Classification of Harish-Chandra modules over some Lie algebras related to the Virasoro algebra *

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Abstract

In this paper, we provide a uniform method to thoroughly classify all Harish-Chandra modules over some Lie algebras related to the Virasoro algebras. We first classify such modules over the Lie algebra $W(\varrho)[s]$ for $s = 0, \frac{1}{2}$. With this result and method, we can also do such works for some Lie algebras related to the Virasoro algebra, including the several kinds of Schrödinger-Virasoro Lie algebras, which are open up to now.

Keywords: Viraosoro algebra, Schrödinger-Virasoro algebra, Harish-Chandra module

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1 Introduction

Recently many infinite dimensional Lie algebras related to the Virasoro algebra were studied sufficiently. Among them, the twisted Heisenberg-Virasoro algebra was first studied by Arbarello et al. in [1], where a connection is established between the second cohomology of certain moduli spaces of curves and the second cohomology of the Lie algebra of differential operators of order at most one. The W-algebra $W(2,2)$ was introduced in [34] for the study of classification of vertex operator algebras generated by weight 2 vectors. Schrödinger-Virasoro algebras, playing important roles in mathematics and statistical physics, were first introduced by M. Henkle in [8] by studying the free Schrödinger equation.

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Since all the above Lie algebras are closely related to the Virasoro Lie algebra, it is highly expected that their Hraish-Chandra modules structures are well classified as the Virasoro algebra in [22] (also see [12], [21], [23], [27], etc.), the high rank Virasoro algebra in [19] and [28], the Weyl algebra in [29]. However, up to now, there are few results about it. In [20] all Harish-Chandra modules over the twisted Heisenberg-Virasoro algebra were classified. However their calculations are very complicated and cannot to be used in general. In [14], such study was considered for the original and twisted Schrödinger-Virasoro Lie algebras. However all irreducible uniform bounded modules over them are yet not classified, it is the key point to classify all Harish-Chandra modules and is open up to now for many Lie algebras. So it is a very important question that how to classify all Harish-Chandra modules, especially all irreducible uniform bounded modules, over some Lie algebras related to the Virasoro algebra. In this paper, we provide a new and uniform method to thoroughly classify all Harish-Chandra modules over some Lie algebras related to the Virasoro algebras, including all the above Lie algebras. Throughout the paper, we shall use $\mathbb{C}, \mathbb{Q}$, $\mathbb{Z}$ and $\mathbb{Z}_+$ to denote the sets of complex numbers, rational numbers, integers and positive integers, respectively. For any set S, we use S^* to denote the set of nonzero elements in S.

First we introduce the following Lie algebras.

Definition 1.1. *For $s = 0, \frac{1}{2}$ and $\varrho \in \mathbb{Q}$, as a vector space over $\mathbb{C}$, the Lie algebra $\mathcal{L}[s] = W(\varrho)[s]$ has a basis $\{L_n, Y_p \mid n \in \mathbb{Z},\ p \in \mathbb{Z} + s\}$ with the following relations*

$$[L_m, L_n] = (n - m)L_{m+n}, \tag{1.1}$$
$$[L_m, Y_p] = (p - m\varrho)Y_{m+p}, \tag{1.2}$$
$$[Y_p, Y_q] = 0 \tag{1.3}$$

for all $m,\ n \in \mathbb{Z}$ and $p,\ q \in \mathbb{Z} + s$.

The Lie algebra $W(\varrho)[s]$ can be realized from the semi-product of the centerless Virasoro algebra Vir and the Vir-module $\mathcal{F}_\varrho$ of the intermediate series. In fact, let Vir= $\mathrm{Span}_{\mathbb{C}}\{L_m \mid m \in \mathbb{Z}\}$ be the centerless Virasoro algebra (also called Witt algebra) and $H = \mathbb{C}\{Y_p \mid p \in \mathbb{Z} + s\}$ (denote by $\mathcal{F}_\varrho$ in [24]) be a Vir-module with actions $L_m \cdot Y_p = (p - m\varrho)Y_{m+p}$ for any $m \in \mathbb{Z},\ p \in \mathbb{Z} + s$, then $W(\varrho)[s]$ is just the Lie algebra Vir $\ltimes H$.

The Lie algebra $\mathcal{L}[s]$ is more connected to the the cohomology of the Virasoro algebra and extensions of some Lie algebras (see [5], [24],[25] and [6]).

As Vir-modules, $\mathcal{F}_0$ and $\mathcal{F}_{-1}$ have $\mathbb{C}v_0$ as a submodule and quotient module respectively. Their corresponding irreducible quotient and submodule are isomorphic (both isomorphic to $\mathcal{A}'_{0,0}$, see Section 2 in detail), so we always suppose that $\varrho \neq -1$ throughout this paper. Moreover if $s = \frac{1}{2}$, we concentrate our study on the case of $\rho = \frac{m}{n}$, $(m, n) = 1$ and n is an even number, which is enough for almost Lie algebras as we known.

The Lie algebra $W(0)[0]$ is just the centerless twisted Heisenberg-Virasoro algebra (see [1]). It can be realized as the Lie algebra of differential operators of order at most 1. The structure and representation theory of this Lie algebra and its universal central extensions were studied in many papers (see [1, 2, 15, 20, 30, 18, 31, 17], etc.).

