D_l}\left(1 + \tfrac{\rho^2}{1-\rho^2}\cdot s\right)\right], \ R_{3-l} \geq \tfrac{1}{2}\log\left[\tfrac{1}{s}\right] \text{ for some } 0 < s \leq 1 \Big\}.$$

Since $\mathcal{R}_{l,2}^*(D_l|\Sigma_{Y^2}),\ l = 1,2$ serve as outer bounds of $\mathcal{R}_2(D^2|\Sigma_{Y^2})$, we have

$$\mathcal{R}_2(D^2|\Sigma_{Y^2}) \subseteq \mathcal{R}_{1,2}^*(D_1|\Sigma_{Y^2}) \cap \mathcal{R}_{2,2}^*(D_2|\Sigma_{Y^2}). \tag{21}$$

Wagner *et al.* [11] derived the condition where the outer bound in the right hand side of (21) is tight. To describe their result set

$$\mathcal{D} \stackrel{\triangle}{=} \Big\{(D_1, D_2) : D_1, D_2 > 0, \ \max\left\{\tfrac{D_1}{\sigma_1^2}, \tfrac{D_2}{\sigma_2^2}\right\} \leq \min\left\{1, \rho^2\min\left\{\tfrac{D_1}{\sigma_1^2}, \tfrac{D_2}{\sigma_2^2}\right\} + 1 - \rho^2\right\}\Big\}.$$

Wagner *et al.* [11] showed that if $D^2 \notin \mathcal{D}$, we have

$$\mathcal{R}_2(D^2|\Sigma_{Y^2}) = \mathcal{R}_{1,2}^*(D_1|\Sigma_{Y^2}) \cap \mathcal{R}_{2,2}^*(D_2|\Sigma_{Y^2}).$$

Next we consider the case of $D^2 \in \mathcal{D}$. In this case by an elementary computation we can show that $\hat{\mathcal{R}}_2^{(\rm in)}(D^2|\Sigma_{Y^2})$ has the following form:

$$\hat{\mathcal{R}}_2^{(\rm in)}(D^2|\Sigma_{Y^2}) = \mathcal{R}_{1,2}^*(D_1|\Sigma_{Y^2}) \cap \mathcal{R}_{2,2}^*(D_2|\Sigma_{Y^2}) \cap \mathcal{R}_{3,2}^*(D^2|\Sigma_{Y^2}),$$

where

$$\mathcal{R}_{3,2}^*(D^2|\Sigma_{Y^2}) \stackrel{\triangle}{=} \Big\{(R_1, R_2) : R_1 + R_2 \geq R_{{\rm sum},2}^{(\rm u)}(D^2|\Sigma_{Y^2})\Big\},$$

$$R_{{\rm sum},2}^{(\rm u)}(D^2|\Sigma_{Y^2}) \stackrel{\triangle}{=} \min_{(R_1,R_2)\in\hat{\mathcal{R}}_2^{(\rm in)}(D^2|\Sigma_{Y^2})} \{R_1 + R_2\} = \tfrac{1}{2}\log\left[\tfrac{1-\rho^2}{2}\cdot\left\{\tfrac{\sigma_1^2\sigma_2^2}{D_1D_2} + \sqrt{\left(\tfrac{\sigma_1^2\sigma_2^2}{D_1D_2}\right)^2 + \tfrac{4\rho^2}{(1-\rho^2)^2}}\right\}\right].$$

The boundary of $\hat{\mathcal{R}}_2^{(\rm in)}(D^2|\Sigma_{Y^2})$ consists of one straight line segment defined by the boundary of $\mathcal{R}_{3,2}^*(D^2|\Sigma_{Y^2})$ and two curved portions defined by the boundaries of $\mathcal{R}_{1,2}^*(D_1|\Sigma_{Y^2})$ and $\mathcal{R}_{2,2}^*(D_2|\Sigma_{Y^2})$. Accordingly, the inner bound established by Berger [4] and Tung [5] partially coincides with $\mathcal{R}_2(D^2|\Sigma_{Y^2})$ at two curved portions of its boundary.

Wagner *et al.* [11] have completed the proof of the optimality of $\hat{\mathcal{R}}_2^{(\rm in)}(D^2|\Sigma_{Y^2})$ by determining the sum rate part $R_{{\rm sum},2}(D^2|\Sigma_{Y^2})$. Their result is as follows.

*Theorem 9 (Wagner et al. [11]):* For any $D^2 \in \mathcal{D}$, we have

$$R_{{\rm sum},2}(D^2|\Sigma_{Y^2}) = R_{{\rm sum},2}^{(\rm u)}(D^2|\Sigma_{Y^2}) = \tfrac{1}{2}\log\left[\tfrac{1-\rho^2}{2}\cdot\left\{\tfrac{\sigma_1^2\sigma_2^2}{D_1D_2} + \sqrt{\left(\tfrac{\sigma_1^2\sigma_2^2}{D_1D_2}\right)^2 + \tfrac{4\rho^2}{(1-\rho^2)^2}}\right\}\right].$$

According to Wagner *et al.* [11], the results of Oohama [16], [17] play an essential role in deriving their result. Their method for the proof depends heavily on the specific property of $L = 2$. It is hard to generalize it to the case of $L \geq 3$. Recently, Wang *et al.* [12] have given an alternative proof of Theorem 9. Their method of the proof is quite different from the previous method employed by Oohama [16], [17] and Wagner *et al.* [11] and also has a great advantage that it is also applicable to the characterization of $R_{\mathrm{sum},L}(D^L|\Sigma_{Y^2})$ for $L \geq 3$. Their result and its relation to our result in the present paper will be discussed in the next subsection.

### *B. New Outer Bounds of Positive Semidefinite Programming*

In this subsection we state our results on the characterizations of $\mathcal{R}_L(\Sigma_d|\Sigma_{Y^L})$, $\mathcal{R}_L(\Gamma, D^L|\Sigma_{Y^L})$, and $\mathcal{R}_L(\Gamma, D|\Sigma_{Y^L})$. Before describing those results we derive an important relation between remote source coding problem and multiterminal source coding problem. We first observe that by an elementary computation we have

$$X^L = \tilde{A}Y^L + \tilde{N}^L\,, \tag{22}$$

where $\tilde{A} = (\Sigma_{X^L}^{-1} + \Sigma_{N^L}^{-1})^{-1}\Sigma_{N^L}^{-1}$ and $\tilde{N}^L$ is a zero mean Gaussian random vector with covariance matrix $\Sigma_{\tilde{N}^L} = (\Sigma_{X^L}^{-1} + \Sigma_{N^L}^{-1})^{-1}$. The random vector $\tilde{N}^L$ is independent of $Y^L$. Set

$$\begin{aligned} B &\stackrel{\triangle}{=} \tilde{A}^{-1}\Sigma_{\tilde{N}^L}{}^{\mathrm{t}}\tilde{A}^{-1} = \Sigma_{N^L} + \Sigma_{N^L}\Sigma_{X^L}^{-1}\Sigma_{N^L}\,, \\ b^L &\stackrel{\triangle}{=} {}^{\mathrm{t}}([B]_{11}, [B]_{22}, \cdots, [B]_{LL})\,, \\ \tilde{B} &\stackrel{\triangle}{=} \Gamma B^{\mathrm{t}}\Gamma\,, \\ \tilde{b}^L &\stackrel{\triangle}{=} {}^{\mathrm{t}}([\tilde{B}]_{11}, [\tilde{B}]_{22}, \cdots, [\tilde{B}]_{LL}). \end{aligned}$$

From (22), we have the following relation between $\boldsymbol{X}^L$ and $\boldsymbol{Y}^L$:

$$\boldsymbol{X}^L = \tilde{A}\boldsymbol{Y}^L + \tilde{\boldsymbol{N}}^L, \tag{23}$$

where $\tilde{\boldsymbol{N}}^L$ is a sequence of $n$ independent copies of $\tilde{N}^L$ and is independent of $\boldsymbol{Y}^L$. Now, we fix $\{(\varphi_1^{(n)}, \varphi_2^{(n)}, \cdots, \varphi_L^{(n)}, \psi^{(n)})\}_{n=1}^{\infty}$, arbitrarily. For each $n = 1, 2, \cdots$, the estimation $\hat{\boldsymbol{X}}^L$ of $\boldsymbol{X}^L$ is given by

$$\hat{\boldsymbol{X}}^L = \begin{bmatrix} \psi_1^{(n)}(\varphi^{(n)}(\boldsymbol{Y}^L)) \\ \psi_2^{(n)}(\varphi^{(n)}(\boldsymbol{Y}^L)) \\ \vdots \\ \psi_L^{(n)}(\varphi^{(n)}(\boldsymbol{Y}^L)) \end{bmatrix}.$$

Using this estimation, we construct an estimation $\hat{\boldsymbol{Y}}^L$ of $\boldsymbol{Y}^L$ by $\hat{\boldsymbol{Y}}^L = \tilde{A}^{-1}\hat{\boldsymbol{X}}^L$, which is equivalent to

$$\hat{\boldsymbol{X}}^L = \tilde{A}\hat{\boldsymbol{Y}}^L. \tag{24}$$

From (23) and (24), we have

$$\boldsymbol{X}^L - \hat{\boldsymbol{X}}^L = \tilde{A}(\boldsymbol{Y}^L - \hat{\boldsymbol{Y}}^L) + \tilde{\boldsymbol{N}}^L. \tag{25}$$

Since $\hat{\boldsymbol{Y}}^L$ is a function of $\boldsymbol{Y}^L$, $\hat{\boldsymbol{Y}}^L - \boldsymbol{Y}^L$ is independent of $\tilde{\boldsymbol{N}}^L$. Based on (25), we compute $\frac{1}{n}\Sigma_{\boldsymbol{X}^L - \hat{\boldsymbol{X}}^L}$ to obtain

$$\frac{1}{n}\Sigma_{\boldsymbol{X}^L-\hat{\boldsymbol{X}}^L} = \tilde{A}\left(\frac{1}{n}\Sigma_{\boldsymbol{Y}^L-\hat{\boldsymbol{Y}}^L}\right){}^{\mathrm{t}}\tilde{A} + \Sigma_{\tilde{N}^L}. \tag{26}$$

From (26), we have

$$\begin{aligned} \frac{1}{n}\Sigma_{\boldsymbol{Y}^L-\hat{\boldsymbol{Y}}^L} &= \tilde{A}^{-1}\left(\frac{1}{n}\Sigma_{\boldsymbol{X}^L-\hat{\boldsymbol{X}}^L} - \Sigma_{\tilde{N}^L}\right){}^{\mathrm{t}}\tilde{A}^{-1} \\ &= \tilde{A}^{-1}\left(\frac{1}{n}\Sigma_{\boldsymbol{X}^L-\hat{\boldsymbol{X}}^L}\right){}^{\mathrm{t}}\tilde{A}^{-1} - B. \end{aligned} \tag{27}$$

Conversely, we fix $\{(\varphi_1^{(n)}, \varphi_2^{(n)}, \cdots, \varphi_L^{(n)}, \phi^{(n)})\}_{n=1}^{\infty}$, arbitrarily. For each $n = 1, 2, \cdots$, using the estimation $\hat{\boldsymbol{Y}}^L$ of $\boldsymbol{Y}^L$ given by

$$\hat{\boldsymbol{Y}}^L = \begin{bmatrix} \phi_1^{(n)}(\varphi^{(n)}(\boldsymbol{Y}^L)) \\ \phi_2^{(n)}(\varphi^{(n)}(\boldsymbol{Y}^L)) \\ \vdots \\ \phi_L^{(n)}(\varphi^{(n)}(\boldsymbol{Y}^L)) \end{bmatrix},$$

we construct an estimation $\hat{\boldsymbol{X}}^L$ of $\boldsymbol{X}^L$ by (24). Then using (23) and (24), we obtain (25). Hence we have the relation (26).

The following proposition provides an important strong connection between remote source coding problem and multiterminal source coding problem.

*Proposition 1:* For any positive definite $\Sigma_d$, we have

$$\mathcal{R}_L(\Sigma_d|\Sigma_{Y^L}) = \mathcal{R}_L(\tilde{A}(\Sigma_d + B)^{\mathrm{t}}\tilde{A}|\Sigma_{X^LY^L}).$$

For any invertible $\Gamma$ and any $D^L > 0$, we have

$$\mathcal{R}_L(\Gamma, D^L|\Sigma_{Y^L}) = \mathcal{R}_L(\Gamma\tilde{A}^{-1}, D^L + \tilde{b}^L|\Sigma_{X^LY^L}).$$

For any invertible $\Gamma$ and any $D > 0$, we have

$$\mathcal{R}_L(\Gamma, D|\Sigma_{Y^L}) = \mathcal{R}_L(\Gamma\tilde{A}^{-1}, D + \mathrm{tr}[\tilde{B}]|\Sigma_{X^LY^L}).$$

*Proof:* Suppose that $R^L \in \mathcal{R}_L(\tilde{A}(\Sigma_d + B)^{\mathrm{t}}\tilde{A}|\Sigma_{X^LY^L})$. Then there exists $\{(\varphi_1^{(n)}, \varphi_2^{(n)}, \cdots, \varphi_L^{(n)}, \psi^{(n)})\}_{n=1}^{\infty}$ such that

$$\begin{aligned} &\limsup_{n\to\infty} R^{(n)} \leq R_l, \text{ for } l \in \Lambda_L\,, \\ &\limsup_{n\to\infty} \frac{1}{n}\Sigma_{\boldsymbol{X}^L-\hat{\boldsymbol{X}}^L} \preceq \tilde{A}(\Sigma_d + B)^{\mathrm{t}}\tilde{A}. \end{aligned}$$

Using $\hat{\boldsymbol{X}}^L$, we construct an estimation $\hat{\boldsymbol{Y}}^L$ of $\boldsymbol{Y}^L$ by $\hat{\boldsymbol{Y}}^L = \tilde{A}^{-1}\hat{\boldsymbol{X}}^L$. Then from (27), we have

$$\begin{aligned} &\limsup_{n\to\infty} \frac{1}{n}\Sigma_{\boldsymbol{Y}^L-\hat{\boldsymbol{Y}}^L} \\ &= \limsup_{n\to\infty} \tilde{A}^{-1}\left(\frac{1}{n}\Sigma_{\boldsymbol{X}^L-\hat{\boldsymbol{X}}^L}\right){}^{\mathrm{t}}\tilde{A}^{-1} - B \\ &\preceq \tilde{A}^{-1}\tilde{A}(\Sigma_d + B)^{\mathrm{t}}\tilde{A}^{\mathrm{t}}\tilde{A}^{-1} - B = \Sigma_d\,, \end{aligned}$$

which implies that $R^L \in \mathcal{R}_L(\tilde{A}(\Sigma_d + B)^{\mathrm{t}}\tilde{A}|\Sigma_{X^LY^L})$. Thus

$$\mathcal{R}_L(\Sigma_d|\Sigma_{Y^L}) \supseteq \mathcal{R}_L(\tilde{A}(\Sigma_d + B)^{\mathrm{t}}\tilde{A}|\Sigma_{X^LY^L})$$

is proved. Next we prove the reverse inclusion. Suppose that $R^L \in \mathcal{R}_L(\Sigma_d|\Sigma_{Y^L})$. Then there exists $\{(\varphi_1^{(n)}, \varphi_2^{(n)}, \cdots, \varphi_L^{(n)}, \phi^{(n)})\}_{n=1}^{\infty}$ such that

$$\begin{aligned} &\limsup_{n\to\infty} R^{(n)} \leq R_l, \text{ for } l \in \Lambda_L\,, \\ &\limsup_{n\to\infty} \frac{1}{n}\Sigma_{\boldsymbol{Y}^L-\hat{\boldsymbol{Y}}^L} \preceq \Sigma_d. \end{aligned}$$

Using $\hat{\boldsymbol{Y}}^L$, we construct an estimation $\hat{\boldsymbol{X}}^L$ of $\boldsymbol{X}^L$ by $\hat{\boldsymbol{X}}^L = \tilde{A}\hat{\boldsymbol{Y}}^L$. Then from (26), we have

$$\begin{aligned}
&\limsup_{n\to\infty} \tfrac{1}{n}\Sigma_{\boldsymbol{X}^L-\hat{\boldsymbol{X}}^L} \\
&= \limsup_{n\to\infty} \tilde{A}\left(\tfrac{1}{n}\Sigma_{\boldsymbol{Y}^L-\hat{\boldsymbol{Y}}^L}\right){}^{\rm t}\tilde{A} + \Sigma_{\tilde{N}^L} \\
&\preceq \tilde{A}\Sigma_d{}^{\rm t}\tilde{A}^{\rm t} + \Sigma_{\tilde{N}^L} = \tilde{A}(\Sigma_d + B)^{\rm t}\tilde{A}^{\rm t}\,,
\end{aligned}$$

which implies that $R^L \in \mathcal{R}_L(\tilde{A}(\Sigma_d+B)^{\rm t}\tilde{A}\,|\Sigma_{X^LY^L})$. Thus,

$$\mathcal{R}_L(\Sigma_d|\Sigma_{Y^L}) \subseteq \mathcal{R}_L(\tilde{A}(\Sigma_d+B)^{\rm t}\tilde{A}|\Sigma_{X^LY^L})$$

is proved. Next we prove the second equality. We have the following chain of equalities:

$$\begin{aligned}
&\mathcal{R}_L(\Gamma, D^L|\Sigma_{Y^L}) = \bigcup_{\Gamma\Sigma_d{}^{\rm t}\Gamma\in\mathcal{S}_L(D^L)} \mathcal{R}_L(\Sigma_d|\Sigma_{Y^L}) \\
&= \bigcup_{\Gamma\Sigma_d{}^{\rm t}\Gamma\in\mathcal{S}_L(D^L)} \mathcal{R}_L(\Gamma\tilde{A}(\Sigma_d+B)^{\rm t}\tilde{A}|\Sigma_{X^LY^L}) \\
&= \bigcup_{\substack{\Gamma\tilde{A}^{-1}\tilde{A}(\Sigma_d+B)^{\rm t}\tilde{A}^{\rm t}\tilde{A}^{-1\rm t}\Gamma \\ -\Gamma B^{\rm t}\Gamma\in\mathcal{S}_L(D^L)}} \mathcal{R}_L(\tilde{A}(\Sigma_d+B)^{\rm t}\tilde{A}|\Sigma_{X^LY^L}) \\
&= \bigcup_{\substack{\Gamma\tilde{A}^{-1}\tilde{A}(\Sigma_d+B)^{\rm t}\tilde{A}^{\rm t}(\Gamma\tilde{A}^{-1}) \\ \in\mathcal{S}_L(D^L+\tilde{b}^L)}} \mathcal{R}_L(\tilde{A}(\Sigma_d+B)^{\rm t}\tilde{A}|\Sigma_{X^LY^L}) \\
&= \bigcup_{\substack{\hat{\Sigma}_d=\tilde{A}(\Sigma_d+B)^{\rm t}\tilde{A}\succ\Sigma_{X^L|Y^L}, \\ \Gamma\tilde{A}^{-1}\hat{\Sigma}_d{}^{\rm t}(\Gamma\tilde{A}^{-1})\in\mathcal{S}_L(D^L+\tilde{b}^L)}} \mathcal{R}_L(\hat{\Sigma}_d|\Sigma_{X^LY^L}) \\
&= \mathcal{R}_L(\Gamma\tilde{A}^{-1}, D^L+\tilde{b}^L|\Sigma_{X^LY^L}).
\end{aligned}$$

Thus the second equality is proved. Finally we prove the third equality. We have the following chain of equalities:

$$\begin{aligned}
&\mathcal{R}_L(\Gamma, D|\Sigma_{Y^L}) = \bigcup_{{\rm tr}[\Gamma\Sigma_d{}^{\rm t}\Gamma]\le D} \mathcal{R}_L(\Sigma_d|\Sigma_{Y^L}) \\
&= \bigcup_{{\rm tr}[\Gamma\Sigma_d{}^{\rm t}\Gamma]\le D} \mathcal{R}_L(\Gamma\tilde{A}(\Sigma_d+B)^{\rm t}\tilde{A}|\Sigma_{X^LY^L}) \\
&= \bigcup_{\substack{{\rm tr}[\Gamma\tilde{A}^{-1}\tilde{A}(\Sigma_d+B)^{\rm t}\tilde{A}^{\rm t}\tilde{A}^{-1\rm t}\Gamma] \\ -{\rm tr}[\Gamma B^{\rm t}\Gamma]\le D}} \mathcal{R}_L(\tilde{A}(\Sigma_d+B)^{\rm t}\tilde{A}|\Sigma_{X^LY^L}) \\
&= \bigcup_{\substack{{\rm tr}[\Gamma\tilde{A}^{-1}\tilde{A}(\Sigma_d+B)^{\rm t}\tilde{A}^{\rm t}(\Gamma\tilde{A}^{-1})] \\ \le D+{\rm tr}[\tilde{B}]}} \mathcal{R}_L(\tilde{A}(\Sigma_d+B)^{\rm t}\tilde{A}|\Sigma_{X^LY^L}) \\
&= \bigcup_{\substack{\hat{\Sigma}_d=\tilde{A}(\Sigma_d+B)^{\rm t}\tilde{A}\succ\Sigma_{X^L|Y^L}, \\ {\rm tr}[\Gamma\tilde{A}^{-1}\hat{\Sigma}_d{}^{\rm t}(\Gamma\tilde{A}^{-1})]\le D+{\rm tr}[\tilde{B}]}} \mathcal{R}_L(\hat{\Sigma}_d|\Sigma_{X^LY^L}) \\
&= \mathcal{R}_L(\Gamma\tilde{A}^{-1}, D+{\rm tr}[\tilde{B}]|\Sigma_{X^LY^L}).
\end{aligned}$$

Thus the third equality is proved. ∎

Proposition 1 implies that all results on the rate distortion regions for the remote source coding problems can be converted into those on the multiterminal source coding problems. In the following we derive inner and outer bounds of $\mathcal{R}_L(\Sigma_d|\Sigma_{Y^L})$, $\mathcal{R}_L(\Gamma, D^L|\Sigma_{Y^L})$, and $\mathcal{R}_L(\Gamma, D|\Sigma_{Y^L})$ using Proposition 1. We first derive inner and outer bounds of $\mathcal{R}_L(\Sigma_d|\Sigma_{Y^L})$. For each $l\in\Lambda_L$ and for $r_l \ge 0$, let $V_l(r_l)$, $l\in\Lambda_L$ be a Gaussian random variable with mean 0 and variance $\sigma_{N_l}^2/({\rm e}^{2r_l}-1)$. We assume that $V_l(r_l), l\in\Lambda_L$ are independent. When $r_l=0$, we formally think that the inverse value $\sigma_{V_l(0)}^{-1}$ of $V_l(0)$ is zero. Let $\Sigma_{V^L(r^L)}$ be a covariance matrix of the random vector $V^L(r^L)$. When $r_S=\boldsymbol{0}$, we formally define

$$\Sigma_{V_{S^{\rm c}}(r_{S^{\rm c}})}^{-1} \triangleq \Sigma_{V^L(r^L)}^{-1}\Big|_{r_S=\boldsymbol{0}}\,.$$

Fix nonnegative vector $r^L$. For $\theta>0$ and for $S\subseteq\Lambda_L$, define

$$\underline{\tilde{J}}_S(\theta, r_S|r_{S^{\rm c}}) \triangleq \frac{1}{2}\log^+\left[\frac{|\Sigma_{Y^L}+B|\prod_{l=1}^{L}{\rm e}^{2r_i}}{\theta|\Sigma_{Y^L}|\left|\Sigma_{Y^L}^{-1}+\Sigma_{V_{S^{\rm c}}(r_{S^{\rm c}})}^{-1}\right|}\right],$$

$$\tilde{J}_S(r_S|r_{S^{\rm c}}) \triangleq \frac{1}{2}\log\frac{\left|\Sigma_{Y^L}^{-1}+\Sigma_{V^L(r^L)}^{-1}\right|}{\left|\Sigma_{Y^L}^{-1}+\Sigma_{V_{S^{\rm c}}(r_{S^{\rm c}})}^{-1}\right|}.$$

Set

$$\tilde{\mathcal{A}}_L(\Sigma_d) \triangleq \left\{r^L\ge 0: \left[\Sigma_{Y^L}^{-1}+\Sigma_{V^L(r^L)}^{-1}\right]^{-1}\preceq\Sigma_d\right\}.$$

Define four regions by

$$\begin{aligned}
\mathcal{R}_L^{(\rm out)}(\theta, r^L|\Sigma_{Y^L}) &\triangleq \Big\{R^L : \sum_{l\in S}R_l \ge \underline{\tilde{J}}_S(\theta, r_S|r_{S^{\rm c}}) \\
&\qquad \text{for any } S\subseteq\Lambda_L.\Big\}\,, \\
\mathcal{R}_L^{(\rm out)}(\Sigma_d|\Sigma_{Y^L}) &\triangleq \bigcup_{r^L\in\tilde{\mathcal{A}}_L(\Sigma_d)} \mathcal{R}_L^{(\rm out)}(|\Sigma_d+B|, r^L|\Sigma_{Y^L})\,, \\
\mathcal{R}_L^{(\rm in)}(r^L|\Sigma_{Y^L}) &\triangleq \Big\{R^L : \sum_{l\in S}R_l \ge J_S(r_S|r_{S^{\rm c}}) \\
&\qquad \text{for any } S\subseteq\Lambda_L.\Big\}\,, \\
\mathcal{R}_L^{(\rm in)}(\Sigma_d|\Sigma_{Y^L}) &\triangleq {\rm conv}\left\{\bigcup_{r^L\in\tilde{\mathcal{A}}_L(\Sigma_d)} \mathcal{R}_L^{(\rm in)}(r^L|\Sigma_{Y^L})\right\}.
\end{aligned}$$

The functions and sets defined above have properties shown in the following.

