ON MAXIMAL REGULARITY AND SEMIVARIATION OF α -TIMES RESOLVENT FAMILIES

FU-BO LI AND MIAO LI

ABSTRACT. Let $1 < \alpha < 2$ and A be the generator of an α -times resolvent family $\{S_{\alpha}(t)\}_{t\geq 0}$ on a Banach space X. It is shown that the fractional Cauchy problem $\mathbf{D}_t^{\alpha}u(t) = Au(t) + f(t)$, $t \in [0, r]; u(0), u'(0) \in D(A)$ has maximal regularity on C([0, r]; X) if and only if $S_{\alpha}(\cdot)$ is of bounded semivariation on [0, r].

1. INTRODUCTION

Many initial and boundary value problems can be reduced to an abstract Cauchy problem of the form

(1.1)
$$u'(t) = Au(t) + f(t), \quad t \in [0, r] \\ u(0) = x \in D(A)$$

where A is the generator of a C_0 -semigroup. One says that (1.1) has maximal regularity on C([0,r];X) if for every $f \in C([0,r];X)$ there exists a unique $u \in C^1([0,r];X)$ satisfying (1.1). From the closed graph theorem it follows easily that if there is maximal regularity on C([0,r];X), then there exists a constant C > 0 such that

$$\|u'\|_{C([0,r];X)} + \|Au\|_{C([0,r];X)} \le \|f\|_{C([0,r];X)}.$$

Travis [5] proved that the maximal regularity is equivalent to the C_0 -semigroup generated by A being of bounded semivariation on [0, r].

Chyan, Shaw and Piskarev [2] gave similar results for second order Cauchy problems. More precisely, they showed that the second order Cauchy problem

(1.2)
$$u''(t) = Au(t) + f(t), \quad t \in [0, r] \\ u(0) = x, \ u'(0) = y, \quad x, y \in D(A)$$

has maximal regularity on [0, r] if and only if the cosine opeator function generated by A is of bounded semivariation on [0, r].

In this paper we will consider the maximal regularity for fractional Cauchy problem

(1.3)
$$\mathbf{D}_t^{\alpha} u(t) = A u(t) + f(t), \quad t \in [0, r] \\ u(0) = x, \ u'(0) = y, \qquad x, y \in D(A)$$

where $\alpha \in (1,2)$, A is the generator of an α -times resolvent family (see Definition 2.2 below) and $\mathbf{D}_t^{\alpha} u$ is understood in the Caputo sense. We show that (1.3) has maximal regularity on C([0,r]; X) if and only if the corresponding α -times resolvent family is of bounded semivariation on [0, r].

²⁰⁰⁰ Mathematics Subject Classification. Primary 45N05; Secondary 26A33, 34G10.

Key words and phrases. α -times resolvent family, maximal regularity, semivariation

The authors were supported by the NSFC-RFBR Programme (Grant No. 108011120015).

FU-BO LI AND MIAO LI

2. Preliminaries

Let $1 < \alpha < 2$, $g_0(t) := \delta(t)$ and $g_\beta(t) := \frac{t^{\beta-1}}{\Gamma(\beta)}(\beta > 0)$ for t > 0. Recall the Caputo fractional derivative of order $\alpha > 0$

$$\mathbf{D}_t^{\alpha} f(t) := \int_0^t g_{2-\alpha}(t-s) \frac{d^2}{ds^2} f(s) ds, \quad t \in [0,r]$$

for $f \in C^2([0,r];X)$. The condition that $f \in C^2([0,r];X)$ can be relaxed to $f \in C^1([0,r];X)$ and $g_{2-\alpha} * (f - f(0) - f'(0)g_2) \in C^2([0,r];X)$, for details and further properties see [1] and references therein. And in the above we denote by

$$(g_{\beta} * f)(t) = \int_0^t g_{\beta}(t-s)f(s)ds$$

the convolution of g_{β} with f. Note that $g_{\alpha} * g_{\beta} = g_{\alpha+\beta}$.

Consider a closed linear operator A densely defined in a Banach space X and the fractional evolution equation (1.3).

Definition 2.1. A function $u \in C([0, r]; X)$ is called a *strong solution* of (1.3) if

$$u \in C([0,r]; D(A)) \cap C^1([0,r]; X), \quad g_{2-\alpha} * (u(t) - x - ty) \in C^2([0,r]; X)$$

and (1.3) holds on [0, r]. $u \in C([0, r]; X)$ is called a *mild solution* of (1.3) if $g_{\alpha} * u \in D(A)$ and

$$u(t) - x - ty = A(g_{\alpha} * u)(t) + (g_{\alpha} * f)(t)$$

for $t \in [0, r]$.