The universal central extension of $W(1)[0]$ (named $W(2,2)$ in [34]) can be also realized from the so-called *loop-Virasoro algebra* $\mathcal{G}$ (see [7]). Let $\mathbb{C}[t,t^{-1}]$ be the Laurents polynomial ring over $\mathbb{C}$, then the $W(2,2)=\mathcal{G}/(t^2)$.

The universal central extension of $W(1)[0]$ and its highest weight modules enter the picture naturally during the discussion on $L(\frac{1}{2},0)\otimes L(\frac{1}{2},0)$. Its highest weight modules produce a new class of vertex operator algebras. Contrast to the Virasoro algebra case, this class of vertex operator algebras are always irrational. Several papers studied its representation theory (see [34, 16, 11, 7], etc.).

Schrödinger-Virasoro algebras (see Section 5 for their definitions) are more connected to $W(\varrho)[s]$. In fact, $W(0)[0]$ (resp. $W(\frac{1}{2})[s]$) is a subalgebra (resp. quotient algebra) of the original or twisted Schrödinger-Virasoro algebra. Recently many researches on the structure and representation theory of the original, twisted, deformative Schrödinger-Virasoro algebras (see [8], [3], [9], [10], [26], [14], [32], [33], [4], etc.).

Therefore Lie algebras $\mathcal{L}[s]$ play very important roles on the study of Lie algebras related to the Virasoro algebra. The present paper is devoted to determining all Harish-Chandra modules (i.e. irreducible weight modules with finite dimensional weight spaces) over Lie algebras $W(\varrho)[s]$. More precisely we prove that there are three different classes of Harish-Chandra modules over them. One class is formed by simple modules of intermediate series, the other two classes consist of the highest, lowest weight modules. It is consistent with that for the Virasoro algebra case in [22]. Here we get a key lemma (Lemma 3.1 in Section 3) for the irreducible uniform bounded weight modules and then obtain the main result over the Lie algebra $W(\varrho)[s]$. With this method and result, we get a beautiful application to determine irreducible uniform bounded weight modules over Schrödinger-Virasoro algebras.

The paper is arranged as follows. In Section 2, we recall some notations and collect known facts about irreducible, indecomposable modules over the Virasoro algebra. In Section 3, we determine all irreducible uniformly bounded weight modules over $\mathcal{L}[s]$ which turn out to be modules of intermediate series. Moreover we obtain the main result of this paper: the classification of irreducible weight $\mathcal{L}[s]$-modules with finite dimensional weight space. As we mentioned, they are irreducible highest, lowest weight modules, or irreducible modules of the intermediate series. In Section 4, using the methods and results in Section 3, we classified all Harish-Chandra modules over some Lie algebra related the Virasoro algebra, including the original, twisted and deformative Schrödinger-Virasoro algebras.

For convenient, all modules considered in this paper are nontrivial. We always denote by $U(L)$ the universal enveloping algebra of a given Lie algebra L.

2 Basics

In this section, we collect some known facts for later use.

Introduce a $(1-s)\mathbb{Z}$-gradation on $\mathcal{L}[s] = W(\varrho)[s]$ by defining the degrees: deg $L_n = n$, deg $Y_p = p$. Set

$$\mathcal{L}[s]_+ = \sum_{n,p>0} (\mathbb{C}L_n + \mathbb{C}Y_p), \ \mathcal{L}[s]_- = \sum_{n,p<0} (\mathbb{C}L_n + \mathbb{C}Y_p),$$

and

$$\mathcal{L}[s]_0 = \mathbb{C}L_0 + \mathbb{C}(1-2s)Y_0.$$

The $(1-s)\mathbb{Z}$-gradation of $\mathcal{L}[s] = \oplus_{p\in(1-s)\mathbb{Z}}\mathcal{L}[s]_p$ induces $(1-s)\mathbb{Z}$-gradations on the universal enveloping algebra $U(\mathcal{L}[s]) = \oplus_{p\in(1-s)\mathbb{Z}}U(\mathcal{L}[s])_p$ and the universal enveloping algebra $U(H) = \oplus_{p\in(1-s)\mathbb{Z}}U(H)_p$, where H is the abelian subalgebra of $\mathcal{L}[s]$ spanned by Y_r for all $r \in \mathbb{Z} + s$.

The universal central extensions of $\mathcal{L}[0]$ were given in [6]. Such work for $\mathcal{L}[\frac{1}{2}]$ can also be easily done as [6].

Proposition 2.1. *[6]*

$$H^2(W(\varrho)[0], \mathbb{C}) = \begin{cases} \mathbb{C}\gamma_0 + \mathbb{C}\gamma_{01} + \mathbb{C}\gamma_{02}, & \varrho = 0; \\ \mathbb{C}\gamma_0 + \mathbb{C}\gamma_{11}, & \varrho = 1; \\ \mathbb{C}\gamma_0, & \textit{otherwise} \end{cases}$$

with 2-cocycles as follows:

$\gamma_0(L_m, L_n) = \frac{1}{12}(m^3 - m)\delta_{m+n,0}$;

$\gamma_{01}(L_m, I_n) = (m^2 - m)\delta_{m+n,0}$; $\gamma_{02}(I_m, I_n) = n\delta_{m+n,0}$;

$\gamma_{11}(L_m, I_n) = \frac{1}{12}(m^3 - m)\delta_{m+n,0}$.