*Property 6:*

a) For any positive definite $\Sigma_d$, $\tilde{\mathcal{G}}(\Sigma_d) = \mathcal{G}(\tilde{A}(\Sigma_d+B)^{\rm t}\tilde{A})$.

b) For any positive definite $\Sigma_d$, we have
$$\hat{\mathcal{R}}_L^{(\rm in)}(\Sigma_d|\Sigma_{Y^L}) = \hat{\mathcal{R}}_L^{(\rm in)}(\tilde{A}(\Sigma_d+B)^{\rm t}\tilde{A}|\Sigma_{X^LY^L}).$$

c) For any positive definite $\Sigma_d$ and any $S\subseteq\Lambda_L$, we have
$$\begin{aligned}
&\underline{\tilde{J}}_S(|\Sigma_d+B|, r_S|r_{S^{\rm c}}) = \underline{J}_S(|\tilde{A}(\Sigma_d+B)^{\rm t}\tilde{A}|, r_S|r_{S^{\rm c}}), \\
&\tilde{J}_S(r_S|r_{S^{\rm c}}) = J_S(r_S|r_{S^{\rm c}}).
\end{aligned}$$

d) For any positive definite $\Sigma_d$, $\tilde{\mathcal{A}}_L(\Sigma_d) = \mathcal{A}_L(\tilde{A}(\Sigma_d+B)^{\rm t}\tilde{A})$.

e) For any positive definite $\Sigma_d$, we have
$$\begin{aligned}
&\mathcal{R}_L^{(\rm out)}(\Sigma_d|\Sigma_{Y^L}) = \mathcal{R}_L^{(\rm out)}(\tilde{A}(\Sigma_d+B)^{\rm t}\tilde{A}|\Sigma_{X^LY^L})\,, \\
&\mathcal{R}_L^{(\rm in)}(\Sigma_d|\Sigma_{Y^L}) = \mathcal{R}_L^{(\rm in)}(\tilde{A}(\Sigma_d+B)^{\rm t}\tilde{A}|\Sigma_{X^LY^L}).
\end{aligned}$$

From Theorem 3, Proposition 1 and Property 6, we have the following.

*Theorem 10:* For any positive definite $\Sigma_d$, we have

$$\mathcal{R}_L^{(\rm in)}(\Sigma_d|\Sigma_{Y^L}) = \hat{\mathcal{R}}_L^{(\rm in)}(\Sigma_d|\Sigma_{Y^L}) \subseteq \mathcal{R}_L(\Sigma_d|\Sigma_{Y^L}) \subseteq \mathcal{R}_L^{(\rm out)}(\Sigma_d|\Sigma_{Y^L}).$$

Next, we derive inner and outer bounds of $\mathcal{R}_L(\Gamma,D^K|\Sigma_{Y^L})$ and $\mathcal{R}_L(\Gamma,D|\Sigma_{Y^L})$. Set

$$\tilde{\mathcal{A}}_L(r^L) \triangleq \{\Sigma_d : \Sigma_d \succeq (\Sigma_{Y^L}^{-1} + \Sigma_{V^L(r^L)}^{-1})^{-1}\},$$

$$\tilde{\theta}(\Gamma, D^L, r^L) \triangleq \max_{\substack{\Sigma_d:\Sigma_d\in\tilde{\mathcal{A}}_L(r^L),\\ \Gamma\Sigma_d{}^{\rm t}\Gamma\in\mathcal{S}_L(D^L)}} |\Sigma_d + B|,$$

$$\tilde{\theta}(\Gamma, D, r^L) \triangleq \max_{\substack{\Sigma_d:\Sigma_d\in\tilde{\mathcal{A}}_L(r^L),\\ {\rm tr}[\Gamma\Sigma_d{}^{\rm t}\Gamma]\le D}} |\Sigma_d + B|.$$

Furthermore, set

$$\tilde{\mathcal{B}}_L(\Gamma, D^L) \triangleq \left\{r^L \ge 0 : \Gamma(\Sigma_{Y^L}^{-1} + \Sigma_{V^L(r^L)}^{-1})^{-1\,{\rm t}}\Gamma \in \mathcal{S}_L(D^L)\right\},$$

$$\tilde{\mathcal{B}}_L(\Gamma, D) \triangleq \left\{r^L \ge 0 : {\rm tr}\left[\Gamma(\Sigma_{Y^L}^{-1} + \Sigma_{V^L(r^L)}^{-1})^{-1\,{\rm t}}\Gamma\right] \le D\right\}.$$

Define four regions by

$$\mathcal{R}_L^{(\rm out)}(\Gamma, D^L|\Sigma_{Y^L}) \triangleq \bigcup_{r^L\in\tilde{\mathcal{B}}_L(\Gamma,D^L)} \mathcal{R}_L^{(\rm out)}(\tilde{\theta}(\Gamma, D^L, r^L), r^L|\Sigma_{Y^L}),$$

$$\mathcal{R}_L^{(\rm in)}(\Gamma, D^L|\Sigma_{Y^L}) \triangleq {\rm conv}\left\{\bigcup_{r^L\in\tilde{\mathcal{B}}_L(\Gamma,D^L)} \mathcal{R}_L^{(\rm in)}(r^L|\Sigma_{Y^L})\right\},$$

$$\mathcal{R}_L^{(\rm out)}(\Gamma, D|\Sigma_{Y^L}) \triangleq \bigcup_{r^L\in\tilde{\mathcal{B}}_L(\Gamma,D)} \mathcal{R}_L^{(\rm out)}(\tilde{\theta}(\Gamma, D, r^L), r^L|\Sigma_{Y^L}),$$

$$\mathcal{R}_L^{(\rm in)}(\Gamma, D|\Sigma_{Y^L}) \triangleq {\rm conv}\left\{\bigcup_{r^L\in\tilde{\mathcal{B}}_L(\Gamma,D)} \mathcal{R}_L^{(\rm in)}(r^L|\Sigma_{Y^L})\right\}.$$

It can easily be verified that the functions and sets defined above have the properties shown in the following.

*Property 7:*

a) For any invertible $\Gamma$ and any $D^L > 0$, we have

$$\hat{\mathcal{R}}_L^{(\rm in)}(\Gamma, D^L|\Sigma_{Y^L}) = \hat{\mathcal{R}}_L^{(\rm in)}(\Gamma\tilde{A}^{-1}, D^L + \tilde{b}^L|\Sigma_{X^LY^L}).$$

For any invertible $\Gamma$ and any $D > 0$, we have

$$\hat{\mathcal{R}}_L^{(\rm in)}(\Gamma, D|\Sigma_{Y^L}) = \hat{\mathcal{R}}_L^{(\rm in)}(\Gamma\tilde{A}^{-1}, D + {\rm tr}[\tilde{B}]|\Sigma_{X^LY^L}).$$

b) For any $r^L \ge 0$, we have

$$\Sigma_d \in \tilde{\mathcal{A}}(r^L) \Leftrightarrow \tilde{A}(\Sigma_d + B)^{\rm t}\tilde{A} \in \mathcal{A}(r^L),$$

$$\tilde{\theta}(\Gamma, D^L, r^L) = \left|\tilde{A}\right|^{-2} \theta(\Gamma\tilde{A}^{-1}, D^L, r^L),$$

$$\tilde{\theta}(\Gamma, D, r^L) = \left|\tilde{A}\right|^{-2} \theta(\Gamma\tilde{A}^{-1}, D, r^L).$$

c) For any invertible $\Gamma$ and any $D^L > 0$, we have

$$\mathcal{R}_L^{(\rm out)}(\Gamma, D^L|\Sigma_{Y^L}) = \mathcal{R}_L^{(\rm out)}(\Gamma\tilde{A}^{-1}, D^L + \tilde{b}^L|\Sigma_{X^LY^L}),$$

$$\mathcal{R}_L^{(\rm in)}(\Gamma, D^L|\Sigma_{Y^L}) = \mathcal{R}_L^{(\rm in)}(\Gamma\tilde{A}^{-1}, D^L + \tilde{b}^L|\Sigma_{X^LY^L}).$$

For any invertible $\Gamma$ and any $D > 0$, we have

$$\mathcal{R}_L^{(\rm out)}(\Gamma, D|\Sigma_{Y^L}) = \mathcal{R}_L^{(\rm out)}(\Gamma\tilde{A}^{-1}, D + {\rm tr}[\tilde{B}]|\Sigma_{X^LY^L}),$$

$$\mathcal{R}_L^{(\rm in)}(\Gamma, D|\Sigma_{Y^L}) = \mathcal{R}_L^{(\rm in)}(\Gamma\tilde{A}^{-1}, D + {\rm tr}[\tilde{B}]|\Sigma_{X^LY^L}).$$

From Corollary 1, Proposition 1 and Property 7, we have the following theorem.

*Theorem 11:* For any invertible $\Gamma$ and any $D > 0$, we have

$$\mathcal{R}_L^{(\rm in)}(\Gamma, D^L|\Sigma_{Y^L}) = \hat{\mathcal{R}}_L^{(\rm in)}(\Gamma, D^L|\Sigma_{Y^L}) \subseteq \mathcal{R}_L(\Gamma, D^L|\Sigma_{Y^L}) \subseteq \mathcal{R}_L^{(\rm out)}(\Gamma, D^L|\Sigma_{Y^L}).$$

For any invertible $\Gamma$ and any $D > 0$, we have

$$\mathcal{R}_L^{(\rm in)}(\Gamma, D|\Sigma_{Y^L}) = \hat{\mathcal{R}}_L^{(\rm in)}(\Gamma, D|\Sigma_{Y^L}) \subseteq \mathcal{R}_L(\Gamma, D|\Sigma_{Y^L}) \subseteq \mathcal{R}_L^{(\rm out)}(\Gamma, D|\Sigma_{Y^L}).$$

The outer bound $\mathcal{R}_L^{(\rm out)}(\Gamma, D^L|\Sigma_{Y^L})$ has a form of positive semidefinite programming. To find a matching condition for inner and outer bounds to match, we must examine a property of the solution to this positive semidefinite programming. On the sum rate part of the rate distortion region in the case of vector distortion criterion we have the following corollary from Theorem 11.

*Corollary 2:* For any $D^L > 0$, we have

$$R_{{\rm sum},L}^{(\rm l)}(D^L|\Sigma_{Y^L}) \le R_{{\rm sum},L}(D^L|\Sigma_{Y^L}) \le R_{{\rm sum},L}^{(\rm u)}(D^L|\Sigma_{Y^L}),$$

where

$$\begin{aligned} R_{{\rm sum},L}^{(\rm u)}(D^L|\Sigma_{Y^L}) &\triangleq \min_{\substack{r^L:(\Sigma_{Y^L}^{-1}+\Sigma_{V^L(r^L)}^{-1})^{-1}\\ \in\mathcal{S}_L(D^L)}} \frac{1}{2}\log|I + \Sigma_{Y^L}\Sigma_{V^L(r^L)}^{-1}| \\ &= \min_{\substack{(r^L,\Sigma_d):\\ \Sigma_d\in\mathcal{S}_L(D^L),\\ \Sigma_d=(\Sigma_{Y^L}^{-1}+\Sigma_{V^L(r^L)}^{-1})^{-1}}} \frac{1}{2}\log\frac{|\Sigma_{Y^L}|}{|\Sigma_d|} \\ &= \min_{\substack{(r^L,\Sigma_d):\\ \Sigma_d\in\mathcal{S}_L(D^L),\\ \Sigma_d=(\Sigma_{Y^L}^{-1}+\Sigma_{V^L(r^L)}^{-1})^{-1}}} \left\{\frac{1}{2}\log\frac{|\Sigma_{Y^L}+B|}{|\Sigma_d+B|} + \sum_{l=1}^{L} r_i\right\}, \end{aligned}$$

$$R_{{\rm sum},L}^{(\rm l)}(D^L|\Sigma_{Y^L}) \triangleq \min_{\substack{(r^L,\Sigma_d):\\ \Sigma_d\in\mathcal{S}_L(D^L),\\ \Sigma_d\succeq(\Sigma_{Y^L}^{-1}+\Sigma_{V^L(r^L)}^{-1})^{-1}}} \left\{\frac{1}{2}\log\frac{|\Sigma_{Y^L}+B|}{|\Sigma_d+B|} + \sum_{l=1}^{L} r_i\right\}.$$

A lower bound of $R_{\mathrm{sum},L}(D^L|\Sigma_{Y^L})$ in a form of positive semidefinite programming was first obtained by Wang *et al.* [12]. Their lower bound denoted by $\tilde{R}^{(\mathrm{l})}_{\mathrm{sum},L}(D^L|\Sigma_{Y^L})$ is as follows. Let $\delta^L \triangleq (\delta_1,\delta_2,\cdots,\delta_L)$ be a positive vector whose components $\delta_l$, $l\in\Lambda_L$ belong to $(0,\sigma^2_{N_l}]$. Let $\mathrm{Diag.}(\delta^L)$ be a diagonal matrix whose $(l,l)$ element is $\delta_l, l\in\Lambda_L$. Then $\tilde{R}^{(\mathrm{l})}_{\mathrm{sum},L}(D^L|\Sigma_{Y^L})$ is given by

$$
\begin{aligned}
&\tilde{R}^{(\mathrm{l})}_{\mathrm{sum},L}(D^L|\Sigma_{Y^L})\\
\triangleq & \min_{\substack{(\delta^L,\Sigma_d):\\ \Sigma_d\in\mathcal{S}_L(D^L),\\ \delta_l\in(0,\sigma^2_{N_l}],l\in\Lambda_L,\\ (\Sigma_d^{-1}+B^{-1})^{-1}\succeq \mathrm{Diag.}(\delta^L)}} \left\{\frac{1}{2}\log\frac{|\Sigma_{Y^L}+B|}{|\Sigma_d+B|}+\sum_{l=1}^{L}\frac{1}{2}\log\frac{\sigma^2_{N_l}}{\delta_l}\right\}.
\end{aligned}
$$

By simple computation we can show that $\tilde{R}^{(\mathrm{l})}_{\mathrm{sum},L}(D^L|\Sigma_{Y^L})$ $= R^{(\mathrm{l})}_{\mathrm{sum},L}(D^L|\Sigma_{Y^L})$. Although the lower bound $\tilde{R}^{(\mathrm{l})}_{\mathrm{sum},L}(D^L|\Sigma_{Y^L})$ of Wang *et al.* [12] is equal to our lower bound $R^{(\mathrm{l})}_{\mathrm{sum},L}(D^L|\Sigma_{Y^L})$, their method to derive $\tilde{R}^{(\mathrm{l})}_{\mathrm{sum},L}(D^L|\Sigma_{Y^L})$ is essentially different from our method. They derived the lower bound by utilizing the semidefinite partial order of the covariance matrices associated with MMSE estimation. Unlike our method, the method of Wang *et al.* is not directly applicable to the characterization of the entire rate distortion region.

When $L=2$, Wang *et al.* [12] solved the positive semidefinite programming describing $\tilde{R}^{(\mathrm{l})}_{\mathrm{sum},2}(D^2|\Sigma_{Y^2})$ to obtain the following result.

*Lemma 5 (Wang et al. [12]):* For any covariance matrix $\Sigma_{Y^2}$, there exist a pair $(\Sigma_{X^2},\Sigma_{N^2})$ of covariance and diagonal covariance matrices such that $\Sigma_{Y^2}=\Sigma_{X^2}+\Sigma_{N^2}$ and

$$\tilde{R}^{(\mathrm{l})}_{\mathrm{sum},2}(D^2|\Sigma_{Y^2}) = R^{(\mathrm{u})}_{\mathrm{sum},2}(D^2|\Sigma_{Y^2}).$$

From Corollary 2 and Lemma 5, we have the following corollary.

*Corollary 3:*

$$
\begin{aligned}
\tilde{R}^{(\mathrm{l})}_{\mathrm{sum},2}(D^2|\Sigma_{Y^2}) &= R^{(\mathrm{l})}_{\mathrm{sum},2}(D^2|\Sigma_{Y^2})\\
= R_{\mathrm{sum},2}(D^2|\Sigma_{Y^2}) &= R^{(\mathrm{u})}_{\mathrm{sum},2}(D^2|\Sigma_{Y^2}).
\end{aligned}
$$

Our method to derive $R^{(\mathrm{l})}_{\mathrm{sum},2}(D^2|\Sigma_{Y^2})\le R_{\mathrm{sum},2}(D^2|\Sigma_{Y^2})$ in Corollary 2 essentially differs from the method of Wang *et al.* [12] to derive $\tilde{R}^{(\mathrm{l})}_{\mathrm{sum},2}(D^2|\Sigma_{Y^2})\le R_{\mathrm{sum},2}(D^2|\Sigma_{Y^2})$. Our method to obtain Corollary 3 is also quite different from that of Wagner *et al.* [11] to prove Theorem 9. Hence, Corollary 3 provides the second alternative proof of Theorem 9.

### C. Matching Condition Analysis

In this subsection, we derive a matching condition for $\mathcal{R}^{(\mathrm{out})}_L(\Gamma,D|\Sigma_{Y^L})$ to coincide with $\mathcal{R}^{(\mathrm{in})}_L(\Gamma,D|\Sigma_{Y^L})$. Using the derived matching condition we derive more explicit matching condition when $\Gamma$ is a positive semidefinite diagonal matrix. Furthermore we apply this result to the analysis of matching condition in the case of vector distortion criterion.

By the third equality of Proposition 1, the determination problem of $\mathcal{R}_L(\Gamma,D|\Sigma_{Y^L})$ can be converted into the determination problem of $\mathcal{R}_L(\Gamma\tilde{A}^{-1},\, D+\mathrm{tr}[\tilde{B}]|\,\Sigma_{X^LY^L})$. Using Theorem 5, we derive a matching condition for $\mathcal{R}^{(\mathrm{in})}_L(\Gamma\tilde{A}^{-1},$ $D+\mathrm{tr}[\tilde{B}]|\Sigma_{X^LY^L})$ to coincide with $\mathcal{R}^{(\mathrm{out})}_L(\Gamma\tilde{A}^{-1},\ D+\mathrm{tr}[\tilde{B}]|\Sigma_{X^LY^L})$. For simplicity of our analysis we use the second simplified matching condition (10) in Theorem 5. Note that

$$
\begin{aligned}
&\left[{}^{\mathrm{t}}(\Gamma\tilde{A}^{-1})^{-1}(\Sigma_{X^L}^{-1}+\Sigma_{N^L}^{-1})(\Gamma\tilde{A}^{-1})^{-1}\right]^{-1}\\
&=\Gamma\tilde{A}^{-1}(\Sigma_{X^L}^{-1}+\Sigma_{N^L}^{-1})^{-1}{}^{\mathrm{t}}(\Gamma\tilde{A}^{-1})=\tilde{B}. \qquad (28)
\end{aligned}
$$

By (28), the second matching condition in Theorem 5, the third equality of Proposition 1, and Property 7 part c), we establish the following.

*Theorem 12:* Let $\mu^*_{\min}$ be the minimum eigenvalue of

$$\tilde{B}=\Gamma\left(\Sigma_{N^L}+\Sigma_{N^L}\Sigma_{X^L}^{-1}\Sigma_{N^L}\right){}^{\mathrm{t}}\Gamma.$$

If we have

$$0<D\le(L+1)\mu^*_{\min}-\mathrm{tr}\left[\Gamma(\Sigma_{N^L}+\Sigma_{N^L}\Sigma_{X^L}^{-1}\Sigma_{N^L}){}^{\mathrm{t}}\Gamma\right],$$

then

$$
\begin{aligned}
\mathcal{R}^{(\mathrm{in})}_L(\Gamma,D|\Sigma_{Y^L}) &= \hat{\mathcal{R}}^{(\mathrm{in})}_L(\Gamma,D|\Sigma_{Y^L})\\
=\mathcal{R}_L(\Gamma,D|\Sigma_{Y^L}) &= \mathcal{R}^{(\mathrm{out})}_L(\Gamma,D|\Sigma_{Y^L}).
\end{aligned}
$$

An important feature of the multiterminal rate distortion problem is that the rate distortion region $\mathcal{R}_L(\Gamma,D|\Sigma_{Y^L})$ remains the same for any choice of covariance matrix $\Sigma_{X^L}$ and diagonal covariance matrix $\Sigma_{N^L}$ satisfying $\Sigma_{Y^L}=\Sigma_{X^L}+\Sigma_{N^L}$. Using this feature and Theorem 12, we find a good pair $(\Sigma_{X^L},\Sigma_{N^L})$ to provide an explicit strong sufficient condition for $\mathcal{R}^{(\mathrm{in})}_L(\Gamma,\,D|\Sigma_{Y^L})$ and $\mathcal{R}^{(\mathrm{out})}_L(\Gamma,\,D|\Sigma_{Y^L})$ to match.

In the following argument we consider the case where $\Gamma$ is the following positive definite diagonal matrix:

$$
\Gamma=\begin{pmatrix}\gamma_1 & & & \mathbf{0}\\ & \gamma_2 & & \\ & & \ddots & \\ \mathbf{0} & & & \gamma_L\end{pmatrix},\quad \gamma_l\in[1,+\infty). \qquad (29)
$$

Set $\gamma^L\triangleq(\gamma_1,\gamma_2,\cdots,\gamma_L)\in[1,+\infty)^L$. We call $\gamma^L$ the weight vector. Since $\Gamma$ is specified by the weight vector $\gamma^L$, we write $\mathcal{R}_L(\Gamma,D|\Sigma_{Y^L})$ as $\mathcal{R}_L(\gamma^L,D|\Sigma_{Y^L})$. Similar notations are adopted for other regions.

