

The ‘Holiverse’: holistic eversion of the 2-sphere in \mathbb{R}^3

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Dedicated to Stephen Smale, who eschewed spin for deeper principles

Abstract

We give a short, simple and conceptual proof, based on spin structures, of sphere eversion: an embedded 2-sphere in \mathbb{R}^3 can be turned inside out by regular homotopy. Ingredients of this eversion are seamlessly connected. We also give the mathematical origins of the proof: the Hopf fibration, and the topological structure of real-projective 3-space.

1 Introduction

In 1957 Smale proved that any two smooth immersions of the 2-sphere in \mathbb{R}^3 are regularly homotopic. As a corollary, any smoothly embedded sphere can be smoothly turned inside out by regular homotopy. Smale’s proof is abstract, and far-reaching, involving liftings and projections of homotopy fibrations, and for 50 years has inspired a number of attempts to conceptualize directly how such an eversion can be achieved, as well as several beautiful computer animations of actual eversions. These animations have also provided the benchmark in computer-graphic realization of mathematical concepts.

We present here the first truly holistic eversion, by which we mean that each stage of the eversion is conceptually a natural part of a seamless whole. The origins of the eversion lie in the simple interplay of the Hopf fibration and the antipodal map on the 3-sphere, and the fact that an embedded 2-sphere in real projective 3-space $\mathbb{R}P^3$ can be everted essentially trivially. However, the actual eversion in \mathbb{R}^3 does not require a conceptual understanding of these origins to be understood in its entirety, and has its own intrinsic integrity. Accordingly we describe the actual eversion, and its mathematical origins, in two independent sections.

To provide conceptual context, we include a very brief historical account, mentioning several of the ingredients of previous proofs. Essentially all of these proofs critically depend, at some stage, on direct visualization of part of the process, and in this regard differ from the proof offered here. Although visualization plays an important role in the communication of the essential ideas, this holistic eversion can be grasped conceptually, and as such is far less reliant on the need for pictures or computer animation.

2010 Mathematics Subject Classification. Primary: 57R42 ; secondary: 57M60, 00A66.

Essentially all existing animations rely at some stage in explaining the local topological changes occurring: for the holiverse, these details are unnecessary. Moreover, the mathematical prerequisites for completely understanding this holiversion are generally taught to students in advanced undergraduate or early graduate level topology classes. In spirit, this proof is closest to the first specific eversion described by Arnold Shapiro [7], although much simpler. We thank George Francis for suggesting the name ‘holiverse’.

A brief outline of the holiversion: The eversion arises from a simple immersed disc, illustrated in Figure 1:

1. **The Disc:** (a) Take a planar embedded disc, with two small distinguished arcs on the boundary: push these towards each other, twisting slightly while doing so – as if to construct an annulus (0-twist) or Mobius band (π twist). The boundary circle is drawn: you can imagine the disc conveniently wrapping around inside a torus.

(b) Push the arcs through each other slightly to create the simplest immersed disc, with double-point set a single arc. The projection of the disc to the plane is still an immersion: the boundary of this disc is still unknotted. However, a collar neighbourhood of the boundary circle has two full negative twists when its core circle is stretched out as a round planar circle.

(c) Smoothly unwist to remove the innermost extraneous crossing, revealing the boundary circle as a $(2, -1)$ -torus knot; the collar annulus in the disc twists exactly as an annular neighbourhood of the circle on the torus.

(d) This introduces a ‘curtain-like’ bend in the disc, which can be done smoothly – like the trace of the homotopy $(x, x^3 + tx, t)$ viewed from the side x axis. (The disc can be adjusted to meet the torus from outside, and is slightly trickier to visualize.)

2. **The Immersed Sphere:** (a) Thicken the disc to create an immersed sphere, which is essentially a union of two parallel copies of the disc, and an annulus following along the boundary circle, twisting to follow a collar in the disc orthogonally, with interior inside the torus.

(b) The sphere is regularly homotopic to an embedded sphere (shrink the disc).

(c) The discs and annulus have boundary two parallel $(2, -1)$ circles.

3. **The Eversion:** (a) Spin each disc rigidly in opposite directions around the torus, so that they coincide as they pass through each other, and then return interchanged.

(b) The two boundary circles bound an annulus, which sweeps across the torus, double covering a Möbius band as the discs coincide, and returning to the initial configuration, inside out with circles interchanged (and with a Dehn twist).

(c) The two discs and annulus remain smoothly glued together: thus spinning turns the original immersed neighbourhood sphere inside out. Shrink to an embedding, to obtain an inside-out version of 2(b) above.

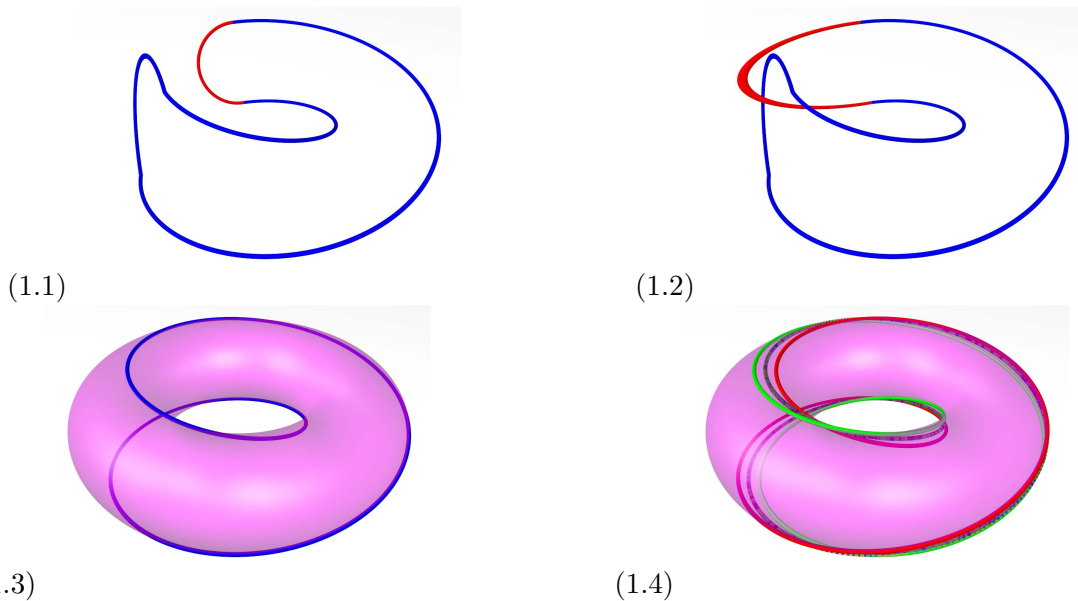


Figure 1: (1.1) An embedded disc (1.2) An immersed disc, unknotted boundary circle. (1.3) Untwisting gives a $(2, -1)$ torus knot (1.4) Rotating two discs rigidly in opposite directions.