An $\mathcal{L}[s]$-module V is called a weight module if V is the sum of all its weight spaces $V^\lambda = \{v \in V \mid L_0 v = \lambda v\}$. For a weight module V we define

$$\mathrm{Supp}(V) := \{\lambda \in \mathbb{C} \mid V^\lambda \neq 0\},$$

which is called the weight set (or the support) of V.

An irreducible weight $\mathcal{L}[s]$-module V is called the intermediate series if all its weight space are one dimensional.

A weight $\mathcal{L}[s]$-module V is called a highest (resp. lowest) weight module with highest weight (resp. lowest weight) $\lambda \in \mathbb{C}$, if there exists a nonzero weight vector $v \in V^\lambda$ such that

1) V is generated by v as $\mathcal{L}[s]$-module;

2) $\mathcal{L}[s]_+ v = 0$ (resp. $\mathcal{L}[s]_- v = 0$).

Obviously, if V is an irreducible weight $\mathcal{L}[s]$-module, then there exists $\lambda \in \mathbb{C}$ such that $\mathrm{Supp}(V) \subset \lambda + (1-s)\mathbb{Z}$. So $V = \sum_{p \in (1-s)\mathbb{Z}} V_p$ is a $(1-s)\mathbb{Z}$-graded module, where $V_p = V^{\lambda+p}$.

If, in addition, all weight spaces V^λ of a weight $\mathcal{L}[s]$-module V are finite dimensional, the module V is called a *Harish-Chandra module*. Clearly a highest (lowest) weight module is a Harish-Chandra module.

Kaplansky-Santharoubane [13] in 1983 gave a classification of Vir-modules of the intermediate series. There are three families of indecomposable modules with each weight space is one-dimensional:

(1) $\mathcal{A}_{a,\,b} = \sum_{i \in \mathbb{Z}} \mathbb{C} v_i$: $L_m v_i = (a + i + bm) v_{m+i}$;

(2) $\mathcal{A}(a) = \sum_{i \in \mathbb{Z}} \mathbb{C} v_i$: $L_m v_i = (i + m) v_{m+i}$ if $i \neq 0$, $L_m v_0 = m(m+a) v_m$;

(3) $\mathcal{B}(a) = \sum_{i \in \mathbb{Z}} \mathbb{C} v_i$: $L_m v_i = i v_{m+i}$ if $i \neq -m$, $L_m v_{-m} = -m(m+a) v_0$, for some $a, b \in \mathbb{C}$.

It is well-known that $\mathcal{A}_{a,\,b} \cong \mathcal{A}_{a+1,\,b}, \forall a, b \in \mathbb{C}$, then we can always suppose that $a \notin \mathbb{Z}$ or $a = 0$ in $\mathcal{A}_{a,\,b}$. Moreover the module $\mathcal{A}_{a,\,b}$ is simple if $a \notin \mathbb{Z}$ or $b \neq 0, 1$. In the opposite case the module contains two simple subquotients namely the trivial module and $\mathbb{C}[t, t^{-1}]/\mathbb{C}$. It is also clear that $\mathcal{A}_{0,0}$ and $\mathcal{B}(a)$ both have $\mathbb{C} v_0$ as a submodule, and their corresponding quotients are isomorphic, which we denote by $\mathcal{A}'_{0,0}$. Dually, $\mathcal{A}_{0,1}$ and $\mathcal{A}(a)$ both have $\mathbb{C} v_0$ as a quotient module, and their corresponding submodules are isomorphic to $\mathcal{A}'_{0,0}$. For convenience we simply write $\mathcal{A}'_{a,b} = \mathcal{A}_{a,b}$ when $\mathcal{A}_{a,b}$ is irreducible.

All Harish-Chandra modules over the Virasoro algebra were classified in [22] (also in [21] and [27]). Since then such works were done on the twisted Heisenberg-Virasoro algebra in [20].

Theorem 2.2. *([22], etc) Let V be an irreducible weight Vir-module with finite dimensional weight spaces. Then V is a highest weight module, lowest weight module, or Harish-Chandra module of intermediate series.*

Theorem 2.3. *[20] Let V be an irreducible weight module over $W(0)[0]$ with all weight spaces finite-dimensional. Then V is a highest weight module, a lowest weight module, or a Harish-Chandra module of intermediate series.*

Remark. For $a, b, c \in \mathbb{C}$, let $\mathcal{A}_{a,\,b,\,c}$ be the $W(0)[0]$-module induced from the Vir-module $\mathcal{A}_{a,\,b}$ with $Y_p v_k = c v_{p+k}$ for all $p, k \in \mathbb{Z}$. Clearly $\mathcal{A}_{a,\,b,\,c}$ is irreducible if and only if $a \notin \mathbb{Z}$ or $b \neq 0, 1$ or $c \neq 0$. We also use $\mathcal{A}'_{a,\,b,\,c}$ to denote by the nontrivial simple subquotient of $\mathcal{A}_{a,\,b,\,c}$ (or itself if it is irreducible). From [20] or [15] we know that any Harish-Chandra $W(0)[0]$-module of intermediate series is isomorphic to $\mathcal{A}'_{a,\,b,\,c}$ for some $a, b, c \in \mathbb{C}$.

