ON MAXIMAL REGULARITY AND SEMIVARIATION OF α -TIMES RESOLVENT FAMILIES

FU-BO LI AND MIAO LI

ABSTRACT. Let $1 < \alpha < 2$ and A be the generator of an α -times resolvent family $\{S_{\alpha}(t)\}_{t>0}$ on a Banach space X. It is shown that the fractional Cauchy problem $\mathbf{D}_t^{\alpha}u(t) = Au(t) + f(t)$, $t \in [0, r]$; $u(0), u'(0) \in D(A)$ has maximal regularity on $C([0, r]; X)$ if and only if $S_{\alpha}(\cdot)$ is of bounded semivariation on $[0, r]$.

1. INTRODUCTION

Many initial and boundary value problems can be reduced to an abstract Cauchy problem of the form

(1.1)
$$
u'(t) = Au(t) + f(t), \quad t \in [0, r]
$$

$$
u(0) = x \in D(A)
$$

where A is the generator of a C_0 -semigroup. One says that (1.1) has maximal regularity on $C([0, r]; X)$ if for every $f \in C([0, r]; X)$ there exists a unique $u \in C^1([0, r]; X)$ satisfying [\(1.1\)](#page-0-0). From the closed graph theorem it follows easily that if there is maximal regularity on $C([0, r]; X)$, then there exists a constant $C > 0$ such that

$$
||u'||_{C([0,r];X)} + ||Au||_{C([0,r];X)} \leq ||f||_{C([0,r];X)}.
$$

Travis [\[5\]](#page-6-0) proved that the maximal regularity is equivalent to the C_0 -semigroup generated by A being of bounded semivariation on $[0, r]$.

Chyan, Shaw and Piskarev [\[2\]](#page-6-1) gave similar results for second order Cauchy problems. More precisely, they showed that the second order Cauchy problem

(1.2)
$$
u''(t) = Au(t) + f(t), \quad t \in [0, r]
$$

$$
u(0) = x, u'(0) = y, \quad x, y \in D(A)
$$

has maximal regularity on $[0, r]$ if and only if the cosine opeator function generated by A is of bounded semivariation on $[0, r]$.

In this paper we will consider the maximal regularity for fractional Cauchy problem

(1.3)
$$
\mathbf{D}_{t}^{\alpha}u(t) = Au(t) + f(t), \quad t \in [0, r] u(0) = x, u'(0) = y, \qquad x, y \in D(A)
$$

where $\alpha \in (1, 2)$, A is the generator of an α -times resolvent family (see Definition [2.2](#page-1-0) below) and $\mathbf{D}_t^{\alpha} u$ is understood in the Caputo sense. We show that [\(1.3\)](#page-0-1) has maximal regularity on $C([0, r]; X)$ if and only if the corresponding α -times resolvent family is of bounded semivariation on $[0, r]$.

²⁰⁰⁰ Mathematics Subject Classification. Primary 45N05; Secondary 26A33, 34G10.

Key words and phrases. α-times resolvent family, maximal regularity, semivariation

The authors were supported by the NSFC-RFBR Programme (Grant No. 108011120015).

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2. Preliminaries

Let $1 < \alpha < 2$, $g_0(t) := \delta(t)$ and $g_\beta(t) := \frac{t^{\beta-1}}{\Gamma(\beta)}$ $\frac{t^{p-1}}{\Gamma(\beta)}(\beta > 0)$ for $t > 0$. Recall the Caputo fractional derivative of order $\alpha > 0$

$$
\mathbf{D}_{t}^{\alpha} f(t) := \int_{0}^{t} g_{2-\alpha}(t-s) \frac{d^{2}}{ds^{2}} f(s) ds, \quad t \in [0, r]
$$

for $f \in C^2([0,r];X)$. The condition that $f \in C^2([0,r];X)$ can be relaxed to $f \in C^1([0,r];X)$ and $g_{2-\alpha} * (f - f(0) - f'(0)g_2) \in C^2([0, r]; X)$, for details and further properties see [\[1\]](#page-5-0) and references therein. And in the above we denote by

$$
(g_{\beta} * f)(t) = \int_0^t g_{\beta}(t - s) f(s) ds
$$

the convolution of g_β with f. Note that $g_\alpha * g_\beta = g_{\alpha+\beta}$.

