SEPARATING INVARIANTS FOR THE KLEIN FOUR GROUP AND CYCLIC GROUPS

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Abstract. We consider indecomposable representations of the Klein four group over a field of characteristic 2 and of a cyclic group of order pm with p, m coprime over a field of characteristic p. For each representation we explicitly describe a separating set in the corresponding ring of invariants. Our construction is recursive and the separating sets we obtain consist of almost entirely orbit sums and products.

1. introduction

Let V be a finite dimensional representation of a group G over an algebraically closed field F. In the sequel we will also call V a G-module. There is an induced action on the symmetric algebra $F[V] := S(V^*)$ given by $\sigma(f) = f \circ \sigma^{-1}$ for $\sigma \in G$ and $f \in F[V]$. We let $F[V]^G$ denote the subalgebra of invariant polynomials in $F[V]$. A subset $A \subseteq F[V]^G$ is said to be separating for V if for any pair of vectors $u, w \in V$, we have: If $f(u) = f(w)$ for all $f \in A$, then $f(u) = f(w)$ for all $f \in F[V]^G$. Goals in invariant theory include finding generators and studying properties of invariant rings. In the study of separating invariants the goal is rather to find and describe a subalgebra of the ring of invariants which separates the group orbits. Although separating invariants have been object of study since the early times of invariant theory, they have regained particular attention following the influential textbook of Derksen and Kemper [5]. The invariant ring is often too complicated and it is difficult to describe explicit generators and relations. Meanwhile, there have been several papers within the last decade that demonstrate that one can construct separating subalgebras with nice properties that make them more accessible. For instance Noether's (relative) bound holds for separating invariants independently of the characteristic of the field [5, Corollary 3.9.14]. For more results on separating algebras we direct the reader to [6, 7, 8, 9, 10, 11, 13, 14].

If the order of the group is divisible by the characteristic of the field, then the degrees of the generators increase unboundedly as the dimension of the representation increases. Therefore computing the invariant ring in this case is particularly difficult. Even in the simplest situation of a cyclic group of prime order acting through Jordan blocks, explicit generating sets are known only for a handful of cases. This rather short list of cases consists of indecomposable representations up to dimension nine and decomposable ones whose indecomposable summands

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have dimension at most four. See [17] for a classical work and [18] for the most recent advances in this matter which also gives a good taste of the difficulty of the problem. On the other hand separating invariants for these representations have a surprisingly simple theory. In [15, 16] it is observed that a separating set for a indecomposable representation of a cyclic p-group over a field of characteristic p can be obtained by adding some explicitly defined invariant polynomials to a separating set for a certain quotient representation. The main ingredient of the proofs of these results is the efficient use of the surjection of a representation to a quotient representation to establish a link between the respective separating sets that generating sets do not have. In this paper we build on this technique to construct separating invariants for the indecomposable representations of the Klein four group over a field of characteristic 2 and of a cyclic group of order pm with p, m coprime over a field of characteristic p. Despite being the immediate follow ups of the cyclic p-groups, their invariant rings have not been computed yet. Therefore these groups (and representations) appear to be the natural cases to consider. As in the case for cyclic p-groups, we describe a finite separating set recursively. We remark that in [5, Theorem 3.9.13], see also [12, Corollary 19], a way is given for calculating separating invariants explicitly for any finite group. This is done by presenting a large polynomial whose coefficients form a separating set. On the other hand, the separating sets we compute consist of invariant polynomials that are almost exclusively orbit sums and products. These are "basic" invariants which are easier to obtain. Additionally, our approach respects the inductive structure of the considered modules. Also, the size of the set we give for the cyclic group of order pm depends only on the dimension of the representation while the size in [5, Theorem 3.9.13] depends on the group order as well. Hence, for large p and m our separating set is much smaller for this group.

The strategy of our construction is based on the following theorem.