In [14], Harish-Chandra modules over the original and twisted Schrödinger-Virasoro Lie algebras were studied. However all irreducible uniform bounded modules over them are yet not classified, it is the key point to classify all Harish-Chandra modules and is open up to now.

Theorem 2.4. *[14] Let V be an irreducible weight module over the Schrödinger-Virasoro Lie algebra $\mathfrak{sv}[s]$ with all weight spaces finite-dimensional. Then V is a highest weight module, a lowest weight module, or a uniformly bounded irreducible weight module.*

3 Harish-Chandra modules over $\mathcal{L}[s]$

In this section, we shall classify all irreducible weight module with finite dimensional weight spaces over $\mathcal{L}[s]$.

Lemma 3.1. *Let V be a uniformly bounded nontrivial irreducible weight module over $\mathcal{L}[s]$. Suppose that there exist a nonzero $u_{i_0} \in V_{i_0}$ and Y_{r_0}, $r_0 \neq 0$ such that $Y_{r_0}u_{i_0} = 0$. Then $HV = 0$, where H be the abelian subalgebra of $\mathcal{L}[s]$ spanned by Y_r for all $r \in \mathbb{Z} + s$.*

Proof.

Assume that $V = \sum V_p$ is a uniformly bounded nontrivial irreducible weight module over $\mathcal{L}[s]$, where $\mathrm{Supp}(V) = \{a + (1-s)\mathbb{Z}\}$ and $V_p = \{v \in V \mid L_0 v_p = (a+p)v\}$ for some $a \in \mathbb{C}$.

From representation theory of Vir (see [13]), $\dim V_p = n$ for all $a + p \neq 0$. If $a \in \mathbb{Z}$, we can assume that $a = 0$. Moreover, as a Vir-module, V has a Vir-submodule filtration

$$0 = W^{(0)} \subset W^{(1)} \subset W^{(2)} \subset \cdots \subset W^{(p)} = V,$$

where $W^{(1)}, \cdots, W^{(p)}$ are Vir-submodules of V, and the quotient modules $W^{(i)}/W^{(i-1)} \cong \mathcal{A}'_{a_i,b_i}$ for some $a_i, b_i \in \mathbb{C}$.

Set $U = U(\mathrm{Vir})u_{i_0}$. Clearly U is a nontrivial Vir-module (if not, U is a trivial $\mathcal{L}[s]$-module). So U is also a uniform bounded Vir-module and then there exists an irreducible Vir-submodule $V' \cong \mathcal{A}'_{a,b}$ of U for some $a, b \in \mathbb{C}$ by the above statement and Theorem 2.2. Moreover $V' = \sum_{i\in\mathbb{Z}} \mathbb{C}v_i$ and $V = U(H)V' = \sum_{i\in\mathbb{Z}} U(H)v_i$ (If $V' \cong \mathcal{A}'_{0,0}$, then $V = U(H)V' = \sum_{i\in\mathbb{Z}^*} U(H)v_i$).

Fixed $i \in \mathbb{Z}$ and $v_i \in U(\mathrm{Vir})u_{i_0}$, there exists $n \in \mathbb{Z}_+$ such that for every $v_i = f_i u_{i_0}$, where f_i is a finite sum of terms $c_i L_{m_1} L_{m_2} \cdots L_{m_k}$ with all $k \leq n-1$ and $c_i \in \mathbb{C}$.

Since $Y_{r_0}u_{i_0} = 0$ then $Y_{r_0}U(H)u_{i_0} = 0$. Moreover by $Y_{r_0}L_m u_{i_0} = L_m Y_{r_0} u_{i_0} - (r_0 - m\varrho)Y_{m+r_0}u_{i_0}$ we have $Y_{r_0}^2 L_m u_{i_0} = 0$. Similarly $Y_{r_0}^{k+1} L_{m_1} L_{m_2} \cdots L_{m_k} u_{i_0} = 0$.

So we have $Y_{r_0}^n v_i = 0$. Now since $L_j v_i = (a + bj + i)v_{i+j}$ for any $j \in \mathbb{Z}$, then there exists $N \in \mathbb{Z}_+$ such that $Y_{r_0}^N V' = 0$.