Consider a closed linear operator A densely defined in a Banach space X and the fractional evolution equation [\(1.3\)](#page-0-1).

Definition 2.1. A function $u \in C([0, r]; X)$ is called a *strong solution* of [\(1.3\)](#page-0-1) if

$$
u \in C([0, r]; D(A)) \cap C^1([0, r]; X), \quad g_{2-\alpha} * (u(t) - x - ty) \in C^2([0, r]; X)
$$

and [\(1.3\)](#page-0-1) holds on [0, r]. $u \in C([0, r]; X)$ is called a *mild solution* of (1.3) if $g_{\alpha} * u \in D(A)$ and

$$
u(t) - x - ty = A(g_{\alpha} * u)(t) + (g_{\alpha} * f)(t)
$$

for $t \in [0, r]$.

Definition 2.2. Assume that A is a closed, densely defined linear operator on X. A family $\{S_{\alpha}(t)\}_{t>0} \subset B(X)$ is called an α -times resolvent family generated by A if the following conditions are satisfied:

- (a) $S_{\alpha}(\cdot)$ is strongly continuous on \mathbb{R}_{+} and $S_{\alpha}(0) = I$;
- (b) $S_{\alpha}(t)D(A) \subset D(A)$ and $AS_{\alpha}(t)x = S_{\alpha}(t)Ax$ for all $x \in D(A), t \geq 0;$
- (c) For all $x \in D(A)$ and $t \geq 0$, $S_{\alpha}(t)x = x + (g_{\alpha} * S_{\alpha})(t)Ax$.

Remark 2.3. Since A is closed and densely defined, it is easy to show that for all $x \in X$, $(g_{\alpha} * S_{\alpha})(t)x \in D(A)$ and $A(g_{\alpha} * S_{\alpha})(t)x = S_{\alpha}x - x$.

The alpha-times resolvent families are closely related to the solutions of [\(1.3\)](#page-0-1). It was shown in [\[1\]](#page-5-0) that if A generates an α -times resolvent family $S_{\alpha}(\cdot)$, then [\(1.3\)](#page-0-1) has a unique strong solution given by $S_{\alpha}(t)x + \int_0^t S_{\alpha}(s)yds$.

Next we recall the definition of functions of bounded semivariation (see e.g. [\[3\]](#page-6-2)). Given a closed interval [a, b] of the real line, a subdivision of [a, b] is a finite sequence $d : a = d_0 < d_1 <$ $\cdots < d_n = b$. Let $D[a, b]$ denote the set of all subdivisions of $[a, b]$.

Definition 2.4. For $G : [a, b] \to B(X)$ and $d \in D[a, b]$, define

$$
SV_d[G] = \sup\{\|\sum_{n=1}^n [G(d_i) - G(d_{i-1})]x_i\| : x_i \in X, \|x_i\| \le 1\}
$$

and $SV[G] = \sup\{SV_d[G] : d \in D[a, b]\}.$ We say G is of bounded sevivariation if $SV[G] < \infty$.

3. Main results

We begin with some properties on α -times resolvent families which will be needed in the sequel.

Proposition 3.1. *Let* $1 < \alpha < 2$ *and* $\{S_{\alpha}(t)\}_{t\geq 0}$ *be the* α -times resolvent family with generator A*. Define*

$$
P_{\alpha}(t)x = (g_{\alpha-1} * S_{\alpha})(t)x = \int_0^t g_{\alpha-1}(t-s)S_{\alpha}(s)xds, \quad x \in X,
$$

then the following statements are true.