Definition 2.2. Assume that A is a closed, densely defined linear operator on X. A family $\{S_{\alpha}(t)\}_{t\geq 0} \subset B(X)$ is called an α -times resolvent family generated by A if the following conditions are satisfied:

- (a) $S_{\alpha}(\cdot)$ is strongly continuous on \mathbb{R}_+ and $S_{\alpha}(0) = I$;
- (b) $S_{\alpha}(t)D(A) \subset D(A)$ and $AS_{\alpha}(t)x = S_{\alpha}(t)Ax$ for all $x \in D(A), t \ge 0$;
- (c) For all $x \in D(A)$ and $t \ge 0$, $S_{\alpha}(t)x = x + (g_{\alpha} * S_{\alpha})(t)Ax$.

Remark 2.3. Since A is closed and densely defined, it is easy to show that for all $x \in X$, $(g_{\alpha} * S_{\alpha})(t)x \in D(A)$ and $A(g_{\alpha} * S_{\alpha})(t)x = S_{\alpha}x - x$.

The alpha-times resolvent families are closely related to the solutions of (1.3). It was shown in [1] that if A generates an α -times resolvent family $S_{\alpha}(\cdot)$, then (1.3) has a unique strong solution given by $S_{\alpha}(t)x + \int_{0}^{t} S_{\alpha}(s)yds$.

Next we recall the definition of functions of bounded semivariation (see e.g. [3]). Given a closed interval [a, b] of the real line, a subdivision of [a, b] is a finite sequence $d : a = d_0 < d_1 < \cdots < d_n = b$. Let D[a, b] denote the set of all subdivisions of [a, b].

Definition 2.4. For $G : [a, b] \to B(X)$ and $d \in D[a, b]$, define

$$SV_d[G] = \sup\{\|\sum_{n=1}^n [G(d_i) - G(d_{i-1})]x_i\| : x_i \in X, \|x_i\| \le 1\}$$

and $SV[G] = \sup\{SV_d[G] : d \in D[a, b]\}$. We say G is of bounded servivariation if $SV[G] < \infty$.

3. Main results

We begin with some properties on α -times resolvent families which will be needed in the sequel.

Proposition 3.1. Let $1 < \alpha < 2$ and $\{S_{\alpha}(t)\}_{t \geq 0}$ be the α -times resolvent family with generator A. Define

$$P_{\alpha}(t)x = (g_{\alpha-1} * S_{\alpha})(t)x = \int_0^t g_{\alpha-1}(t-s)S_{\alpha}(s)xds, \quad x \in X,$$

then the following statements are true.

(a) For every $x \in X$, $\int_0^t P_\alpha(s) x ds \in D(A)$ and

$$A\int_0^t P_\alpha(s)xds = S_\alpha(t)x - x;$$

(b) For every $x \in X$, $0 \le a, b \le t$, $\int_a^b s P_\alpha(t-s) x dx \in D(A)$ and

$$A\int_{a}^{b} sP_{\alpha}(t-s)xds = aS_{\alpha}(t-a)x - bS_{\alpha}(t-b)x + \int_{a}^{b} S_{\alpha}(t-s)xds;$$

(c) For every $x \in X$, $\int_0^t g_\alpha(t-s)sP_\alpha(s)xds \in D(A)$ and

$$A\Big(\int_0^t g_\alpha(t-s)sP_\alpha(s)xds\Big) = -\alpha(g_\alpha * S_\alpha)(t)x + tP_\alpha(t)x;$$

(d) If $f \in C([0,r];X)$, then $g_{\alpha} * S_{\alpha} * f \in D(A)$ and

(3.1)
$$A(g_{\alpha} * S_{\alpha} * f) = (S_{\alpha} - 1) * f.$$

Proof. (a) follows from the fact that $\int_0^t P_\alpha(s)xds = (g_1 * g_{\alpha-1} * S_\alpha)(t)x = (g_\alpha * S_\alpha)(t)x \in D(A)$ and $A(g_\alpha * S_\alpha)(t)x = S_\alpha(t)x - x$ by Remark 2.3.