2 Historical context: underlying constructs

We present a chronological synopsis of previous work on sphere eversion: this draws heavily on information organized by John Sullivan, who credits George Francis as primary source [18], and on correspondence with George Francis.

- 1924 J. W. Alexander [2] proves that every embedded 2-sphere in \mathbb{R}^3 is isotopic to the standard round sphere.
- 1957: Stephen Smale announced his classification of immersions of the 2-sphere in \mathbb{R}^3 up to regular homotopy, published in [16], and generalizing the Whitney-Graustein classification of immersions of the circle in the plane. Smale's work involves normalizing a disc, homotopy fibrations, and homotopy classes of the tangent mapping into Stiefel and Grassman manifolds. Smale's approach immediately generalizes to higher dimension (Hirsch-Smale), and for immersions of arbitrary smooth manifolds. The original case implicitly exploits $\pi_2(G) = 0$ for any Lie group G , in particular when $G = SO(3) \cong \mathbb{R}P^3 \cong L(2, -1)$, the Lens space. These spaces play a role underlying the holiverse: the author learned immersion theory from Smale in a graduate course at Berkeley in the early 1980's.
- 1960. Arnold Shapiro described, but did not publish, an eversion using a neigh-

bourhood of Boy’s immersion of the real projective plane $\mathbb{R}P^2$ into \mathbb{R}^3 , requiring a non-intuitive regularly homotopy, in several stages, of an embedded sphere to the boundary of a twisted I -bundle neighbourhood of the immersed $\mathbb{R}P^2$. The projective plane is the union of a disc and a Möbius band, a decomposition which features in the holiverse. An exposition was given in 1979 by Francis and Morin [7]: the holiverse has several features in common, such as the appearance of the $(2, -1)$ torus knot, the need for an immersed disc bounding this knot, and the fact that a neighbourhood of an immersed disc immediately gives an immersed sphere regularly homotopic to an embedded sphere. Nonetheless, Shapiro’s eversion remained hard to visualize, especially so at the time it was announced: there were no computer aids for graphical representation.

- 1966. Tony Phillips [15] attempted to describe Shapiro’s eversion pictorially, and in so doing obtained a new purely visual description of an eversion. Phillips’ rendering of his procedure exploits sequences of pictures of immersed ribbons/annuli, in part with stages emulating the more familiar Whitney-Graustein regular homotopies of circles in the plane.

Around this time, in interaction with Froissart, Bernard Morin utilized the symmetry of a symmetric 4-lobed sphere immersion as half-way surface, as an alternative to an immersed projective plane; this makes manifest the equivalence of inside and out, with the symmetry interchanging these. Again, an explicit non-intuitive regular homotopy from an embedded sphere must be demonstrated. François Apéry and Morin subsequently showed that Morin’s eversion has the minimal number of topological events [13].

- 1968 Bryce De Witt [5] gave an outline of another pictorial scheme for an eversion, although details making this explicit have not appeared.
- c. 1970 Charles Pugh, Smale’s colleague at Berkeley, constructed by hand a sequence of chicken-wire models showing the stages of Morin’s eversion, based in turn on Morin’s clay models. These were proudly suspended in the Mathematics Department until stolen sometime in the mid-late 1970’s.
- 1974. William Thurston, Hirsch’s student at Berkeley, conceptualized an eversion using the so-called ‘belt trick’, with origins in $Spin(3) = SU(2)$: a long strip can be given a full twist by either twisting the ends, or interchanging them by parallel transport along a straight line. The resulting eversions use ideas of corrugations and symmetry, again with more general implications, but there also remains some difficulty in explicitly describing or following every stage of the eversion.
- 1977. Nelson Max spent six years digitalizing Charles Pugh’s chicken-wire models, with coordinates for points on the models calculated by hand and entered as data for

his landmark animated movie [12]. This was an international sensation, and a tour de force in the dawning age of computer graphics: it motivated the author’s interest.

- 1979. George Francis and Bernard Morin [7] published a description of Arnold Shapiro’s eversion of the sphere, pointing out the desire for a holistic eversion in a similar spirit, with each stage understandable in terms of familiar topological concepts.
- 1987. George Francis’ ‘A Topological Picturebook’ [6] is published, with many hand-drawn pictures of stages of eversions.
- 1992. François Apéry, with Bernard Morin, describe an algebraic halfway model for the eversion of the sphere [3]. Being able to describe surfaces in such a way facilitates the production of computer graphics and software such as Povray which render mathematical equations, rather than objects constructed from points in a large database.
- 1994. Silvio Levy, Delle Maxwell, and Tamara Munzner create the animation ‘Outside In’, describing and explaining Thurston’s ideas [14].
- 1995. John Sullivan, Bob Kusner and George Francis implemented a minimax eversion based on Kusner and Bryant’s work on Willmore energies: Kusner, another Berkeley student, had found an analytic surface which reminded him of Morin’s halfway surface. The use of a natural energy minimizing ‘Willmore flow’ as a guiding principle recaptures earlier approaches more beautifully, providing a heuristic justifying Morin’s initial viewpoint. The Willmore flow clearly converges to an embedded sphere when run on Brakke’s ‘Surface Evolver’ program [4], but only recently have results on existence and uniqueness for the corresponding 4th-order PDE flow equations begun to emerge [18].
- 1998. ‘The Optiverse’, an animation of the Willmore flow eversion created by Sullivan, Francis, and Stuart Levy, won a prize for animation at the 1998 Berlin International Congress of Mathematicians. [8, 9, 10, 17]