We choose $\Sigma_{N^L}$ so that $\Sigma_{N^L}=\delta\Gamma^{-2}$. Set $\tilde{\Sigma}_{X^L}\triangleq\Gamma\Sigma_{X^L}\Gamma$ and $\tilde{\Sigma}_{Y^L}\triangleq\Gamma\Sigma_{Y^L}\Gamma$. Then, we have

$$
\left.\begin{aligned}
\tilde{B}&=\delta I_L+\delta^2\tilde{\Sigma}_{X^L}^{-1},\\
\tilde{\Sigma}_{X^L}&=\tilde{\Sigma}_{Y^L}-\delta I_L.
\end{aligned}\right\} \qquad (30)
$$

Let $\eta_{\min}\triangleq\eta_1\le\eta_2\le\cdots\le\eta_L\triangleq\eta_{\max}$ be the ordered list of $L$ eigenvalues of $\Sigma_{Y^L}$ and let $\tilde{\eta}_{\min}\triangleq\tilde{\eta}_1\le\tilde{\eta}_2\le\cdots\le\tilde{\eta}_L\triangleq\tilde{\eta}_{\max}$ be the ordered list of $L$ eigenvalues of $\tilde{\Sigma}_{Y^L}$. Set $\gamma_{\max}\triangleq\max_{1\le l\le L}\gamma_i$. Since $\eta_{\min}I_L\preceq\Sigma_{Y^L}\preceq\eta_{\max}I_L$, we have

$$\eta_{\min}I_L\preceq\eta_{\min}\Gamma^2\preceq\tilde{\Sigma}_{Y^L}\preceq\eta_{\max}\Gamma^2\preceq\gamma^2_{\max}\eta_{\max}I_L,$$

from which we obtain

$$\eta_{\min}\le\tilde{\eta}_{\min}\le\tilde{\eta}_{\max}\le\gamma^2_{\max}\eta_{\max}. \qquad (31)$$

We choose $\delta$ so that $0<\delta<\tilde{\eta}_{\min}$. Then, by (30), we have

$$\left.\begin{aligned}\mu^*_{\min} &= \delta+\frac{\delta^2}{\tilde{\eta}_{\max}-\delta},\\ \mathrm{tr}[\tilde{B}] &= \mathrm{tr}\left[\delta I_L+\delta^2\tilde{\Sigma}^{-1}_{X^L}\right]=L\delta+\sum_{l=1}^{L}\frac{\delta^2}{\tilde{\eta}_l-\delta}.\end{aligned}\right\} \quad (32)$$

From (32), we have

$$\begin{aligned}&(L+1)\mu^*_{\min}-\mathrm{tr}[\tilde{B}]=\delta+\frac{(L+1)\delta^2}{\tilde{\eta}_{\max}-\delta}-\sum_{l=1}^{L}\frac{\delta^2}{\tilde{\eta}_l-\delta}\\ &=\delta+\frac{L\delta^2}{\tilde{\eta}_{\max}-\delta}-\sum_{l=1}^{L-1}\frac{\delta^2}{\tilde{\eta}_l-\delta}\\ &\geq\delta+L\frac{\delta^2}{\tilde{\eta}_{\max}-\delta}-(L-1)\frac{\delta^2}{\tilde{\eta}_{\min}-\delta}\\ &=L\tilde{\eta}_{\max}\left(\frac{\tilde{\eta}_{\max}}{\tilde{\eta}_{\max}-\delta}-1\right)\\ &\quad-(L-1)\tilde{\eta}_{\min}\left(\frac{\tilde{\eta}_{\min}}{\tilde{\eta}_{\min}-\delta}-1\right).\end{aligned} \quad (33)$$

By an elementary computation we can show that the right member of (33) takes the maximum value

$$\begin{aligned}&(\sqrt{L}-\sqrt{L-1})^2\cdot\frac{\tilde{\eta}_{\max}\tilde{\eta}_{\min}}{\tilde{\eta}_{\max}-\tilde{\eta}_{\min}}\\ &=\frac{1}{(\sqrt{L}+\sqrt{L-1})^2}\cdot\frac{\tilde{\eta}_{\max}\tilde{\eta}_{\min}}{\tilde{\eta}_{\max}-\tilde{\eta}_{\min}}\end{aligned}$$

at

$$\delta=\frac{(\sqrt{L}-\sqrt{L-1})\tilde{\eta}_{\max}\tilde{\eta}_{\min}}{\sqrt{L}\tilde{\eta}_{\max}-\sqrt{L-1}\tilde{\eta}_{\min}}.$$

Furthermore, taking (31) into account, we obtain

$$\begin{aligned}\frac{\tilde{\eta}_{\max}\tilde{\eta}_{\min}}{\tilde{\eta}_{\max}-\tilde{\eta}_{\min}}&=\left[\tilde{\eta}^{-1}_{\min}-\tilde{\eta}^{-1}_{\max}\right]^{-1}\geq\left[\eta^{-1}_{\min}-\gamma^{-2}_{\max}\eta^{-1}_{\max}\right]^{-1}\\ &=\frac{\eta_{\max}\eta_{\min}}{\eta_{\max}-\gamma^{-2}_{\max}\eta_{\min}}.\end{aligned}$$

Hence if

$$0<D\leq\frac{1}{(\sqrt{L}+\sqrt{L-1})^2}\cdot\frac{\eta_{\max}\eta_{\min}}{\eta_{\max}-\gamma^{-2}_{\max}\eta_{\min}},$$

then the matching condition holds. Summarizing the above argument, we obtain the following corollary from Theorem 12.

*Corollary 4:* Let $\gamma^L\in[1,+\infty)^L$ be a weight vector and let $\gamma_{\max}=\max_{1\leq l\leq L}\gamma_l$. If

$$0<D\leq\frac{1}{(\sqrt{L}+\sqrt{L-1})^2}\cdot\frac{\eta_{\max}\eta_{\min}}{\eta_{\max}-\gamma^{-2}_{\max}\eta_{\min}},$$

then we have

$$\begin{aligned}&\mathcal{R}^{(\mathrm{in})}_L(\gamma^L,D|\Sigma_{Y^L})=\hat{\mathcal{R}}^{(\mathrm{in})}_L(\gamma^L,D|\Sigma_{Y^L})\\ &=\mathcal{R}_L(\gamma^L,D|\Sigma_{Y^L})=\mathcal{R}^{(\mathrm{out})}_L(\gamma^L,D|\Sigma_{Y^L}).\end{aligned} \quad (34)$$

In particular, if

$$0<D\leq\frac{1}{(\sqrt{L}+\sqrt{L-1})^2}\cdot\eta_{\min},$$

then we have (34) for any weight vector $\gamma^L\in[1,\infty)^L$. If $\gamma_{\max}=1$ and

$$0<D\leq\frac{1}{(\sqrt{L}+\sqrt{L-1})^2}\cdot\frac{\eta_{\max}\eta_{\min}}{\eta_{\max}-\eta_{\min}},$$

then we have

$$\begin{aligned}&\mathcal{R}^{(\mathrm{in})}_L(D|\Sigma_{Y^L})=\hat{\mathcal{R}}^{(\mathrm{in})}_L(D|\Sigma_{Y^L})\\ &=\mathcal{R}_L(D|\Sigma_{Y^L})=\mathcal{R}^{(\mathrm{out})}_L(D|\Sigma_{Y^L}).\end{aligned}$$

Fix $\gamma^L\in[1,+\infty)^L$ arbitrarily. Consider the region $\mathcal{R}_L(\gamma^L|\Sigma_{Y^L})$ and the minimum distortion $D_L(\gamma^L,R^L|\Sigma_{Y^L})$ induced by $\mathcal{R}_L(\gamma^L,D|\Sigma_{Y^L})$. Those are formally defined by

$$\begin{aligned}\mathcal{R}_L(\gamma^L|\Sigma_{Y^L})&\triangleq\left\{(R^L,D):R^L\in\mathcal{R}_L(\gamma^L,D|\Sigma_{Y^L})\right\},\\ D_L(\gamma^L,R^L|\Sigma_{Y^L})&\triangleq\inf\left\{D:(R^L,D)\in\mathcal{R}_L(\gamma^L|\Sigma_{Y^L})\right\}.\end{aligned}$$

Similarly, we define

$$\begin{aligned}&\mathcal{R}^{(\mathrm{in})}_L(\gamma^L|\Sigma_{Y^L})\\ &\triangleq\left\{(R^L,D):R^L\in\mathcal{R}^{(\mathrm{in})}_L(\gamma^L,D|\Sigma_{Y^L})\right\},\\ &\mathcal{R}^{(\mathrm{out})}_L(\gamma^L|\Sigma_{Y^L})\\ &\triangleq\left\{(R^L,D):R^L\in\mathcal{R}^{(\mathrm{out})}_L(\gamma^L,D|\Sigma_{Y^L})\right\},\\ &D^{(\mathrm{u})}_L(\gamma^L,R^L|\Sigma_{Y^L})\\ &\triangleq\inf\left\{D:(R^L,D)\in\mathcal{R}^{(\mathrm{in})}_L(\gamma^L|\Sigma_{Y^L})\right\},\\ &D^{(\mathrm{l})}_L(\gamma^L,R^L|\Sigma_{Y^L})\\ &\triangleq\inf\left\{D:(R^L,D)\in\mathcal{R}^{(\mathrm{out})}_L(\gamma^L|\Sigma_{Y^L})\right\}.\end{aligned}$$

From Theorem 11 and Corollary 4, we obtain the following corollary.

*Corollary 5:* For any $R^L\geq 0$ and any $\gamma^L\in[1,+\infty)^L$, we have

$$\begin{aligned}D^{(\mathrm{u})}_L(\gamma^L,R^L|\Sigma_{Y^L})&\geq D_L(\gamma^L,R^L|\Sigma_{Y^L})\\ &\geq D^{(\mathrm{l})}_L(\gamma^L,R^L|\Sigma_{Y^L}).\end{aligned}$$

For each $\gamma^L\in[1,+\infty)^L$, if we have

$$0<D^{(\mathrm{u})}_L(\gamma^L,R^L|\Sigma_{Y^L})\leq\frac{1}{(\sqrt{L}+\sqrt{L-1})^2}\cdot\eta_{\min},$$

then

$$\begin{aligned}D^{(\mathrm{u})}_L(\gamma^L,R^L|\Sigma_{Y^L})&=D_L(\gamma^L,R^L|\Sigma_{Y^L})\\ &=D^{(\mathrm{l})}_L(\gamma^L,R^L|\Sigma_{Y^L}).\end{aligned}$$

We apply Corollary 5 to the derivation of matching condition in the case of vector distortion criterion. We consider the region $\mathcal{R}_L(\Sigma_{Y^L})$ and the distortion rate region $\mathcal{D}_L(R^L|\Sigma_{Y^L})$ induced by $\mathcal{R}(D^L|\Sigma_{Y^L})$. Those two regions are formally defined by

$$\begin{aligned}\mathcal{R}_L(\Sigma_{Y^L})&\triangleq\left\{(R^L,D^L):R^L\in\mathcal{R}_L(D^L|\Sigma_{Y^L})\right\},\\ \mathcal{D}_L(R^L|\Sigma_{Y^L})&\triangleq\left\{D^L:(R^L,D^L)\in\mathcal{R}_L(\Sigma_{Y^L})\right\}.\end{aligned}$$

Similarly, we define

$$\mathcal{R}_L^{(\mathrm{in})}(\Sigma_{Y^L}) \triangleq \left\{(R^L, D^L) : R^L \in \mathcal{R}_L^{(\mathrm{in})}(D^L|\Sigma_{Y^L})\right\},$$
$$\mathcal{D}_L^{(\mathrm{in})}(R^L|\Sigma_{Y^L}) \triangleq \left\{D^L : (R^L, D^L) \in \mathcal{R}_L^{(\mathrm{in})}(\Sigma_{Y^L})\right\}.$$

Although the distortion rate region is merely an alternative characterization of the rate distortion region, the former is more convenient than the latter for our analysis of matching condition. We examine a part of the boundary of $\mathcal{D}^{(\mathrm{in})}(R^L|\Sigma_{Y^L})$ which coincides with the boundary of $\mathcal{D}(R^L|\Sigma_{Y^L})$. By definition of $D_L(\gamma^L, R^L|\Sigma_{Y^L})$ and $D_L^{(\mathrm{u})}(\gamma^L, R^L|\Sigma_{Y^L})$, we have

$$D_L(\gamma^L, R^L|\Sigma_{Y^L}) = \min_{D^L \in \mathcal{D}_L(R^L|\Sigma_{Y^L})} \sum_{l=1}^{L} \gamma_i^2 D_l, \quad (35)$$

$$D_L^{(\mathrm{u})}(\gamma^L, R^L|\Sigma_{Y^L}) = \min_{D^L \in \mathcal{D}_L^{(\mathrm{in})}(R^L|\Sigma_{Y^L})} \sum_{l=1}^{L} \gamma_l^2 D_l. \quad (36)$$

Consider the following two hyperplanes:

$$\Pi_L(\gamma^L) \triangleq \left\{D^L : \sum_{l=1}^{L} \gamma_l^2 D_l = D_L(\gamma^L, R^L|\Sigma_{Y^L})\right\},$$
$$\Pi_L^{(\mathrm{u})}(\gamma^L) \triangleq \left\{D^L : \sum_{l=1}^{L} \gamma_l^2 D_l = D_L^{(u)}(\gamma^L, R^L|\Sigma_{Y^L})\right\}.$$

It can easily be verified that the region $\mathcal{D}_L(R^L|\Sigma_{Y^L})$ is a closed convex set. Then by (35), $\Pi_L(\gamma^L)$ becomes a supporting hyperplane of $\mathcal{D}_L(R^L|\Sigma_{Y^L})$ and every $D^L \in \Pi_L(\gamma^L) \cap \mathcal{D}_L(R^L|\Sigma_{Y^L})$ is on the boundary of $\mathcal{D}_L(R^L|\Sigma_{Y^L})$. On the other hand, by its definition the region $\mathcal{D}_L^{(\mathrm{in})}(R^L|\Sigma_{Y^L})$ is also a closed convex set. Then by (36), $\Pi_L^{(\mathrm{u})}(\gamma^L)$ becomes a supporting hyperplane of $\mathcal{D}_L^{(\mathrm{in})}(R^L|\Sigma_{Y^L})$ and every $D^L \in \Pi_L^{(\mathrm{u})}(\gamma^L) \cap \mathcal{D}_L^{(\mathrm{in})}(R^L|\Sigma_{Y^L})$ is on the boundary of $\mathcal{D}_L^{(\mathrm{in})}(R^L|\Sigma_{Y^L})$. Set

$$\zeta_L \triangleq \frac{1}{(\sqrt{L} + \sqrt{L-1})^2} \eta_{\min},$$
$$\mathcal{T}_L(\zeta_L) \triangleq \left\{\gamma^L \in [1, +\infty)^L : D_L^{(u)}(\gamma^L, R^L|\Sigma_{Y^L}) \le \zeta_L\right\}.$$

Then by Corollary 5, for any $\gamma^L \in \mathcal{T}_L(\zeta_L)$, we have $\Pi_L^{(\mathrm{u})}(\gamma^L) = \Pi_L(\gamma^L)$, which together with $\mathcal{D}_L^{(\mathrm{in})}(R^L|\Sigma_{Y^L}) \subseteq \mathcal{D}_L(R^L|\Sigma_{Y^L})$ implies that every $D^L \in \Pi_L^{(\mathrm{u})}(\gamma^L) \cap \mathcal{D}_L^{(\mathrm{in})}(R^L|\Sigma_{Y^L})$ must belong to $\Pi_L(\gamma^L) \cap \mathcal{D}_L(R^L|\Sigma_{Y^L})$. Hence this $D^L$ must be on the boundary of $\mathcal{D}_L(R^L|\Sigma_{Y^L})$. It can easily be verified that an existence of $\Pi_L^{(\mathrm{u})}(\gamma^L)$ satisfying $\gamma^L \in \mathcal{T}_L(\zeta_L)$ is equivalent to $\Pi_L^{(\mathrm{u})}(\gamma^L) \cap \{D^L \ge 0\} \subseteq \mathcal{D}_L^{(+)}(\zeta_L)$, where

$$\mathcal{D}_L^{(+)}(\zeta_L) \triangleq \left\{D^L : D^L \ge 0, \sum_{l=1}^{L} D_l \le \zeta_L\right\}.$$

Summarizing the above argument, we establish the following.

*Theorem 13:* The distortion rate region $\mathcal{D}_L(R^L|\Sigma_{Y^L})$ and its inner bound $\mathcal{D}_L^{(\mathrm{in})}(R^L|\Sigma_{Y^L})$ share their boundaries at $\mathcal{D}_L^{*}(\zeta_L) \cap \mathcal{D}_L^{(\mathrm{in})}(R^L|\Sigma_{Y^L})$, where

$$\mathcal{D}_L^{*}(\zeta_L) \triangleq \bigcup_{\gamma^L \in \mathcal{T}_L(\zeta_L)} \Pi_L^{(\mathrm{u})}(\gamma^L)$$
$$= \bigcup_{\Pi_L^{(\mathrm{u})}(\gamma^L) \cap \{D^L \ge 0\} \subseteq \mathcal{D}_L^{(+)}(\zeta_L)} \Pi_L^{(\mathrm{u})}(\gamma^L).$$

When $L = 3$, we show $\mathcal{D}_3^{(\mathrm{in})}(R^3|\Sigma_{Y^3})$, $\mathcal{D}_3^{(+)}(\zeta_3)$, and $\Pi_3^{(\mathrm{u})}(\gamma^3) \cap \{D^3 \ge 0\}$ in Fig. 3.

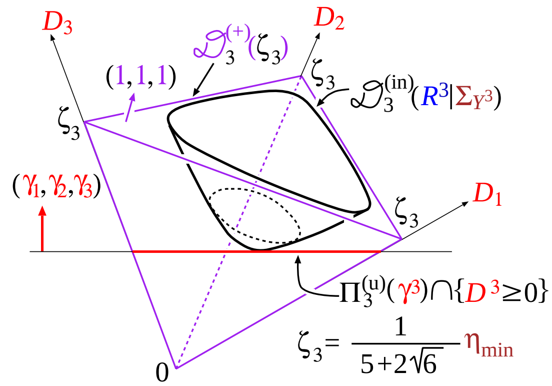


Fig. 3. $\mathcal{D}_3^{(\mathrm{in})}(R^3|\Sigma_{Y^3})$, $\Pi_L^{(\mathrm{u})}(\gamma^3) \cap \{D^3 \ge 0\}$, and $\mathcal{D}_3^{(+)}(\zeta_3)$ in the case of $L = 3$. In this figure we are in a position so that we can view the supporting hyperplane $\Pi_3^{(\mathrm{u})}(\gamma^3)$ as a horizontal line.

### *D. Sum Rate Characterization for the Cyclic Shift Invariant Source*

In this subsection we further examine an explicit characterization of $R_{\mathrm{sum},L}(D|\Sigma_{Y^l})$ when the source has a certain symmetrical property. Let

$$\tau = \begin{pmatrix} 1 & 2 & \cdots & l & \cdots & L \\ \tau(1) & \tau(2) & \cdots & \tau(l) & \cdots & \tau(L) \end{pmatrix}$$

be a cyclic shift on $\Lambda_L$, that is,

$$\tau(1) = 2, \tau(2) = 3, \cdots, \tau(L-1) = L, \tau(L) = 1.$$

Let $p_{X_{\Lambda_L}}(x_{\Lambda_L}) = p_{X_1 X_2 \cdots X_L}(x_1, x_2, \cdots, x_L)$ be a probability density function of $X^L$. The source $X^L$ is said to be cyclic shift invariant if we have

$$p_{X_{\Lambda_L}}(x_{\tau(\Lambda_L)}) = p_{X_1 X_2 \cdots X_L}(x_2, x_3, \cdots, x_L, x_1)$$
$$= p_{X_1 X_2 \cdots X_L}(x_1, x_2, \cdots, x_{L-1}, x_L)$$

for any $(x_1, x_2, \cdots, x_L) \in \mathcal{X}^L$. In the following argument we assume that $X^L$ satisfies the cyclic shift invariant property. We further assume that $N_l, l \in \Lambda_L$ are i.i.d. Gaussian random variables with mean 0 and variance $\epsilon$. Then, the observation $Y^L = X^L + N^L$ also satisfies the cyclic shift invariant property. We assume that the covariance matrix $\Sigma_{N^L}$ of $N^L$ is given by $\epsilon I_L$. Then $\tilde{A}$ and $B$ are given by

$$\tilde{A} = \left(\epsilon \Sigma_{X^L}^{-1} + I_L\right)^{-1}, \ B = \epsilon \left(I_L + \epsilon \Sigma_{X^L}^{-1}\right).$$

Fix $r > 0$, let $N_l(r)$, $l \in \Lambda_L$ be $L$ i.i.d. Gaussian random variables with mean 0 and variance $\epsilon/(1 - \mathrm{e}^{-2r})$. The covariance matrix $\Sigma_{N^L(r)}$ for the random vector $N^L(r)$ is given by

$$\Sigma_{N^L(r)} = \frac{1-\mathrm{e}^{-2r}}{\epsilon} I_L.$$

Let $\mu_l, l \in \Lambda_L$ be $L$ eigenvalues of the matrix $\Sigma_{Y^L}$ and let $\beta_l = \beta_l(r), l \in \Lambda_L$ be $L$ eigenvalues of the matrix

$${}^{\mathrm{t}}\tilde{A}\left(\Sigma_{X^L}^{-1} + \frac{1-\mathrm{e}^{-2r}}{\epsilon} I_L\right)\tilde{A}.$$

Using the eigenvalues of $\Sigma_{Y^L}$, $\beta_l(r), l \in \Lambda_L$ can be written as

$$\beta_l(r) = \frac{1}{\epsilon}\left[1 - \frac{\epsilon}{\mu_l} - \left(1 - \frac{\epsilon}{\mu_l}\right)^2 \mathrm{e}^{-2r}\right].$$

Let $\xi$ be a nonnegative number that satisfies

$$\sum_{l=1}^{L}\{[\xi - \beta_l^{-1}]^+ + \beta_l^{-1}\} = D + \mathrm{tr}[B].$$

Define

$$\tilde{\omega}(D,r) \triangleq \prod_{l=1}^{L}\left\{[\xi - \beta_l^{-1}]^+ + \beta_l^{-1}\right\}.$$

The function $\tilde{\omega}(D,r)$ has an expression of the so-called water filling solution to the following optimization problem:

$$\tilde{\omega}(D,r) = \max_{\substack{\xi_l\beta_l \geq 1, l \in \Lambda_L, \\ \sum_{l=1}^{L}\xi_l \leq D + \mathrm{tr}[B]}} \prod_{l=1}^{L}\xi_l. \tag{37}$$

Set

$$\underline{\tilde{J}}(D,r) \triangleq \frac{1}{2}\log\left[\frac{\mathrm{e}^{2Lr}\,|\Sigma_{Y^L} + B|}{\tilde{\omega}(D,r)}\right],$$

$$\pi(r) \triangleq \mathrm{tr}\left[\tilde{A}^{-1}\left(\Sigma_{X^L}^{-1} + \frac{1-\mathrm{e}^{-2r}}{\epsilon} I_L\right)^{-1}{}^{\mathrm{t}}\tilde{A}^{-1}\right].$$

By definition we have

$$\pi(r) = \sum_{l=1}^{L}\frac{1}{\beta_l(r)}. \tag{38}$$

Since $\pi(r)$ is a monotone decreasing function of $r$, there exists a unique $r$ such that $\pi(r) = D + \mathrm{tr}[B]$, we denote it by $r^*(D+\mathrm{tr}[B])$. We can show that $\tilde{\omega}(D,r)$ satisfies the following property.

*Property 8:*

a) For $D > 0$,

$$\underbrace{(r, r, \cdots, r)}_{L} \in \mathcal{B}_L(\tilde{A}^{-1}, D + \mathrm{tr}[B])$$
$$\Leftrightarrow \pi(r) \leq D + \mathrm{tr}[B] \Leftrightarrow r \geq r^*(D + \mathrm{tr}[B]),$$
$$\tilde{\omega}(D, r^*) = |\tilde{A}|^{-2}\left|\Sigma_{X^L}^{-1} + \frac{1-\mathrm{e}^{-2r^*}}{\epsilon} I_L\right|^{-1}.$$

b) The function $\tilde{\omega}(D,r)$ is a convex function of $r \in [r^*(D+\mathrm{tr}[B]), \infty)$.