(a) For every $x \in X$, $\int_0^t P_\alpha(s)x ds \in D(A)$ and

$$
A\int_0^t P_\alpha(s)xds = S_\alpha(t)x - x;
$$

(b) For every $x \in X$, $0 \le a, b \le t$, $\int_a^b s P_\alpha(t-s) x dx \in D(A)$ and

$$
A\int_a^b sP_\alpha(t-s)xds = aS_\alpha(t-a)x - bS_\alpha(t-b)x + \int_a^b S_\alpha(t-s)xds;
$$

(c) For every $x \in X$, $\int_0^t g_\alpha(t-s) s P_\alpha(s) x ds \in D(A)$ and

$$
A\Big(\int_0^t g_\alpha(t-s)s P_\alpha(s)x ds\Big) = -\alpha(g_\alpha * S_\alpha)(t)x + tP_\alpha(t)x;
$$

(d) If
$$
f \in C([0, r]; X)
$$
, then $g_{\alpha} * S_{\alpha} * f \in D(A)$ and

(3.1)
$$
A(g_{\alpha} * S_{\alpha} * f) = (S_{\alpha} - 1) * f.
$$

Proof. (a) follows from the fact that $\int_0^t P_\alpha(s)xds = (g_1 * g_{\alpha-1} * S_\alpha)(t)x = (g_\alpha * S_\alpha)(t)x \in D(A)$ and $A(g_{\alpha} * S_{\alpha})(t)x = S_{\alpha}(t)x - x$ by Remark [2.3.](#page-1-1)

(b) By integration by parts we have

$$
\int_{a}^{b} sP_{\alpha}(t-s)xds = \int_{a}^{b} sd_{s} \left[\int_{0}^{s} P_{\alpha}(t-\tau)xd\tau \right]
$$

\n
$$
= \int_{a}^{b} sd_{s} \left[(g_{\alpha} * S_{\alpha})(t-s)x \right]
$$

\n
$$
= -s(g_{\alpha} * S_{\alpha})(t-s)x \Big|_{a}^{b} + \int_{a}^{b} (g_{\alpha} * S_{\alpha})(t-s)xds
$$

\n
$$
= a(g_{\alpha} * S_{\alpha})(t-a)x - b(g_{\alpha} * S_{\alpha})(t-b)x + \int_{a}^{b} (g_{\alpha} * S_{\alpha})(t-s)xds,
$$

since $(g_{\alpha} * S_{\alpha})(t)xds \in D(A)$ by Remark [2.3,](#page-1-1) operating A on both sides of the above identity gives (b).

(c) follows from the fact that

$$
\int_0^t g_\alpha(t-s)sP_\alpha(s)xds
$$
\n
$$
= \int_0^t g_\alpha(t-s)(s-t)P_\alpha(s)xds+t \int_0^t g_\alpha(t-s)P_\alpha(s)xds
$$
\n
$$
= -\alpha \int_0^t g_{\alpha+1}(t-s)P_\alpha(s)xds+t(g_\alpha*P_\alpha)(t)x
$$
\n
$$
= -\alpha(g_{\alpha+1}*P_\alpha)(t)x+t(g_\alpha*P_\alpha)(t)x
$$
\n
$$
= -\alpha(g_{\alpha+1}*g_{\alpha-1}*S_\alpha)(t)x+t(g_{\alpha}*g_{\alpha-1}*S_\alpha)(t)x
$$
\n
$$
= -\alpha(g_\alpha*g_\alpha*S_\alpha)(t)x+t(g_{\alpha-1}*g_\alpha*S_\alpha)(t)x
$$

belongs to $D(A)$ and

$$
A(\int_0^t g_\alpha(t-s)s P_\alpha(s)x ds) = -\alpha(g_\alpha * A(g_\alpha * S_\alpha))(t)x + t(g_{\alpha-1} * A(g_\alpha * S_\alpha))(t)x
$$

\n
$$
= -\alpha(g_\alpha * (S_\alpha - 1))(t)x + t(g_{\alpha-1} * (S_\alpha - 1))(t)x
$$

\n
$$
= -\alpha(g_\alpha * S_\alpha)(t)x + \alpha g_{\alpha+1}(t)x + t(g_{\alpha-1} * S_\alpha)(t) - tg_\alpha(t)x
$$

\n
$$
= -\alpha(g_\alpha * S_\alpha)(t)x + tP_\alpha(t)x.
$$

(d) [\(3.1\)](#page-2-0) is true for step functions, and then for continuous functions by the closedness of $A.$

The following two lemmas can be proved similarly as that in [\[2,](#page-6-1) [5\]](#page-6-0).