3 Holistic eversion in \mathbb{R}^3 : Explicit details

Let D_{zx} be the unit disc in the zx -plane with equation $(x - \sqrt{2})^2 + z^2 = 1$, with $S_{zx}^1 := \partial D_{zx}$ meeting the $+x$ -axis at points $d_1 < d_2$. Denote angle measure from the centre of D_{zx} by ϕ , measured anticlockwise from the $+x$ -axis. Let $SD_z \cong S^1 \times D^2$ denote the solid donut obtained by revolving D_{zx} around the z -axis in R_{xyz}^3 , while rotating the disc about its centre uniformly by π anticlockwise, so that $\{d_1, d_2\}$ creates a $(2, -1)$ torus knot K_0 . Foliate $D_{zx} - \{d_1, d_2\}$ by arcs of circles whose centres lie on the x -axis, and which are orthogonal to S_{zx}^1 . Let α_ϕ denote such an arc with endpoints on S_{zx}^1 at angles $\pm\phi$, $0 < \phi < \pi$, and let $A_\phi \subset SD_{zx}$ denote the annulus obtained by rotating $\{\alpha_\phi, \alpha_{\pi-\phi}\}$; for $\phi = \pi/2$, this

covers a Möbius band M . Let θ denote standard angle measure in the xy -plane, and let K_θ denote the image of K_0 after θ -rotation about the z -axis, $-\pi \leq \theta \leq \pi$. K_θ meets S_{zx}^1 in two antipodal points S_θ^0 , at angles $\phi = -\theta/2, \pi - \theta/2$; $K_\theta \cup K_{-\theta} = \partial A_{\theta/2} \subset \partial SD_z$.

Proposition 1. *There exists a smoothly immersed disc $D_0 \subset R_{xyz}^3$, $\partial D_0 = K_0$, orthogonal to ∂SD_z , meeting from the outside.*

Proof. Take an equilateral triangle in R_{xy}^2 , with one vertex on the $+x$ -axis and one edge on the y -axis. Fold over a square attached to the top edge, on the $+z$ side; fold under another square attached to the bottom edge, on the $-z$ side, to create an embedded disc which projects to the plane with two fold curves. Change the crossing between the two edges of folded squares which project to intersect on the $-x$ -axis. Smooth the construction to obtain an immersed disc D' with unknotted boundary and a proper arc of intersection, and observe that each of (i) a collar neighbourhood of the boundary circle of D' , (ii) a collar neighbourhood of $K_{\pm\pi}$ in M , and (iii) a neighbourhood of K_0 on ∂SD_z , is an unknotted annulus with four negative half-twists. Using a collar of ∂SD_z and a regular homotopy/isotopy of D' , we obtain the desired immersed disc D_0 , with boundary-circle-collar embedded in $R_{xyz}^3 - \text{int}(SD_z)$. Let D_θ be the θ -rotation of D_0 , with $\partial D_\theta = K_\theta$, and define $S_\theta := D_{-\theta} \cup A_{\theta/2} \cup D_\theta$.

Proposition 2. *S_θ is regularly homotopic to an embedded sphere in R^3 , for (non-zero) $\theta \in (-\pi, \pi)$.*

Proof. Let S_t^1 be the circle of radius t in the uv -plane, $0 < \epsilon \leq t \leq 1 + \epsilon$, bounding the disc D_t^2 , so that $D_{1+\epsilon}^2$ gives a neighbourhood of the unit disc D_1^2 . The 2-spheres $S_t^2 = \partial(D_t^2 \times [-t, t]) := \partial N_t \subset R_{uvw}^3$ are concentric with the boundary $S_{1+\epsilon}^2$ of a tubular neighbourhood of D_1^2 in R_{uvw}^3 . These spheres have a natural splitting as two discs D_t^\pm and an annulus \bar{A}_t : the construction can naturally be done smoothly. Doing so, let $\rho : D_1^2 \rightarrow R_{xyz}^3$ be a smooth immersion with image D_0 , and $\rho^* : B^3 \cong N_{1+\epsilon} = D_1^2 \times [-1 - \epsilon, 1 + \epsilon] \rightarrow R_{xyz}^3$ be an immersion with image an immersed neighbourhood of the immersed disc D_0 , with $\rho^*(\bar{A}_{1+\epsilon}) = A_{\theta_0/2}$ for some small θ_0 . We may assume that $\rho^*(S_\epsilon^2)$ is smoothly embedded; $\rho^*(S_t^2)$ gives a regular homotopy to $\rho^*(S_{1+\epsilon}^2) := D_{1+\epsilon}^- \cup A_{\theta_0/2} \cup D_{1+\epsilon}^+$. Both $D_{1+\epsilon}^\pm$ are regularly homotopic to D_0 , and have boundary $K_{\pm\theta_0}$; they are regularly homotopic keeping $K_{\pm\theta_0}$ fixed, to $D_{\pm\theta_0}$. Hence $\rho^*(S_{1+\epsilon}^2)$ is regularly homotopic to $S_{\theta_0}^2$.

Proposition 3. *S_{θ_0} is regularly homotopic to itself with orientation reversed.*

Proof. The regular homotopy is provided by the sequence of spheres S_θ , $\theta \in [\theta_0, 2\pi - \theta_0]$. The intermediate sphere S_π double covers the immersed projective plane $M \cup D_\pi$. Doing this continuously, observe that the two discs $D_{\pm\theta_0}$ are interchanged after being rotated in opposite directions; the annulus $A_{\theta_0/2}$ undergoes an additional Dehn twist.

Theorem 1. (Smale [16]) *An embedded sphere in R^3 is regularly homotopic to itself, reversing orientation.*

Proof. This is an immediate application of the preceding propositions: An embedded disc expands to create a single arc of self-intersection, with the boundary of the immersed disc still unknotted but embedded on the torus as a $(2, -1)$ torus knot. An immersed 2-sphere neighbourhood of the disc is regularly homotopic to an embedded sphere, using neighbourhoods of immersed discs as they shrink back to the original disc. This immersed sphere is naturally the union of two ‘parallel’ copies of the disc, and an equatorial annulus which is an embedded annulus in the solid torus, with boundary two parallel copies of the $(2, -1)$ torus knot. Rotating both of these circles in opposite directions, we can rotate the two immersed hemispheres of the disc to interchange. Simultaneously the annulus enlarges inside the solid donut, passes through itself as a double-covered Möbius band, and then returns to the original annulus (after a Dehn twist). The sphere has now turned itself inside out smoothly, and can be shrunk back to an inside-out embedded sphere, completing the eversion. There is an immersed projective plane at the half way stage.

The theorem is a corollary of the more general results first published in [16].