Theorem 1. *Let V and W be G-modules, $\phi : V \to W$ a G-equivariant surjection, and $\phi^* : F[W] \hookrightarrow F[V]$ the corresponding inclusion. Let $S \subseteq F[W]^G$ be a separating set for W and let $T \subseteq F[V]^G$ be a set of invariant polynomials such that if $v_1, v_2 \in V$ are in different G-orbits and if $\phi(v_1) = \phi(v_2)$, then there is a polynomial $f \in T$ such that $f(v_1) \neq f(v_2)$. Then $\phi^*(S) \cup T$ is a separating set for V.*

Proof. Pick two vectors $v_1, v_2 \in V$ in different G-orbits. If $\phi(v_1)$ and $\phi(v_2)$ are in different G-orbits, then there exits a polynomial $f \in S$ that separates these vectors, so $\phi^*(f)$ separates v_1, v_2. So we may assume that $\phi(v_1)$ and $\phi(v_2)$ are in the same G-orbit. Furthermore, by replacing v_2 with a suitable vector in its orbit we may take $\phi(v_1) = \phi(v_2)$. Hence, by construction, T contains an invariant that separates v_1 and v_2 as desired. □

Before we finish this section we recall the definitions of a transfer and a norm. For a subgroup $H \subseteq G$ and $f \in F[V]^H$, the relative transfer $\mathrm{Tr}_H^G(f)$ is defined to be $\sum_{\sigma \in G/H} \sigma(f)$. We also denote $\mathrm{Tr}_{\{\iota\}}^G(f) = \mathrm{Tr}^G(f)$, where ι is the identity element of G. Also for $f \in F[V]$, the norm $N_H(f)$ is defined to be the product $\prod_{\sigma \in H} \sigma(f)$.

2. The Klein four group

For the Klein four group $G = \{\iota, \sigma_1, \sigma_2, \sigma_3\}$ over an algebraically closed field F of characteristic 2, the complete list of indecomposable G-modules is given in Benson [2, Theorem 4.3.3]. For each module in the list, we will explicitly construct a finite

separating set. The modules in this list come in five "types". We use the same enumeration as in [2]. The first type (i) is just the regular representation FG of G, and a separating set or even the invariant ring can be computed with MAGMA [3]. In the following, we will thus concentrate on the remaining four types, where each type consists of an infinite series of indecomposable representations. Let I_n denote the identity matrix of $F^{n\times n}$, and J_λ denote an upper triangular Jordan block of size n with eigenvalue $\lambda \in F$. Let $H_i = \{\iota, \sigma_i\}$ for $i = 1, 2, 3$ be the three subgroups of order 2.

2.1. **Type (ii).** Let G act on $V_{2n} = F^{2n}$ by the representation $\sigma_1 \mapsto \begin{pmatrix} I_n & I_n \\ 0 & I_n \end{pmatrix}$ and $\sigma_3 \mapsto \begin{pmatrix} I_n & J_\lambda \\ 0 & I_n \end{pmatrix}$. We write $F[V_{2n}] = F[x_1, \ldots, x_{2n}]$. We then have

$$\begin{array}{rcll} \sigma_1 x_i & = & x_i + x_{n+i} & \text{for} \quad 1 \le i \le n, \\ \sigma_3 x_i & = & x_i + \lambda x_{n+i} + x_{n+i+1} & \text{for} \quad 1 \le i \le n-1, \\ \sigma_3 x_n & = & x_n + \lambda x_{2n}, & \\ x_{n+i} & \in & F[V_{2n}]^G & \text{for} \quad 1 \le i \le n. \end{array}$$

We start by computing several transfers and norms modulo some subspaces of $F[V_{2n}]$. Define $R := F[x_2, \ldots, x_n]$. Note that $S := F[x_1, \ldots, x_{n-1}, x_{n+1}, \ldots, x_{2n}]$ is a G-subalgebra of $F[V_{2n}]$, and the congruence in Lemma 2(a) also holds modulo $S \cap R = F[x_2, \ldots, x_{n-1}, x_{n+1}, \ldots, x_{2n}]$. This will be needed for type (v), so we mark this result with a star.

Lemma 2. *We have*

(a)* $\mathrm{Tr}^G(x_1 x_i x_j) \equiv x_1(x_{n+i}x_{n+j+1} + x_{n+i+1}x_{n+j}) \mod R$ *for* $2 \le i, j \le n-1$.