Lemma 3.2. *If* $f \in C([0, r]; X)$ *and the* α -times resolvent family $S_{\alpha}(t)$ *is of bounded semivariation on* [0, r], then $(P_\alpha * f)(t) \in D(A)$ and

$$
A(P_{\alpha}*f)(t) = -\int_0^t d_s[S_{\alpha}(t-s)]f(s).
$$

Lemma 3.3. *If* $f \in C([0, r]; X)$ *and the* α -times resolvent family $S_{\alpha}(t)$ *is of bounded semivariation on* $[0, r]$ *, then* $\int_0^t d_s [S_\alpha(t-s)]f(s)$ *is continuous in t on* $[0, r]$ *.*

We next turn to the solution of

(3.2)
$$
\mathbf{D}_{t}^{\alpha}u(t) = Au(t) + f(t), \quad t \in [0, r],
$$

$$
u(0) = 0, u'(0) = 0,
$$

where A is the generator of an α -times resolvent family. If $v(t)$ is a mild solution of [\(3.2\)](#page-3-0), then by Definition [2.1](#page-1-2) $(g_{\alpha} * v)(t) \in D(A)$ and $v(t) = A(g_{\alpha} * v)(t) + (g_{\alpha} * f)(t)$. It then follows from the properties of α -times resolvent family that

 $1 * v = (S_{\alpha} - A(g_{\alpha} * S_{\alpha})) * v = S_{\alpha} * v - S_{\alpha} * A(g_{\alpha} * v) = S_{\alpha} * (v - A(g_{\alpha} * v)) = S_{\alpha} * g_{\alpha} * f,$ which implies that $g_{\alpha}*S_{\alpha}*f$ is differentiable and

$$
v(t) = \frac{d}{dt}(g_{\alpha} * S_{\alpha} * f)(t) = (g_{\alpha - 1} * S_{\alpha} * f)(t) = (P_{\alpha} * f)(t).
$$

Therefore, the mild solution of [\(1.3\)](#page-0-1) is given by

(3.3)
$$
u(t) = S_{\alpha}(t)x + \int_0^t S_{\alpha}(s)yds + (P_{\alpha}*f)(t).
$$

Proposition 3.4. Let A be the generator of an α -times resolvent family $S_{\alpha}(\cdot)$, and let $f \in$ $C([0, r]; X)$ *and* $x, y \in D(A)$ *. Then the following statements are equivalent:*

- *(a) [\(1.3\)](#page-0-1) has a strong solution;*
- *(b)* $(S_{\alpha}*f)(\cdot) \in C^1([0,r];X);$
- (c) $(P_{\alpha}*f)(t) \in D(A)$ *for* $0 \le t \le r$ *and* $A(P_{\alpha}*f)(t)$ *is continuous in* t *on* [0, *r*].

Proof. (a) If $u(t)$ is a strong solution of [\(1.3\)](#page-0-1), then u is given by [\(3.3\)](#page-3-1) since every strong solution is a mild solution. Therefore, by the definition of strong solutions, $g_{2-\alpha} * P_{\alpha} * f = g_1 * S_{\alpha} * f \in$ $C^2([0,r];X)$; it then follows that $S_\alpha * f \in C^1([0,r];X)$, this is (b).

 $(b) \Rightarrow (c)$. Suppose that $S_{\alpha}*f \in C^1([0,r];X)$. Since $g_1 * P_{\alpha}*f = g_{\alpha}*S_{\alpha}*f$, by Proposition [3.1\(](#page-2-1)d), $g_1 * P_\alpha * f \in D(A)$ and

(3.4)
$$
A(g_1 * P_\alpha * f) = A(g_\alpha * S_\alpha * f) = (S_\alpha - 1) * f.
$$

Since A is closed and $S_{\alpha}*f \in C^{1}([0,r];X)$, we have $P_{\alpha}*f \in D(A)$ and $A(P_{\alpha}*f) = (S_{\alpha}*f)' - f$ is continuous.