Therefore

$$Y_{r_0}^N V = 0. \tag{3.1}$$

By actions of suitable L_m for some $m \in \mathbb{Z}$ on (3.1) we can deduce that

$$H^N V = 0, \tag{3.2}$$

where $H^N := \mathbb{C}\{Y_{r_1}Y_{r_2}\cdots Y_{r_N} \mid r_i \in \mathbb{Z}+s, i = 1, 2, \cdots, N\}$ (just as a vector space over $\mathbb{C}$). Since HV is a $\mathcal{L}[s]$-submodule of V, then $HV = V$ or $HV = 0$. In the first case we also have $HV = 0$ by (3.2). $\square$

Theorem 3.2. *Let V be a uniformly bounded irreducible $\mathcal{L}[s] = W(\varrho)[s]$-module. If $\varrho \neq 0$, then $V \simeq \mathcal{A}'_{a,\,b,\,0}$ for some $a, b \in \mathbb{C}$, where $\mathcal{A}_{a,\,b,\,0}$ is the $\mathcal{L}[0]$-module induced from the Vir-module $\mathcal{A}_{a,\,b}$ with $Y_p v_k = 0$ for all $p, k \in \mathbb{Z}$, and $\mathcal{A}'_{a,\,b,\,0}$ is nontrivial simple subquotient of $\mathcal{A}_{a,\,b,\,0}$ (or itself if it is irreducible). Consequently, V is an irreducible weight module over the Virasoro algebra.*

Proof. We will use the notations from Lemma 3.1.

By Lemma 3.1 we only need to consider the case that all $Y_i,\ i \in \mathbb{Z}+s, i \neq 0$, are nonsingular on V. In the case of $s = 0$, if there exists a nonzero $v \in V$ such that $Y_0 v = 0$, then as in the proof of Lemma 3.1, we can also deduce that $H^n V = 0$. It is contradict to the hypothetical conditions. So Y_0 is also nonsingular on V.

So we can suppose that all $Y_i, i \in \mathbb{Z}+s$, are nonsingular on V. Then all $V_p = \{v \in V \mid L_0 v = (a+p)v\}, p \in (1-s)\mathbb{Z}$, have a same finite dimension.

Now choose an irreducible Vir-submodule $V' = \sum \mathbb{C}v_i \cong \mathcal{A}'_{a,b}$ of V and then $V = U(H)V' = \sum U(H)v_i$.

Let $h = Y_i Y_{-i} \in U(H)_0$ and $k \in (1-s)\mathbb{Z}$, then h is a linear transformation over the finite dimensional vector space V_k. So there exists a minimal polynomial $m(x)$ of h such that $m(h)V_k = 0$ (if $s = \frac{1}{2}$, then we can regard h as a linear transformation over the finite dimensional vector space $V_k + V_{k+\frac{1}{2}}$). By actions of some Y_j we can get

$$m(h)V = 0. \tag{3.3}$$

Since $\varrho \in \mathbb{Q}^*$, so there exist $m_0 \in \mathbb{Z}$ and $i_0 \in \mathbb{Z}+s$ such that $i_0 - m_0\varrho = 0$ (where we suppose that if $s = \frac{1}{2}$, then $\rho = \frac{m}{n}$, $(m, n) = 1$ and n is an even number, see Section 1). By action of L_{m_0} on (3.3) for $h = Y_{i_0}Y_{-i_0}$, we have $Y_{i_0}Y_{m_0-i_0}m'(h)V = 0$ then $m'(h)V = 0$. It gets a contradiction. $\square$

Remark. The above method is not suitable for the case of $\varrho = 0$.

Proposition 3.3. *Let V be an irreducible weight module over $W(\varrho)[s]$ ($\varrho \neq 0$) with all weight spaces finite-dimensional. If V is not uniformly bounded, then V is either a highest weight module or a lowest weight module.*

Proof. The proof is essentially same as that in the case of $\varrho = 0$ in [20] (also see [14] for the $\varrho = \frac{1}{2}$ case). $\square$

With Theorems 3.2 and Proposition 3.3, we obtain the main result of this paper.

Theorem 3.4. *If V is an irreducible weight module over $W(\varrho)[s]$ ($\varrho \neq 0$), with finite dimensional weight spaces, then V is a highest or lowest weight module or the Harish-Chandra module of the intermediate series.*

4 Harish-Chandra modules over some Lie algebras

In this section, we use the results and methods in Section 3 to classify all Harish-Chandra modules over some Lie algebras related to the Virasoro algebra. *Especially, we get a beautiful application to the Schrödinger algebras for such work, which was conjectured several years.*

4.1 The Schrödinger-Virasoro Lie algebra

For $s = 0, \frac{1}{2}$, the Schrödinger-Virasoro Lie algebra $\mathfrak{sv}[s]$ introduced in [9, 10, 26], in the context of non-equilibrium statistical physics as a by-product of the computation of n-point functions that are covariant under the action of the Schrödinger group, is the infinite-dimensional Lie algebra with $\mathbb{C}$-basis $\{L_n,\, M_n,\, Y_p \mid n \in \mathbb{Z},\, p \in \mathbb{Z}+s\}$ and Lie brackets,

$$[L_m,\, L_n] = (n-m)L_{m+n}, \tag{4.1}$$
$$[L_m,\, Y_p] = (p-\frac{m}{2})Y_{p+m}, \tag{4.2}$$
$$[L_m,\, M_n] = nM_{n+m}, \tag{4.3}$$
$$[Y_p,\, Y_q] = (q-p)M_{q+p}, \tag{4.4}$$
$$[Y_p,\, M_n] = [M_n,\, M_{n'}] = 0. \tag{4.5}$$

The Lie algebras $\mathfrak{sv}[\frac{1}{2}]$ and $\mathfrak{sv}[0]$ are called the original and twisted Schrödinger-Virasoro algebra respectively in [26]. We also denote by $H = \mathbb{C}\{Y_p, p \in \mathbb{Z}+s\}$.