Proof of Property 8 part a) is easy. We omit the detail. Proof of Property 8 part b) will be given in Section V. Set

$$\begin{aligned}
R_{\mathrm{sum},L}^{(\mathrm{u})}(D|\Sigma_{Y^L}) &\triangleq \underline{\tilde{J}}(D, r^*) \\
&= \frac{1}{2}\log\left[|\Sigma_{Y^L} + B|\mathrm{e}^{2Lr*}\prod_{l=1}^{L}\beta_l(r)\right] \\
&= \sum_{l=1}^{L}\frac{1}{2}\log\left\{\frac{\mu_l}{\epsilon}\left[\mathrm{e}^{2r*} - 1\right] + 1\right\}
\end{aligned}$$

$$R_{\mathrm{sum},L}^{(\mathrm{l})}(D|\Sigma_{Y^L}) \triangleq \min_{r \geq r^*(D+\mathrm{tr}[B])}\underline{\tilde{J}}(D,r).$$

Then we have the following.

*Theorem 14:* Assume that the source $X^L$ and its noisy version $Y^L = X^L + N^L$ are cyclic shift invariant. Then, we have

$$R_{\mathrm{sum},L}^{(\mathrm{l})}(D|\Sigma_{Y^L}) \leq R_{\mathrm{sum},L}(D|\Sigma_{Y^L}) \leq R_{\mathrm{sum},L}^{(\mathrm{u})}(D|\Sigma_{Y^L}).$$

Proof of this theorem will be stated in Section V. We next examine a necessary and sufficient condition for $R_{\mathrm{sum},L}^{(\mathrm{l})}(D|\Sigma_{Y^L})$ to coincide with $R_{\mathrm{sum},L}^{(\mathrm{u})}(D|\Sigma_{Y^L})$. It is obvious that this condition is equivalent to the condition that the function $\underline{\tilde{J}}(D,r)$, $r \geq r^* = r^*(D + \mathrm{tr}[B])$, attains the minimum at $r = r^*$. Set

$$\mu_{\min} \triangleq \min_{1\leq l\leq L}\mu_l, \mu_{\max} \triangleq \max_{1\leq l\leq L}\mu_l.$$

Let $l_0 \in \Lambda_L$ be the largest integer such that $\mu_{\max} = \mu_{l_0}$ and let $l_1 = l_1(r) \in \Lambda_L$ be the largest integer such that

$$\beta_{l_1}(r) = \max_{1\leq l\leq L}\beta_l(r).$$

The following is a basic lemma to derive our necessary and sufficient matching condition on $R_{\mathrm{sum},L}^{(\mathrm{l})}(D|\Sigma_{Y^L}) = R_{\mathrm{sum},L}^{(\mathrm{u})}(D|\Sigma_{Y^L})$.

*Lemma 6:* The function $\underline{\tilde{J}}(D,r)$, $r \in [r^*(D + \mathrm{tr}[B]), \infty)$ attains the minimum at $r = r^*$ if and only if

$$\begin{aligned}
&\frac{1}{2}\left(\frac{\mathrm{d}}{\mathrm{d}r}\underline{\tilde{J}}(D,r)\right)_{r=r^*} \\
&= \sum_{l=1}^{L}\frac{\mathrm{e}^{2r^*}\left[\mathrm{e}^{2r^*} - 1 + \frac{\epsilon}{\mu_l}\right] - \left(1 - \frac{\epsilon}{\mu_{l_1}}\right)\left[\mathrm{e}^{2r^*} - 1 + \frac{\epsilon}{\mu_{l_1}}\right]}{\left[\mathrm{e}^{2r^*} - 1 + \frac{\epsilon}{\mu_{l_1}}\right]^2} \\
&\geq 0.
\end{aligned} \tag{39}$$

Proof of Lemma 6 will be given in Section V. Note that for any $l \in \Lambda_L$, we have

$$\begin{aligned}
&\mathrm{e}^{2r^*}\left[\mathrm{e}^{2r^*} - 1 + \frac{\epsilon}{\mu_l}\right] - \left(1 - \frac{\epsilon}{\mu_{l_1}}\right)\left[\mathrm{e}^{2r^*} - 1 + \frac{\epsilon}{\mu_{l_1}}\right] \\
&\geq \mathrm{e}^{2r^*}\left[\mathrm{e}^{2r^*} - 1 + \frac{\epsilon}{\mu_{l_0}}\right] - \left(1 - \frac{\epsilon}{\mu_{l_1}}\right)\left[\mathrm{e}^{2r^*} - 1 + \frac{\epsilon}{\mu_{l_1}}\right] \\
&\geq \epsilon\left(\frac{1}{\mu_{l_0}} - \frac{1}{\mu_{l_1}}\right).
\end{aligned} \tag{40}$$

From (39) in Lemma 6 and (40), we can see that $l_0 = l_1$ is a sufficient matching condition for $R_{\mathrm{sum},L}^{(\mathrm{l})}(D|\Sigma_{Y^L}) = R_{\mathrm{sum},L}^{(\mathrm{u})}(D|\Sigma_{Y^L})$. Let $\tilde{\mu}$ be the second largest eigenvalue of $\Sigma_{Y^L}$ and

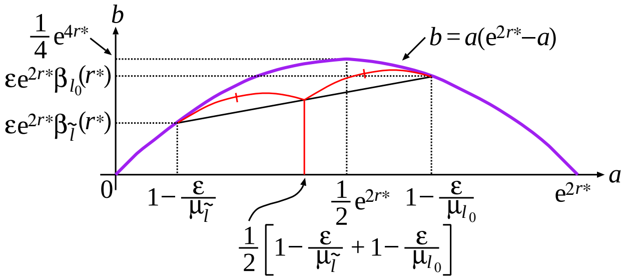


Fig. 4. The graph of $b = a(\mathrm{e}^{2r^*} - a)$.

let $\tilde{l} \in \Lambda_L$ be the largest integer such that $\tilde{\mu} = \mu_{\tilde{l}}$. From the graph of $b = a(\mathrm{e}^{2r^*} - a)$ shown in Fig. 4, we can see that

$$\frac{1}{2}\left[1 - \frac{\epsilon}{\tilde{\mu}} + 1 - \frac{\epsilon}{\mu_{\max}}\right] \le \frac{1}{2}\mathrm{e}^{2r^*}$$

or equivalent to

$$\mathrm{e}^{2r^*} - 1 \ge \left[1 - \epsilon\left(\frac{1}{\tilde{\mu}} + \frac{1}{\mu_{\max}}\right)\right] \tag{41}$$

is a necessary and sufficient condition for $l_0 = l_1$. Hence (41) is a sufficient matching condition. Next, we derive another simple matching condition. Note that

$$\begin{aligned} &\mathrm{e}^{2r^*}\left[\mathrm{e}^{2r^*} - 1 + \frac{\epsilon}{\mu_l}\right] - \left(1 - \frac{\epsilon}{\mu_{l_1}}\right)\left[\mathrm{e}^{2r^*} - 1 + \frac{\epsilon}{\mu_{l_1}}\right] \\ &\ge \mathrm{e}^{2r^*}\left[\mathrm{e}^{2r^*} - 1 + \frac{\epsilon}{\mu_{\max}} - \frac{1}{4}\mathrm{e}^{2r^*}\right] \\ &= \frac{3}{4}\mathrm{e}^{2r^*}\left[\mathrm{e}^{2r^*} - 1 - \frac{1}{3}\left(1 - \frac{4\epsilon}{\mu_{\max}}\right)\right]. \end{aligned}$$

Hence, if we have

$$\mathrm{e}^{2r^*} - 1 \ge \frac{1}{3}\left(1 - \frac{4\epsilon}{\mu_{\max}}\right), \tag{42}$$

then the condition (39) holds. For $\epsilon \in (0, \mu_{\min})$, define

$$\begin{aligned} s(\epsilon) \triangleq \frac{1}{2}\log\Bigg\{1 + \min\Bigg\{&\left[1 - \epsilon\left(\frac{1}{\tilde{\mu}} + \frac{1}{\mu_{\max}}\right)\right]^+, \\ &\frac{1}{3}\left[1 - \frac{4\epsilon}{\mu_{\max}}\right]^+\Bigg\}\Bigg\}. \end{aligned}$$

Then the condition (41) or (42) is equivalent to $r^* \ge s(\epsilon)$. Furthermore, this condition is equivalent to $0 \le D \le D_{\mathrm{th}}(\epsilon)$, where

$$D_{\mathrm{th}}(\epsilon) \triangleq \sum_{l=1}^{L} \frac{1}{\beta_l(s(\epsilon))} - \mathrm{tr}[B] = \sum_{l=1}^{L} \frac{\mu_l \epsilon}{\mu_l\left[\mathrm{e}^{2s(\epsilon)} - 1\right] + \epsilon}.$$

Summarizing the above argument we have the following.

*Theorem 15:* We suppose that $Y^L$ is cyclic shift invariant. Fix $\epsilon \in (0, \mu_{\min})$ arbitrary. If $0 \le D \le D_{\mathrm{th}}(\epsilon)$, then we have

$$R^{(\mathrm{l})}_{\mathrm{sum},L}(D|\Sigma_{Y^L}) = R_{\mathrm{sum},L}(D|\Sigma_{Y^L}) = R^{(\mathrm{u})}_{\mathrm{sum},L}(D|\Sigma_{Y^L}).$$

Furthermore, the curve $R = R_{\mathrm{sum},L}(D|\Sigma_{Y^L})$ has the following parametric form:

$$\left.\begin{aligned} R &= \sum_{l=1}^{L} \frac{1}{2}\log\left\{\frac{\mu_l}{\epsilon}\left[\mathrm{e}^{2r} - 1\right] + 1\right\}, \\ D &= \sum_{l=1}^{L} \frac{1}{\beta_l(r)} - \mathrm{tr}[B] = \sum_{l=1}^{L} \frac{\mu_l \epsilon}{\mu_l(\mathrm{e}^{2r} - 1) + \epsilon} \\ &\text{for } r \in [s(\epsilon), \infty). \end{aligned}\right\} \tag{43}$$

Since $D_{\mathrm{th}}(\epsilon)$ is a monotone increasing function of $\epsilon$, to choose $\epsilon$ arbitrary close to $\mu_{\min}$ is a choice yielding the best matching condition. Note here that we can not choose $\epsilon = \mu_{\min}$ because $\pi(r)$ becomes infinity in this case. Letting $\epsilon$ arbitrary close to $\mu_{\min}$ and considering the continuities of $D_{\mathrm{th}}(\epsilon)$ and the functions in the right hand side of (43) with respect to $\epsilon$, we have the following.

*Theorem 16:* We suppose that $Y^L$ is cyclic shift invariant. If $0 \le D \le D_{\mathrm{th}}(\mu_{\min})$, then we have

$$R^{(\mathrm{l})}_{\mathrm{sum},L}(D|\Sigma_{Y^L}) = R_{\mathrm{sum},L}(D|\Sigma_{Y^L}) = R^{(\mathrm{u})}_{\mathrm{sum},L}(D|\Sigma_{Y^L}).$$

Furthermore, the curve $R = R_{\mathrm{sum},L}(D|\Sigma_{Y^L})$ has the following parametric form:

$$\left.\begin{aligned} R &= \sum_{l=1}^{L} \frac{1}{2}\log\left\{\frac{\mu_l}{\mu_{\min}}\left[\mathrm{e}^{2r} - 1\right] + 1\right\}, \\ D &= \sum_{l=1}^{L} \frac{\mu_l \mu_{\min}}{\mu_l(\mathrm{e}^{2r} - 1) + \mu_{\min}}, \ \text{for } r \in [s(\mu_{\min}), \infty). \end{aligned}\right\}$$

Let $1^L \triangleq (1, 1, \cdots, 1)$ be a $L$ dimensional vector whose $L$ components are all 1. We consider the characterization of $R_{\mathrm{sum},L}(D \cdot 1^L|\Sigma_{Y^L})$. From Theorem 16, we obtain the following corollary.

*Corollary 6:* Suppose that $Y^L$ is cyclic shift invariant. If $0 \le D \le \frac{1}{L}D_{\mathrm{th}}(\mu_{\min})$, then we have

$$\begin{aligned} &R^{(\mathrm{l})}_{\mathrm{sum},L}(D \cdot 1^L|\Sigma_{Y^L}) \\ &= R_{\mathrm{sum},L}(D \cdot 1^L|\Sigma_{Y^L}) = R^{(\mathrm{u})}_{\mathrm{sum},L}(D \cdot 1^L|\Sigma_{Y^L}). \end{aligned}$$

Furthermore, the curve $R = R_{\mathrm{sum},L}(D \cdot 1^L|\Sigma_{Y^L})$ has the following parametric form:

$$\left.\begin{aligned} R &= \sum_{l=1}^{L} \frac{1}{2}\log\left\{\frac{\mu_l}{\mu_{\min}}\left[\mathrm{e}^{2r} - 1\right] + 1\right\}, \\ D &= \frac{1}{L}\sum_{l=1}^{L} \frac{\mu_l \mu_{\min}}{\mu_l(\mathrm{e}^{2r} - 1) + \mu_{\min}}, \ \text{for } r \in [s(\mu_{\min}), \infty). \end{aligned}\right\}$$

Here we consider the case where $\Sigma_{Y^L}$ has at most two eigenvalues. In this case we have $\tilde{\mu} = \mu_{\min}$. Then we have $s(\mu_{\min}) = 0$ and $D_{\mathrm{th}}(0) = \mathrm{tr}[\Sigma_{Y^L}]$. This implies that $R = R_{\mathrm{sum},L}(D \cdot 1^L|\Sigma_{Y^L})$ is determined for all $0 \le D \le \frac{1}{L}\mathrm{tr}[\Sigma_{Y^L}]$. Wagner *et al.* [11] determined $R = R_{\mathrm{sum},L}(D \cdot 1^L|\Sigma_{Y^L})$ in a special case where $\Sigma_{Y^L}$ satisfies $[\Sigma_{Y^L}]_{ll} = \sigma^2$ for $l \in \Lambda_L$ and $[\Sigma_{Y^L}]_{ll'} = c\sigma^2, 0 < c < 1$ for $l \ne l' \in \Lambda_L$. In this special case $\Sigma_{Y^L}$ has two distinct eigenvaules. Hence our result includes their result as a special case.

Yang and Xiong [25] determined $R_{\mathrm{sum},L}(D\cdot 1^L|\Sigma_{Y^L})$ in the case where $\Sigma_{Y^L}$ has two distinct eigenvalues. Wang *et al.* [12] determined $R_{\mathrm{sum},L}(D\cdot 1^L|\Sigma_{Y^L})$ for another case of $\Sigma_{Y^L}$. The class of information sources satisfying the cyclic shift invariant property is different from the class of information sources investigated by Yang and Xiong [25] and Wang *et al.* [12] although we have some overlap between them.

## V. Proofs of the Results

### A. Derivation of the Outer Bounds

In this subsection we prove the results on outer bounds of the rate distortion region. We first state two important lemmas which are mathematical cores of the converse coding theorem. For $l\in\Lambda_L$, set

$$W_l \stackrel{\triangle}{=} \varphi_l(\boldsymbol{Y}_l), r_l^{(n)} \stackrel{\triangle}{=} \frac{1}{n}I(\boldsymbol{Y}_l;W_l|\boldsymbol{X}^K). \tag{44}$$

For $Q\in\mathcal{O}_K$, set $Z^K \stackrel{\triangle}{=} QX^K$. For

$$\boldsymbol{X}^K=(X^K(1),X^K(2),\cdots,X^K(n))$$

we set

$$\boldsymbol{Z}^K \stackrel{\triangle}{=} Q\boldsymbol{X}^K=(QX^K(1),QX^K(2),\cdots,QX^K(n)).$$

Furthermore, for $\hat{\boldsymbol{X}}^K=(\hat{X}^K(1),\ \hat{X}^K(2),\ \cdots,\hat{X}^K(n))$, we set

$$\hat{\boldsymbol{Z}}^K=Q\hat{\boldsymbol{X}}^K \stackrel{\triangle}{=} (Q\hat{X}^K(1),Q\hat{X}^K(2),\cdots,Q\hat{X}^K(n)).$$

We have the following two lemmas.

*Lemma 7:* For any $k\in\Lambda_K$ and any $Q\in\mathcal{O}_K$, we have

$$\begin{aligned}
&h(\boldsymbol{Z}_k|\,\boldsymbol{Z}_{[k]}^K W^L)\le h(\boldsymbol{Z}_k-\hat{\boldsymbol{Z}}_k\mid \boldsymbol{Z}_{[k]}^K-\hat{\boldsymbol{Z}}_{[k]}^K)\\
&\le\frac{n}{2}\log\left\{(2\pi\mathrm{e})\left[Q\left(\tfrac{1}{n}\Sigma^{-1}_{\boldsymbol{X}^K-\hat{\boldsymbol{X}}^K}\right)^{-1}{}^{\mathrm{t}}Q\right]^{-1}_{kk}\right\},
\end{aligned}$$

where $h(\cdot)$ stands for the differential entropy.

*Lemma 8:* For any $k\in\Lambda_K$ and any $Q\in\mathcal{O}_K$, we have

$$\begin{aligned}
&h(\boldsymbol{Z}_k|\boldsymbol{Z}_{[k]}^K W^L)\\
&\ge\frac{n}{2}\log\left\{(2\pi\mathrm{e})\left[Q\left(\Sigma^{-1}_{X^K}+{}^{\mathrm{t}}A\Sigma^{-1}_{N_{\Lambda_L}(r^{(n)}_{\Lambda_L})}A\right){}^{\mathrm{t}}Q\right]^{-1}_{kk}\right\}.
\end{aligned}$$

Proofs of Lemmas 7 and 8 will be stated in Appendixes A and B, respectively. The following lemma immediately follows from Lemmas 7 and 8.

*Lemma 9:* For any $\Sigma_{X^KY^L}$ and for any $(\varphi_1^{(n)},\ \varphi_2^{(n)},\cdots,\varphi_L^{(n)},\ \psi^{(n)})$, we have

$$\left(\tfrac{1}{n}\Sigma_{\boldsymbol{X}^K-\hat{\boldsymbol{X}}^K}\right)^{-1}\preceq\Sigma^{-1}_{X^K}+{}^{\mathrm{t}}A\Sigma^{-1}_{N_{\Lambda_L}(r^{(n)}_{\Lambda_L})}A.$$

From Lemma 8, we obtain the following lemma.

*Lemma 10:* For any $S\subseteq\Lambda_L$, we have

$$I(\boldsymbol{X}^K;W_S)\le\frac{n}{2}\log\left|I+\Sigma_{X^K}{}^{\mathrm{t}}A\Sigma^{-1}_{N_S(r_S^{(n)})}A\right|. \tag{45}$$

*Proof:* For each $l\in\Lambda_L-S$, we choose $W_l$ so that it takes a constant value. In this case we have $r_l^{(n)}=0$ for $l\in\Lambda_L-S$. Then by Lemma 8, for any $k\in\Lambda_K$, we have

$$\begin{aligned}
&h(\boldsymbol{Z}_k|\boldsymbol{Z}_{[k]}^K W_S)\\
&\ge\frac{n}{2}\log\left\{(2\pi\mathrm{e})\left[Q\left(\Sigma^{-1}_{X^K}+{}^{\mathrm{t}}A\Sigma^{-1}_{N_S(r_S^{(n)})}A\right){}^{\mathrm{t}}Q\right]^{-1}_{kk}\right\}.
\end{aligned}\tag{46}$$

We choose an orthogonal matrix $Q\in\mathcal{O}_K$ so that

$$Q\left(\Sigma^{-1}_{X^K}+{}^{\mathrm{t}}A\Sigma^{-1}_{N_S(r_S^{(n)})}A\right){}^{\mathrm{t}}Q$$

becomes the following diagonal matrix:

$$Q\left(\Sigma^{-1}_{X^K}+{}^{\mathrm{t}}A\Sigma^{-1}_{N_S(r_S^{(n)})}A\right){}^{\mathrm{t}}Q=\begin{pmatrix}\lambda_1 & & & \mathbf{0}\\ & \lambda_2 & & \\ & & \ddots & \\ \mathbf{0} & & & \lambda_K\end{pmatrix}. \tag{47}$$

Then we have the following chain of inequalities:

$$\begin{aligned}
&I(\boldsymbol{X}^K;W_S)=h(\boldsymbol{X}^K)-h(\boldsymbol{X}^K|W_S)\\
&\stackrel{\text{(a)}}{=}h(\boldsymbol{X}^K)-h(\boldsymbol{Z}^K|W_S)\le h(\boldsymbol{X}^K)-\sum_{k=1}^K h(\boldsymbol{Z}_k|\boldsymbol{Z}_{[k]}^K W_S)\\
&\stackrel{\text{(b)}}{\le}\frac{n}{2}\log\left[(2\pi\mathrm{e})^K|\Sigma_{X^K}|\right]\\
&\quad+\sum_{k=1}^K\frac{n}{2}\log\left\{\frac{1}{2\pi\mathrm{e}}\left[Q\left(\Sigma^{-1}_{X^K}+{}^{\mathrm{t}}A\Sigma^{-1}_{N_S(r_S^{(n)})}A\right){}^{\mathrm{t}}Q\right]_{kk}\right\}\\
&\stackrel{\text{(c)}}{=}\frac{n}{2}\log|\Sigma_{X^K}|+\sum_{k=1}^K\frac{n}{2}\log\lambda_l\\
&=\frac{n}{2}\log|\Sigma_{X^K}|+\frac{n}{2}\log\left|\Sigma^{-1}_{X^K}+{}^{\mathrm{t}}A\Sigma^{-1}_{N_S(r_S^{(n)})}A\right|\\
&=\frac{n}{2}\log\left|I+\Sigma_{X^K}{}^{\mathrm{t}}A\Sigma^{-1}_{N_S(r_S^{(n)})}A\right|.
\end{aligned}$$

Step (a) follows from the rotation invariant property of the (conditional) differential entropy. Step (b) follows from (46). Step (c) follows from (47). ■

We first prove the inclusion $\mathcal{R}_L(\Sigma_d|\,\Sigma_{X^KY^L})\subseteq\mathcal{R}_L^{(\mathrm{out})}(\Sigma_d|\Sigma_{X^KY^L})$ stated in Theorem 3. Using Lemmas 7, 8, 10 and a standard argument on the proof of converse coding theorems, we can prove the above inclusion.