(b) $\mathrm{Tr}^G(x_1 x_{n-1} x_n) \equiv x_1 x_{2n}^2 \mod R$.

Proof. (a*) We only have to care for terms containing x_1, so

$$\begin{array}{rcl} \mathrm{Tr}^G(x_1 x_i x_j) & \equiv & x_1 x_i x_j + x_1(x_i + x_{n+i})(x_j + x_{n+j}) \\ & & +x_1(x_i + \lambda x_{n+i} + x_{n+i+1})(x_j + \lambda x_{n+j} + x_{n+j+1}) \\ & & +x_1(x_i + (\lambda+1)x_{n+i} + x_{n+i+1})(x_j + (\lambda+1)x_{n+j} + x_{n+j+1}) \\ & \equiv & x_1 x_{n+i} x_{n+j+1} + x_1 x_{n+i+1} x_{n+j} \mod R. \end{array}$$

(b) Follows from above with $i = n-1,\ j = n$ and setting $x_{2n+1} := 0$. □

Lemma 3. *For* $n \ge 3$ *we have*

(a) $\mathrm{Tr}^G(x_1 x_2^3) \equiv \lambda(\lambda+1) x_1 x_{n+2}^3 \mod (R + x_{n+3}F[V_{2n}])$.

(b) For $\lambda \in \{0, 1\}$*, we have the invariant*

$$N_{H_2}(x_1 x_{n+2} + x_2 x_{n+1}) \equiv x_1^2 x_{n+2}^2 + x_1 x_{n+2}(x_{n+2}^2 + x_{n+1}x_{n+3}) \mod R.$$

Proof. (a) We only care for terms containing x_1 and not x_{n+3}, so

$$\begin{array}{rcl} \mathrm{Tr}^G(x_1 x_2^3) & \equiv & x_1 x_2^3 + x_1(x_2 + x_{n+2})^3 + x_1(x_2 + \lambda x_{n+2})^3 \\ & & +x_1(x_2 + (\lambda+1)x_{n+2})^3 \\ & \equiv & \lambda(\lambda+1) x_1 x_{n+2}^3 \mod (R + x_{n+3}F[V_{2n}]). \end{array}$$

(b) Just note that $x_1 x_{n+2} + x_2 x_{n+1}$ is H_1-invariant, so the norm is G-invariant. □

Let $(a_1,\dots,a_n,a_{n+1},\dots,a_{2n}) \in F^{2n}$. We have a G-equivariant surjection $V_{2n} \to V_{2n-2}$ given by

$$\phi : (a_1,\dots,a_n,a_{n+1},\dots,a_{2n}) \to (a_2,\dots,a_n,a_{n+2},\dots,a_{2n}) \in F^{2n-2}.$$

Therefore $F[V_{2n-2}] = F[x_2,\cdots,x_n,x_{n+2},\cdots x_{2n}]$ is a G-subalgebra of $F[V_{2n}] = F[x_1,\cdots,x_n,x_{n+1},\cdots,x_{2n}]$.

Proposition 4. *Let $n \geq 3$ and $S \subseteq F[V_{2n-2}]^G$ be a separating set for V_{2n-2}. Then $\phi^*(S)$ together with the set T consisting of*

$$x_{n+1}, \quad N_G(x_1), \quad f_\lambda := \begin{cases} \mathrm{Tr}^G(x_1x_2^3) & \text{for } \lambda \neq 0,1 \\ N_{H_2}(x_1x_{n+2}+x_2x_{n+1}) & \text{for } \lambda \in \{0,1\} \end{cases}$$

$$\mathrm{Tr}^G(x_1x_ix_{i+1}) \quad \text{for } 2 \leq i \leq n-1,$$

is a separating set for V_{2n}.