Let V be an irreducible weight $\mathfrak{sv}[s]$-module with finite dimensional weight spaces. If V is not a highest (lowest) weight module, then it is uniformly bounded (see [14]). However, it is the most difficulty in classifying Harish-Chandra modules to classify all irreducible uniformly bounded modules. It is opened up to now. With the methods and results in Section 3, we can easily do such works.

Proposition 4.1. *Let V be a uniformly bounded irreducible $\mathfrak{sv}[s]$-module. Then $M_nV = Y_{n+s}V = 0$ for all $n \in \mathbb{Z}$.*

Proof. Set $\mathcal{T} = \mathbb{C}\{L_m, M_n \mid n \in \mathbb{Z}\}$ is the subalgebra of $\mathfrak{sv}[s]$, then $\mathcal{T} \cong W(0)[0]$. Choose an irreducible $\mathcal{T}$-submodule $V' \cong \mathcal{A}'_{a,b,c} = \sum_{i\in\mathbb{Z}} \mathbb{C}v_i$ for some $a, b, c \in \mathbb{C}$, which is existed by the statements in [20] and Theorem 2.3. Then $V = U(H)V'$, where $U(H) = \mathbb{C}\{Y_{p_1}Y_{p_2}\cdots Y_{p_n} \mid \forall p_1, \cdots, p_n \in \mathbb{Z}+s, \forall n \in \mathbb{Z}_+\}$, as a vector space over $\mathbb{C}$.

Case 1. $c = 0$. In this case $M_nV = 0$. The irreducibility of V as $\mathfrak{sv}[s]$-modules is equivalent to that of V as a $W(\frac{1}{2})[s]$-modules, where $W(\frac{1}{2})[s] = \mathbb{C}\{L_m, Y_{m+s} \mid m \in \mathbb{Z}\}$ since $[Y_i, Y_j]v = (j-i)M_{i+j}v = 0$. So by Theorem 3.2, V is the Harish-Chandra module of intermediate series.

Case 2. $c \neq 0$.

Now set $U_i = U(H)v_i$ for any $i \in \mathbb{Z}$.

Claim 1. $U_i = U_j$ for all $i, j \in \mathbb{Z}$. It means $V = U(H)v_i$ for any $i \in \mathbb{Z}$.

In fact, for any $i \neq j \in \mathbb{Z}$, $v_j = \frac{1}{c}M_{j-i}v_i = \frac{1}{c(j-i-2p)}[Y_p, Y_{j-i-p}]v_i \in U_i$ for some p such that $j-i-2p \neq 0$, then $U_i \subset U_j$. Similarly we have $U_j \subset U_i$. Then $U_i = U_j$. So $V = U(H)v_i$ for any $i \in \mathbb{Z}$. So we get the Claim 1. Moreover M_j is nonsingular on V.

Claim 2. All Y_i are nonsingular on V.

In fact, if there exists $u_{i_0} \in V_i$ such that $Y_{p_0}u_{i_0} = 0$ for some $p_0 \in \mathbb{Z}+s$. Then we can easily to prove that Y_{p_0} is nilpotent on V_i and then it is nilpotent on V since any M_j is commutative with Y_{p_0}. Suppose that n is least positive integer such that $Y_{p_0}^n V = 0$. By action of $Y_p\,(p \neq p_0)$ we have $(p_0 - p)Y_{p_0}^{n-1}M_{p+p_0}V = 0$. Then $Y_{p_0}^{n-1}V = 0$. It is contradict to the choice of n.

Now for any $h = Y_p^2 M_{-2p} \in U(H)_0$ and $k \in (1-s)\mathbb{Z}$, then h is a linear transformation over the finite dimensional vector space V_k. So there exists a minimal polynomial $m(x)$ of h such that $m(h)V_k = 0$ (if $s = \frac{1}{2}$, then we can regard h as a linear transformation over the finite dimensional vector space $V_k + V_{k+\frac{1}{2}}$). By action some M_j we have

$$m(h)V = 0. \tag{4.6}$$

By action of $Y_q, (q \neq p)$, on (4.6), we get Y_p is not nonsingular on V. So we get a contradiction. □

Remark. Let $\mathcal{S}$ be the Lie subalgebra of $\mathfrak{sv}[s]$ generated by M_i, Y_p for all $i \in \mathbb{Z}$ and $p \in \mathbb{Z}+s$, then $\mathcal{S}$ is just the infinite-dimensional Schrödinger algebra (see [8]). Clearly any irreducible uniformed bounded $\mathfrak{sv}[s]$-module with all Y_i nonsingular is just an irreducible uniform bounded S-module with all Y_i nonsingular and $M_i u_j = cu_{i+j}$ for any $u_i \in V_i$. From the above proof in the Case 2, $\mathcal{S}$ has no such modules.

