

# A NOTE ON SCHRÖDINGER EQUATION WITH LINEAR POTENTIAL AND HITTING TIMES

GERARDO HERNÁNDEZ-DEL-VALLE

ABSTRACT. In this note we derive a solution to the Schrödinger-type backward equation which satisfies a necessary boundary condition used in hitting-time problems [as described in Hernández-del-Valle (2010a)]. We do so by using an idea introduced by Bluman and Shtelen (1996) which is worked out in Hernández-del-Valle (2010b). This example is interesting since it is independent of the parameter  $\lambda$ , namely:

$$\kappa(s, x) = \frac{x}{\sqrt{2\pi s}} \exp \left\{ -\frac{(x + \int_0^s f'(u))^2}{2s} \right\}.$$

and suggest a procedure to generating more vanishing solutions and  $x = 0$ .

## 1. THE EXAMPLE.

In Hernández-del-Valle (2010b) the author finds solutions to a Schrödinger-type backward equation which satisfy some necessary boundary condition used in hitting-time problems [see Hernandez-del-Valle (2010a)]. Namely, the PDE of interest is

$$(1) \quad -\frac{\partial w}{\partial t}(t, x) + x f''(t) w(t, x) = \frac{1}{2} \frac{\partial^2 w}{\partial x^2}(t, x)$$

which alternatively is related to the following expectation:

$$w(t, x) = \tilde{\mathbb{E}} \left[ \exp \left\{ -\int_t^s f''(u) \tilde{X}_u du \right\} \middle| \tilde{X}_t = x \right]$$

where process  $\tilde{X}$  is the so-called three-dimensional Bessel bridge and  $f$  corresponds to a moving boundary. Furthermore it was shown in Hernandez-del-Valle (2010a) that  $w$  should satisfy the following inequality

$$(2) \quad 0 \leq w(t, x) \leq h(s - t, x) \quad \forall 0 \leq t < s, x \geq 0,$$

where  $h$  is the so-called derived heat source solution defined as

$$h(s, x) := \frac{x}{\sqrt{2\pi s^3}} \exp \left\{ -\frac{x^2}{2s} \right\}.$$

(Which is also the density of the first hitting time of one-dimensional standard Brownian motion to the fixed boundary  $x$ .) Thus, for all time  $t$  it follows that  $w$  should satisfy the following boundary condition:

$$(3) \quad \lim_{x \rightarrow 0} w(t, x) = 0.$$

Yet, for the specific problem of finding the density of hitting a moving boundary we only need the previous inequality (2) and boundary condition (3) to hold at  $t = 0$ :

$$(4) \quad w(0, x) = \tilde{\mathbb{E}} \left[ \exp \left\{ - \int_0^s f''(u) \tilde{X}_u du \right\} \middle| \tilde{X}_0 = x \right].$$

This is accomplished by using an idea introduced in Bluman and Shtelen (1996) which relates the following PDE

$$-\frac{\partial u}{\partial t}(t, x) + V_1(t, x)u(t, x) = \frac{1}{2} \frac{\partial^2 u}{\partial x^2}(t, x)$$

and its adjoint

$$(5) \quad \frac{\partial \Phi}{\partial t}(t, x) + V_1(t, x)\Phi(t, x) = \frac{1}{2} \frac{\partial^2 \Phi}{\partial x^2}(t, x)$$

with the following backward equation

$$-\frac{\partial w}{\partial t}(t, x) + V_2(t, x)w(t, x) = \frac{1}{2} \frac{\partial^2 w}{\partial x^2}(t, x)$$

where

$$V_2(t, x) = V_1(t, x) - \frac{\partial^2}{\partial x^2} \log \Phi.$$

It is done so through the following expression:

$$(6) \quad w(t, x) = \frac{1}{\Phi(t, x)} \left[ \int_0^x u(t, \xi) \Phi(t, \xi) d\xi + B_2(t) \right]$$

with  $B_2(t)$  satisfying the condition

$$(7) \quad \frac{dB_2}{dt} = \frac{1}{2} \left( \frac{\partial \Phi}{\partial x}(t, 0)u(t, 0) - \Phi(t, 0) \frac{\partial u}{\partial x}(t, 0) \right).$$

