Higher Spin String States Scattered from D-particle in the Regge Regime and Factorized Ratios of Fixed Angle Scatterings

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Abstract

We study scattering of higher spin closed string states at arbitrary mass levels from D-particle in the Regge regime. We extract the *complete* infinite ratios among high-energy amplitudes of different string states in the fixed angle regime from these Regge string scattering amplitudes. In this calculation, we have used an identity proved recently based on a signless Stirling number identity in combinatorial theory. The complete ratios calculated by this indirect method include a subset of ratios calculated previously by direct fixed angle calculation [19]. Moreover, we discover that in spite of the non-factorizability of the closed string D-particle scattering amplitudes, the complete ratios derived for the fixed angle regime are found to be factorized. These ratios are consistent with the decoupling of high-energy zero norm states calculated previously.

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I. INTRODUCTION

Recently high-energy, fixed angle behavior of string scattering amplitudes [1–3] was intensively investigated for massive higher-spin string states at arbitrary mass levels [4–12]. The motivation was to uncover the fundamental hidden stringy spacetime symmetry. An important new ingredient of this calculation was the zero-norm states (ZNS) [13–15] in the old covariant first quantized string spectrum, in particular, the identification of inter-particle symmetries induced by the inter-particle ZNS [13] in the spectrum. An infinite number of linear relations among high-energy fixed angle scattering amplitudes of different string states at each fixed but arbitrary mass levels can be derived. Moreover, these linear relations can be used to fix the ratios among high-energy scattering amplitudes of different string states at each fixed mass level. On the other hand, 2D discrete zero-norm states were also shown [14] to carry the spacetime ω_∞ symmetry charges of toy 2D string theory. Furthermore, in the high-energy limit, these discrete zero-norm states approach to [8, 9] the discrete Polyakov positive-norm states which generate the well-known ω_∞ symmetry of the 2D string [16–18]. This strongly suggests that the linear relations obtained from zero-norm states are indeed related to the hidden symmetry of the 26 dimensional string.

The calculation above was extended to scatterings of bosonic massive closed string states at arbitrary mass levels from D-brane in [19, 20]. The scattering of massless string states from D-brane was well studied in the literature and can be found in [21]. Since the mass of D-brane scales as the inverse of the string coupling constant $1/g$, it was assumed that it is infinitely heavy to leading order in g and does not recoil. It was discovered [19] that all the scattering amplitudes at arbitrary energy can be expressed in terms of the generalized hypergeometric function ${}_3F_2$ with special arguments, which terminates to a finite sum and, as a result, the whole scattering amplitudes consistently reduce to the usual beta function. For the simple case of D-particle, the authors of [19] explicitly calculated high-energy limit of a series of the above scattering amplitudes for arbitrary mass levels, and derive infinite linear relations among them for each fixed mass level. The ratios of these high-energy scattering amplitudes were found to be consistent with the decoupling of high-energy zero-norm states of the previous works. [4–11]. However, these ratios form only a subset of the complete ratios for general high-energy vertex in the fixed angle.

In this paper, we calculate the general high-energy scattering amplitudes of arbitrary higher spin massive closed string states scattered from D-particle in the small angle or Regge regime (RR). We will assume as before that the mass of the D-particle is infinitely heavy and so does not recoil. For Regge string-string scatterings, see [22–27]. See also [28–30]. Regge string-string scatterings for arbitrary higher spin massive states were intensively studied recently in [31–35]. In contrast to the case of scatterings in the fixed angle regime, we will see that there is no linear relation among string D-particle scatterings in the RR. However, as in the case of Regge string-string scattering amplitude calculation [31–33], we can extract the infinite fixed angle ratios of string D-particle scatterings from these Regge string D-particle scattering amplitudes. In this calculation, we have used a set of identities proved recently in [34] to extract the fixed angle ratios from the Regge scattering amplitudes.