(b) By integration by parts we have

$$\begin{aligned} \int_{a}^{b} sP_{\alpha}(t-s)xds &= \int_{a}^{b} sd_{s}[\int_{0}^{s} P_{\alpha}(t-\tau)xd\tau] \\ &= \int_{a}^{b} sd_{s}[(g_{\alpha}*S_{\alpha})(t-s)x] \\ &= -s(g_{\alpha}*S_{\alpha})(t-s)x\Big|_{a}^{b} + \int_{a}^{b}(g_{\alpha}*S_{\alpha})(t-s)xds \\ &= a(g_{\alpha}*S_{\alpha})(t-a)x - b(g_{\alpha}*S_{\alpha})(t-b)x + \int_{a}^{b}(g_{\alpha}*S_{\alpha})(t-s)xds, \end{aligned}$$

since $(g_{\alpha} * S_{\alpha})(t)xds \in D(A)$ by Remark 2.3, operating A on both sides of the above identity gives (b).

(c) follows from the fact that

$$\int_0^t g_\alpha(t-s)sP_\alpha(s)xds$$

$$= \int_0^t g_\alpha(t-s)(s-t)P_\alpha(s)xds + t\int_0^t g_\alpha(t-s)P_\alpha(s)xds$$

$$= -\alpha \int_0^t g_{\alpha+1}(t-s)P_\alpha(s)xds + t(g_\alpha * P_\alpha)(t)x$$

$$= -\alpha (g_{\alpha+1} * P_\alpha)(t)x + t(g_\alpha * P_\alpha)(t)x$$

$$= -\alpha (g_{\alpha+1} * g_{\alpha-1} * S_\alpha)(t)x + t(g_\alpha * g_{\alpha-1} * S_\alpha)(t)x$$

$$= -\alpha (g_\alpha * g_\alpha * S_\alpha)(t)x + t(g_{\alpha-1} * g_\alpha * S_\alpha)(t)x$$

belongs to D(A) and

$$\begin{aligned} A(\int_0^t g_\alpha(t-s)sP_\alpha(s)xds) &= -\alpha(g_\alpha * A(g_\alpha * S_\alpha))(t)x + t(g_{\alpha-1} * A(g_\alpha * S_\alpha))(t)x \\ &= -\alpha(g_\alpha * (S_\alpha - 1))(t)x + t(g_{\alpha-1} * (S_\alpha - 1))(t)x \\ &= -\alpha(g_\alpha * S_\alpha)(t)x + \alpha g_{\alpha+1}(t)x + t(g_{\alpha-1} * S_\alpha)(t) - tg_\alpha(t)x \\ &= -\alpha(g_\alpha * S_\alpha)(t)x + tP_\alpha(t)x. \end{aligned}$$

(d) (3.1) is true for step functions, and then for continuous functions by the closedness of A.

The following two lemmas can be proved similarly as that in [2, 5].

Lemma 3.2. If $f \in C([0, r]; X)$ and the α -times resolvent family $S_{\alpha}(t)$ is of bounded semivariation on [0, r], then $(P_{\alpha} * f)(t) \in D(A)$ and

$$A(P_{\alpha} * f)(t) = -\int_0^t d_s [S_{\alpha}(t-s)]f(s).$$

Lemma 3.3. If $f \in C([0, r]; X)$ and the α -times resolvent family $S_{\alpha}(t)$ is of bounded semivariation on [0, r], then $\int_0^t d_s [S_{\alpha}(t-s)] f(s)$ is continuous in t on [0, r].

We next turn to the solution of

(3.2)
$$\mathbf{D}_{t}^{\alpha}u(t) = Au(t) + f(t), \quad t \in [0, r], \\ u(0) = 0, \ u'(0) = 0,$$

where A is the generator of an α -times resolvent family. If v(t) is a mild solution of (3.2), then by Definition 2.1 $(g_{\alpha} * v)(t) \in D(A)$ and $v(t) = A(g_{\alpha} * v)(t) + (g_{\alpha} * f)(t)$. It then follows from the properties of α -times resolvent family that

 $1 * v = (S_{\alpha} - A(g_{\alpha} * S_{\alpha})) * v = S_{\alpha} * v - S_{\alpha} * A(g_{\alpha} * v) = S_{\alpha} * (v - A(g_{\alpha} * v)) = S_{\alpha} * g_{\alpha} * f,$ which implies that $g_{\alpha} * S_{\alpha} * f$ is differentiable and

$$v(t) = \frac{d}{dt}(g_{\alpha} * S_{\alpha} * f)(t) = (g_{\alpha-1} * S_{\alpha} * f)(t) = (P_{\alpha} * f)(t).$$