*Proof of $\mathcal{R}_L(\Sigma_d|\Sigma_{X^KY^L})\subseteq\mathcal{R}_L^{(\mathrm{out})}(\Sigma_d|\Sigma_{X^KY^L})$:* We first observe that

$$W_S\to\boldsymbol{Y}_S\to\boldsymbol{X}^K\to\boldsymbol{Y}_{S^{\mathrm{c}}}\to W_{S^{\mathrm{c}}} \tag{48}$$

hold for any subset $S$ of $\Lambda_L$. Assume $(R_1,R_2,\cdots,R_L)\in\mathcal{R}_L(\Sigma_d|\Sigma_{X^KY^L})$. Then, there exists a sequence $\{(\varphi_1^{(n)},\varphi_2^{(n)},\cdots,\varphi_L^{(n)},\psi^{(n)}\}_{n=1}^{\infty}$ such that

$$\left.\begin{aligned}
&\limsup_{n\to\infty}R_l^{(n)}\le R_l, l\in\Lambda_L,\\
&\limsup_{n\to\infty}\frac{1}{n}\Sigma_{\boldsymbol{X}^K-\hat{\boldsymbol{X}}^K}\preceq\Sigma_d.
\end{aligned}\right\}\tag{49}$$

We set

$$r_l\stackrel{\triangle}{=}\limsup_{n\to\infty}r_l^{(n)}=\limsup_{n\to\infty}\frac{1}{n}I(\boldsymbol{Y}_l;W_S|\boldsymbol{X}^K). \tag{50}$$

For any subset $S \subseteq \Lambda_L$, we have the following chain of inequalities:

$$
\begin{aligned}
&\sum_{l\in S} nR_l^{(n)} \geq \sum_{l\in S} \log M_l \geq \sum_{l\in S} H(W_l) \geq H(W_S|W_{S^c})\\
&= I(\boldsymbol{X}^K; W_S|W_{S^c}) + H(W_S|W_{S^c}\boldsymbol{X}^K)\\
&\overset{\text{(a)}}{=} I(\boldsymbol{X}^K; W_S|W_{S^c}) + \sum_{l\in S} H(W_l|\boldsymbol{X}^K)\\
&\overset{\text{(b)}}{=} I(\boldsymbol{X}^K; W_S|W_{S^c}) + \sum_{l\in S} H(W_l|\boldsymbol{X}^K)\\
&\overset{\text{(c)}}{=} I(\boldsymbol{X}^K; W_S|W_{S^c}) + n\sum_{l\in S} r_l^{(n)}, \qquad (51)
\end{aligned}
$$

where steps (a),(b) and (c) follow from (48). We estimate a lower bound of $I(\boldsymbol{X}^K; W_S|W_{S^c})$. Observe that

$$
I(\boldsymbol{X}^K; W_S|W_{S^c}) = I(\boldsymbol{X}^K; W^L) - I(\boldsymbol{X}^K; W_{S^c}). \quad (52)
$$

Since an upper bound of $I(\boldsymbol{X}_{S^c}; W_{S^c})$ is derived by Lemma 10, it suffices to estimate a lower bound of $I(\boldsymbol{X}^K;\, W^L)$. We have the following chain of inequalities:

$$
\begin{aligned}
&I(\boldsymbol{X}^K; W^L) = h(\boldsymbol{X}^K) - h(\boldsymbol{X}^K|W^L)\\
&\geq h(\boldsymbol{X}^K) - h(\boldsymbol{X}^K|\hat{\boldsymbol{X}}^K) \geq h(\boldsymbol{X}^K) - h(\boldsymbol{X}^K - \hat{\boldsymbol{X}}^K)\\
&\geq \frac{n}{2}\log\left[(2\pi \mathrm{e})^K \left|\Sigma_{X^K}\right|\right] - \frac{n}{2}\log\left[(2\pi\mathrm{e})^K\left|\tfrac{1}{n}\Sigma_{\boldsymbol{X}^K-\hat{\boldsymbol{X}}^K}\right|\right]\\
&= \frac{n}{2}\log\frac{|\Sigma_{X^K}|}{\left|\frac{1}{n}\Sigma_{\boldsymbol{X}^K-\hat{\boldsymbol{X}}^K}\right|}. \qquad (53)
\end{aligned}
$$

Combining (52), (53), and Lemma 10, we have

$$
\begin{aligned}
&I(\boldsymbol{X}^K; W_S|W_{S^c}) + n\sum_{l\in S} r_l^{(n)}\\
&\geq \frac{n}{2}\log\left[\frac{\prod_{l\in S}\mathrm{e}^{2r_l^{(n)}}\,|\Sigma_{X^K}|}{\left|I + \Sigma_{X^K}{}^{\mathrm{t}}A\Sigma^{-1}_{N_{S^c}(r^{(n)}_{S^c})}A\right|\left|\frac{1}{n}\Sigma_{\boldsymbol{X}^K-\hat{\boldsymbol{X}}^K}\right|}\right]\\
&= \frac{n}{2}\log\left[\frac{\prod_{l\in S}\mathrm{e}^{2r_l^{(n)}}}{\left|\Sigma_{X^K}^{-1} + {}^{\mathrm{t}}A\Sigma^{-1}_{N_{S^c}(r^{(n)}_{S^c})}A\right|\left|\frac{1}{n}\Sigma_{\boldsymbol{X}^K-\hat{\boldsymbol{X}}^K}\right|}\right].
\end{aligned}
$$

Note here that $I(\boldsymbol{X}^K; W_S|W_{S^c}) + n\sum_{i\in S} r_i^{(n)}$ is nonnegative. Hence, we have

$$
\begin{aligned}
&I(\boldsymbol{X}^K; W_S|W_{S^c}) + n\sum_{i\in S} r_i^{(n)}\\
&\geq n\underline{J}_S\left(\left|\tfrac{1}{n}\Sigma_{\boldsymbol{X}^K-\hat{\boldsymbol{X}}^K}\right|, r_S^{(n)}\middle| r_{S^c}^{(n)}\right). \qquad (54)
\end{aligned}
$$

Combining (51) and (54), we obtain

$$
\sum_{l\in S} R_l^{(n)} \geq \underline{J}_S\left(\left|\tfrac{1}{n}\Sigma_{\boldsymbol{X}^K-\hat{\boldsymbol{X}}^K}\right|, r_S^{(n)}\middle| r_{S^c}^{(n)}\right) \qquad (55)
$$

for $S \subseteq \Lambda_L$. On the other hand, by Lemma 9, we have

$$
\Sigma_{X^K}^{-1} + {}^{\mathrm{t}}A\Sigma^{-1}_{N_{\Lambda_L}(r^{(n)}_{\Lambda_L})}A \succeq \tfrac{1}{n}\Sigma^{-1}_{\boldsymbol{X}^K-\hat{\boldsymbol{X}}^K}. \qquad (56)
$$

By letting $n\to\infty$ in (55) and (56) and taking (49) into account, we have for any $S\subseteq\Lambda_L$

$$
\sum_{l\in S} R_l \geq \underline{J}_S(|\Sigma_d|, r_S|r_{S^c}), \qquad (57)
$$

and

$$
\Sigma_{X^K}^{-1} + {}^{\mathrm{t}}A\Sigma^{-1}_{N^L(r^L)}A \succeq \Sigma_d^{-1}. \qquad (58)
$$

From (57) and (58), $\mathcal{R}_L(\Sigma_d|\Sigma_{X^KY^L}) \subseteq \mathcal{R}_L^{(\mathrm{out})}(\Sigma_d|\Sigma_{X^KY^L})$ is concluded. ■

*Proof of Theorem 4:* We choose an orthogonal matrix $Q\in\mathcal{O}_K$ so that

$$
Q\Gamma^{-1}\left(\Sigma_{X^K}^{-1} + {}^{\mathrm{t}}A\Sigma^{-1}_{N^L(r^L)}A\right){}^{\mathrm{t}}\Gamma^{-1\mathrm{t}}Q
= \begin{bmatrix} \alpha_1 & & & \text{\Large 0}\\ & \alpha_2 & & \\ & & \ddots & \\ \text{\Large 0} & & & \alpha_K \end{bmatrix}.
$$

Then we have

$$
Q\Gamma\left(\Sigma_{X^K}^{-1} + {}^{\mathrm{t}}A\Sigma^{-1}_{N^L(r^L)}A\right)^{-1}{}^{\mathrm{t}}\Gamma^{\mathrm{t}}Q
= \begin{bmatrix} \alpha_1^{-1} & & & \text{\Large 0}\\ & \alpha_2^{-1} & & \\ & & \ddots & \\ \text{\Large 0} & & & \alpha_K^{-1} \end{bmatrix}. \qquad (59)
$$

For $\Sigma_d\in\mathcal{A}(r^L)$, set

$$
\tilde{\Sigma}_d \overset{\triangle}{=} Q\Gamma\Sigma_d{}^{\mathrm{t}}\Gamma^{\mathrm{t}}Q, \quad \xi_k \overset{\triangle}{=} \left[\tilde{\Sigma}_d\right]_{kk}.
$$

Since

$$
\Gamma\Sigma_d{}^{\mathrm{t}}\Gamma \succeq \Gamma(\Sigma_{X^L}^{-1} + {}^{\mathrm{t}}A\Sigma^{-1}_{N^L(r^L)}A)^{-1\mathrm{t}}\Gamma,
$$

(59), and $\mathrm{tr}[\Gamma\Sigma_d{}^{\mathrm{t}}\Gamma]\leq D$, we have

$$
\left.
\begin{aligned}
&\xi_k \geq \alpha_k^{-1}, \text{ for } k\in\Lambda_K,\\
&\sum_{k=1}^{K}\xi_k = \mathrm{tr}\left[\tilde{\Sigma}_d\right] = \mathrm{tr}\left[\Gamma\Sigma_d{}^{\mathrm{t}}\Gamma\right] \leq D.
\end{aligned}
\right\} \qquad (60)
$$

Furthermore, by Hadamard's inequality we have

$$
|\Sigma_d| = |\Gamma|^{-2}|\tilde{\Sigma}_d| \leq |\Gamma|^{-2}\prod_{k=1}^{K}[\tilde{\Sigma}_d]_{kk} = |\Gamma|^{-2}\prod_{k=1}^{K}\xi_k. \quad (61)
$$

Combining (60) and (61), we obtain

$$
\begin{aligned}
\theta(\Gamma, D, r^L) &= \max_{\substack{\Sigma_d:\Sigma_d\in\mathcal{A}_L(r^L),\\ \mathrm{tr}[\Gamma\Sigma_d{}^{\mathrm{t}}\Gamma]\leq D}} |\Sigma_d|\\
&\leq |\Gamma|^{-2}\max_{\substack{\xi_k\alpha_k\geq 1, k\in\Lambda_K,\\ \sum_{k=1}^{K}\xi_k\leq D}}\prod_{k=1}^{K}\xi_k = \omega(\Gamma, D, r^L).
\end{aligned}
$$

The equality holds when $\tilde{\Sigma}_d$ is a diagonal matrix. ■

*Proof of Theorem 14:* Assume that $(R_1, R_2, \cdots, R_L) \in \mathcal{R}_L(D|\Sigma_{Y^L})$. Then, there exists a sequence $\{(\varphi_1^{(n)}, \varphi_2^{(n)}, \cdots, \varphi_L^{(n)}, \phi^{(n)}\}_{n=1}^{\infty}$ such that

$$\left.\begin{array}{l}\limsup_{n\to\infty} R_l^{(n)} \le R_l, l \in \Lambda_L \\ \limsup_{n\to\infty} \frac{1}{n}\Sigma_{\boldsymbol{Y}_{\Lambda_L}-\hat{\boldsymbol{Y}}_{\Lambda_L}} \preceq \Sigma_d,\ \mathrm{tr}[\Sigma_d] \le D \\ \text{for some } \Sigma_d. \end{array}\right\} \tag{62}$$

For each $j = 0, 1, \cdots, L-1$, we use $(\varphi_{\tau^j(1)}^{(n)}, \varphi_{\tau^j(2)}^{(n)}, \cdots, \varphi_{\tau^j(L)}^{(n)})$ for the encoding of $(\boldsymbol{Y}_1, \boldsymbol{Y}_2, \cdots, \boldsymbol{Y}_L)$. For $l \in \Lambda_L$ and for $j = 0, 1, \cdots, L-1$, set

$$W_{j,l} \triangleq \varphi_{\tau^j(l)}^{(n)}(\boldsymbol{Y}_l), r_{j,l}^{(n)} \triangleq \frac{1}{n} I(\boldsymbol{Y}_l; W_{j,l}|\boldsymbol{X}^L).$$

In particular,

$$r_{0,l}^{(n)} = r_l^{(n)} = \frac{1}{n} I(\boldsymbol{Y}_l; W_l|\boldsymbol{X}_i), \quad \text{for } l \in \Lambda_L.$$

Furthermore, set

$$r_{\tau^j(\Lambda_L)}^{(n)} \triangleq (r_{j,1}^{(n)}, r_{j,2}^{(n)}, \cdots, r_{j,L}^{(n)}), \text{ for } j = 0, 1, \cdots, L-1,$$
$$r^{(n)} \triangleq \frac{1}{L}\sum_{l=1}^{L} r_l^{(n)}.$$

By the cyclic shift invariant property of $\boldsymbol{X}_{\Lambda_L}$ and $\boldsymbol{Y}_{\Lambda_L}$, we have for $j = 0, 1, \cdots, L-1$,

$$\frac{1}{L}\sum_{l=1}^{L} r_{j,l}^{(n)} = \frac{1}{L}\sum_{l=1}^{L} r_{0,i}^{(n)} = r^{(n)}. \tag{63}$$

For $j = 0, 1, \cdots, L-1$ and for $l \in \Lambda_L$, set

$$\hat{\boldsymbol{Y}}_{j,l} \triangleq \phi_{\tau^j(l)}(\varphi_{\tau^j(1)}(\boldsymbol{Y}_1), \varphi_{\tau^j(2)}(\boldsymbol{Y}_2), \cdots, \varphi_{\tau^j(L)}(\boldsymbol{Y}_L)),$$
$$\hat{\boldsymbol{Y}}_{\tau^j(\Lambda_L)} \triangleq \begin{bmatrix} \hat{\boldsymbol{Y}}_{j,1} \\ \hat{\boldsymbol{Y}}_{j,2} \\ \vdots \\ \hat{\boldsymbol{Y}}_{j,L} \end{bmatrix}.$$

By the cyclic shift invariant property of $\boldsymbol{Y}_{\Lambda_L}$, we have

$$\begin{aligned} &\mathrm{E}\langle \boldsymbol{Y}_l - \hat{\boldsymbol{Y}}_{j,l}, \boldsymbol{Y}_{l'} - \hat{\boldsymbol{Y}}_{j,l'} \rangle \\ &= \mathrm{E}\langle \boldsymbol{Y}_{\tau(l)} - \hat{\boldsymbol{Y}}_{j,l}, \boldsymbol{Y}_{\tau(l')} - \hat{\boldsymbol{Y}}_{j,l'} \rangle \end{aligned} \tag{64}$$

for $(l, l') \in \Lambda_L^2$ and for $j = 0, 1, \cdots, L-1$. For $\Sigma_d = [d_{ll'}]$, set

$$\tau^j(\Sigma_d) \triangleq [d_{\tau^j(l)\tau^j(l')}], \ \overline{\Sigma_d} \triangleq \frac{1}{L}\sum_{j=0}^{L-1} \tau^j(\Sigma_d).$$

Then, we have

$$\begin{aligned} &\limsup_{n\to\infty} \frac{1}{L}\sum_{j=0}^{L-1} \tfrac{1}{n}\Sigma_{\boldsymbol{Y}_{\Lambda_L}-\hat{\boldsymbol{Y}}_{\tau^j(\Lambda_L)}} \\ &\overset{\text{(a)}}{=} \limsup_{n\to\infty} \frac{1}{L}\sum_{j=0}^{L-1} \tfrac{1}{n}\Sigma_{\boldsymbol{Y}_{\tau^j(\Lambda_L)}-\hat{\boldsymbol{Y}}_{\tau^j(\Lambda_L)}} \\ &\overset{\text{(b)}}{\preceq} \frac{1}{L}\sum_{j=0}^{L-1} \tau^j(\Sigma_d) \overset{\text{(c)}}{=} \overline{\Sigma_d}. \end{aligned} \tag{65}$$

Step (a) follows from (64). Step (b) follows from (62). Step (c) follows from the definition of $\overline{\Sigma_d}$. From $\boldsymbol{Y}_{\Lambda_L}$, we construct an estimation $\hat{\boldsymbol{X}}_{\Lambda_L}$ of $\boldsymbol{X}_{\Lambda_L}$ by $\hat{\boldsymbol{X}}_{\Lambda_L} = \tilde{A}\hat{\boldsymbol{Y}}_{\Lambda_L}$. Then for $j = 0, 1, \cdots, L-1$, we have the following:

$$\begin{aligned} &\Sigma_{X_{\Lambda_L}}^{-1} + \Sigma_{N_{\tau^j(\Lambda_L)}(r_{\tau^j(\Lambda_L)}^{(n)})}^{-1} \\ &\overset{\text{(a)}}{=} \Sigma_{X_{\tau^j(\Lambda_L)}}^{-1} + \Sigma_{N_{\tau^j(\Lambda_L)}(r_{\tau^j(\Lambda_L)}^{(n)})}^{-1} \\ &\overset{\text{(b)}}{\succeq} \tfrac{1}{n}\Sigma_{\boldsymbol{X}_{\tau^j(\Lambda_L)}-\hat{\boldsymbol{X}}_{\tau^j(\Lambda_L)}}^{-1} \overset{\text{(c)}}{=} \tfrac{1}{n}\Sigma_{\boldsymbol{X}_{\Lambda_L}-\hat{\boldsymbol{X}}_{\tau^j(\Lambda_L)}}^{-1} \\ &= \left[\tilde{A}\left(\tfrac{1}{n}\Sigma_{\boldsymbol{Y}_{\Lambda_L}-\hat{\boldsymbol{Y}}_{\tau^j(\Lambda_L)}}\right){}^{\mathrm{t}}\tilde{A} + \Sigma_{X_{\Lambda_L}|Y_{\Lambda_L}}\right]^{-1}. \end{aligned} \tag{66}$$

Steps (a) and (c) follow from the cyclic shift invariant property of $X_{\Lambda_L}$ and $\boldsymbol{X}_{\Lambda_L}$, respectively. Step (b) follows from Lemma 9. From (66), we have

$$\begin{aligned} &\frac{1}{L}\sum_{j=0}^{L-1}\left[\Sigma_{X_{\Lambda_L}}^{-1} + \Sigma_{N_{\tau^j(\Lambda_L)}(r_{\tau^m(\Lambda_L)}^{(n)})}^{-1}\right] \\ &\succeq \frac{1}{L}\sum_{j=0}^{L-1}\left[\tilde{A}\left(\tfrac{1}{n}\Sigma_{\boldsymbol{Y}_{\Lambda_L}-\hat{\boldsymbol{Y}}_{\tau^j(\Lambda_L)}}\right){}^{\mathrm{t}}\tilde{A} + \Sigma_{X_{\Lambda_L}|Y_{\Lambda_L}}\right]^{-1} \\ &\overset{\text{(a)}}{\succeq} \left[\tilde{A}\left(\frac{1}{L}\sum_{j=0}^{L-1}\tfrac{1}{n}\Sigma_{\boldsymbol{Y}_{\Lambda_L}-\hat{\boldsymbol{Y}}_{\tau^j(\Lambda_L)}}\right){}^{\mathrm{t}}\tilde{A} + \Sigma_{X_{\Lambda_L}|Y_{\Lambda_L}}\right]^{-1} \\ &= \left[\tilde{A}\left(\frac{1}{L}\sum_{j=0}^{L-1}\tfrac{1}{n}\Sigma_{\boldsymbol{Y}_{\Lambda_L}-\hat{\boldsymbol{Y}}_{\tau^j(\Lambda_L)}} + B\right){}^{\mathrm{t}}\tilde{A}\right]^{-1}. \end{aligned} \tag{67}$$

Step (a) follows form that $(\tilde{A}\Sigma{}^{\mathrm{t}}\tilde{A} + \Sigma_{X_{\Lambda_L}|Y_{\Lambda_L}})^{-1}$ is convex with respect to $\Sigma$. On the other hand, we have

$$\begin{aligned} &\frac{1}{L}\sum_{j=0}^{L-1}\left[\Sigma_{X_{\Lambda_L}}^{-1} + \Sigma_{N_{\tau^j(\Lambda_L)}(r_{\tau^j(\Lambda_L)}^{(n)})}^{-1}\right] \\ &= \Sigma_{X_{\Lambda_L}}^{-1} + \left(\frac{1}{L}\sum_{l=1}^{L}\frac{1-\mathrm{e}^{-2r_l^{(n)}}}{\epsilon}\right) I_L \\ &\overset{\text{(a)}}{\preceq} \Sigma_{X_{\Lambda_L}}^{-1} + \left(\frac{1-\mathrm{e}^{-2\frac{1}{L}\sum_{l=1}^{L} r_l^{(n)}}}{\epsilon}\right) I_L \\ &= \Sigma_{X_{\Lambda_L}}^{-1} + \left(\frac{1-\mathrm{e}^{-2r^{(n)}}}{\epsilon}\right) I_L. \end{aligned} \tag{68}$$

Step (a) follows from that $1 - \mathrm{e}^{-2a}$ is a concave function of $a$. Combining (67) and (68), we obtain

$$\begin{aligned} &\Sigma_{X_{\Lambda_L}}^{-1} + \left(\frac{1-\mathrm{e}^{-2r^{(n)}}}{\epsilon}\right) I_L \\ &\succeq \left[\tilde{A}\left(\frac{1}{L}\sum_{j=0}^{L-1}\tfrac{1}{n}\Sigma_{\boldsymbol{Y}_{\Lambda_L}-\hat{\boldsymbol{Y}}_{\tau^j(\Lambda_L)}} + B\right){}^{\mathrm{t}}\tilde{A}\right]^{-1}, \end{aligned}$$

from which we obtain

$$\begin{aligned} &\frac{1}{L}\sum_{j=0}^{L-1}\tfrac{1}{n}\Sigma_{\boldsymbol{Y}_{\Lambda_L}-\hat{\boldsymbol{Y}}_{\tau^j(\Lambda_L)}} + B \\ &\succeq \left[{}^{\mathrm{t}}\tilde{A}\left\{\Sigma_{X_{\Lambda_L}}^{-1} + \left(\frac{1-\mathrm{e}^{-2r^{(n)}}}{\epsilon}\right) I_L\right\}\tilde{A}\right]^{-1}. \end{aligned} \tag{69}$$

Next we derive a lower bound of the sum rate part. For each $j=0,1,\cdots,L-1$, we have the following chain of inequalities:

$$\begin{aligned}
&\sum_{l\in\Lambda_L} nR_l^{(n)} \geq \sum_{l\in\Lambda_L}\log M_l \geq \sum_{l\in\Lambda_L} H(W_{j,l})\\
&\geq H(W_{\tau^j(\Lambda_L)}) = I(\boldsymbol{X}_{\Lambda_L};W_{\tau^j(\Lambda_L)}) + H(W_{\tau^l(\Lambda_L)}|\boldsymbol{X}_{\Lambda_L})\\
&\overset{\text{(a)}}{=} I(\boldsymbol{X}_{\Lambda_L};W_{\tau^j(\Lambda_L)}) + \sum_{l\in\Lambda_L} H(W_{j,l}|\boldsymbol{X}_{\Lambda_L})\\
&= I(\boldsymbol{X}_{\Lambda_L};W_{\tau^j(\Lambda_L)}) + \sum_{l\in\Lambda_L} I(\boldsymbol{Y}_{\Lambda_L};W_{j,l}|\boldsymbol{X}_{\Lambda_L})\\
&\overset{\text{(b)}}{=} I(\boldsymbol{X}_{\Lambda_L};W_{\tau^j(\Lambda_L)}) + nLr^{(n)}\\
&\overset{\text{(c)}}{\geq} \frac{n}{2}\log\left[\frac{\left|\Sigma_{X_{\Lambda_L}}\right|}{\left|\frac{1}{n}\Sigma_{\boldsymbol{X}_{\Lambda_L}-\hat{\boldsymbol{X}}_{\tau^j(\Lambda_L)}}\right|}\right] + nLr^{(n)}\\
&= \frac{n}{2}\log\frac{\left|\tilde{A}\Sigma_{Y_{\Lambda_L}}{}^{\rm t}\tilde{A}+\Sigma_{X_{\Lambda_L}|Y_{\Lambda_L}}\right|}{\left|\tilde{A}\left(\frac{1}{n}\Sigma_{\boldsymbol{Y}_{\Lambda_L}-\hat{\boldsymbol{Y}}_{\tau^j(\Lambda_L)}}\right){}^{\rm t}\tilde{A}+\Sigma_{X_{\Lambda_L}|Y_{\Lambda_L}}\right|} + nLr^{(n)}\\
&= \frac{n}{2}\log\frac{\left|\Sigma_{Y_{\Lambda_L}}+B\right|}{\left|\frac{1}{n}\Sigma_{\boldsymbol{Y}_{\Lambda_L}-\hat{\boldsymbol{Y}}_{\tau^j(\Lambda_L)}}+B\right|} + nLr^{(n)}. \qquad (70)
\end{aligned}$$