We stress that the fixed angle ratios calculated in the present paper by this indirect method from the Regge calculation are for the most general high-energy vertex rather than only a subset of ratios [19] obtained directly from the fixed angle calculation previously. More importantly, we discover that the amplitudes calculated in this paper for closed string D-particle scatterings can not be factorized and thus are different from amplitudes for the high-energy closed string-string scattering calculated previously [32]. Amplitudes for the high-energy closed string-string scattering can be factorized into two open string scattering amplitudes by using a calculation [11, 32] based on the KLT formula [36]. Presumably, this non-factorization is due to the non-existence of a KLT-like formula for the string D-brane scattering amplitudes. There is no physical picture for open string D-particle tree scattering amplitudes and thus no factorizaion for closed string D-particle scatterings into two channels of open string D-particle scatterings. However, we discover that in spite of the non-factorizability of the closed string D-particle scattering amplitudes, the complete ratios derived for the fixed angle regime are found to be *factorized*. These ratios are consistent with the decoupling of high-energy zero norm states calculated previously [4–11].

This paper is organized as follows. In section II, we first set up the kinematics. In section III, we calculate the general string D-particle scatterings in the RR. In section IV, we extract the ratios of string D-particle fixed angle scattering amplitudes from RR amplitudes. We also discuss and compare the ratios of string D-particle and string-string scatterings. Finally, we give a brief conclusion in section V.

II. KINEMATICS SET-UP

In this paper, we consider an incoming string state with momentum k_2 scattered from an infinitely heavy D-particle and end up with string state with momentum $k_1$in the RR. The high-energy scattering plane will be assumed to be the $X-Y$ plane, and the momenta are arranged to be

$$k_1 = (E, \mathrm{k}_1 \cos\phi, -\mathrm{k}_1 \sin\phi), \tag{2.1}$$

$$k_2 = (-E, -\mathrm{k}_2, 0) \tag{2.2}$$

where

$$E = \sqrt{\mathrm{k}_2^2 + M_2^2} = \sqrt{\mathrm{k}_1^2 + M_1^2}, \tag{2.3}$$

and ϕ is the scattering angle. For simplicity, we will calculate the disk amplitude in this paper. The relevant propagators for the left-moving string coordinate $X^\mu(z)$ and the right-moving one $\tilde{X}^\nu(\bar{w})$ are

$$\langle X^\mu(z), X^\nu(w)\rangle = -\eta^{\mu\nu}\langle X(z), X(w)\rangle = -\eta^{\mu\nu}\ln(z-w), \tag{2.4}$$

$$\left\langle \tilde{X}^\mu(\bar{z}), \tilde{X}^\nu(\bar{w})\right\rangle = -\eta^{\mu\nu}\left\langle \tilde{X}(\bar{z}), \tilde{X}(\bar{w})\right\rangle = -\eta^{\mu\nu}\ln(\bar{z}-\bar{w}), \tag{2.5}$$

$$\left\langle X^\mu(z), \tilde{X}^\nu(\bar{w})\right\rangle = -D^{\mu\nu}\left\langle X(z), \tilde{X}(\bar{w})\right\rangle = -D^{\mu\nu}\ln(1-z\bar{w}) \quad \text{(for Disk)} \tag{2.6}$$

where matrix D has the standard form for the fields satisfying Neumann boundary condition, while D reverses the sign for the fields satisfying Dirichlet boundary condition. Instead of the Mandelstam variables used in the string-string scatterings, we define

$$a_0 \equiv k_1 \cdot D \cdot k_1 = -E^2 - \mathrm{k}_1^2 \sim -2E^2, \tag{2.7}$$

$$a_0' \equiv k_2 \cdot D \cdot k_2 = -E^2 - \mathrm{k}_2^2 \sim -2E^2, \tag{2.8}$$

$$b_0 \equiv 2k_1 \cdot k_2 + 1 = 2\left(E^2 - \mathrm{k}_1\mathrm{k}_2\cos\phi\right) + 1 = fixed, \tag{2.9}$$

$$c_0 \equiv 2k_1 \cdot D \cdot k_2 + 1 = 2\left(E^2 + \mathrm{k}_1\mathrm{k}_2\cos\phi\right) + 1, \tag{2.10}$$

so that

$$2a_0 + b_0 + c_0 = 2M_1^2 + 2. \tag{2.11}$$

Since we are going to calculate Regge scattering amplitudes, $b_0 = fixed$. We can use Eq.(2.3) and Eq.(2.9) to calculate

$$\cos\phi \sim 1 - \frac{b_0 - M_1^2 - M_2^2 - 1}{2\mathrm{k}_1^2} \tag{2.12}$$

$$\sin\phi \sim \frac{\sqrt{b_0 - M_1^2 - M_2^2 - 1}}{\mathrm{k}_1} \equiv \frac{\sqrt{\tilde{b}_0}}{\mathrm{k}_1} \tag{2.13}$$