Thus at  $t = 0$ , see equation (4),

$$w(0, x) = \frac{1}{\Phi(0, x)} \left[ \int_0^x u(0, \xi) \Phi(0, \xi) d\xi \right]$$

and hence

$$\lim_{x \rightarrow 0} w(0, x) = 0$$

as long as

$$\lim_{x \rightarrow 0} \Phi(0, x) \neq 0.$$

Of course, in general we will not be solving the same PDE unless

$$(8) \quad \frac{\partial^2}{\partial x^2} \log \Phi = 0,$$

in which case  $V_2(t, x) = V_1(t, x)$ , that is the form of the PDE is preserved. The reader may check [or consult at Hernandez-del-Valle (2010a) and (2010b)] that given  $V_1(t, x) = xf''(t)$  a solution to (5) which also satisfies condition (8) is for instance:

$$\begin{aligned} \Phi(t, x) \\ = \exp \left\{ \frac{1}{2} \int_0^t (f'(u))^2 du - xf'(t) - \frac{1}{2} \lambda^2 t - i\lambda \left( x - \int_0^t f'(u) du \right) \right\} \end{aligned}$$

where  $i = \sqrt{-1}$  and  $\lambda$  is some scalar.

In the remainder of this note we derive a solution to (1), which is independent of  $\lambda$ , and also satisfies boundary condition (3) at  $t = 0$ . We do so by using Bluman and Shtelen's representation, equations (6) and (7), and a solution to (5) given by:

$$(9) \quad u(t, x) = \exp \left\{ \frac{1}{2} \int_t^s (f'(u))^2 du + xf'(t) \right\}$$

$$(10) \quad \times \exp \left\{ -\frac{1}{2} \lambda^2 (s - t) + i\lambda \left( x + \int_t^s f'(u) du \right) \right\}.$$

[The reader may consult Hernández-del-Valle (2007).] It follows that:

$$\Phi(t, x)u(t, x) = \exp \left\{ \frac{1}{2} \int_0^s (f'(u))^2 du - \frac{1}{2} \lambda^2 s + i\lambda \int_0^s f'(u) du \right\}$$

and

$$\begin{aligned} \frac{dB_2}{dt}(t) &= \frac{1}{2} \Phi u \left( \frac{\Phi_x}{\Phi} - \frac{u_x}{u} \right) \\ &= -(f'(t) + i\lambda) u \Phi. \end{aligned}$$

Hence,  $B_2$  might be written in the two following ways

$$(11) \quad B_2(t) = - \left( \int_0^t f'(u) du + i\lambda t \right) u \Phi$$

or

$$(12) \quad B_2'(t) = \left( \int_t^s f'(u) du + i\lambda(s - t) \right) u \Phi.$$

From (6)  $w$  satisfies

$$\begin{aligned} w(t, x) &= \frac{1}{\Phi} \left[ \int_0^x u \Phi dy + B_2(t) \right] \\ &= x \frac{u\Phi}{\Phi} + \frac{B_2(t)}{\Phi}, \end{aligned}$$

and hence for  $B_2$  as in (11)

$$(13) \quad w = \left( \left\{ x - \int_0^t f'(u) du \right\} - i\lambda t \right) u,$$

and for  $B'_2$  as in (12)

$$(14) \quad w = \left( \left\{ x + \int_t^s f'(u) du \right\} + i\lambda(s-t) \right) u.$$

Finally, recall the following Fourier representations:

$$(15) \quad \begin{aligned} k(t, x) &:= \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}\lambda^2 t + i\lambda x} d\lambda \end{aligned}$$

$$(16) \quad \begin{aligned} h(t, x) &:= \frac{x}{\sqrt{2\pi t^3}} e^{-\frac{x^2}{2t}} \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} (-i\lambda) e^{-\frac{1}{2}\lambda^2 t + i\lambda x} d\lambda \end{aligned}$$

also known as the source and *derived* source heat equations respectively.