The mathematics underlying this proof simply combines basic facts about $Spin(3)$, the Hopf fibration, the Clifford torus, $SO(3)$, $\mathbb{R}P^3$, the Lens space $L(2, -1)$, the simplest embedded Möbius band and the simplest (non-trivial) immersed disc in \mathbb{R}^3 , and S^3 as -1 surgery on the unknot: this is explained in the next section.

4 Eversion by spin: the origin of the eversion in $\mathbb{R}P^3$

A 2-sphere can be trivially turned inside out in $\mathbb{R}P^3 = B^3 \cup N(\mathbb{R}P^2)$, the union of a ball and a twisted I -bundle neighbourhood of a projective plane $\mathbb{R}P^2$: radially expand an embedded sphere metrically centered at the centre of the ball until it becomes the boundary of $N(\mathbb{R}P^2)$. Push the sphere across the projective plane, so that it ceases to be an embedding only at the instant it double covers the projective plane. It then becomes the boundary of the ball, but with its orientation reversed. Since the 2-sphere is simply connected, this regular homotopy lifts to the universal (2-fold) cover S^3 of $\mathbb{R}P^3 = S^3/a$, with exactly two lifts since $\pi_1(\mathbb{R}P^3)$ is $\mathbb{Z}/2\mathbb{Z}$: we see two 2-spheres, centered respectively at 0 and ∞ in $S^3 \cong \mathbb{R}^3 \cup \infty$. These bound $S^2 \times I$, and $\{S_t\} := \{S^2 \times \{t\}\}$ determines the regular homotopy to interchange the 2-spheres $\{S_{\pm\epsilon}\}$ by isotopy, turning a single 2-sphere inside-out in $\mathbb{R}P^3$.

We modify the radial expansion of S^2 by twisting upper and lower hemispheres as we radially expand, in a manner prescribed by the Hopf fibration. This will allow the regular homotopy in $\mathbb{R}P^3$ to be realized in \mathbb{R}^3 , by first constructing a map from $\mathbb{R}P^3$ into R^3 whose composition with the regular homotopy is a sphere eversion in \mathbb{R}^3 . This map, naturally constructed using the Lens space structure of $\mathbb{R}P^3$, intertwines the Hopf flow with the standard rotation of R^3 round the z -axis, making the complete eversion comprehensible and describable in ‘one shot’.

$S^3 = \partial B^4 \subset \mathbb{R}^4 \cong \mathbb{C}^2$ is defined by $\{(\alpha, \beta) \mid \alpha\bar{\alpha} + \beta\bar{\beta} = 1\}$. The subset $\alpha\bar{\alpha} = \beta\bar{\beta}$ is the Clifford torus $CT \cong S^1_{1/2} \times S^1_{1/2}$. CT defines two complementary ‘solid donuts’ $SD, SD^* \subset S^3$ with core circles $\{(e^{i\phi}, 0)\}, \{(0, e^{i\psi})\}$ in the α and β complex lines, giving the standard genus 1 Heegaard decomposition of S^3 . Define oriented circles on CT by $\lambda := \{(e^{i\phi}, 1)/\sqrt{2}\}, \mu := \{(1, e^{i\psi})/\sqrt{2}\}$, meeting at $P \in CT$, and which give homology basis $[\lambda] = (1, 0), [\mu] = (0, 1)$ respectively for $H_1(CT; \mathbb{Z})$. Thus $SD \cong \lambda \times D$, where $\partial D = \mu$, and $SD^* \cong D^* \times \mu$, with $\partial D^* = \lambda$, where D, D^* are 2-dimensional discs: solid donuts are parametrized by S^1 -actions on S^3 with circle-of-fixed-points λ or μ . Identify $e^{i\psi}$ with $(1, e^{i\psi})/\sqrt{2} \in \mu \subset \mathbb{C}^2$, corresponding to the S^1 -action $(\alpha, \beta) \rightarrow e^{i\psi}(\alpha, \beta) := (\alpha, e^{i\psi}\beta)$, and define $D_\psi^* := (D^*, e^{i\psi}) \subset SD^*$. Thus P has orbit μ , and $\partial D_\psi^* := K_\psi^* \subset CT$ is a circle isotopic to $K_0^* = \lambda$. SD^* constitutes a 2-handle and 3-handle attached to SD to give S^3 : the 2-handle $\cup_{|\psi| \leq \epsilon} D_\psi^*$ has core disc D_0^* attached along λ .

One lift of a radial regular homotopy of S^2 in $\mathbb{R}P^3$ begins with radial expansion of a small sphere \bar{S}_* centered at $0 := D_0^* \cap \{\beta = 1\}$, until it becomes \bar{S}_0^2 , tangent to CT along K_0^* . Let \bar{S}_ψ^* , $0 < \psi < \pi$, denote the expanded sphere intersecting CT in the pair of circles $K_{\pm\psi}^*$: these circles also bound an annulus \bar{A}_ψ^* in SD , which decomposes the 2-sphere into $\bar{S}_\psi^* = \bar{D}_\psi^2 \cup \bar{A}_\psi^* \cup \bar{D}_{-\psi}^2$. The equatorial sphere $\bar{S}_{\pi/2}^2 \subset S^3$ lies midway in the isotopy of \bar{S}_* to the antipodal small sphere $a(\bar{S}_*)$, double-covers a projective plane in $\mathbb{R}P^3$, and bounds a fundamental domain for $a : S^3 \rightarrow S^3$.

The circles $K_{\pm\psi}^*$ also bound the pair of discs $D_{\pm\psi}^* \subset SD^*$, meeting CT orthogonally. In SD , choose annuli A_ψ^* , $\psi \in (0, \pi)$, meeting CT orthogonally along $K_{\pm\psi}^*$, intersecting each disc of SD in a circular arc, and define $S_\psi^* := D_\psi^* \cup A_\psi^* \cup D_{-\psi}^*$: each sphere S_ψ^* is the boundary of a 3-ball neighbourhood of the disc D_0^* , the core disc of the 2-handle, and there is a natural collapse of the disjoint family $\{S_\psi^*\}$ onto D_0^* . This family also projects to give an eversion in $\mathbb{R}P^3$: We use the Hopf fibration to modify this eversion.