Proof. Let $v_1 = (a_1,\dots,a_n,a_{n+1},\dots,a_{2n})$ and $v_2 = (b_1,\dots,b_n,b_{n+1},\dots,b_{2n})$ be two vectors in V_{2n} in different G-orbits with $\phi(v_1) = \phi(v_2)$, so $a_i = b_i$ except for $i = 1, n+1$. To apply Theorem 1, we assume for a contradiction that all elements of T take the same values on v_1 and v_2. Since $x_{n+1} \in T$, we have $a_{n+1} = b_{n+1}$, hence we have $v_2 = (b_1,a_2,\dots,a_n,a_{n+1},\dots,a_{2n})$. Because of Lemma 2 (b) we can assume $a_{2n} = 0$. Since $\mathrm{Tr}^G(x_1x_ix_{i+1}) \equiv x_1(x_{n+i}x_{n+i+2} + x_{n+i+1}^2) \mod R$ for $2 \leq i \leq n-2$, we succesively get $a_{2n-1} = a_{2n-2} = \ldots = a_{n+3} = 0$. In case $\lambda \neq 0,1$ we can also assume $a_{n+2} = 0$ by Lemma 3(a). In case $\lambda \in \{0,1\}$ and $a_{n+2} \neq 0$, $N_{H_2}(x_1x_{n+2}+x_2x_{n+1})$ taking the same value on v_1, v_2 implies $a_1 = b_1 + a_{n+2}$, hence $v_1 = \sigma_3 v_2$ for $\lambda = 0$ and $v_1 = \sigma_2 v_2$ for $\lambda = 1$ respectively. So now assume $a_{n+2} = 0$. Then $N_G(x_1)(v_1) = N_G(x_1)(v_2)$ implies $a_1 + b_1 \in \{a_{n+1}, \lambda a_{n+1}, (\lambda+1)a_{n+1}\}$, hence $v_1 = \sigma_i v_2$ for some $i \in \{1,2,3\}$. □

We give the induction start for $n = 2$ and $\lambda \neq 0,1$ - the case $\lambda \in \{0,1\}$ is left to the reader (or to Magma).

Lemma 5. *A separating set for $\lambda \neq 0,1$ and $n = 2$ is given by the invariants*

$$f_1 := x_1x_4 + \frac{1}{\lambda(\lambda+1)}x_2^2 + x_2(x_3 + \frac{1}{\lambda(\lambda+1)}x_4),$$

$$N_G(x_1), \quad N_G(x_2), \quad x_3, \quad x_4$$

Note that since G is not a reflection group, we need at least 5 separating invariants by [8, Theorem 1.1].

Proof. The invariants x_3, x_4 allow us to consider two points $v_1 = (a_1,a_2,a_3,a_4)$ and $v_2 = (b_1,b_2,a_3,a_4)$ in different orbits. If $N_G(x_2)(v_1) = N_G(x_2)(v_2)$, then $a_2 + b_2 \in \{0, a_4, \lambda a_4, (\lambda+1)a_4\}$, so after replacing v_2 by an element in its orbit we can assume $a_2 = b_2$. If $a_4 \neq 0$, then f_1 separates v_1, v_2, so assume $a_4 = 0$. Then $N_G(x_1)(v_1) = N_G(x_1)(v_2)$ implies $a_1 + b_1 \in \{0, a_3, \lambda a_3, (\lambda+1)a_3\}$, so v_1, v_2 are in the same orbit. □

2.2. **Type (iii).** Let G act on $V_{2n} = F^{2n}$ by the representation $\sigma_1 \mapsto \begin{pmatrix} I_n & J_0 \\ 0 & I_n \end{pmatrix}$ and $\sigma_3 \mapsto \begin{pmatrix} I_n & I_n \\ 0 & I_n \end{pmatrix}$. This leads to the same invariants as in type (ii) with $\lambda = 0$, just σ_1 and σ_3 are interchanged.