So from the above proposition, we have

Theorem 4.2. *Let V be an irreducible weight $\mathfrak{sv}[s]$-module with finite dimensional weight spaces. Then V is a highest weight module, lowest weight module, or Harish-Chandra module of intermediate series.*

4.2 The deformative twisted Schrödinger-Virasoro Lie algebra

For any $\varrho \in \mathbb{Q}$, [26] introduced a family of infinite-dimensional Lie algebras called twisted deformative Schrödinger-Virasoro Lie algebras $\mathcal{D}(\varrho)$, admitting $\mathbb{C}$-basis $\{L_n$,

Y_n, $M_n \mid n \in \mathbb{Z}\}$ and the following Lie brackets

$$\begin{aligned}
&[L_m, L_n] = (n-m)L_{n+m}, \\
&[L_m, Y_n] = (n - \frac{\varrho+1}{2}m)Y_{n+m}, \quad [Y_n, Y_m] = (m-n)M_{n+m}, \\
&[L_m, M_n] = (n-\varrho m)M_{n+m}, \qquad [Y_n, M_m] = [M_n, M_m] = 0.
\end{aligned}$$

Clearly, $\mathcal{D}(\varrho)$ contain $W(\varrho)$ as a subalgebra. If $\varrho = 0$, then $\mathcal{D}(\varrho)$ is just the above twisted Schrödinger-Virasoro Lie algebra. Now we suppose that $\varrho \neq 0, -1, -3$ in this subsection.

Proposition 4.3. *Let V be a uniformly bounded irreducible $\mathcal{D}(\varrho)$-module. Then $M_nV = Y_nV = 0$. Then V is the Harish-Chandra module of intermediate series.*

Proof. Since $\varrho \neq 0$, the Lie subalgebra generated by $\{L_n, M_n \mid n \in \mathbb{Z}\}$ is isomorphic to $W(\varrho)[0]$. Choose an irreducible $W(\varrho)[0]$-module V' of V and $M_nV' = 0$ by Theorem 2.3.

Certainly as a $\mathcal{D}(\varrho)$-module $V = U(H)V'$, where $H = \mathbb{C}\{Y_n, n \in \mathbb{Z}\}$. By $[M_n, Y_m] = 0$ for all $m, n \in \mathbb{Z}$, we have $M_nV = 0$ and then all $[Y_m, Y_n]$ act as zero's since $[Y_n, Y_m] = (m-n)M_{n+m}$. The irreducibility of V as $\mathcal{D}(\varrho)$-modules is equivalent to that of V as a $W(\frac{\varrho+1}{2})[0]$-modules, where $W(\frac{\varrho+1}{2})[0] = \mathbb{C}\{L_n, Y_n \mid n \in \mathbb{Z}\}$ with $[Y_m, Y_n] = 0$ for all $m, n \in \mathbb{Z}$. So by Theorem 3.2, V is the Harish-Chandra module of intermediate series as an $\mathcal{D}(\varrho)$-module. □

Similarly we can also easily prove that an irreducible weight module V is a uniform bounded module if it is not a highest weight module and a lowest weight module over $\mathcal{D}(\varrho)$. So we obtain the following result.

Theorem 4.4. *Let V be an irreducible weight $\mathcal{D}(\varrho)$-module with finite dimensional weight spaces. Then V is a highest weight module, lowest weight module, or Harish-Chandra module of intermediate series.* □

Remarks. 1. The Lie algebra $\mathcal{D}(1)$ is just the truncated Virasoro-loop algebra Vir$\otimes$ $\mathbb{C}[t, t^{-1}]/(t^3)$. Its Harish-Chandra modules are classified in [7].

2. Due to the central elements in the universal central extensions of the above Lie algebras are trivially act on the irreducible uniform bounded modules, so we have classified all Harish-Chandra modules over the universal central extensions of all the above Lie algebras.

3. With the methods and results in Section 3, we can classify Harish-Chandra modules over many Lie algebras, which consist of the Virasoro operators and some intertwining operators, such as Lie algebras listed in Section 4.

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REFERENCES

[1] E. Arbarello; C. DeConcini; V.G. Kac; C. Procesi, *Moduli spaces of curves and representation theory*, Commun. Math. Phys., **117** (1988), 1-36.

[2] Y. Billig, *Representations of the twisted Heisenberg-Virasoro algebra at level zero*, Canad. Math. Bull., 46(2003), no.4, 529-537.

[3] V.K. Dobrev; H.-D. Doebner; C. Mrugalla, *Lowest weight representations of the Schrödinger algebra and generalized heat equations*, Rep. Math. Phys. **39** (1997) 201-218.

[4] V.K. Dobrev; S. Stoimenov, *Singular vectors and invariant equations for the Schrödinger algebra in $n \geq 3$ space dimensions: the General Case*, Physics of Atomic Nuclei, **73**, No. 11 (2010), 1916-1924.

[5] D. B. Fuks, *Cohomology of Infinite-Dimensional Lie Algebras* [in Russian], Nauka, Moscow (1984); English transl., Plenum, New York (1986).

[6] S. Gao; C. Jiang; Y. Pei, *Low dimensional cohomology groups of the Lie algebras $W(a,b)$*, Commun. Alg. 39 (2011), no. 2, 397-423.

[7] X. Guo; R. Lv; K. Zhao, *Simple Harish-Chandra modules,intermediate series modules, and Verma modules over the loop-Virasoro algebra*Forum Math. 23(2011), 1029-1052.