The Hopf flow and Hopf fibration is determined by the circle action $e_{\text{Hopf}}^{i\theta}(\alpha, \beta) := (e^{i\theta}\alpha, e^{i\theta}\beta)$. Each of SD, CT , and SD^* is invariant under the Hopf flow, and hence under the antipodal map a , which corresponds to $\theta = \pi$.

The Hopf circles on the Clifford torus are orbits of this free circle action, and in coordinates, are $(1, 1)$ circles. The Hopf circle through P intersects K_ψ^* at $P_\psi^* = (e^{i\psi}, e^{i\psi})/\sqrt{2}$. Let D_θ^h denote the rotated disc $e_{\text{Hopf}}^{i\theta}(D_\theta^*)$: the point $P \in D^*$ follows P_θ^* , inducing a twisted reparametrization A_θ^h of A_θ^* as we isotope the spheres S_θ^* across SD .

Change parametrization of SD by a left-hand Dehn twist, so that the Hopf flow on $SD \cong S^1 \times D^2$ is $e_{\text{rot}}^{i\theta}(x, y) = (e^{i\theta}x, y)$. (This corresponds to constructing S^3 by (-1) -surgery on the unknot.) The $(1, 1)$ Hopf circle through P becomes a $(1, 0)$ circle, P_θ^* lies on this circle, and λ becomes a $(1, -1)$ circle, so that the discs $D_\theta^h \cong D_\theta^*$ constituting SD^* are attached as Hopf-rotated discs $e_{\text{rot}}^{i\theta}(D_0^h) = D_0^h$. The rotated discs $D_{\pm\pi/2}^h$ are equivariantly interchanged by a , and the annulus $A_{\pm\pi/2}^h$ double covers a Möbius band in $\mathbb{R}P^3$. Let $S_\theta^h = D_\theta^h \cup A_\theta^h \cup D_{-\theta}^h$ denote the twist-parametrized spheres, and $S_\theta^2 = D_\theta^2 \cup A_{\theta/2}^2 \cup D_{-\theta}^2$

their projection to $\mathbb{R}P^3 = SD/a \cup SD^*/a$. The eversion in $\mathbb{R}P^3$ can be constructed from a ball neighbourhood of D_0^2 and $\{S_\theta^2\}$, $\theta \in (0, \pi)$. Observe $A_{\theta/2}^2 \subset SD/a$; $D_\theta^2 = e_{rot}^{i\theta} D_0^2$.

The solid donut SD/a also embeds in \mathbb{R}^3 realizing the Hopf circle action as rotation around the z -axis; the attaching circle for the 2-handle arising from SD^*/a is a $(2, -1)$ -torus knot K_0 , which bounds a disc D_0 immersed in \mathbb{R}^3 . Thus the discs D_θ are all embedded by rotating D_0 , and the spheres S_θ^2 all simultaneously smoothly immerse in $\mathbb{R}P^3$. Eversion in an abstract $\mathbb{R}P^3$ is emulated in the corresponding (non-immersed) image of $\mathbb{R}P^3$ in \mathbb{R}^3 described in the previous section.

5 Additional explanatory figures

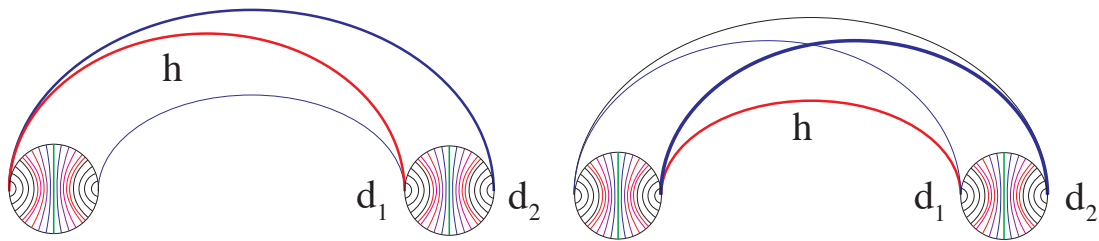


Figure 2: The solid donut SD_z obtained from $I \times D^2$ by identifying two end-discs after π -rotation. The two points d_1, d_2 and arcs in the foliated disc sweep out the $(2, -1)$ knot K_0 , annuli A_ϕ wrapping twice, and a Möbius band M created from the vertical arc. (h stands for ‘Hopf circle’ - see later)

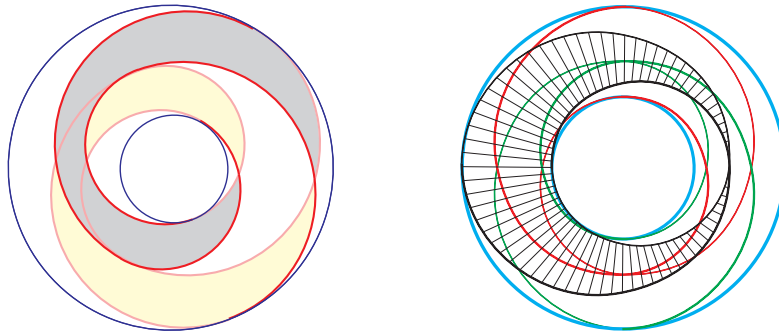


Figure 3: An annulus A_ϕ on the left; on the right, two knots $K_{\pm\pi/2}$ and M , with $\partial M = K_{+\pi} \equiv K_{-\pi}$.

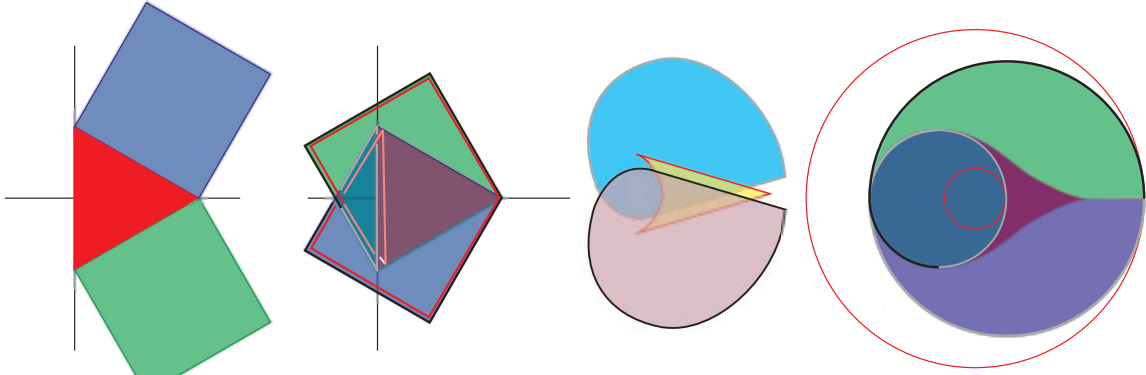


Figure 4: Folding two squares over a triangle, and intersecting, before smoothing to create D' and D_0 . Alternatively, create an intersection first, and then 'fold', as in Figure 1.