The normalized polarization vectors on the high-energy scattering plane of the k_2 string state are defined to be [4, 5]

$$e_P = \frac{1}{M_2}(-E, -\mathrm{k}_2, 0) = \frac{k_2}{M_2}, \tag{2.14}$$

$$e_L = \frac{1}{M_2}(-\mathrm{k}_2, -E, 0), \tag{2.15}$$

$$e_T = (0, 0, 1). \tag{2.16}$$

One can then easily calculate the following kinematics

$$\begin{aligned}
e^T \cdot k_2 &= 0, \\
e^T \cdot k_1 &= -\mathrm{k}_1 \sin\phi \sim -\sqrt{\tilde{b}_0}, \\
e^T \cdot D \cdot k_1 &= \mathrm{k}_1 \sin\phi \sim \sqrt{\tilde{b}_0}, \\
e^T \cdot D \cdot k_2 &= 0, \\
e^P \cdot k_2 &= -M_2, \\
e^P \cdot k_1 &= \frac{1}{M_2}\left[E^2 - \mathrm{k}_1\mathrm{k}_2\cos\phi\right] = \frac{b_0 - 1}{2M_2}, \\
e^P \cdot D \cdot k_1 &= \frac{1}{M_2}\left[E^2 + \mathrm{k}_1\mathrm{k}_2\cos\phi\right] = \frac{c_0 - 1}{2M_2}, \\
e^P \cdot D \cdot k_2 &= \frac{1}{M_2}\left[-E^2 - \mathrm{k}_2^2\right] = \frac{a_0'}{M_2} \sim \frac{a_0}{M_2}, \\
e^T \cdot D \cdot e^T &= -1, \\
e^T \cdot D \cdot e^P &= e^P \cdot D \cdot e^T = 0, \\
e^P \cdot D \cdot e^P &= \frac{1}{M_2^2}\left[-E^2 - \mathrm{k}_2^2\right] = \frac{a_0'}{M_2^2} \sim \frac{a_0}{M_2^2},
\end{aligned} \tag{2.17}$$

which will be useful in the amplitude calculation in the next section.

III. REGGE STRING D-PARTICLE SCATTERINGS

We now begin to calculate the scattering amplitudes. For simplicity, we will take k_1 to be the tachyon and k_2 to be the tensor states. One can easily argue that a class of high-energy string states for k_2 in the RR are [31, 33]

$$\left|p_n, p_n', q_m, q_m'\right\rangle = \left[\prod_{n>0}\left(\alpha_{-n}^T\right)^{p_n}\prod_{m>0}\left(\alpha_{-m}^P\right)^{q_m}\right]\left[\prod_{n>0}\left(\tilde{\alpha}_{-n}^T\right)^{p_n'}\prod_{m>0}\left(\tilde{\alpha}_{-m}^P\right)^{q_m'}\right]|0,k\rangle \tag{3.1}$$

with

$$\sum_n n\left(p_n - p_n'\right) + \sum_m m\left(q_m - q_m'\right) = 0, \tag{3.2}$$

$$\sum_n n\left(p_n + p_n'\right) + \sum_m m\left(q_m + q_m'\right) = N = \text{const} \tag{3.3}$$

where $M_2^2 = (N-2)$.

A. An example

Before calculating the string D-particle scattering amplitudes for general cases, we take an example and illustrate the method of calculation. We consider the case

$$p_1 = p_1' = q_1 = q_1' = q_2 = q_2' = 1, \quad \text{others} = 0. \tag{3.4}$$

As we will see in the next subsection, the string D-particle scattering amplitudes with the general states (3.1) are reduced to simple forms in the Regge limit, in which most of the ways of contracting the operators are discarded as subleading. For a fixed number of the contractions between ∂X^P and $\bar{\partial}\tilde{X}^P$, the ways of contracting the other factors are determined by the following rules.