Step (a) follows from (48). Step (b) follows from (65). Step (c) follows from (53). From (70), we have

$$\begin{aligned}
&\sum_{l\in\Lambda_L} R_l^{(n)} = \frac{1}{L}\sum_{j=0}^{L-1}\sum_{l\in\Lambda_L} R_l^{(n)}\\
&\geq \frac{1}{L}\sum_{j=0}^{L-1}\frac{1}{2}\log\frac{\left|\Sigma_{Y_{\Lambda_L}}+B\right|}{\left|\frac{1}{n}\Sigma_{\boldsymbol{Y}_{\Lambda_L}-\hat{\boldsymbol{Y}}_{\tau^j(\Lambda_L)}}+B\right|} + Lr^{(n)}\\
&\overset{\text{(a)}}{\geq} \frac{1}{2}\log\frac{\left|\Sigma_{Y_{\Lambda_L}}+B\right|}{\left|\frac{1}{L}\sum_{j=0}^{L-1}\frac{1}{n}\Sigma_{\boldsymbol{Y}_{\Lambda_L}-\hat{\boldsymbol{Y}}_{\tau^j(\Lambda_L)}}+B\right|} + Lr^{(n)}. \qquad (71)
\end{aligned}$$

Step (a) follows from that $-\log|\Sigma+B|$ is convex with respect to $\Sigma$. Letting $n\to\infty$ in (69) and (71) and taking (65) into account, we have

$$\sum_{l\in\Lambda_L} R_l \geq \frac{1}{2}\log\frac{\left|\Sigma_{Y_{\Lambda_L}}+B\right|}{\left|\overline{\Sigma_d}+B\right|}+Lr, \qquad (72)$$

$$\overline{\Sigma_d}+B \succeq \left[{}^{\rm t}\tilde{A}\left\{\Sigma_{X_{\Lambda_L}}^{-1}+\left(\frac{1-{\rm e}^{-2r}}{\epsilon}\right)I_L\right\}\tilde{A}\right]^{-1}, \qquad (73)$$

$${\rm tr}[\overline{\Sigma_d}+B] = {\rm tr}[\Sigma_d]+{\rm tr}[B] \leq D+{\rm tr}[B]. \qquad (74)$$

Now we choose an orthogonal matrix $Q\in\mathcal{O}_L$ so that

$$Q^{\rm t}\tilde{A}\left\{\Sigma_{X_{\Lambda_L}}^{-1}+\left(\frac{1-{\rm e}^{-2r}}{\epsilon}\right)I_L\right\}\tilde{A}^{\rm t}Q = \begin{bmatrix}\beta_1 & & & \mbox{\Large 0}\\ & \beta_2 & & \\ & & \ddots & \\ \mbox{\Large 0} & & & \beta_L\end{bmatrix}.$$

Set

$$\hat{\Sigma}_d \triangleq Q\Sigma_d{}^{\rm t}Q, \hat{B}_d \triangleq QB^{\rm t}Q, \xi_l \triangleq \left[\hat{\Sigma}_d+\hat{B}\right]_{ll}.$$

From (73) and (74), we have

$$\left.\begin{aligned}
&\xi_l \geq \beta_l^{-1}(r), l\in\Lambda_L,\\
&\sum_{l=1}^L \xi_l = {\rm tr}\left[\hat{\Sigma}_d+\hat{B}\right] = {\rm tr}\left[\Sigma_d+B\right] \leq D+{\rm tr}[B].
\end{aligned}\right\} \qquad (75)$$

From (75), we have

$$\begin{aligned}
\sum_{l=1}^L\frac{1}{\beta_l(r)} &\leq \sum_{l=1}^L \xi_l = {\rm tr}[\hat{\Sigma_d}+\hat{B}] \leq D+{\rm tr}[B]\\
&\Leftrightarrow r \geq r^*(D+{\rm tr}[B]). \qquad (76)
\end{aligned}$$

Furthermore, by Hadamard's inequality we have

$$|\Sigma_d+B| = |\hat{\Sigma}_d+\hat{B}| \leq \prod_{l=1}^L[\hat{\Sigma}_d+\hat{B}]_{ll} = \prod_{l=1}^L \xi_l. \qquad (77)$$

Combining (75) and (77), we obtain

$$|\Sigma_d+B| \leq \max_{\substack{\xi_l\beta_l\geq 1, l\in\Lambda_L,\\ \sum_{l=1}^L\xi_l\leq D+{\rm tr}[B]}}\prod_{l=1}^L\xi_l = \tilde{\omega}(D,r). \qquad (78)$$

Hence, from (72), (76), and (78), we have

$$\begin{aligned}
\sum_{l=1}^L R_l &\geq \min_{r\geq r^*(D+{\rm tr}[B])}\frac{1}{2}\log\left[\frac{{\rm e}^{Lr}|\Sigma_Y+B|}{\tilde{\omega}(D,r)}\right]\\
&= \min_{r\geq r^*(D+{\rm tr}[B])}\underline{\tilde{J}}(D,r) = R_{{\rm sum},L}(D|\Sigma_{Y^L}),
\end{aligned}$$

completing the proof. ■

### B. Derivation of the Inner Bound

In this subsection we prove $\mathcal{R}_L^{(\rm in)}(\Sigma_d\,|\Sigma_{X^KY^L})\subseteq\mathcal{R}_L(\Sigma_d\,|\Sigma_{X^KY^L})$ stated in Theorem 3.

*Proof of* $\mathcal{R}_L^{(\rm in)}(\Sigma_d|\Sigma_{X^KY^L})\subseteq\mathcal{R}_L(\Sigma_d|\Sigma_{X^KY^L})$*:* Since $\hat{\mathcal{R}}_L^{(\rm in)}(\,\Sigma_d|\Sigma_{X^KY^L})\subseteq\mathcal{R}_L(\Sigma_d|\Sigma_{X^KY^L})$ is proved by Theorem 1, it suffices to show $\mathcal{R}_L^{(\rm in)}(\Sigma_d|\Sigma_{X^KY^L})=\hat{\mathcal{R}}_L^{(\rm in)}(\Sigma_d|\Sigma_{X^KY^L})$ to prove $\mathcal{R}_L^{(\rm in)}(\Sigma_d|\Sigma_{X^KY^L})\subseteq\mathcal{R}_L(\Sigma_d|\Sigma_{X^KY^L})$. We assume that $R^L\in\mathcal{R}_L^{(\rm in)}\,(\Sigma_d|\Sigma_{X^KY^L})$. Then, there exists nonnegative vector $r^L$ such that

$$\left(\Sigma_{X^K}^{-1}+{}^{\rm t}A\Sigma_{N^L(r^L)}^{-1}A\right)^{-1}\preceq\Sigma_d$$

and

$$\sum_{i\in S}R_l \geq J_S(r_S|r_{S^{\rm c}}) \text{ for any } S\subseteq\Lambda_L. \qquad (79)$$

Let $V_l, l\in\Lambda_L$ be $L$ independent zero mean Gaussian random variables with variance $\sigma_{V_l}^2$. Define Gaussian random variables $U_i, l\in\Lambda_L$ by $U_l=X_l+N_l+V_l$. By definition it is obvious that

$$\left.\begin{aligned}
&U^L\to Y^L\to X^K\\
&U_S\to Y_S\to X^K\to Y_{S^{\rm c}}\to U_{S^{\rm c}}\\
&\text{for any } S\subseteq\Lambda_L.
\end{aligned}\right\} \qquad (80)$$

For given $r_l\geq 0, l\in\Lambda_L$, choose $\sigma_{V_l}^2$ so that $\sigma_{V_l}^2=\sigma_{N_l}^2/({\rm e}^{2r_l}-1)$ when $r_l>0$. When $r_l=0$, we choose $U_l$ so that $U_l$ takes constant value zero. In the above choice the covariance matrix of $N^L + V^L$ becomes $\Sigma_{N^L(r^L)}$. Define the linear function $\psi$ of $U^L$ by

$$\psi\left(U^L\right) = (\Sigma_{X^K}^{-1} + {}^{\rm t}A\Sigma_{N^L(r^L)}^{-1}A)^{-1}{}^{\rm t}A\Sigma_{N^L(r^L)}^{-1}U^L.$$

Set $\hat{X}^L = \psi\left(U^L\right)$ and

$$d_{kk} \stackrel{\triangle}{=} {\rm E}\left[||X_k - \hat{X}_k||^2\right], 1 \leq k \leq K,$$
$$d_{kk'} \stackrel{\triangle}{=} {\rm E}\left[\left(X_k - \hat{X}_k\right)\left(X_{k'} - \hat{X}_{k'}\right)\right], 1 \leq k \neq k' \leq K.$$

Let $\Sigma_{X^K - \hat{X}^K}$ be a covariance matrix with $d_{kk'}$ in its $(k, k')$ element. By simple computations we can show that

$$\Sigma_{X^K - \hat{X}^K} = (\Sigma_{X^K}^{-1} + {}^{\rm t}A\Sigma_{N^L(r^L)}^{-1}A)^{-1} \preceq \Sigma_d \quad (81)$$

and that for any $S \subseteq \Lambda_L$,

$$J_S(r_S|r_{S^{\rm c}}) = I(Y_S; U_S|U_{S^{\rm c}}). \quad (82)$$

From (80) and (81), we have $U^L \in \mathcal{G}(\Sigma_d)$. Thus, from (82) $\mathcal{R}_L^{(\rm in)}(\Sigma_d|\Sigma_{X^KY^L}) \subseteq \hat{\mathcal{R}}_L^{(\rm in)}(\Sigma_d|\Sigma_{X^KY^L})$ is concluded. ■

### *C. Proofs of the Results on Matching Conditions*

We first observe that the condition

$${\rm tr}\left[\Gamma\left(\Sigma_{X^K}^{-1} + {}^{\rm t}A\Sigma_{N^L(r^L)}^{-1}A\right)^{-1}{}^{\rm t}\Gamma\right] \leq D$$

is equivalent to

$$\sum_{k=1}^{K} \frac{1}{\alpha_k(r^L)} \leq D. \quad (83)$$

*Proof of Lemma 3:* Let $\Lambda_K = \{1, 2, \cdots, K\}$ and let $S \subseteq \Lambda_K$ be a set of integers that satisfies $\alpha_i^{-1} \geq \xi$ in the definition of $\theta(\Gamma, D, r^L)$. Then, $\theta(\Gamma, D, r^L)$ is computed as

$$\theta(\Gamma, D, r^L) = \tfrac{1}{(K-|S|)^{K-|S|}}\left(\prod_{j\in S}\frac{1}{\alpha_j}\right)\left(D - \sum_{k\in S}\frac{1}{\alpha_k}\right)^{K-|S|}.$$

Fix $l \in \Lambda_L$ arbitrarily and set $\Psi_l \stackrel{\triangle}{=} 2r_l - \log\theta(\Gamma, D, r^L)$. Computing the partial derivative of $\Psi_l$ by $r_l$, we obtain

$$\begin{aligned}
\frac{\partial\Psi_l}{\partial r_l} &= \sum_{j\in S}\left(\frac{\partial\alpha_j}{\partial r_l}\right)\left[\frac{1}{\alpha_j} - \frac{K-|S|}{D - \sum_{k\in S}\frac{1}{\alpha_k}}\frac{1}{\alpha_j^2}\right] + 2 \\
&\stackrel{\rm (a)}{\geq} \sum_{j\in S}\left(\frac{\partial\alpha_j}{\partial r_l}\right)\left[\frac{1}{\alpha_j} - \frac{K-|S|}{\sum_{k\in\Lambda_K - S}\frac{1}{\alpha_k}}\frac{1}{\alpha_j^2}\right] + 2 \\
&\geq \sum_{j\in S}\left(\frac{\partial\alpha_j}{\partial r_l}\right)\left[\frac{1}{\alpha_j} - \frac{\alpha_{\max}}{\alpha_j^2}\right] + 2 \\
&\stackrel{\rm (b)}{=} \sum_{j\in S}\left(\frac{\partial\alpha_j}{\partial r_l}\right)\left[\frac{\alpha_j - \alpha_{\max}}{\alpha_j^2}\right] + \frac{\sigma_{N_l}^2{\rm e}^{2r_l}}{||\hat{\boldsymbol{a}}_l||^2}\sum_{j=1}^{L}\left(\frac{\partial\alpha_j}{\partial r_l}\right) \\
&\geq \sum_{j\in S}\left(\frac{\partial\alpha_j}{\partial r_l}\right)\left[\frac{\sigma_{N_l}^2{\rm e}^{2r_l}}{||\hat{\boldsymbol{a}}_l||^2} - \frac{\alpha_{\max}}{\alpha_j}\left(\frac{1}{\alpha_{\max}} - \frac{1}{\alpha_j}\right)\right] \\
&\geq \left[\frac{\sigma_{N_l}^2{\rm e}^{2r_l}}{||\hat{\boldsymbol{a}}_l||^2} - \frac{\alpha_{\max}}{\alpha_{\min}}\left(\frac{1}{\alpha_{\max}} - \frac{1}{\alpha_{\min}}\right)\right]\sum_{j\in S}\left(\frac{\partial\alpha_j}{\partial r_l}\right). \quad (84)
\end{aligned}$$

Step (a) follows from the following inequality which is equivalent to (83):

$$D - \sum_{k\in S}\frac{1}{\alpha_k(r^L)} \geq \sum_{k\in\Lambda_K - S}\frac{1}{\alpha_k(r^L)}.$$

Step (b) follows from Lemma 2. Hence, by (84) and Lemma 2, $\frac{\partial\Psi_l}{\partial r_l}$ is nonnegative if

$$\frac{\sigma_{N_l}^2{\rm e}^{2r_l}}{||\hat{\boldsymbol{a}}_l||^2} - \frac{\alpha_{\max}}{\alpha_{\min}}\left(\frac{1}{\alpha_{\min}} - \frac{1}{\alpha_{\max}}\right) \geq 0,$$

completing the proof. ■

*Proof of Lemma 4:* Without loss of generality we may assume $k = 1$. For $T \in \mathcal{O}_K(\hat{\boldsymbol{a}}_l, k)$, the matrix $C^*(\Gamma^{-1}T, r_l)$ has the form:

$$C^*(\Gamma^{-1}T, r_l) = \left[\begin{array}{c|c} c_{11}^*(\Gamma^{-1}T, r_l) & \boldsymbol{c}_{1[1]}^*(\Gamma^{-1}T) \\ \hline {}^{\rm t}\boldsymbol{c}_{1[1]}^*(\Gamma^{-1}T) & C_{22}^*(\Gamma^{-1}T)\end{array}\right],$$

where $C_{22}^*(\Gamma^{-1}T)$ is a $(K-1)\times(K-1)$ matrix with $c_{kk'}^*(\Gamma^{-1}T)$, $(k, k') \in (\Lambda_K - \{1\})^2$ in its $(k, k')$ element. Since $C^*(\Gamma^{-1}T, r_l) \preceq \alpha_{\max}^*(r_l)I_K$, we must have $C_{22}^*(\Gamma^{-1}T) \preceq \alpha_{\max}^*(r_l)I_{K-1}$. Then we have

$$C^*(\Gamma^{-1}T, r_l) \preceq \left[\begin{array}{c|c} c_{11}^*(\Gamma^{-1}T, r_l) & \boldsymbol{c}_{1[1]}^*(\Gamma^{-1}T) \\ \hline {}^{\rm t}\boldsymbol{c}_{1[1]}^*(\Gamma^{-1}T) & \alpha_{\max}^*(r_l)I_{K-1}\end{array}\right]. \quad (85)$$

Let $\lambda$ be the minimum eigenvalue of the matrix in the right hand side of (85). Then, by (85), we have $\lambda \geq \alpha_{\min}^*(r_l)$ and $\lambda$ satisfies the following:

$$(\lambda - c_{11}^*(\Gamma^{-1}T, r_l))(\lambda - \alpha_{\max}^*(r_l)) - ||\boldsymbol{c}_{1[1]}^*(\Gamma^{-1}T)||^2 = 0. \quad (86)$$

From (86), we have

$$\begin{aligned}
c_{11}^*(\Gamma^{-1}T, r_l) &= \lambda + \frac{||\boldsymbol{c}_{1[1]}^*(\Gamma^{-1}T)||^2}{\alpha_{\max}^*(r_l) - \lambda} \\
&\geq \alpha_{\min}^*(r_l) + \frac{||\boldsymbol{c}_{1[1]}^*(\Gamma^{-1}T)||^2}{\alpha_{\max}^*(r_l) - \alpha_{\min}^*(r_l)} \\
&\geq \alpha_{\min}(r^L) + \frac{||\boldsymbol{c}_{1[1]}^*(\Gamma^{-1}T)||^2}{\alpha_{\max}^* - \alpha_{\min}(r^L)},
\end{aligned}$$

completing the proof. ■

Next we prove Theorems 5 and 6. For simplicity of notation we set

$$a(r^L) \stackrel{\triangle}{=} \frac{1}{\alpha_{\min}(r^L)}, b(r^L) \stackrel{\triangle}{=} \frac{1}{\alpha_{\max}(r^L)}, b^* \stackrel{\triangle}{=} \frac{1}{\alpha_{\max}^*}.$$

Then the condition (6) in Lemma 3 is rewritten as

$$a(r^L)\left[\frac{a(r^L)}{b(r^L)} - 1\right] \leq \frac{\sigma_{N_l}^2{\rm e}^{2r_l}}{||\hat{\boldsymbol{a}}_l||^2}. \quad (87)$$

*Proof of Theorem 5:* For $(l, k) \in \Lambda_L \times \Lambda_K$, we choose $T \in \mathcal{O}_K(\hat{\boldsymbol{a}}_l, k)$. By Lemma 4, we have

$$\frac{\sigma_{N_l}^2{\rm e}^{2r_l}}{||\hat{\boldsymbol{a}}_l||^2} \geq \left[\chi_k^* - \frac{1}{a(r^L)} - \frac{a(r^L)b^*||\boldsymbol{c}_{k[k]}^*||^2}{a(r^L) - b^*}\right]^{-1}. \quad (88)$$

It follows from (87), (88), and Lemma 3 that if for any $l \in \Lambda_L$, there exist $k \in \Lambda_K$ and $T \in \mathcal{O}_K(\hat{\boldsymbol{a}}_l, k)$ such that

$$a(r^L)\left[\frac{a(r^L)}{b(r^L)}-1\right] \leq \left[\chi_k^* - \frac{1}{a(r^L)} - \frac{a(r^L)b^*||\boldsymbol{c}_{k[k]}^*||^2}{a(r^L)-b^*}\right]^{-1} \tag{89}$$

holds for $r^L \in \mathcal{B}(\Gamma, D)$, then $\theta(\Gamma, D, r^L)$ satisfies the MD condition on $\mathcal{B}_L(\Gamma, D)$. Since the left hand side of (89) is a monotone decreasing function of $b(r^L)$ and $b(r^L) \geq b^*$,

$$a(r^L)\left[\frac{a(r^L)}{b^*}-1\right] \leq \left[\chi_k^* - \frac{1}{a(r^L)} - \frac{a(r^L)b^*||\boldsymbol{c}_{k[k]}^*||^2}{a(r^L)-b^*}\right]^{-1} \tag{90}$$

implies (89). Observe that (90) is equivalent to

$$\begin{aligned} & a(r^L)\left[\frac{a(r^L)}{b^*}-1\right] \cdot \left[\chi_k^* - \frac{1}{a(r^L)} - \frac{a(r^L)b^*||\boldsymbol{c}_{k[k]}^*||}{a(r^L)-b^*}\right] \leq 1 \\ & \Leftrightarrow \left(\frac{a(r^L)}{b^*}-1\right)\chi_k^* - \frac{1}{b^*} - a(r^L)||\boldsymbol{c}_{k[k]}^*||^2 \leq 0. \end{aligned} \tag{91}$$

Solving (91) with respect to $a(r^L)$, we have

$$\begin{aligned} a(r^L) &\leq \frac{\chi_k^* + \frac{1}{b^*}}{\frac{1}{b^*}\chi_k^* - ||\boldsymbol{c}_{k[k]}^*||^2} = \frac{b^*\chi_k^*+1}{\chi_k^* - b^*||\boldsymbol{c}_{k[k]}^*||^2} \\ &= b^* + \frac{1+(b^*)^2||\boldsymbol{c}_{k[k]}^*||^2}{\chi_k^* - b^*||\boldsymbol{c}_{k[k]}^*||^2}. \end{aligned} \tag{92}$$

On the other hand, by (83), we have

$$a(r^L) \leq D-(K-1)b(r^L) \leq D-(K-1)b^*. \tag{93}$$

Then we have the following.

$$\begin{aligned} & D \leq Kb^* + \frac{1+(b^*)^2||\boldsymbol{c}_{k[k]}^*||^2}{\chi_k^* - b^*||\boldsymbol{c}_{k[k]}^*||^2}. \\ & \Leftrightarrow D-(K-1)b^* \leq b^* + \frac{1+(b^*)^2||\boldsymbol{c}_{k[k]}^*||^2}{\chi_k^* - b^*||\boldsymbol{c}_{k[k]}^*||^2}. \\ & \Rightarrow \text{(92) holds under (93).} \\ & \Rightarrow \text{(92) holds for } r^L \in \mathcal{B}(\Gamma, D). \\ & \Leftrightarrow \text{(90) holds for } r^L \in \mathcal{B}(\Gamma, D). \\ & \Rightarrow \text{(89) holds for } r^L \in \mathcal{B}(\Gamma, D). \end{aligned}$$

Hence, if for any $l \in \Lambda_L$, there exist $k \in \Lambda_K$ and $T \in \mathcal{O}_K(\hat{\boldsymbol{a}}_l, k)$ such that

$$D \leq \frac{K}{\alpha_{\max}^*} + \frac{1+\frac{||\boldsymbol{c}_{k[k]}^*(\Gamma^{-1}T)||^2}{(\alpha_{\max}^*)^2}}{\chi_k^*(\Gamma^{-1}T) - \frac{||\boldsymbol{c}_{k[k]}^*(\Gamma^{-1}T)||^2}{\alpha_{\max}^*}},$$

then $\theta(\Gamma, D, r^L)$ satisfies the MD condition on $\mathcal{B}_L(\Gamma, D)$. Thus, by Lemma 1,

$$\begin{aligned} D \leq & \frac{K}{\alpha_{\max}^*} \\ & + \min_{l\in\Lambda_L} \max_{\substack{k\in\Lambda_K \\ T\in\mathcal{O}_K(\hat{\boldsymbol{a}}_l,k)}} \frac{1+\frac{||\boldsymbol{c}_{k[k]}^*(\Gamma^{-1}T)||^2}{(\alpha_{\max}^*)^2}}{\chi_k^*(\Gamma^{-1}T) - \frac{||\boldsymbol{c}_{k[k]}^*(\Gamma^{-1}T)||^2}{\alpha_{\max}^*}} \end{aligned}$$

is a sufficient matching condition. ■

*Proof of Theorem 6:* The inequality (6) in Lemma 3 is rewritten as

$$[a(r^L)-b(r^L)]\frac{a(r^L)}{b(r^L)} \leq \tau_l \mathrm{e}^{2r_l}. \tag{94}$$

From (94), we can see that if we have

$$[a(r^L)-b(r^L)]\frac{a(r^L)}{b(r^L)} \leq \tau^* \tag{95}$$

on $\mathcal{B}_L(\Gamma, D)$, then $\theta(\Gamma, D, r^L)$ satisfies the MD condition on $\mathcal{B}_L(\Gamma, D)$. On the other hand, from (83), we obtain

$$a(r^L) \leq D-(K-1)b(r^L). \tag{96}$$

Under (96), we have

$$\begin{aligned} & [a(r^L)-b(r^L)]\frac{a(r^L)}{b(r^L)} \\ & \leq \left[D-Kb(r^L)\right]\frac{D-(K-1)b(r^L)}{b(r^L)}. \end{aligned}$$

Hence the following is a sufficient condition for (95) to hold:

$$\left[D-Kb(r^L)\right]\frac{D-(K-1)b(r^L)}{b(r^L)} \leq \tau^*. \tag{97}$$

Solving (97) with respect to $D$, we obtain

$$D \leq Kb(r^L) + \frac{1}{2}\left[\sqrt{b^2(r^L)+4\tau^* b(r^L)} - b(r^L)\right]. \tag{98}$$

Since the right hand side of (98) is a monotone increasing function of $b(r^L)$ and $b(r^L) \geq 1/\alpha_{\max}^*$ by Lemma 2, the condition

$$D \leq \frac{K}{\alpha_{\max}^*} + \frac{1}{2\alpha_{\max}^*}\left\{\sqrt{1+4\alpha_{\max}^*\tau^*} - 1\right\}$$

is a sufficient condition for (95) to hold. ■

Next, we prove Lemma 6. To prove this lemma we prepare a lemma shown below.