(c) ⇒ (a). By [\(3.4\)](#page-4-0), $g_1 * A(P_\alpha * f) = A(g_1 * P_\alpha * f) = (S_\alpha - 1) * f$, therefore $S_\alpha * f$ is differentiable and thus $g_{2-\alpha} * P_\alpha * f = g_1 * S_\alpha * f$ is in $C^2([0,r];X)$. It is easy to check that $u(t)$ defined by (3.3) is a strong solution of (1.3) .

Now we are in the position to give the main result of this paper. The proof is similar to that of Proposition 3.1 in [\[5\]](#page-6-0) or Theorem 4.2 in [\[2\]](#page-6-1), we write it out for completeness.

Theorem 3.5. Suppose that A generates an α -times resolvent family $\{S_{\alpha}(t)\}_t>0$. Then the *function* [\(3.3\)](#page-3-1) *is a strong solution of the Cauchy problem* [\(1.3\)](#page-0-1) *for every pair* $x, y \in D(A)$ *and continuous function* f *if and only if* $S_\alpha(\cdot)$ *is of bounded semivariation on* [0, r].

Proof. The sufficiency follows from Lemmas [3.2](#page-3-2) and [3.3.](#page-3-3)

Conversely, suppose that for $x, y \in D(A)$ and continuous function f, $u(t)$ given by [\(3.3\)](#page-3-1) is a strong solution for [\(1.3\)](#page-0-1). Define the bounded linear operator $L : C([0,r]; X) \to X$ by $L(f) = (P_{\alpha} * f)(r)$. By Proposition [3.4](#page-4-1) (c) $Lf \in D(A)$, it thus follows from the closedness of A that $AL: C([0,r];X) \to X$ is bounded.

Let $\{d_i\}_{i=0}^n$ be a subdivision of $[0, r]$ and $\epsilon > 0$ such that $\epsilon < \min_{1 \leq i \leq n} \{|d_i - d_{i-1}|\}$. For $x_i \in X$ with $||x_i|| \leq 1$ $(i = 1, 2, \dots, n + 1)$, define $f_{d,\epsilon} \in C([0, r]; X)$ by

$$
f_{d,\epsilon}(\tau) = \begin{cases} x_i, & d_{i-1} \leq \tau \leq d_i - \epsilon \\ x_{i+1} + \frac{\tau - d_i}{\epsilon} (x_{i+1} - x_i), & d_i - \epsilon \leq \tau \leq d_i \end{cases}
$$

then $||f_{d,\epsilon}||_{C([0,r];X)} \leq 1$. By Proposition [3.1,](#page-2-1)

$$
AL(f_{d,\epsilon}) = A \int_0^r P_{\alpha}(r-s) f_{d,\epsilon}(s) ds
$$

=
$$
\sum_{i=1}^n \left[A \int_{d_{i-1}}^{d_i-\epsilon} P_{\alpha}(r-s) x_i ds + A \int_{d_i-\epsilon}^{d_i} \frac{s-d_i}{\epsilon} P_{\alpha}(r-s) (x_{i+1}-x_i) dx \right]
$$