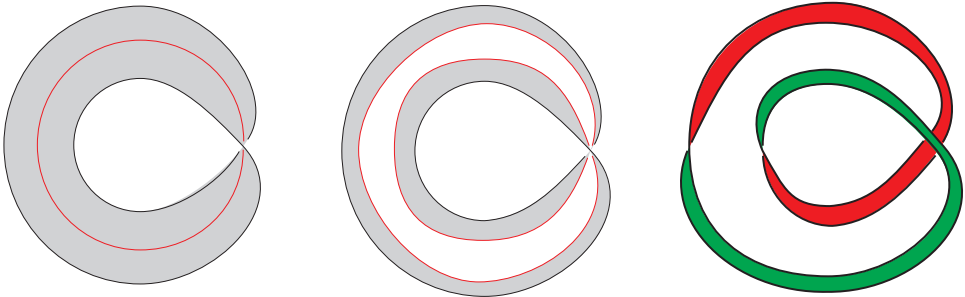


Figure 5: The (-4) -half-twisted annulus as the collar of the Möbius band, also appearing in Figure 4 as the collar of the immersed disc, and as an annular neighbourhood of the $(2, -1)$ -knot on the torus.

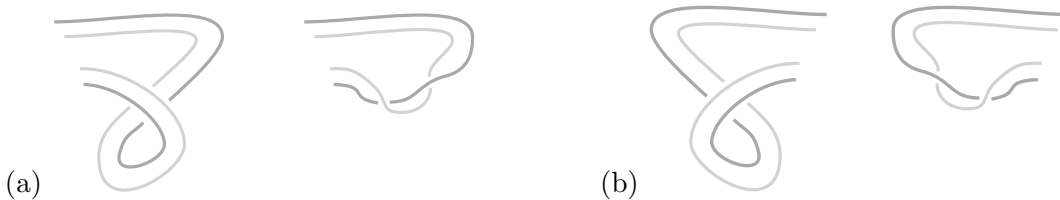


Figure 6: A band in \mathbb{R}^3 with loops can always be pulled tight, converting loops into an even number of half-twists. (b) A loop with a negative crossing yields two negative half-twists. Positive and negative loops are regularly homotopic.

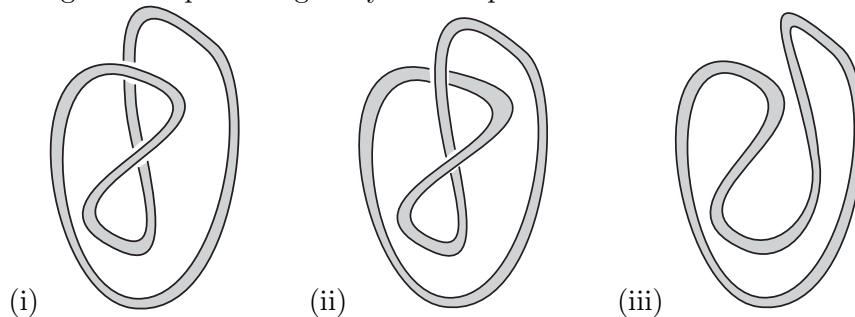


Figure 7: Untwisting the (-4) -half-twisted annulus in (ii) by regular homotopy in \mathbb{R}^3 , maintaining planarity of projection and creating a disc with a single clasp intersection: (i) vertical crossing change homotopy (iii) horizontal homotopy

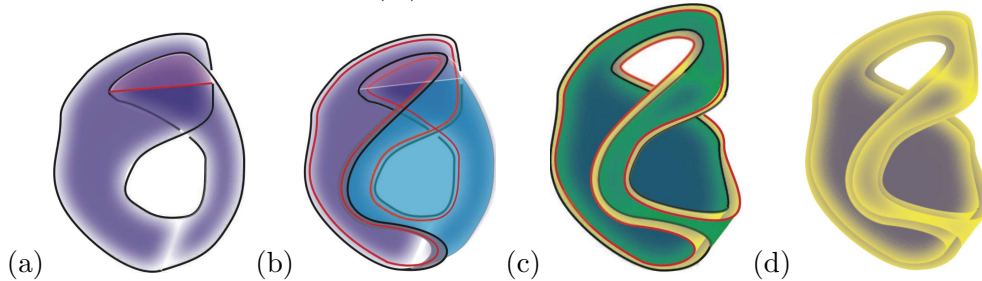


Figure 8: (a) A disc with a single clasp intersection (b) with right edge folded over and curled, revealing a (-4) -half-twisted boundary annulus (c), (d) The same annulus on a thickened Möbius band as 2-fold cover. This is a twisted I -bundle in \mathbb{R}^3 : a solid donut.

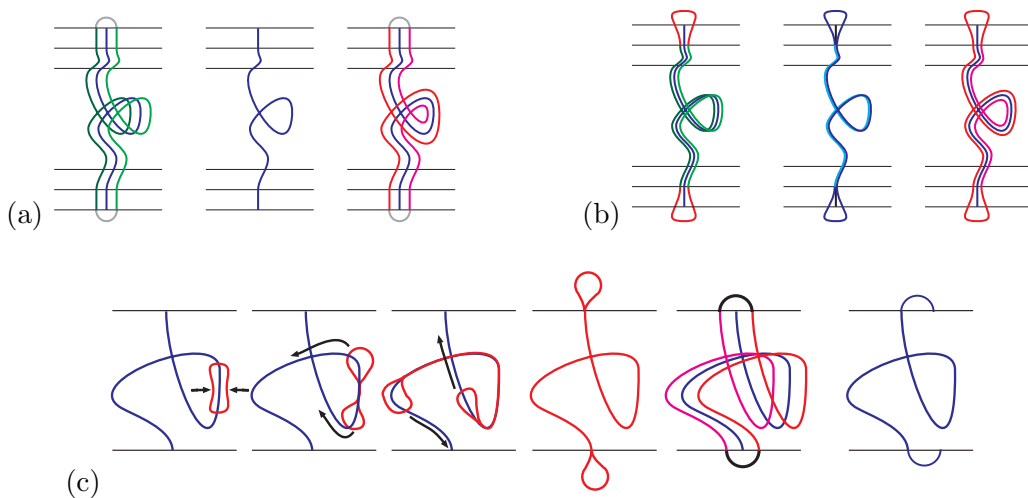


Figure 9: (a) (Spin pictures to obtain:/Cross sections of:) An immersed disc (centre); an immersed sphere, by capping two rotated (translated) copies with an annulus (left); an immersed sphere as the boundary of an immersed regular neighbourhood (right). (b) Such spheres are regularly homotopic. (c) An immersed disc has an immersed neighbourhood whose boundary sphere is regularly homotopic to an embedding. The last picture indicates how to adjust an immersion so that it meets from the other side.