*Lemma 11:* A necessary and sufficient condition for $\underline{\tilde{J}}(D, r)$ to take the maximum at $r = r^*$ is

$$\left(\frac{\mathrm{d}}{\mathrm{d}r}\underline{\tilde{J}}(D,r)\right)_{r=r^*} \geq 0.$$

*Proof:* For simplicity of notation we set $\underline{\tilde{J}}(r) \triangleq \underline{\tilde{J}}(D, r)$. Suppose that

$$\left(\frac{\mathrm{d}\underline{\tilde{J}}(r)}{\mathrm{d}r}\right)_{r=r^*} \geq 0. \tag{99}$$

Under (99), we assume that $\underline{\tilde{J}}(r)$ does not take the minimum at $r = r^*$. Then there exists $\epsilon > 0$ and $\tilde{r} > r^*$ such that $\underline{\tilde{J}}(\tilde{r}) \leq \underline{\tilde{J}}(r^*) - \epsilon$. Since $\underline{\tilde{J}}(r)$ is a convex function of $r \geq r^*$, we have

$$\begin{aligned} & \underline{\tilde{J}}(\tau\tilde{r}+(1-\tau)r^*) \leq \tau\underline{\tilde{J}}(\tilde{r}) + (1-\tau)\underline{\tilde{J}}(r^*) \\ & \leq \tau(\underline{\tilde{J}}(r^*)-\epsilon) + (1-\tau)\underline{\tilde{J}}(r^*) = \underline{\tilde{J}}(r^*) - \tau\epsilon \end{aligned} \tag{100}$$

for any $\tau \in (0, 1]$. From (100), we obtain

$$\frac{\underline{\tilde{J}}(r^*+\tau(\tilde{r}-r^*)) - \underline{\tilde{J}}(r^*)}{\tau(\tilde{r}-r^*)} \leq -\frac{\epsilon}{\tilde{r}-r^*} \tag{101}$$

for any $\tau \in (0,1]$. By letting $\tau \to 0$ in (101), we have

$$\left(\frac{\mathrm{d}\underline{\tilde{J}}(r)}{\mathrm{d}r}\right)_{r=r^*} \le -\frac{\epsilon}{\tilde{r}-r^*} < 0,$$

which contradicts (99). Hence under (99), $\underline{\tilde{J}}(r)$ takes the minimum at $r = r^*$. It is obvious that when $\left(\frac{\mathrm{d}\underline{\tilde{J}}(r)}{\mathrm{d}r}\right)_{r=r^*} < 0$, $\underline{\tilde{J}}(r)$ does not take the minimum at $r = r^*$. ∎

*Proof of Lemma 6:* We first derive expression of $\tilde{\omega}(D,r)$ using $\beta_l = \beta_l(r)$, $l \in \Lambda_L$ in a neighborhood of $r = r^*$. Let $S(r) = \{l : \beta_l(r) < \beta_{l_1}(r)\}$. By definition, $L - |S(r)|$ is equal to the multiplicity of the $\beta_{l_1}(r)$. In particular, for $r = r^*$, we have

$$\frac{1}{\beta_{l_1}(r^*)} = \frac{1}{L-|S(r^*)|}\left(D + \mathrm{tr}[B] - \sum_{l \in S(r^*)} \frac{1}{\beta_l(r^*)}\right). \tag{102}$$

Since $\beta_l(r), l \in \Lambda_L$ are strictly monotone increasing functions of $r$, there exists small positive number $\delta$ such that for any $r \in [r^*, r^*+\delta)$, we have

$$\begin{aligned}
&S(r) = S(r^*),\\
&\frac{1}{\beta_{l_1}(r)} < \frac{1}{L-|S(r)|}\left(D + \mathrm{tr}[B] - \sum_{l \in S(r)} \frac{1}{\beta_l(r)}\right)\\
&\qquad < \frac{1}{\beta_k(r)} \quad \text{for } k \notin S(r^*).
\end{aligned}$$

The function $\tilde{\omega}(D,r)$,$r \in [r^*, r^*+\delta)$ is computed as

$$\begin{aligned}
\tilde{\omega}(D,r) &= \tfrac{1}{(L-|S(r^*)|)^{L-|S(r^*)|}}\left(\prod_{l \in S(r^*)} \frac{1}{\beta_l(r)}\right)\\
&\quad \times \left(D + \mathrm{tr}[B] - \sum_{l \in S(r^*)} \frac{1}{\beta_l(r)}\right)^{L-|S(r^*)|}.
\end{aligned}$$

In the following we use the simple notations $\beta_l$ and $S$ for $\beta_l(r^*)$ and $S(r^*)$, respectively. Computing the derivative of $\underline{\tilde{J}}(D,r)$ at $r = r^*$, we obtain

$$\begin{aligned}
&\frac{1}{2}\left(\frac{\mathrm{d}}{\mathrm{d}r}\underline{\tilde{J}}(D,r)\right)_{r=r^*}\\
&= \frac{1}{\epsilon \mathrm{e}^{2r^*}} \sum_{l \in S}\left(1-\frac{\epsilon}{\mu_l}\right)^2 \left[\frac{1}{\beta_l} - \frac{L-|S|}{D+\mathrm{tr}[B]-\sum_{l\in S}\frac{1}{\beta_l}}\frac{1}{\beta_l^2}\right] + L\\
&\overset{\text{(a)}}{=} \frac{1}{\epsilon \mathrm{e}^{2r^*}} \sum_{l \in S}\left(1-\frac{\epsilon}{\mu_l}\right)^2 \left[\frac{1}{\beta_l} - \frac{\beta_{l_1}}{\beta_l^2}\right] + L\\
&= \frac{1}{\epsilon \mathrm{e}^{2r^*}} \sum_{l=1}^{L}\left(1-\frac{\epsilon}{\mu_l}\right)^2 \left[\frac{1}{\beta_l} - \frac{\beta_{l_1}}{\beta_l^2}\right] + L\\
&= \sum_{l=1}^{L}\left\{\frac{\left(1-\frac{\epsilon}{\mu_l}\right)\left[\mathrm{e}^{2r^*}-1+\frac{\epsilon}{\mu_l}\right]}{\left[\mathrm{e}^{2r^*}-1+\frac{\epsilon}{\mu_l}\right]^2} - \frac{\left(1-\frac{\epsilon}{\mu_{l_1}}\right)\left[\mathrm{e}^{2r^*}-1+\frac{\epsilon}{\mu_{l_1}}\right]}{\left[\mathrm{e}^{2r^*}-1+\frac{\epsilon}{\mu_l}\right]^2}\right\} + L\\
&= \sum_{l=1}^{L} \frac{\mathrm{e}^{2r^*}\left[\mathrm{e}^{2r^*}-1+\frac{\epsilon}{\mu_l}\right] - \left(1-\frac{\epsilon}{\mu_{l_1}}\right)\left[\mathrm{e}^{2r^*}-1+\frac{\epsilon}{\mu_{l_1}}\right]}{\left[\mathrm{e}^{2r^*}-1+\frac{\epsilon}{\mu_l}\right]^2}\\
&\ge 0.
\end{aligned}$$

Step (a) follows from (102). ∎

## VI. Conclusion

We have considered the distributed source coding of correlated Gaussian sources $Y_l, l \in \Lambda_L$ which are $L$ observations of $K$ remote sources $X_k, k \in \Lambda_K$. We have studied the remote source coding problem where the decoder wish to reconstruct $X^K$ and have derived explicit outer bounds $\mathcal{R}_L^{(\mathrm{out})}(\Gamma, D^L|\Sigma_{X^KY^L})$ and $\mathcal{R}_L^{(\mathrm{out})}(\Gamma, D|\Sigma_{X^KY^L})$ of $\mathcal{R}_L(\Gamma, D^L|\Sigma_{X^KY^L})$ and $\mathcal{R}_L(\Gamma, D|\Sigma_{X^KY^L})$, respectively. Those outer bounds are described in a form of positive semi definite programming. On the outer bound $\mathcal{R}_L^{(\mathrm{out})}(\Gamma, D|\Sigma_{X^KY^L})$, we have shown that it has a form of the water filling solution. Using this form, we have derived two different matching conditions for $\mathcal{R}_L^{(\mathrm{out})}(\Gamma, D|\Sigma_{X^KY^L})$ to coincide with $\mathcal{R}_L(\Gamma, D|\Sigma_{X^KY^L})$.

In the case of $K = L, A = I_L$, we have considered the multiterminal source coding problem where the decoder wishes to reconstruct $Y^L = X^L + N^L$. Using the strong relation between the remote source coding problem and the multiterminal source coding problem, we have obtained the outer bounds $\mathcal{R}_L^{(\mathrm{out})}(\Gamma, D^L|\Sigma_{Y^L})$ and $\mathcal{R}_L^{(\mathrm{out})}(\Gamma, D|\Sigma_{Y^L})$, of $\mathcal{R}_L(\Gamma, D^L|\Sigma_{Y^L})$ and $\mathcal{R}_L(\Gamma, D|\Sigma_{Y^L})$, respectively. Furthermore, using this relation, we have obtained the matching condition for $\mathcal{R}_L^{(\mathrm{out})}(\Gamma, D|\Sigma_{Y^L})$ to coincide with $\mathcal{R}_L(\Gamma, D|\Sigma_{Y^L})$.

In the remote source coding problem, finding an explicit condition for $\mathcal{R}_L^{(\mathrm{out})}(\Gamma, D^L|\Sigma_{X^KY^L})$ to be tight is left to us as a future work. Similarly, in the multiterminal source coding problem, finding an explicit condition for $\mathcal{R}_L^{(\mathrm{out})}(\Gamma, D^L|\Sigma_{Y^L})$ to be tight is also left to us as a future work. To investigate those problems we must examine the solutions to the problems of positive semi definite programming describing those two outer bounds. Those analysis are rather mathematical problems in the field of convex optimization.

## Appendix

*Proof of Property 8 part b):* Since

$$\underline{\tilde{J}}(D,r) = Lr - \log\tilde{\omega}(D,r) + \frac{1}{2}\log|\Sigma_{Y^L} + B|,$$

it suffices to prove the concavity of $\log\tilde{\omega}(D,r)$ with respect to $r \ge r^*$. We first observe that $\log\tilde{\omega}(D,r)$ has the following expression:

$$\log\tilde{\omega}(D,r) = \max_{\substack{\sum_{l=1}^{L}\xi_l \le D+\mathrm{tr}[B],\\ \xi_l\beta_l(r)\ge 1}} \sum_{l=1}^{L}\log\xi_l$$

For each $j \in \{1,2\}$, let $\xi_l^{(j)}, l = 1,2,\cdot,L$ be $L$ positive numbers that attain $\log\tilde{\omega}(D,r^{(j)})$. Let $t_1, t_2$ be a pair of nonnegative numbers such that $t_1+t_2=1$. Then we have

$$\begin{aligned} & t_1\log\tilde{\omega}(D,r^{(1)})+t_2\log\tilde{\omega}(D,r^{(2)}) \\ &=\sum_{i=1}^{L}\left(t_1\log\xi_i^{(1)}+t_2\log\xi_i^{(2)}\right) \\ &\overset{\rm (a)}{\leq}\sum_{i=1}^{L}\log\left(t_1\xi_i^{(1)}+t_2\xi_i^{(2)}\right). \end{aligned} \tag{103}$$

Step (a) follows from the concavity of the logarithm functions. Since

$$\begin{aligned} \{\beta_l(r)\}^{-1}&=\frac{\mu_l\epsilon}{\mu_l-\epsilon}\frac{{\rm e}^{2r}}{\mu_l[{\rm e}^{2r}-1]+\epsilon} \\ &=\frac{\mu_l\epsilon}{\mu_l-\epsilon}+\frac{\mu_l\epsilon}{\mu_l[{\rm e}^{2r}-1]+\epsilon} \end{aligned}$$

$\{\beta_l(r)\}^{-1}$ is a convex function of $r\geq r^*$. Then we have

$$\begin{aligned} t_1\xi_i^{(1)}+t_2\xi_i^{(2)}&\geq t_1\{\beta_i(r^{(1)})\}^{-1}+t_2\{\beta_i(r^{(2)})\}^{-1} \\ &\geq\{\beta_i(t_1r^{(1)}+t_2r^{(2)})\}^{-1}, \end{aligned} \tag{104}$$

for $l=1,2,\cdots,L$. Furthermore, we have

$$\sum_{l=1}^{L}\left(t_1\xi_l^{(1)}+t_2\xi_l^{(2)}\right)=t_1\sum_{l=1}^{L}\xi_l^{(1)}+t_2\sum_{l=1}^{L}\xi_l^{(2)}\leq D\,. \tag{105}$$

From (104), (105), and the definition of $\log\tilde{\omega}(D,r)$, we have

$$\sum_{l=1}^{L}\log\left(t_1\xi_l^{(1)}+t_2\xi_l^{(2)}\right)\leq\log\tilde{\omega}\left(D,t_1r_1^{(1)}+t_2r_2^{(2)}\right). \tag{106}$$

From (103) and (106), we have

$$\begin{aligned} & t_1\log\tilde{\omega}(D,r^{(1)})+t_2\log\tilde{\omega}(D,r^{(2)}) \\ &\leq\log\tilde{\omega}\left(D,t_1r_1^{(1)}+t_2r_2^{(2)}\right), \end{aligned}$$

completing the proof. ∎

## A. *Proof of Lemma 7*

In this appendix we prove Lemma 7. To prove this lemma we need some preparations. For $k\in\Lambda_K$ and for $Q\in\mathcal{O}_K$, set

$$F_k(\Sigma|Q)\triangleq\sup_{\substack{p_{\tilde{X}^K|X^K}:\\ \Sigma_{X^K-\hat{X}^K}\preceq\Sigma}}h(Z_k-\hat{Z}_k|Z_{[k]}^K-\hat{Z}_{[k]}^K).$$

To compute $F_k(\Sigma|Q)$, define two random variables by

$$\tilde{X}^K\triangleq X^K-\hat{X}^K,\tilde{Z}^K\triangleq Z^K-\hat{Z}^K.$$

Note that by definition we have $\tilde{Z}^K=Q\tilde{X}^K$. Let $p_{X^K\tilde{X}^K}$ $(x^K,\tilde{x}^K)$ be a density function of $(X^K,\tilde{X}^K)$. Let $q_{Z^K\tilde{Z}^K}$ $(z^K,\tilde{z}^K)$ be a density function of $(Z^K,\tilde{Z}^K)$ induced by the orthogonal matrix $Q$, that is,

$$q_{Z^K\tilde{Z}^K}(z^K,\tilde{z}^K)\triangleq p_{{}^{\rm t}QZ^K\,{}^{\rm t}Q\tilde{Z}^K}({}^{\rm t}Qz^K,{}^{\rm t}Q\tilde{z}^K).$$

Expression of $F_k(\Sigma|Q)$ using the above density functions is the following.

$$\begin{aligned} F_k(\Sigma|Q)&=\sup_{\substack{p_{\tilde{X}^K|X^K}:\\ \Sigma_{\tilde{X}^K}\preceq\Sigma}}h(\tilde{Z}_k|\tilde{Z}_{[k]}^K) \\ &=\sup_{\substack{p_{\tilde{X}^K|X^K}:\\ \Sigma_{\tilde{X}^K}\preceq\Sigma}}-\int q_{\tilde{Z}^K}(z^K)\log q_{\tilde{Z}_k|\tilde{Z}_{[k]}^K}(z_k|z_{[k]}^K){\rm d}z^K \\ &=\sup_{\substack{p_{\tilde{X}^K|X^K}:\\ \Sigma_{\tilde{X}^K}\preceq\Sigma}}-\int q_{\tilde{Z}^K}(z^K)\log\frac{q_{\tilde{Z}^K}(z^K)}{q_{\tilde{Z}_{[k]}^K}(z_{[k]}^K)}{\rm d}z^K. \end{aligned}$$

The following two properties on $F_k(\Sigma|Q)$ are useful for the proof of Lemma 7.

*Lemma 12:* $F_k(\Sigma|Q)$ is concave with respect to $\Sigma$.

*Lemma 13:*

$$F_k(\Sigma|Q)=\frac{1}{2}\log\left\{(2\pi{\rm e})\left[Q\Sigma^{-1}{}^{\rm t}Q\right]_{kk}^{-1}\right\}.$$

We first prove Lemma 7 using those two lemmas and next prove Lemmas 12 and 13.

*Proof of Lemma 7:* We have the following chain of inequalities:

$$\begin{aligned} & h(\boldsymbol{Z}_k|\,\boldsymbol{Z}_{[k]}^KW^K)\leq h(\boldsymbol{Z}_k-\hat{\boldsymbol{Z}}_k\mid\boldsymbol{Z}_{[k]}^K-\hat{\boldsymbol{Z}}_{[k]}^K) \\ &\leq\sum_{t=1}^{n}h(Z_k(t)-\hat{Z}_k(t)\mid Z_{[k]}^K(t)-\hat{Z}_{[k]}^K(t)) \\ &\overset{\rm (a)}{\leq}\sum_{t=1}^{n}F_k\left(\Sigma_{X^K(t)-\hat{X}^K(t)}\Big|Q\right) \\ &\overset{\rm (b)}{\leq}nF_k\left(\frac{1}{n}\sum_{t=1}^{n}\Sigma_{X^K(t)-\hat{X}^K(t)}\Bigg|Q\right) \\ &=nF_k\left(\tfrac{1}{n}\Sigma_{\boldsymbol{X}^K-\hat{\boldsymbol{X}}^K}\Big|Q\right) \\ &\overset{\rm (c)}{=}\frac{n}{2}\log\left\{(2\pi{\rm e})\left[Q\left(\tfrac{1}{n}\Sigma_{\boldsymbol{X}^K-\hat{\boldsymbol{X}}^K}\right)^{-1}{}^{\rm t}Q\right]_{kk}^{-1}\right\}. \end{aligned}$$

Step (a) follows from the definition of $F_k(\Sigma|Q)$. Step (b) follows from Lemma 12. Step (c) follows from Lemma 13. ∎

*Proof of Lemma 12:* For given covariance matrices $\Sigma^{(0)}$ and $\Sigma^{(1)}$, let $p_{\tilde{X}^K|X^K}^{(0)}$ and $p_{\tilde{X}^K|X^K}^{(1)}$ be conditional densities achieving $F_k(\Sigma^{(0)}|Q)$ and $F_k(\Sigma^{(1)}|Q)$, respectively. For $0\leq\alpha\leq 1$, define a conditional density parameterized with $\alpha$ by

$$p_{\tilde{X}^K|X^K}^{(\alpha)}=(1-\alpha)p_{\tilde{X}^K|X^K}^{(0)}+\alpha p_{\tilde{X}^K|X^K}^{(1)}.$$

Let $p_{X^K\tilde{X}^K}^{(\alpha)}$ be a density function of $(X^K,\tilde{X}^K)$ defined by $(p_{\tilde{X}^K|X^K}^{(\alpha)},p_{X^K}^{(\alpha)})$. Let $\Sigma_{\tilde{X}}^{(\alpha)}$ be a covariance matrix computed from the density $p_{\tilde{X}^K}^{(\alpha)}$. Since

$$p_{\tilde{X}^K}^{(\alpha)}=(1-\alpha)p_{\tilde{X}^K}^{(0)}+\alpha p_{\tilde{X}^K}^{(1)},$$

we have

$$\begin{aligned} \Sigma_{\tilde{X}}^{(\alpha)}&=(1-\alpha)\Sigma_{\tilde{X}}^{(0)}+\alpha\Sigma_{\tilde{X}}^{(1)} \\ &\preceq(1-\alpha)\Sigma^{(0)}+\alpha\Sigma^{(1)}. \end{aligned} \tag{107}$$

Let $q^{(\alpha)}_{Z^K\tilde{Z}^K}$ be a density function of $(Z^K,\tilde{Z}^K)$ induced by the orthogonal matrix $Q$, that is,

$$q^{(\alpha)}_{Z^K\tilde{Z}^K}(z^K,\tilde{z}^K)\triangleq p^{(\alpha)}_{{}^{\rm t}QZ^K{}^{\rm t}Q\tilde{Z}^K}({}^{\rm t}Qz^K,{}^{\rm t}Q\tilde{z}^K).$$

By definition it is obvious that

$$q^{(\alpha)}_{\tilde{Z}^K}=(1-\alpha)q^{(0)}_{\tilde{Z}^K}+\alpha q^{(1)}_{\tilde{Z}^K}.$$

Then we have

$$\begin{aligned}
&(1-\alpha)F_k(\Sigma^{(0)}|Q)+\alpha F_k(\Sigma^{(1)}|Q)\\
&=-(1-\alpha)\int q^{(0)}_{\tilde{Z}^K}(z^K)\log\frac{q^{(0)}_{\tilde{Z}^K}(z^K)}{q^{(0)}_{\tilde{Z}^K_{[k]}}(z^K_{[k]})}{\rm d}z^K\\
&\quad-\alpha\int q^{(1)}_{\tilde{Z}^K}(z^K)\log\frac{q^{(1)}_{\tilde{Z}^K}(z^K)}{q^{(1)}_{\tilde{Z}^K_{[k]}}(z^K_{[k]})}{\rm d}z^K\\
&\overset{\rm (a)}{\leq}-\int q^{(\alpha)}_{\tilde{Z}^K}(z^K)\log\frac{q^{(\alpha)}_{\tilde{Z}^K}(z^K)}{q^{(\alpha)}_{\tilde{Z}^K_{[k]}}(z^K_{[k]})}{\rm d}z^K\\
&=-\int q^{(\alpha)}_{\tilde{Z}^K}(z^K)\log q^{(\alpha)}_{\tilde{Z}_k|\tilde{Z}^K_{[k]}}(z_k|z^K_{[k]}){\rm d}z^K\\
&\overset{\rm (b)}{\leq}F_k\left((1-\alpha)\Sigma^{(0)}+\alpha\Sigma^{(1)}\middle|Q\right).
\end{aligned}$$

Step (a) follows from log sum inequality. Step (b) follows from the definition of $F_k(\Sigma|Q)$ and (107). ■

*Proof of Lemma 13:* Let

$$q^{\rm (G)}_{\tilde{Z}^K}(z^K)\triangleq\frac{1}{(2\pi{\rm e})^{\frac{K}{2}}\left|\Sigma_{\tilde{Z}^K}\right|^{\frac12}}{\rm e}^{-\frac12{}^{\rm t}[z^K]\Sigma^{-1}_{\tilde{Z}^K}[z^K]}$$

and let

$$q^{\rm (G)}_{\tilde{Z}_k|\tilde{Z}^K_{[k]}}(z_k|z^K_{[k]})=\frac{q^{\rm (G)}_{\tilde{Z}^K}(z^K)}{q^{\rm (G)}_{\tilde{Z}^K_{[k]}}(z^K_{[k]})}$$

be a conditional density function induced by $q^{\rm (G)}_{\tilde{Z}^K}(\cdot)$. We first observe that

$$\int q_{\tilde{Z}^K}(z^K)\log\frac{q_{\tilde{Z}_k|\tilde{Z}^K_{[k]}}(z_k|z^K_{[k]})}{q^{\rm (G)}_{\tilde{Z}_k|\tilde{Z}^K_{[k]}}(z_k|z^K_{[k]})}{\rm d}z^K\geq 0. \tag{108}$$