2.3. **Type (iv).** Take $\lambda = 1$ in type (ii). If $e_1, \dots, e_{2n} \in F^{2n}$ denotes the standard basis vectors, we can consider the submodule $V_{2n-1} := \langle e_1, \dots, e_n, e_{n+2}, \dots, e_{2n} \rangle$. Its representation is given by

$$\sigma_1 \mapsto \left(\begin{array}{c|c} I_n & \begin{array}{c} 0_{n-1} \\ \hline I_{n-1} \end{array} \\ \hline 0 & I_{n-1} \end{array}\right) \quad \text{and} \quad \sigma_2 \mapsto \left(\begin{array}{c|c} I_n & \begin{array}{c} I_{n-1} \\ \hline 0_{n-1} \end{array} \\ \hline 0 & I_{n-1} \end{array}\right),$$

which correpsonds to type (iv). Since the restriction map $F[V_{2n}]^G \to F[V_{2n-1}]^G$, $f \mapsto f|_{V_{2n-1}}$ maps separating sets to separating sets by [5, Theorem 2.3.16], we are done by our treatment of type (ii).

2.4. **Type (v).** Again we look at the case $\lambda = 1$ of type (ii). Then $\langle e_n \rangle$ is a G-submodule, and we look at the factor module $V_{2n-1} := V_{2n}/\langle e_n \rangle$ with basis $\tilde{e_i} := e_i + \langle e_n \rangle$, $i \in \{1, \dots, 2n\} \setminus \{n\}$. Its representation is given by

$$\sigma_1 \mapsto \left(\begin{array}{c|c|c} I_{n-1} & I_{n-1} & 0_{n-1} \\ \hline 0 & \multicolumn{2}{c}{I_n} \end{array}\right) \quad \text{and} \quad \sigma_2 \mapsto \left(\begin{array}{c|c|c} I_{n-1} & 0_{n-1} & I_{n-1} \\ \hline 0 & \multicolumn{2}{c}{I_n} \end{array}\right).$$

We have a G-algebra inclusion $F[V_{2n-1}] = F[x_1, \dots, x_{n-1}, x_{n+1}, \dots, x_{2n}] \subset F[V_{2n}]$.

The action on the variables is given by

$$\sigma_1(x_i) = \begin{cases} x_i + x_{n+i} & \text{for } 1 \le i \le n-1 \\ x_i & \text{for } n+1 \le i \le 2n, \end{cases}$$

and

$$\sigma_2(x_i) = \begin{cases} x_i + x_{n+i+1} & \text{for } 1 \le i \le n-1 \\ x_i & \text{for } n+1 \le i \le 2n. \end{cases}$$

Let $(a_1, \dots, a_{n-1}, a_{n+1}, \dots, a_{2n}) \in F^{2n-1} \cong V_{2n-1}$. We have a G-equivariant surjection $V_{2n-1} \to V_{2n-3}$ given by

$$\phi : (a_1, \dots, a_{n-1}, a_{n+1}, \dots, a_{2n}) \to (a_2, \dots, a_{n-1}, a_{n+2}, \dots, a_{2n}) \in F^{2n-3}.$$

Therefore $F[V_{2n-3}] = F[x_2, \cdots, x_{n-1}, x_{n+2}, \cdots x_{2n}]$ is a G-subalgebra of $F[V_{2n-1}] = F[x_1, \cdots, x_{n-1}, x_{n+1}, \cdots, x_{2n}]$. Also, let $R := F[x_2, \cdots, x_{n-1}, x_{n+1}, \cdots x_{2n}]$. We will make computations modulo R, considered as a subvectorspace of $F[V_{2n-1}]$, and we can re-use the equation of Lemma 2(a*).

Lemma 6. *Let $v_1, v_2 \in V_{2n-1}$ be two vectors in different orbits that agree everywhere except the first coordinate. Say, $v_1 = (a_1, \dots, a_{n-1}, a_{n+1}, \dots, a_{2n})$, $v_2 = (b_1, a_2 \dots, a_{n-1}, a_{n+1}, \dots, a_{2n})$. Assume further that one of the following holds.*

(a) $a_{n+2} \neq 0$ and $a_i = 0$ for $n+3 \le i \le 2n$
(b) $a_i = a_{2n} \neq 0$ for $n+2 \le i \le 2n-1$.

Then the invariant

$$f := N_{H_2}(x_1 x_{n+2} + x_2 x_{n+1}) \equiv x_1^2 x_{n+2}^2 + x_1 x_{n+2}(x_{n+2}^2 + x_{n+1} x_{n+3}) \mod R$$

separates v_1 and v_2.