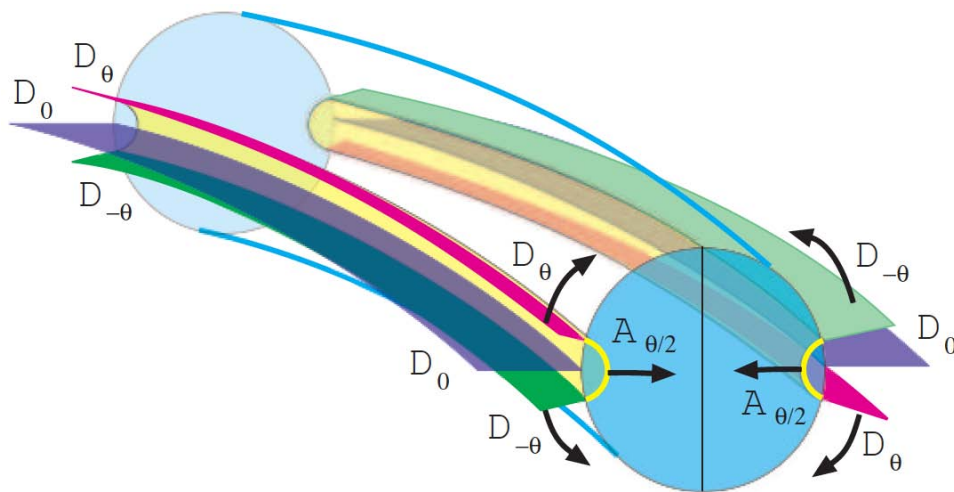


Figure 10: Immersed spheres S_θ consist of two immersed discs D_θ , $D_{-\theta} \equiv D_{2\pi-\theta}$ and an annulus $A_{\theta/2}$. Each annulus $A_{\theta/2}$ wraps twice around the solid donut. The two discs and two sheets of the annulus can be interchanged by increasing θ , rotating the discs in opposite directions: the annulus passes smoothly through itself as it double covers the Möbius band M . Discs inherit opposite orientations from S_θ .

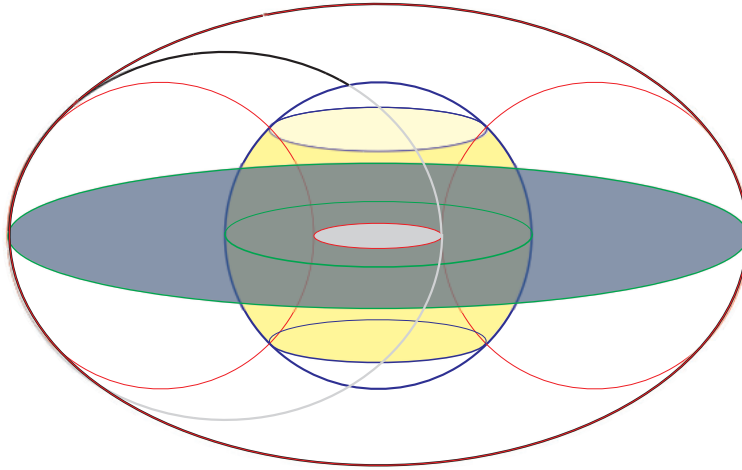


Figure 11: After stereographic projection: View in 3-dimensional space \mathbb{R}^3 of the unit sphere S , Clifford torus CT , Hopf circle and solid donut SD with boundary torus CT in \mathbb{R}^3 : $A = S \cap SD$ and $SD \cap xy$ -plane are annuli meeting along the unit circle. D_0 is the innermost xy -planar disc. The 3-ball and solid donut arise by rotating three coplanar unit discs, whose boundary circles meet orthogonally, around the z -axis. The interior of the solid donut SD can be foliated by annuli parallel to A ; similarly, the complementary solid donut of SD in S^3 can be foliated by discs ‘parallel’ to D_0 , which are caps of spheres, with a ‘disc’ D^* the exterior of the xy -plane centered at infinity. These foliating discs and annuli can be chosen naturally to meet CT orthogonally, with boundary circles parallel to $\partial D_0 := \lambda$. The boundary circle of D_0 can be parametrized in the standard way by angle θ ; each point determines a unique Hopf circle, with H as drawn corresponding to angle $\theta = 0$.

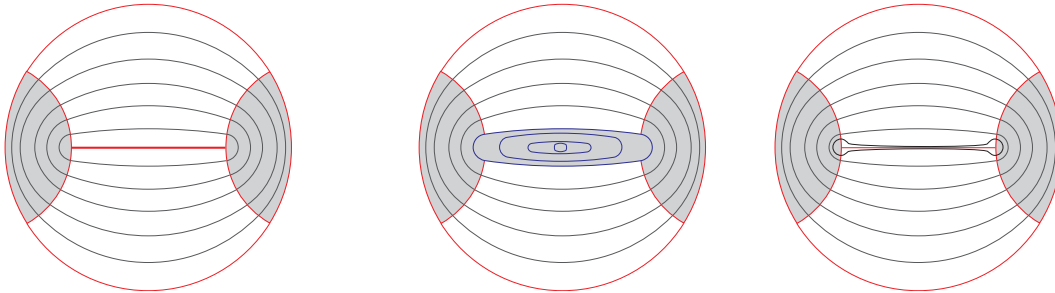


Figure 12: Cross section of the unit ball intersecting the Clifford torus CT , SD and SD^* , with spanning disc D_0 appearing as an interval: spin this around the vertical axis. Concentric spheres can shrink down to D_0 , to a point, or to a shrinking neighbourhoods of a shrinking disc in D_0 .