Therefore, the mild solution of (1.3) is given by

(3.3)
$$u(t) = S_{\alpha}(t)x + \int_{0}^{t} S_{\alpha}(s)yds + (P_{\alpha} * f)(t).$$

Proposition 3.4. Let A be the generator of an α -times resolvent family $S_{\alpha}(\cdot)$, and let $f \in C([0,r];X)$ and $x, y \in D(A)$. Then the following statements are equivalent:

- (a) (1.3) has a strong solution;
- (b) $(S_{\alpha} * f)(\cdot) \in C^1([0, r]; X);$
- (c) $(P_{\alpha} * f)(t) \in D(A)$ for $0 \le t \le r$ and $A(P_{\alpha} * f)(t)$ is continuous in t on [0, r].

Proof. (a) If u(t) is a strong solution of (1.3), then u is given by (3.3) since every strong solution is a mild solution. Therefore, by the definition of strong solutions, $g_{2-\alpha} * P_{\alpha} * f = g_1 * S_{\alpha} * f \in C^2([0, r]; X)$; it then follows that $S_{\alpha} * f \in C^1([0, r]; X)$, this is (b).

 $(b) \Rightarrow (c)$. Suppose that $S_{\alpha} * f \in C^{1}([0, r]; X)$. Since $g_{1} * P_{\alpha} * f = g_{\alpha} * S_{\alpha} * f$, by Proposition 3.1(d), $g_{1} * P_{\alpha} * f \in D(A)$ and

(3.4)
$$A(g_1 * P_\alpha * f) = A(g_\alpha * S_\alpha * f) = (S_\alpha - 1) * f.$$

Since A is closed and $S_{\alpha} * f \in C^{1}([0, r]; X)$, we have $P_{\alpha} * f \in D(A)$ and $A(P_{\alpha} * f) = (S_{\alpha} * f)' - f$ is continuous.

 $(c) \Rightarrow (a).$ By (3.4), $g_1 * A(P_\alpha * f) = A(g_1 * P_\alpha * f) = (S_\alpha - 1) * f$, therefore $S_\alpha * f$ is differentiable and thus $g_{2-\alpha} * P_\alpha * f = g_1 * S_\alpha * f$ is in $C^2([0,r];X)$. It is easy to check that u(t) defined by (3.3) is a strong solution of (1.3).

Now we are in the position to give the main result of this paper. The proof is similar to that of Proposition 3.1 in [5] or Theorem 4.2 in [2], we write it out for completeness.

Theorem 3.5. Suppose that A generates an α -times resolvent family $\{S_{\alpha}(t)\}_{t\geq 0}$. Then the function (3.3) is a strong solution of the Cauchy problem (1.3) for every pair $x, y \in D(A)$ and continuous function f if and only if $S_{\alpha}(\cdot)$ is of bounded semivariation on [0, r].

Proof. The sufficiency follows from Lemmas 3.2 and 3.3.

Conversely, suppose that for $x, y \in D(A)$ and continuous function f, u(t) given by (3.3) is a strong solution for (1.3). Define the bounded linear operator $L : C([0, r]; X) \to X$ by $L(f) = (P_{\alpha} * f)(r)$. By Proposition 3.4 (c) $Lf \in D(A)$, it thus follows from the closedness of A that $AL : C([0, r]; X) \to X$ is bounded.

Let $\{d_i\}_{i=0}^n$ be a subdivision of [0, r] and $\epsilon > 0$ such that $\epsilon < \min_{1 \le i \le n} \{|d_i - d_{i-1}|\}$. For $x_i \in X$ with $||x_i|| \le 1$ $(i = 1, 2, \dots, n+1)$, define $f_{d,\epsilon} \in C([0, r]; X)$ by

$$f_{d,\epsilon}(\tau) = \begin{cases} x_i, & d_{i-1} \le \tau \le d_i - \epsilon \\ x_{i+1} + \frac{\tau - d_i}{\epsilon} (x_{i+1} - x_i), & d_i - \epsilon \le \tau \le d_i \end{cases},$$

then $||f_{d,\epsilon}||_{C([0,r];X)} \leq 1$. By Proposition 3.1,

$$AL(f_{d,\epsilon}) = A \int_0^r P_\alpha(r-s) f_{d,\epsilon}(s) ds$$

=
$$\sum_{i=1}^n \left[A \int_{d_{i-1}}^{d_i-\epsilon} P_\alpha(r-s) x_i ds + A \int_{d_i-\epsilon}^{d_i} P_\alpha(r-s) (x_{i+1}-x_i) dx \right]$$