Proof. Assume the first case. Then $f(v_1) = f(v_2)$ implies $(a_1 + b_1)^2 a_{n+2}^2 = (a_1 + b_1) a_{n+2}^3$, hence $a_1 = b_1 + a_{n+2}$. Since $a_i = 0$ for $i \ge n+3$ this implies that $v_1 = \sigma_2 v_2$ which is a contradiction because v_1 and v_2 are in different orbits.

Next assume the second case. Then $f(v_1) = f(v_2)$ implies $(a_1 + b_1)^2 a_{n+2}^2 = (a_1 + b_1) a_{n+2}^2 (a_{n+1} + a_{n+2})$, hence $a_1 = b_1 + a_{n+1} + a_{n+2}$. Since $a_i = a_{2n}$ for $n+2 \le i \le 2n-1$, this implies that $v_1 = \sigma_3 v_2$ yielding a contradiction. □

Lemma 7. *For $2 \leq i \leq n-1$, we have the following elements in $F[V_{2n-1}]^G$:*

$$\mathrm{Tr}^G(x_1x_i^3) \equiv x_1x_{n+i}x_{n+i+1}(x_{n+i}+x_{n+i+1}) \mod R.$$

Proof.

$$\begin{aligned}\mathrm{Tr}^G(x_1x_i^3) &\equiv x_1x_i^3 + x_1(x_i+x_{n+i})^3 + x_1(x_i+x_{n+i+1})^3 \\ &\quad + x_1(x_i+x_{n+i}+x_{n+i+1})^3 \\ &\equiv x_1x_{n+i}x_{n+i+1}(x_{n+i}+x_{n+i+1}) \mod R.\end{aligned}$$

□

Proposition 8. *Let $n \geq 3$ and $S \subseteq F[V_{2n-3}]^G$ be a separating set for V_{2n-3}. Then $\phi^*(S)$ together with the set T consisting of*

$$x_{n+1}, \quad N_G(x_1), \quad N_{H_2}(x_1x_{n+2}+x_2x_{n+1}), \quad \mathrm{Tr}^G(x_1x_2x_{n-1}),$$

$$\mathrm{Tr}^G(x_1x_ix_{i+1}) \text{ for } 2 \leq i \leq n-2, \qquad \mathrm{Tr}^G(x_1x_i^3) \text{ for } 2 \leq i \leq n-1,$$

is a separating set for V_{2n-1}.

Proof. Let $v_1 = (a_1, \ldots, a_{n-1}, a_{n+1}, \ldots, a_{2n})$ and $v_2 = (b_1, \ldots, b_{n-1}, b_{n+1}, \ldots, b_{2n})$ be two vectors in V_{2n-1} in different G-orbits with $\phi(v_1) = \phi(v_2)$, so $a_i = b_i$ except for $i = 1, n+1$. To apply Theorem 1, we assume for a contradiction that all elements of T take the same values on v_1 and v_2. Since $x_{n+1} \in T$, we have $a_{n+1} = b_{n+1}$, hence we have $v_2 = (b_1, a_2, \ldots, a_{n-1}, a_{n+1}, \ldots, a_{2n})$.

We first assume $a_{n+i} \neq 0$ for $2 \leq i \leq n$. Lemma 7 implies $a_{n+2} = a_{n+3} = \ldots = a_{2n} \neq 0$, a contradiction to Lemma 6 (b). Thus there must be a $2 \leq i \leq n$ with $a_{n+i} = 0$, so let i be maximal with this property. Consider the invariants $f_j := \mathrm{Tr}^G(x_1x_jx_{j+1}) \equiv x_1(x_{n+j}x_{n+j+2} + x_{n+j+1}^2) \mod R$ of T for $2 \leq j \leq n-2$ (see Lemma 2(a*)).

If $i \leq n-2$, then $a_{n+i+1} \neq 0$, and f_i separates v_1, v_2.