After contour integration with respect to  $\lambda$ , (13) becomes:

$$\begin{aligned} w(t, x) &= \exp \left\{ \frac{1}{2} \int_t^s (f'(u))^2 du + x f'(t) \right\} \\ &\quad \times \left\{ \left( x - \int_0^t f'(u) du \right) k \left( s-t, x + \int_t^s f'(u) du \right) \right. \\ &\quad \left. + t \cdot \frac{\left( x + \int_t^s f'(u) du \right)}{(s-t)} k \left( s-t, x + \int_t^s f'(u) du \right) \right\} \end{aligned}$$

and for  $B'_2$  or equation (14)

$$\begin{aligned} w(t, x) &= \exp \left\{ \frac{1}{2} \int_t^s (f'(u))^2 du + x f'(t) \right\} \\ &\quad \times \left\{ \left( x + \int_t^s f'(u) du \right) k \left( s - t, x + \int_t^s f'(u) du \right) \right. \\ &\quad \left. - \left( x + \int_t^s f'(u) du \right) k \left( s - t, x + \int_t^s f'(u) du \right) \right\} \\ &\equiv 0. \end{aligned}$$

The second solution is identically zero, and the first evaluated at  $t = 0$  is

$$w(0, x) = \exp \left\{ \frac{1}{2} \int_0^s (f'(u))^2 du + x f'(0) \right\} x \cdot k \left( s, x + \int_0^s f'(u) du \right)$$

This alternatively implies that an approximation to the first hitting time density is given by:

$$\frac{x}{\sqrt{2\pi s}} \exp \left\{ -\frac{(x + \int_0^s f'(u) du)^2}{2s} \right\}.$$

**1.1. More vanishing solutions at  $x = 0$  and  $t = 0$ .** Observe that if  $u$  is as in (9) which alternatively solves (5) then

$$u'(t, x) = \Gamma(\lambda, s) u(t, x)$$

is also a solution to (5) for an arbitrary function  $\Gamma$ . For instance, suppose that  $\Gamma(\lambda, s) = (-i\lambda)$  then equation (13) becomes:

$$\begin{aligned} w' &= \left( \left\{ x - \int_0^t f'(u) du \right\} - i\lambda t \right) (-i\lambda) u \\ &= \left( \left\{ x - \int_0^t f'(u) du \right\} (-i\lambda) + (-i\lambda)^2 t \right) u. \end{aligned}$$

After contour integration and observing that

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} (-i\lambda)^2 e^{-\frac{1}{2}\lambda^2 t + i\lambda x} d\lambda = \left( \frac{x^2}{t^2} - \frac{1}{t} \right) k(t, x)$$

it follows that the new solution  $w'$  is given by

$$\begin{aligned} w'(t, x) &= \exp \left\{ \frac{1}{2} \int_t^s (f'(u))^2 du + x f'(t) \right\} \\ &\times \left\{ \left( x - \int_0^t f'(u) du \right) h \left( s - t, x + \int_t^s f'(u) du \right) \right. \\ &\quad \left. + t \cdot \left[ \frac{\left( x + \int_t^s f'(u) du \right)^2}{(s-t)^2} - \frac{1}{(s-t)} \right] k \left( s - t, x + \int_t^s f'(u) du \right) \right\} \end{aligned}$$

or

$$w'(0, x) = \exp \left\{ \frac{1}{2} \int_0^s (f'(u))^2 du + x f'(0) \right\} x \cdot h \left( s, x + \int_0^s f'(u) du \right)$$

and  $h$  is as in (16).

#### REFERENCES

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