$$= \sum_{i=1}^{n} \left\{ \left[S_{\alpha}(r-d_{i-1})x_{i} - S_{\alpha}(r-d_{i}+\epsilon)x_{i} \right] + \left[S_{\alpha}(r-d_{i}+\epsilon)x_{i+1} - S_{\alpha}(r-d_{i})x_{i+1} \right] - \frac{d}{\epsilon} \left[S_{\alpha}(r-d_{i}+\epsilon)(x_{i+1}-x_{i}) - S_{\alpha}(r-d_{i})(x_{i+1}-x_{i}) \right] + \frac{1}{\epsilon} \left[(d_{i}-\epsilon)S_{\alpha}(r-d_{i}+\epsilon)(x_{i+1}-x_{i}) - d_{i}S_{\alpha}(r-d_{i})(x_{i+1}-x_{i}) \right] + \frac{1}{\epsilon} \int_{d_{i}-\epsilon}^{d_{i}} S_{\alpha}(r-s)(x_{i+1}-x_{i})ds \right\}$$

$$= \sum_{i=1}^{n} \left\{ \left[S_{\alpha}(r-d_{i-1})x_{i} - S_{\alpha}(r-d_{i})x_{i+1} \right] + \frac{1}{\epsilon} \int_{d_{i}-\epsilon}^{d_{i}} S_{\alpha}(r-s)(x_{i+1}-x_{i})ds \right\}$$

$$= \sum_{i=1}^{n} \left\{ \left[S_{\alpha}(r-d_{i-1}) - S_{\alpha}(r-d_{i}) \right]x_{i} - S_{\alpha}(r-d_{i})(x_{i+1}-x_{i}) + \frac{1}{\epsilon} \int_{d_{i}-\epsilon}^{d_{i}} S_{\alpha}(r-s)(x_{i+1}-x_{i})ds \right\}$$

it then follows that

$$\left\|\sum_{i=1}^{n} [S_{\alpha}(r-d_{i-1}) - S_{\alpha}(r-d_{i})]x_{i}\right\|$$

$$\leq \|AL(f_{d,\epsilon})\| + \sum_{i=1}^{n} \|S_{\alpha}(r-d_{i})(x_{i+1}-x_{i}) - \frac{1}{\epsilon} \int_{d_{i}-\epsilon}^{d_{i}} S_{\alpha}(r-s)(x_{i+1}-x_{i})ds\|.$$

By letting $\epsilon \to 0$, we obtain that S_{α} is of bounded semivariation on [0, r].

Corollary 3.6. Suppose that $\{S_{\alpha}(t)\}_{t\geq 0}$ is an α -times resolvent family with generator A and $S_{\alpha}(\cdot)$ is of bounded semivariation on [0,r] for some r > 0. Then $R(P_{\alpha}(t)) \subset D(A)$ for $t \in [0,r]$ and $||tAP_{\alpha}(t)||$ is bounded on [0,r].

Proof. For $x \in X$, consider $f(t) = \alpha S_{\alpha}(t)x$. By Proposition 3.1(c), $tP_{\alpha}(t)x$ is a mild solution of (3.2). Moreover, it follows from Proposition 3.4 that $P_{\alpha} * f$ is a strong solution of (3.2). Since a strong solution must be a mild solution, we have $(P_{\alpha} * f)(t) = tP_{\alpha}(t)x$. Thus our claim follows from Proposition 3.4.

Remark 3.7. Let $\alpha = 1$. If A generates a C_0 -semigroup $T(\cdot)$, then the condition that tAT(t) is bounded on [0, r] implies that $T(\cdot)$ is analytic (see [4]). When $\alpha = 2$ and A generates a cosine function $C(\cdot)$, then the condition that tAC(t) is bounded on [0, r] implies that A is bounded ([2]). However, since there is no semigroup properties for α -times resolvent family, it is not clear that one can get the analyticity of $S_{\alpha}(\cdot)$ from the local boundedness of $tAP_{\alpha}(t)$.

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DEPARTMENT OF MATHEMATICS, SICHUAN UNIVERSITY, CHENGDU, SICHUAN 610064, P.R. CHINA *E-mail address*: lifubo@scu.edu.cn; mli@scu.edu.cn