If $i = n-1$, then $a_{2n} \neq 0$, and $f_j(v_1) = f_j(v_2)$ for $j = n-3, n-4, \ldots, 2$ implies $a_{n+j} = 0$ for $3 \leq j \leq n-1$. As $\mathrm{Tr}(x_1x_2x_{n-1}) \equiv x_1(x_{n+2}x_{2n} + x_{n+3}x_{2n-1}) \mod R$ takes the same value on v_1, v_2, we also have $a_{n+2} = 0$. Now $N_G(x_1)(v_1) = N_G(x_1)(v_2)$ implies $a_1 = b_1 + a_{n+1}$, thus $v_1 = \sigma_1 v_2$.

If $i = n$, i.e. $a_{2n} = 0$, then since $f_j(v_1) = f_j(v_2)$ for $j = n-2, n-3, \ldots, 2$, we get $a_{n+j} = 0$ for $3 \leq j \leq 2n$. In case $a_{n+2} \neq 0$, we are done by Lemma 6 (a). If $a_{n+2} = 0$, then $N_G(x_1)(v_1) = N_G(x_1)(v_2)$ implies as before $a_1 = b_1 + a_{n+1}$ and $v_1 = \sigma_1 v_2$. □

Remark 9. A separating set for V_3 is formed by $N_G(x_1)$, x_3, x_4.

3. Cyclic groups

Let $G = \boldsymbol{Z}_{p^rm}$ be the cyclic group of order p^rm, where p is a prime number and r, m are non-negative integers with $(m, p) = 1$. Let H and M be the subgroups of G of order p^r and m, respectively. Let V_n be an indecomposable G-module of dimension n.

Lemma 10. *There exists a basis $e_1, e_2, \ldots, e_n$ of V_n such that $\sigma^{-1}(e_i) = e_i + e_{i+1}$ for $1 \leq i \leq n-1$ and $\sigma^{-1}(e_n) = e_n$ for a generator σ of H, and $\alpha(e_i) = \lambda e_i$ for $1 \leq i \leq n$ for a m-th root of unity $\lambda \in F$ and α a generator of M.*

Proof. It is well known that $n \leq p^r$ and there is basis such that a generator ρ of G acts by a Jordan matrix $J_\mu = \mu I_n + N$ with μ a mth root of unity [1, p. 24]. Then ρ^{p^r} is a generator of M acting by $(\mu I_n + N)^{p^r} = \mu^{p^r} I_n$, and ρ^m is a generator of H acting by $(\mu I_n + N)^m = I_n + m\mu^{m-1}N + \binom{m}{2}\mu^{m-2}N^2 + \ldots$. This matrix has Jordan normal form J_1, and the representation matrix of ρ^{p^r} is invariant under base change, which proves the lemma. □

Since we want our representation to be faithful, we will assume that λ is a primitive mth root of unity from now. We also restrict to the case $r = 1$. Let $x_1, x_2, \ldots, x_n$ be the corresponding basis elements in V_n^*. We have $\sigma(x_i) = x_i + x_{i-1}$ for $2 \leq i \leq n$, $\sigma(x_1) = x_1$ and $\alpha(x_i) = \lambda^{-1}x_i$ for $1 \leq i \leq n$. Since α acts by multiplication by a primitive mth root of unity, there exists a non-negative integer k such that $x_n x_{i+1}^{p-1} x_i^k \in F[V_n]^M$ for $1 \leq i \leq n-2$. We may assume that k is the smallest such integer. Let I_i denote the ideal in $F[V_n]$ generated by $x_1, x_2, \ldots, x_i$. Set $f_i = x_n x_{i+1}^{p-1} x_i^k$ for $1 \leq i \leq n-2$.

Lemma 11. *Let a be a positive integer. Then $\sum_{0 \leq l \leq p-1} l^a \equiv -1 \mod p$ if $p-1$ divides a and $\sum_{0 \leq l \leq p-1} l^a \equiv 0 \mod p$, otherwise.*

Proof. See [4, 9.4] for a proof for this statement. □

Now set $R := F[x_1, x_2, \ldots, x_{n-1}]$.