From (108), we have the following chain of inequalities:

$$\begin{aligned}
h(\tilde{Z}_k|\tilde{Z}^K_{[k]})&=-\int q_{\tilde{Z}^K}(z^K)\log q_{\tilde{Z}_k|\tilde{Z}^K_{[k]}}(z_k|z^K_{[k]}){\rm d}z^K\\
&\leq-\int q_{\tilde{Z}^K}(z^K)\log q^{\rm (G)}_{\tilde{Z}_k|\tilde{Z}^K_{[k]}}(z_k|z^K_{[k]}){\rm d}z^K\\
&=-\int q_{\tilde{Z}^K}(z^K)\log\frac{q^{\rm (G)}_{\tilde{Z}^K}(z^K)}{q^{\rm (G)}_{\tilde{Z}^K_{[k]}}(z^K_{[k]})}{\rm d}z^K\\
&=-\int q_{\tilde{Z}^K}(z^K)\log q^{\rm (G)}_{\tilde{Z}^K}(z^K){\rm d}z^K\\
&\quad+\int q_{\tilde{Z}^K}(z^K)\log q^{\rm (G)}_{\tilde{Z}^K_{[k]}}(z^K_{[k]}){\rm d}z^K\\
&\overset{\rm (a)}{=}-\int q^{\rm (G)}_{\tilde{Z}^K}(z^K)\log q^{\rm (G)}_{\tilde{Z}^K}(z^K){\rm d}z^K\\
&\quad+\int q^{\rm (G)}_{\tilde{Z}^K}(z^K)\log q^{\rm (G)}_{\tilde{Z}^K_{[k]}}(z^K_{[k]}){\rm d}z^K\\
&=\frac12\log\left\{(2\pi{\rm e})\frac{|\Sigma_{\tilde{Z}^K}|}{|\Sigma_{\tilde{Z}^K_{[k]}}|}\right\}\overset{\rm (b)}{=}\frac12\log\left\{(2\pi{\rm e})\left[\Sigma^{-1}_{\tilde{Z}^K}\right]^{-1}_{kk}\right\}\\
&=\frac12\log\left\{(2\pi{\rm e})\left[Q\Sigma^{-1}_{\tilde{X}^K}{}^{\rm t}Q\right]^{-1}_{kk}\right\}\\
&\overset{\rm (c)}{\leq}\frac12\log\left\{(2\pi{\rm e})\left[Q\Sigma^{-1}{}^{\rm t}Q\right]^{-1}_{kk}\right\}.
\end{aligned}$$

Step (a) follows from the fact that $q_{\tilde{Z}^L}$ and $q^{\rm (G)}_{\tilde{Z}^L}$ yield the same moments of the quadratic form $\log q^{\rm (G)}_{\tilde{Z}^L}$. Step (b) is a well known formula on the determinant of matrix. Step (c) follows from $\Sigma_{\tilde{X}^L}\preceq\Sigma$. Thus

$$F_k(\Sigma|Q)\leq\frac12\log\left\{(2\pi{\rm e})\left[Q\Sigma^{-1}{}^{\rm t}Q\right]^{-1}_{kk}\right\}$$

is concluded. Reverse inequality holds by letting $p_{\tilde{X}^K|X^K}$ be Gaussian with covariance matrix $\Sigma$. ■

### B. *Proof of Lemma 8*

In this appendix we prove Lemma 8. We write an orthogonal matrix $Q\in\mathcal{O}_K$ as $Q=[q_{kk'}]$, where $q_{kk'}$ stands for the $(k,k')$ element of $Q$. The orthogonal matrix $Q$ transforms $X^K$ into $Z^K=QX^K$. Set $\tilde{Q}=Q^{\rm t}A$ and let $\tilde{q}_{kl}$ be the $(k,l)$ element of $Q^{\rm t}A$. The following lemma states an important property on the distribution of Gaussian random vector $Z^K$. This lemma is a basis of the proof of Lemma 8.

*Lemma 14:* For any $k\in\Lambda_K$, we have the following.

$$Z_k=-\frac{1}{g_{kk}}\sum_{k'\neq k}\nu_{kk'}Z_{k'}+\frac{1}{g_{kk}}\sum_{l=1}^{L}\frac{\tilde{q}_{kl}}{\sigma^2_{N_l}}Y_l+\hat{N}_k, \tag{109}$$

where

$$g_{kk}=\left[Q\Sigma^{-1}_{X^K}{}^{\rm t}Q\right]_{kk}+\sum_{l=1}^{L}\frac{\tilde{q}^2_{kl}}{\sigma^2_{N_l}}, \tag{110}$$

$\nu_{kk'}$, $k'\in\Lambda_K-\{k\}$ are suitable constants and $\hat{N}_k$ is a zero mean Gaussian random variables with variance $\frac{1}{g_{kk}}$. For each $k\in\Lambda_K$, $\hat{N}_k$ is independent of $Z_{k'},k'\in\Lambda_K-\{k\}$ and $Y_l,l\in\Lambda_L$.

*Proof:* Without loss of generality we may assume $k=1$. Since $Y^L=AX^K+N^L$, we have

$$\Sigma_{X^KY^L}=\begin{bmatrix}\Sigma_{X^K} & \Sigma_{X^K}{}^{\rm t}A\\ A\Sigma_{X^K} & A\Sigma_{X^K}{}^{\rm t}A+\Sigma_{N^L}\end{bmatrix}.$$

Since $Z^K=QX^K$, we have

$$\Sigma_{Z^KY^L}=\begin{bmatrix}Q\Sigma_{X^K}{}^{\rm t}Q & Q\Sigma_{X^K}{}^{\rm t}A\\ {}^{\rm t}A\Sigma_{X^K}{}^{\rm t}Q & A\Sigma_{X^K}{}^{\rm t}A+\Sigma_{N^L}\end{bmatrix}.$$

The density function $p_{Z^KY^L}(z^K,y^L)$ of $(Z^K,Y^L)$ is given by

$$\begin{aligned}
&p_{Z^KY^L}(z^K,y^L)\\
&=\frac{1}{(2\pi{\rm e})^{\frac{K+L}{2}}\left|\Sigma_{Z^KY^L}\right|^{\frac12}}{\rm e}^{-\frac12{}^{\rm t}[z^Ky^L]\Sigma^{-1}_{Z^KY^L}\begin{bmatrix}z^K\\ y^L\end{bmatrix}},
\end{aligned}$$

where $\Sigma^{-1}_{Z^KY^L}$ has the following form:

$$\Sigma^{-1}_{Z^KY^L}=\begin{bmatrix}Q(\Sigma^{-1}_{X^K}+{}^{\rm t}A\Sigma^{-1}_{N^L}A){}^{\rm t}Q & -Q^{\rm t}A\Sigma^{-1}_{N^L}\\ -\Sigma^{-1}_{N^L}A^{\rm t}Q & \Sigma^{-1}_{N^L}\end{bmatrix}.$$

For $(k,k')\in\Lambda_K^2$ and $l\in\Lambda_K$, set

$$\left.\begin{aligned}\nu_{kk'} &\triangleq \left[Q(\Sigma_{X^K}^{-1}+{}^{\rm t}A\Sigma_{N^L}^{-1}A)^{\rm t}Q\right]_{kk'}\\ &=\left[Q\Sigma_{X^K}^{-1}{}^{\rm t}Q\right]_{kk'}+\sum_{l=1}^{L}\frac{\tilde{q}_{kl}\tilde{q}_{k'l}}{\sigma_{N_l}^2},\\ \beta_{kl} &\triangleq -\left[Q^{\rm t}A\Sigma_{N^L}^{-1}\right]_{kl}=-\frac{\tilde{q}_{kl}}{\sigma_{N_l}^2}.\end{aligned}\right\} \tag{111}$$

Now, we consider the following partition of $\Sigma_{Z^KY^L}^{-1}$:

$$\begin{aligned}\Sigma_{Z^KY^L}^{-1} &= \begin{bmatrix} Q(\Sigma_{X^K}^{-1}+{}^{\rm t}A\Sigma_{N^L}^{-1}A)^{\rm t}Q & -Q^{\rm t}A\Sigma_{N^L}^{-1}\\ -\Sigma_{N^L}^{-1}A^{\rm t}Q & \Sigma_{N^L}^{-1}\end{bmatrix}\\ &=\left[\begin{array}{c|c} g_{11} & {}^{\rm t}g_{12}\\ \hline g_{12} & G_{22}\end{array}\right],\end{aligned}$$

where $g_{11}$, $g_{12}$, and $G_{22}$ are scalar, $K+L-1$ dimensional column vector, and $(K+L-1)\times(K+L-1)$ matrix, respectively. It is obvious from the above partition of $\Sigma_{Z^KY^L}^{-1}$ that we have

$$\left.\begin{aligned} g_{11} &= \nu_{11}=\left[Q\Sigma_{X^K}^{-1}{}^{\rm t}Q\right]_{11}+\sum_{l=1}^{L}\frac{\tilde{q}_{1l}^2}{\sigma_{N_l}^2},\\ g_{12} &= {}^{\rm t}\left[\nu_{12}\cdots\nu_{1K}\beta_{11}\beta_{12}\cdots\beta_{1L}\right].\end{aligned}\right\} \tag{112}$$

It is well known that $\Sigma_{Z^KY^L}^{-1}$ has the following expression:

$$\begin{aligned}\Sigma_{Z^KY^L}^{-1} &= \left[\begin{array}{c|c} g_{11} & {}^{\rm t}g_{12}\\ \hline g_{12} & G_{22}\end{array}\right]\\ &=\left[\begin{array}{c|c} 1 & {}^{\rm t}0_{12}\\ \hline \frac{1}{g_{11}}g_{12} & I_{L-1}\end{array}\right]\left[\begin{array}{c|c} g_{11} & {}^{\rm t}0_{12}\\ \hline 0_{12} & G_{22}-\frac{1}{g_{11}}{}^{\rm t}g_{12}g_{12}\end{array}\right]\\ &\quad\times\left[\begin{array}{c|c} 1 & \frac{1}{g_{11}}{}^{\rm t}g_{12}\\ \hline 0_{12} & I_{L-1}\end{array}\right].\end{aligned}$$

Set

$$\hat{n}_1\triangleq[z_1|{}^{\rm t}z_{[1]}^K{}^{\rm t}y^L]\left[\begin{array}{c}1\\ \hline \frac{1}{g_{11}}g_{12}\end{array}\right]=z_1+\frac{1}{g_{11}}\left[{}^{\rm t}z_{[1]}^K{}^{\rm t}y^L\right]g_{12}. \tag{113}$$

Then, we have

$$[{}^{\rm t}z^K{}^{\rm t}y^L]\Sigma_{Z^KY^L}\begin{bmatrix}z^K\\ y^L\end{bmatrix}=[z_1|{}^{\rm t}z_{[1]}^K{}^{\rm t}y^L]\left[\begin{array}{c|c} g_{11} & {}^{\rm t}g_{12}\\ \hline g_{12} & G_{22}\end{array}\right]\left[\begin{array}{c} z_1\\ \hline z_{[1]}^K\\ y^L\end{array}\right]$$

$$=[\hat{n}_1|{}^{\rm t}z_{[1]}^K{}^{\rm t}y^L]\left[\begin{array}{c|c} g_{11} & {}^{\rm t}0_{12}\\ \hline 0_{12} & G_{22}-\frac{1}{g_{11}}g_{12}{}^{\rm t}g_{12}\end{array}\right]\left[\begin{array}{c}\hat{n}_1\\ \hline z_{[1]}^K\\ y^L\end{array}\right]. \tag{114}$$

From (111)-(113), we have

$$\begin{aligned}\hat{n}_1 &= z_1+\frac{1}{g_{11}}\sum_{j=2}^{L}\nu_{1j}z_j+\frac{1}{g_{11}}\sum_{l=1}^{L}\beta_{1l}y_l\\ &= z_1+\frac{1}{g_{11}}\sum_{j=2}^{L}\nu_{1j}z_j-\frac{1}{g_{11}}\sum_{l=1}^{L}\frac{\tilde{q}_{1l}}{\sigma_{N_l}^2}y_l. \end{aligned}\tag{115}$$

It can be seen from (114) and (115) that the random variable $\hat{N}_1$ defined by

$$\hat{N}_1\triangleq Z_1+\frac{1}{g_{11}}\sum_{j=2}^{L}\nu_{1j}Z_j-\frac{1}{g_{11}}\sum_{l=1}^{L}\frac{\tilde{q}_{1l}}{\sigma_{N_l}^2}Y_l$$

is a zero mean Gaussian random variable with variance $\frac{1}{g_{11}}$ and is independent of $Z_{[1]}^K$ and $Y^L$. This completes the proof of Lemma 14. ∎

The followings are two variants of the entropy power inequality.

*Lemma 15:* Let $\boldsymbol{U}_i, i=1,2,3$ be $n$ dimensional random vectors with densities and let $T$ be a random variable taking values in a finite set. We assume that $\boldsymbol{U}_3$ is independent of $\boldsymbol{U}_1$, $\boldsymbol{U}_2$, and $T$. Then, we have

$$\tfrac{1}{2\pi{\rm e}}{\rm e}^{\frac{2}{n}h(\boldsymbol{U}_2+\boldsymbol{U}_3|\boldsymbol{U}_1T)}\geq\tfrac{1}{2\pi{\rm e}}{\rm e}^{\frac{2}{n}h(\boldsymbol{U}_2|\boldsymbol{U}_1T)}+\tfrac{1}{2\pi{\rm e}}{\rm e}^{\frac{2}{n}h(\boldsymbol{U}_3)}.$$

*Lemma 16:* Let $\boldsymbol{U}_i$, $i=1,2,3$ be $n$ random vectors with densities. Let $T_1,T_2$ be random variables taking values in finite sets. We assume that those five random variables form a Markov chain $(T_1,\boldsymbol{U}_1)\to\boldsymbol{U}_3\to(T_2,\boldsymbol{U}_2)$ in this order. Then, we have

$$\begin{aligned}&\tfrac{1}{2\pi{\rm e}}{\rm e}^{\frac{2}{n}h(\boldsymbol{U}_1+\boldsymbol{U}_2|\boldsymbol{U}_3T_1T_2)}\\ &\geq\tfrac{1}{2\pi{\rm e}}{\rm e}^{\frac{2}{n}h(\boldsymbol{U}_1|\boldsymbol{U}_3T_1)}+\tfrac{1}{2\pi{\rm e}}{\rm e}^{\frac{2}{n}h(\boldsymbol{U}_2|\boldsymbol{U}_3T_2)}.\end{aligned}$$

*Proof of Lemma 8:* By Lemma 14, we have

$$\boldsymbol{Z}_k=-\frac{1}{g_{kk}}\sum_{k'\neq k}\nu_{kk'}\boldsymbol{Z}_{k'}+\frac{1}{g_{kk}}\sum_{l=1}^{L}\frac{\tilde{q}_{kl}}{\sigma_{N_l}^2}\boldsymbol{Y}_l+\hat{\boldsymbol{N}}_k, \tag{116}$$

where $\hat{\boldsymbol{N}}_k$ is a vector of $n$ independent copies of zero mean Gaussian random variables with variance $\frac{1}{g_{kk}}$. For each $k\in\Lambda_K$, $\hat{\boldsymbol{N}}_k$ is independent of $\boldsymbol{Z}_{k'},k'\in\Lambda_K-\{k\}$ and $\boldsymbol{Y}_l,l\in\Lambda_L$. Set

$$h^{(n)}\triangleq\frac{1}{n}h(\boldsymbol{Z}_k|\boldsymbol{Z}_{[k]}^K,W^L).$$

Furthermore, for $l\in\Lambda_L$, define

$$S_l\triangleq\{l,l+1,\cdots,L\},\Psi_l=\Psi_l(\boldsymbol{Y}_{S_l})\triangleq\sum_{j=l}^{L}\frac{\tilde{q}_{kj}}{\sigma_{N_j}^2}\boldsymbol{Y}_j.$$

Applying Lemma 15 to (116), we have

$$\frac{{\rm e}^{2h^{(n)}}}{2\pi{\rm e}}\geq\frac{1}{(g_{kk})^2}\frac{1}{2\pi{\rm e}}{\rm e}^{\frac{2}{n}h(\Psi_1|\boldsymbol{Z}_{[k]}^K,W^L)}+\frac{1}{g_{kk}}. \tag{117}$$

On the quantity $h(\Psi_1|\boldsymbol{z}_{[k]}^K,W^L)$ in the right member of (117), we have the following chain of equalities:

$$\begin{aligned}&h(\Psi_1|\boldsymbol{Z}_{[k]}^K,W^L)\\ &=I(\Psi_1;\boldsymbol{X}^K|\boldsymbol{Z}_{[k]}^K,W^L)+h(\Psi_1|\boldsymbol{X}^K,\boldsymbol{Z}_{[k]}^K,W^L)\\ &\overset{\rm (a)}{=}I(\Psi_1;\boldsymbol{Z}^K|\boldsymbol{Z}_{[k]}^K,W^L)+h(\Psi_1|\boldsymbol{X}^K,W^L)\\ &=I(\Psi_1;\boldsymbol{Z}_k|\boldsymbol{Z}_{[k]}^K,W^L)+h(\Psi_1|\boldsymbol{X}^K,W^L)\\ &=h(\boldsymbol{Z}_k|\boldsymbol{Z}_{[k]}^K,W^L)-h(\boldsymbol{Z}_k|\Psi_1,\boldsymbol{Z}_{[k]}^K,W^L)\\ &\quad+h(\Psi_1|\boldsymbol{X}^K,W^L)\\ &\overset{\rm (b)}{=}nh^{(n)}-h(\boldsymbol{Z}_k|\Psi_1,\boldsymbol{Z}_{[k]}^K)+h(\Psi_1|\boldsymbol{X}^K,W^L)\\ &=nh^{(n)}-\frac{n}{2}\log\left[2\pi{\rm e}(g_{kk})^{-1}\right]+h(\Psi_1|\boldsymbol{X}^K,W^L).\end{aligned}\tag{118}$$

Step (a) follows from that $\boldsymbol{Z}^K$ can be obtained from $\boldsymbol{X}^K$ by the invertible matrix $Q$. Step (b) follows from the Markov chain

$$\boldsymbol{Z}_k\to(\Psi_1,\boldsymbol{Z}_{[k]}^K)\to\boldsymbol{Y}^L\to W^L.$$

From (118), we have

$$\frac{1}{2\pi \mathrm{e}}\mathrm{e}^{\frac{2}{n}h(\Psi_1|\boldsymbol{Z}_{[k]}^K,W^L)}=\frac{\mathrm{e}^{2h^{(n)}}}{2\pi \mathrm{e}}g_{kk}\cdot\frac{1}{2\pi \mathrm{e}}\mathrm{e}^{\frac{2}{n}h(\Psi_1|\boldsymbol{X}^K,W^L)}. \tag{119}$$

Substituting (119) into (117), we obtain

$$\frac{\mathrm{e}^{2h^{(n)}}}{2\pi \mathrm{e}}\geq\frac{\mathrm{e}^{2h^{(n)}}}{2\pi \mathrm{e}}\frac{1}{g_{kk}}\cdot\frac{1}{2\pi \mathrm{e}}\mathrm{e}^{\frac{2}{n}h(\Psi_1|\boldsymbol{X}^K,W^L)}+\frac{1}{g_{kk}}. \tag{120}$$

Solving (120) with respect to $\frac{\mathrm{e}^{2h^{(n)}}}{2\pi\mathrm{e}}$, we obtain

$$\frac{\mathrm{e}^{2h^{(n)}}}{2\pi \mathrm{e}}\geq\left[g_{kk}-\frac{1}{2\pi \mathrm{e}}\mathrm{e}^{\frac{2}{n}h(\Psi_1|\boldsymbol{X}^K,W^L)}\right]^{-1}. \tag{121}$$

Next, we evaluate a lower bound of $\mathrm{e}^{\frac{2}{n}h(\Psi_1|\boldsymbol{X}^K,W^L)}$. Note that for $l=1,2,\cdots,L-1$ we have the following Markov chain:

$$\left(W_{S_{l+1}},\Psi_{l+1}(\boldsymbol{Y}_{S_{l+1}})\right)\to\boldsymbol{X}^K\to\left(W_l,\frac{\tilde{q}_{kl}}{\sigma_{N_l}^2}\boldsymbol{Y}_l\right). \tag{122}$$

Based on (122), we apply Lemma 16 to $\frac{1}{2\pi\mathrm{e}}\mathrm{e}^{\frac{2}{n}h(\Psi_l|\boldsymbol{X}^K,W^L)}$ for $l=1,2,\cdots,L-1$. Then, for $l=1,2,\cdots,L-1$, we have the following chains of inequalities :

$$\begin{aligned}
&\frac{1}{2\pi \mathrm{e}}\mathrm{e}^{\frac{2}{n}h(\Psi_l|\boldsymbol{X}^K,W^L)}\\
&=\frac{1}{2\pi \mathrm{e}}\mathrm{e}^{\frac{2}{n}h\left(\Psi_{l+1}+\frac{\tilde{q}_{kl}}{\sigma_{N_1}^2}\boldsymbol{Y}_l\middle|\boldsymbol{X}^K,W_{S_{l+1}},W_l\right)}\\
&\geq\frac{1}{2\pi \mathrm{e}}\mathrm{e}^{\frac{2}{n}h\left(\Psi_{l+1}|\boldsymbol{X}^K,W_{S_{l+1}}\right)}+\frac{1}{2\pi \mathrm{e}}\mathrm{e}^{\frac{2}{n}h\left(\frac{\tilde{q}_{kl}}{\sigma_{N_l}^2}\boldsymbol{Y}_l\middle|\boldsymbol{X}^K,W_l\right)}\\
&=\frac{1}{2\pi \mathrm{e}}\mathrm{e}^{\frac{2}{n}h\left(\Psi_{l+1}|\boldsymbol{X}^K,W_{S_{l+1}}\right)}+\tilde{q}_{kl}^2\frac{\mathrm{e}^{-2r_l^{(n)}}}{\sigma_{N_l}^2}.
\end{aligned} \tag{123}$$

Using (123) iteratively for $l=1,2,\cdots,L-1$, we have

$$\frac{1}{2\pi \mathrm{e}}\mathrm{e}^{\frac{2}{n}h(\Psi_1|\boldsymbol{X}^K,W^L)}\geq\sum_{l=1}^{L}\tilde{q}_{kl}^2\frac{\mathrm{e}^{-2r_l^{(n)}}}{\sigma_{N_l}^2}. \tag{124}$$

Combining (110), (121), and (124), we have

$$\begin{aligned}
\frac{\mathrm{e}^{2h^{(n)}}}{2\pi \mathrm{e}}&\geq\left\{\left[Q\Sigma_{X^K}^{-1}\,{}^{\mathrm{t}}Q\right]_{kk}+\sum_{l=1}^{L}\tilde{q}_{kl}^2\frac{1-\mathrm{e}^{-2r_l^{(n)}}}{\sigma_{N_l}^2}\right\}^{-1}\\
&=\left[Q\left(\Sigma_{X^K}^{-1}+{}^{\mathrm{t}}A\Sigma_{N_{\Lambda_L}(r_{\Lambda_L}^{(n)})}^{-1}A\right){}^{\mathrm{t}}Q\right]_{kk}^{-1},
\end{aligned}$$

completing the proof. ∎

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