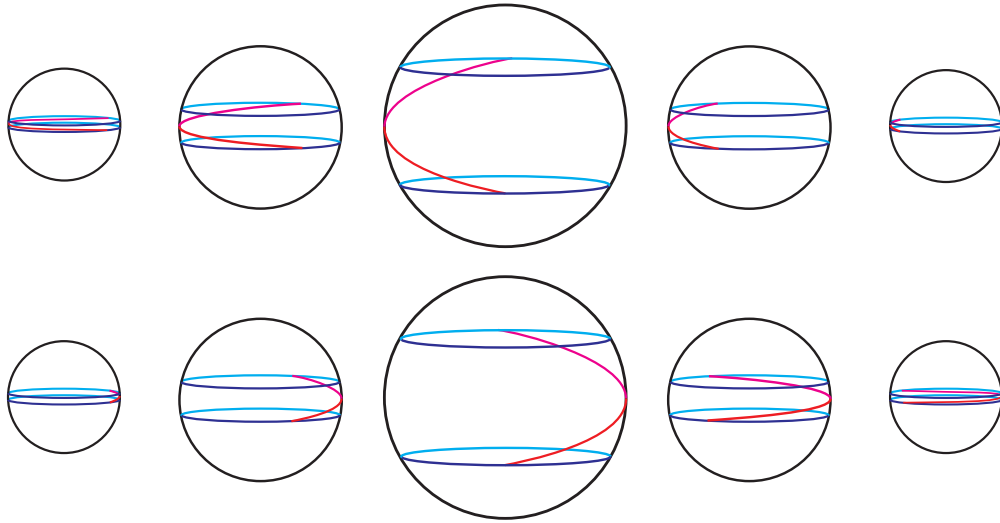


Figure 13: Push D_0 around SD^* in both directions, guided by the Hopf flow. The upper and lower spherical caps rotate in opposite directions. This pair of discs glues to an annulus in SD which undergoes a complete Dehn twist. The top picture is the antipodal image in reverse order to the bottom, which, read left to right, shows the sequence of spheres beginning with a neighbourhood of D_0 .

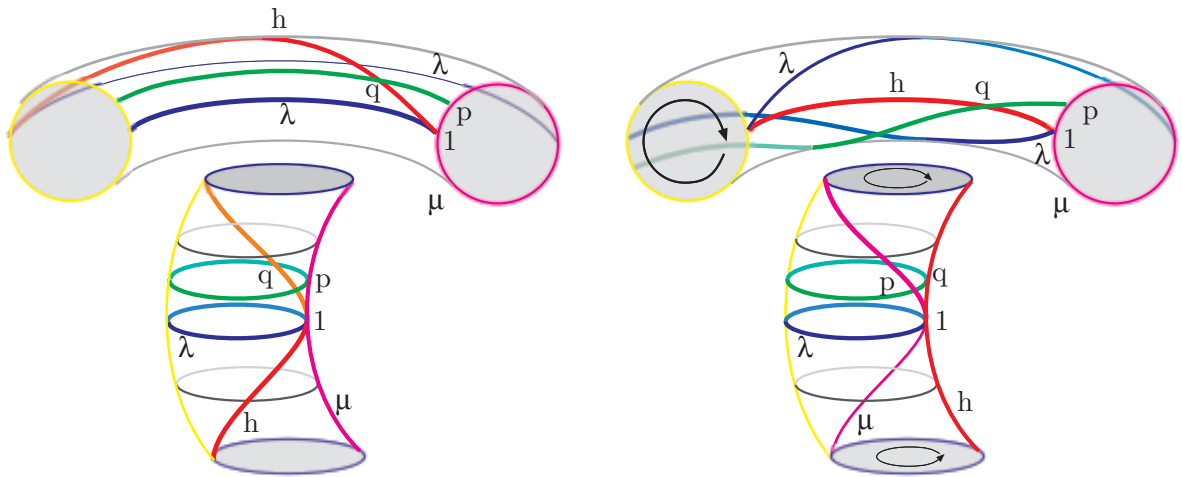


Figure 14: The left picture shows the standard longitude-meridian pair parametrizing the solid donuts SD , SD^* as products $S^1 \times D^2$; the right picture reveals the parametrization by the Hopf flow. Only half of SD is shown; SD^* is shown cut along D_∞ . The point P is indicated as '1'.

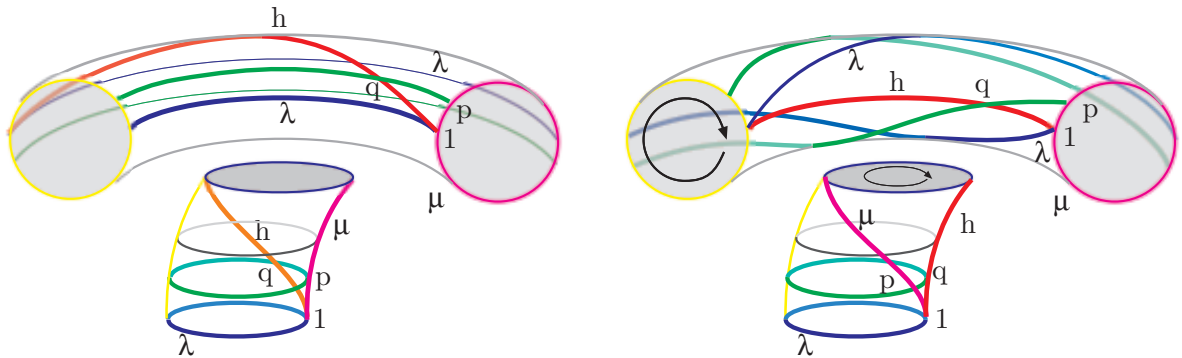


Figure 15: The left picture shows the standard longitude-meridian pair parametrizing the solid donuts SD/a , SD^*/a in RP^3 , as products $S^1 \times D^2$; with π -twist required for identification. the right picture reveals the parametrization by the Hopf flow, enabling mapping RP^3 into R^3 so that the Hopf flow becomes rotation around the z -axis.

Acknowledgement. The author would like to express thanks to Professor Makoto Sakuma and Professor George Francis for their very helpful suggestions, and members of the Department of Mathematics at Hiroshima University for their hospitality.

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