Lemma 12. *Let $1 \leq i \leq n-2$. We have*

$$\mathrm{Tr}_M^G(f_i) \equiv -x_n x_i^{p+k-1} \mod \big(I_{i-1} + R\big).$$

Proof. We only care for terms containing x_n but not $x_1, \ldots, x_{i-1}$, thus we have

$$\begin{aligned}\sigma^l(f_i) &= (x_n + lx_{n-1} + \binom{l}{2}x_{n-2} + \cdots)(x_{i+1} + lx_i + \cdots)^{p-1}(x_i + lx_{i-1} + \cdots)^k \\ &\equiv x_n(x_{i+1} + lx_i)^{p-1}x_i^k \mod \big(I_{i-1} + R\big).\end{aligned}$$

Thus it suffices to show that $\sum_{0 \leq l \leq p-1}(x_{i+1} + lx_i)^{p-1} = -x_i^{p-1}$. Let a and b be non-negative integers such that $a + b = p - 1$. Then the coefficient of $x_{i+1}^a x_i^b$ in $(x_{i+1} + lx_i)^{p-1}$ is $\binom{p-1}{b} l^b$ and so the coefficient of $x_{i+1}^a x_i^b$ in $\sum_{0 \leq l \leq p-1}(x_{i+1} + lx_i)^{p-1}$ is $\sum_{0 \leq l \leq p-1} \binom{p-1}{b} l^b$. Hence the result follows from the previous lemma. □

Let $(c_1, c_2, \ldots c_n)$ be a vector in V_n. There is a G-equivariant surjection $\phi : V_n \to V_{n-1}$ given by $(c_1, c_2, \ldots c_n) \to (c_1, c_2, \ldots c_{n-1})$. Hence $F[V_{n-1}] = F[x_1, \cdots, x_{n-1}]$ is a G-subalgebra of $F[V_n]$. Let l be the smallest non-negative integer such that $N_H(x_n)(N_H(x_{n-1}))^l \in F[V_n]^G$. Note that since $(p, m) = 1$ such an integer exists.

Proposition 13. *Let $S \subseteq F[V_{n-1}]^G$ be a separating set for V_{n-1}. Then $\phi^*(S)$ together with the set T consisting of*

$$N_H(x_n)(N_H(x_{n-1}))^l, \quad N_G(x_n), \quad \mathrm{Tr}_M^G(f_i) \quad \text{for } 1 \leq i \leq n-2,$$

is a separating set for V_n.

Proof. Let $v_1 = (c_1, c_2, \ldots, c_n)$ and $v_2 = (d_1, d_2, \ldots, d_n)$ be two vectors in V_n in different G-orbits with $\phi(v_1) = \phi(v_2)$, so $c_i = d_i$ for $1 \leq i \leq n-1$. To apply Theorem 1, we assume for a contradiction that all elements of T take the same values on v_1 and v_2. Assume that there exists an integer $i \leq n-2$ such that $c_i \neq 0$. Assume further that i is the smallest such integer. Then $\mathrm{Tr}_M^G(f_i)$ separates v_1 and v_2

by the previous lemma. Therefore we have $c_1 = c_2 = \cdots = c_{n-2} = 0$. We consider two cases. First assume that $c_{n-1} = 0$. Then $N_G(x_n)(v_1) = N_G(x_n)(v_2)$, i.e. $c_n^{pm} = d_n^{pm}$, implies that $c_n = \lambda^a d_n$ for some integer a and hence v_1 and v_2 are in the same orbit. If $c_{n-1} \neq 0$, then we see that $N_H(x_n)(N_H(x_{n-1}))^l$ separates v_1 and v_2 as follows. We have $(N_H(x_{n-1}))^l(v_1) = (N_H(x_{n-1}))^l(v_2) \neq 0$. Therefore it suffices to show $N_H(x_n)(v_1) \neq N_H(x_n)(v_2)$. But otherwise $c_n^p - c_n c_{n-1}^{p-1} = d_n^p - d_n c_{n-1}^{p-1}$, which implies $c_n = d_n + l c_{n-1}$ for some $0 \leq l \leq p-1$, so v_1 and v_2 are in the same orbit. □

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