[8] M. Henkel, *Schrödinger invariance and strongly anisotropic critical systems*, Journal of Statistical Physics, 75(5-6)(1994), 1023-1061.

[9] M. Henkel; J. Unterberger, *Schrödinger invariance and spacetime symmetries*, Nucl. Phys. B, **660** (2003), 407–435.

[10] M. Henkel; J. Unterberger, *Supersymmetric extensions of Schrödinger invariance*, Nucl. Phys. B, **746** (2003), 155–201.

[11] W. Jiang; Y. Pei, *Verma modules over the W-algebra $W(2,2)$*, J. Math. Phys. 51(2)(2010).

[12] V. G. Kac, *Some problems of infinite-dimensional Lie algebras and their representations*, Lecture Notes in Mathematics 933, 117-126, Springer, 1982.

[13] I. Kaplansky; L. J. Santharoubane, *Harish-Chandra modules over the Virasoro algebra*, Infinite-dimensional groups with applications (Berkeley, Calif. 1984), 217–231, Math. Sci. Res. Inst. Publ., 4, Springer, New York, 1985.

[14] J. Li; Y. Su, *Representations of the Schrödinger-Virasoro algebras*, J. Math. Phys. **49** (2008), 053512.

[15] D. Liu; C. Jiang, *Harish-Chandra modules over the twisted Heisenberg-Virasoro algebra,* J. Math. Phys., 49(1)(2008), 012901, 13 pp.

[16] D. Liu; S. Gao; L. Zhu, *Classification of irreducible weight modules over W-algebra $W(2,2)$,* J. Math. Phys. 49(11)(2008), 113503 6 pp.

[17] D. Liu; Y. Pei, L. Zhu, *Lie bialgebra structures on the twisted Heisenberg-Virasoro algebra*, J. Alg. 359 (2012) 35-48.

[18] D. Liu; Y. Wu; L. Zhu, *Whittaker modules over the twisted Heisenberg-Virasoro algebra*, J. Math. Phys. 51(2)(2010), 023524, 12pp.

[19] R. Lv; K. Zhao, *Classification of irreducible weight modules over higher rank Virasoro algebras*, Adv. Math. **201**(2)(2006), 630-656.

[20] R. Lv; K. Zhao, *Classification of irreducible weight modules over the twisted Heisenberg-Virasoro algebra*, Commun. Contemp. Math. 12 (2010), no. 2, 183-205

[21] C. Martin; A. Piard, *Indecomposable modules over the Virasoro Lie algebra and a Conjecture of V. Kac*, Commun. Math. Phys. 137,109-132 (1991).

[22] O. Mathieu, *Classification of Harish-Chandra modules over the Virasoro Lie algebra*, Invent. Math. **107**(1992), 225-234.

[23] V. Mazorchuk, K. Zhao, *Classification of simple weight Virasoro modules with a finite-dimensional weight space*, J. Algebra, **307**(2007), 209-214.

[24] V.Yu. Ovsienko; C. Roger, *Extension of Virasoro group and Virasoro algebra by modules of tensor densities on S^1*, Funct. Anal. Appl. 31(4)(1996), 290-291.

[25] V.Yu. Ovsienko;C. Roger, *Generalizations of the Virasoro group and Virasoro algebra through extensions by modules of tensor densities on S1.* Indag. Mathem, N.S. 8, 3 (1998), 277-288

[26] C. Roger, J. Unterberger, *The Schrödinger-Virasoro Lie group and algebra: representation theory and cohomological study*, Ann. Henri Poincaré, **7** (2006), 1477-1529.

[27] Y. Su, *A classification of indecomposable $sl_2(\mathbb{C})$-modules and a conjecture of Kac on irreducible modules over the Virasoro algebra*, J. Alg, 161(1993), 33-46.

[28] Y. Su, *Classification of Harish-Chandra modules over the higher rank Virasoro algebras*, Commun. Math. Phys. **240** (2003), 539-551.

[29] Y. Su, *Classification of quasifinite modules over the Lie algebras of Weyl type*, Adv. Math. **174** (2003), 57-68.

[30] R. Shen; C. Jiang, *The Derivation algebra and automorphism group of the twisted Heisenberg-Virasoro Algebra*, Comm. Alg. 34(2006), 2547-2558.

[31] R. Shen; Y. Su, *Classification of irreducible weight modules with a finite dimensional weight space over twisted Heisenberg-Virasoro algebra*, Acta Mathematica Sinica 23(1)(2007), 189-192.

[32] S. Tan; X. Zhang, *Automorphisms and Verma modules for generalized Schrödinger-Virasoro algebras*. J. Alg. 322 (2009), no. 4, 1379-1394.

[33] W. Wang; J. Li; Y. Xu, *Derivations and automorphisms of twisted deformative Schrödinger-Virasoro Lie algebras*, Comm. Algebra 40 (2012), 3365-3388.

[34] W. Zhang; C. Dong, *W-algebra $W(2,2)$ and the vertex operator algebra* $L(\frac{1}{2},0)\otimes L(\frac{1}{2},0)$, Commun. Math. Phys. 285 (2009), no. 3, 991-1004.