$$
= \sum_{i=1}^{n} \left\{ [S_{\alpha}(r - d_{i-1})x_i - S_{\alpha}(r - d_i + \epsilon)x_i] + [S_{\alpha}(r - d_i + \epsilon)x_{i+1} - S_{\alpha}(r - d_i)x_{i+1}] \right\}- \frac{d}{\epsilon}[S_{\alpha}(r - d_i + \epsilon)(x_{i+1} - x_i) - S_{\alpha}(r - d_i)(x_{i+1} - x_i)] + \frac{1}{\epsilon}[(d_i - \epsilon)S_{\alpha}(r - d_i + \epsilon)(x_{i+1} - x_i) - d_iS_{\alpha}(r - d_i)(x_{i+1} - x_i)] + \frac{1}{\epsilon} \int_{d_i - \epsilon}^{d_i} S_{\alpha}(r - s)(x_{i+1} - x_i)ds \right\}= \sum_{i=1}^{n} \left\{ [S_{\alpha}(r - d_{i-1})x_i - S_{\alpha}(r - d_i)x_{i+1}] + \frac{1}{\epsilon} \int_{d_i - \epsilon}^{d_i} S_{\alpha}(r - s)(x_{i+1} - x_i)ds \right\}= \sum_{i=1}^{n} \left\{ [S_{\alpha}(r - d_{i-1}) - S_{\alpha}(r - d_i)]x_i - S_{\alpha}(r - d_i)(x_{i+1} - x_i) + \frac{1}{\epsilon} \int_{d_i - \epsilon}^{d_i} S_{\alpha}(r - s)(x_{i+1} - x_i)ds \right\},
$$

it then follows that

$$
\Big\| \sum_{i=1}^{n} [S_{\alpha}(r - d_{i-1}) - S_{\alpha}(r - d_i)]x_i \Big\|
$$

\n
$$
\leq \|AL(f_{d,\epsilon})\| + \sum_{i=1}^{n} \|S_{\alpha}(r - d_i)(x_{i+1} - x_i) - \frac{1}{\epsilon} \int_{d_{i-\epsilon}}^{d_i} S_{\alpha}(r - s)(x_{i+1} - x_i) ds \Big\|.
$$

By letting $\epsilon \to 0$, we obtain that S_{α} is of bounded semivariation on [0, r].

Corollary 3.6. Suppose that $\{S_{\alpha}(t)\}_{t\geq 0}$ is an α -times resolvent family with generator A and $S_{\alpha}(\cdot)$ *is of bounded semivariation on* [0, r] *for some* $r > 0$. Then $R(P_{\alpha}(t)) \subset D(A)$ *for* $t \in [0, r]$ *and* $||tAP_\alpha(t)||$ *is bounded on* $[0, r]$ *.*

Proof. For $x \in X$, consider $f(t) = \alpha S_{\alpha}(t)x$. By Proposition [3.1\(](#page-2-1)c), $tP_{\alpha}(t)x$ is a mild solution of [\(3.2\)](#page-3-0). Moreover, it follows from Proposition [3.4](#page-4-1) that $P_{\alpha} * f$ is a strong solution of [\(3.2\)](#page-3-0). Since a strong solution must be a mild solution, we have $(P_{\alpha}*f)(t) = tP_{\alpha}(t)x$. Thus our claim follows from Proposition [3.4.](#page-4-1)

Remark 3.7. Let $\alpha = 1$. If A generates a C_0 -semigroup $T(\cdot)$, then the condition that $tAT(t)$ is bounded on $[0, r]$ implies that $T(\cdot)$ is analytic (see [\[4\]](#page-6-3)). When $\alpha = 2$ and A generates a cosine function $C(\cdot)$, then the condition that $tAC(t)$ is bounded on $[0, r]$ implies that A is bounded ([\[2\]](#page-6-1)). However, since there is no semigroup properties for α -times resolvent family, it is not clear that one can get the analyticity of $S_{\alpha}(\cdot)$ from the local boundedness of $tAP_{\alpha}(t)$.

REFERENCES

[1] E.G. Bajlekova, Fractional Evolution Equations in Banach Spaces, Dissertation, Eindhoven University of Technology, 2001.

- [2] D.K. Chyan, S.Y. Shaw and S. Piskarev, *On maximal regularity and semivariation of cosine operator functions*, J. London Math. Soc. 59 (1999), 1023-1032.
- [3] C.S. Hönig, Volterra Stieltjes Integral Equations, North-Holland, Amsterdam, 1975.
- [4] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Applied Math. Series 44, Springer, New-York, 1984.
- [5] C.C. Travis, Differentiability of weak solutions to an abstract inhomogeneous differential equation, Proc. Amer. Math. Soc. 82 (1981), 425-430.

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