$$\alpha_{-n}^T \quad 1 \text{ term (contraction of } ik_1X \text{ with } \partial_n X^T) \tag{3.5}$$

$$\tilde{\alpha}_{-n}^T \quad 1 \text{ term (contraction of } ik_1\tilde{X} \text{ with } \bar{\partial}_n \tilde{X}^T) \tag{3.6}$$

$$\alpha_{-n}^P \quad \begin{cases} (n>1) \quad 1 \text{ term (contraction of } ik_1X \text{ with } \partial_n X^P) \\ (n=1) \quad 2 \text{ terms (contraction of } ik_1X \text{ and } ik_2X \text{ with } \partial X^P) \end{cases} \tag{3.7}$$

$$\tilde{\alpha}_{-n}^P \quad \begin{cases} (n>1) \quad 1 \text{ term (contraction of } ik_1\tilde{X} \text{ with } \bar{\partial}_n \tilde{X}^P\ ) \\ (n=1) \quad 2 \text{ terms (contraction of } ik_1\tilde{X} \text{ and } ik_2\tilde{X} \text{ with } \bar{\partial}\tilde{X}^P\ ) \end{cases} \tag{3.8}$$

Therefore we take the state Eq.(3.4) as the simplest example for the purpose of this subsection.

We start with the procedure in [36] to treat the vertex operator corresponding to the state (3.4).

$$\begin{aligned} V &= i^6 \varepsilon_{\mu_1 \cdots \mu_6} : \partial X^{\mu_1} \partial X^{\mu_2} \partial^2 X^{\mu_3} e^{ik_2 X} (z) : : \bar{\partial} \tilde{X}^{\mu_4} \bar{\partial} \tilde{X}^{\mu_5} \bar{\partial}^2 \tilde{X}^{\mu_6} e^{ik_2 \tilde{X}} (\bar{z}) : \\ &= i^6 : \partial X^T \partial X^P \partial^2 X^P e^{ik_2 X} (z) : : \bar{\partial} \tilde{X}^T \bar{\partial} \tilde{X}^P \bar{\partial}^2 \tilde{X}^P e^{ik_2 \tilde{X}} (\bar{z}) : \\ &= i^6 \Big[: \exp \Big\{ ik_2 X(z) + \varepsilon_T^{(1)} \partial X^T(z) + \varepsilon_P^{(1)} \partial X^P(z) + \varepsilon_P^{(2)} \partial^2 X^P(z) \Big\} : \\ &\qquad \times : \exp \Big\{ ik_2 \tilde{X}(\bar{z}) + \varepsilon_T'^{(1)} \partial \tilde{X}^T(\bar{z}) + \varepsilon_P'^{(1)} \partial \tilde{X}^P(\bar{z}) + \varepsilon_P'^{(2)} \partial^2 \tilde{X}^P(\bar{z}) \Big\} : \Big]_{\text{linear terms}} \end{aligned} \tag{3.9}$$

In the last equation, we have introduced the dummy variables $\varepsilon_T^{(1)}, \varepsilon_P^{(1)}, \varepsilon_P^{(2)}, \varepsilon_T'^{(1)}, \varepsilon_P'^{(1)}, \varepsilon_P'^{(2)}$ associated with the non-vanishing component ε_{TPPTPP} of the polarization tensor and written the operator in the exponential form. "linear terms" indicate that we take the sum of the terms linear in all of $\varepsilon_T^{(1)}, \varepsilon_P^{(1)}, \varepsilon_P^{(2)}, \varepsilon_T'^{(1)}, \varepsilon_P'^{(1)}$, and $\varepsilon_P'^{(2)}$. This sum can be rephrased as the coefficient of the product $\varepsilon_T^{(1)} \varepsilon_P^{(1)} \varepsilon_P^{(2)} \varepsilon_T'^{(1)} \varepsilon_P'^{(1)} \varepsilon_P'^{(2)}$ because we set the dummy variables to be 1 at the end of calculation.

The string D-particle scattering amplitudes can be calculated to be

$$\begin{aligned} A = &\int d^2 z_1 d^2 z_2 \ i^6 \\ &\cdot \Big\langle : e^{ik_1 X} (z_1) :: e^{ik_1 \tilde{X}} (\bar{z}_1) :: \partial X^T \partial X^P \partial^2 X^P e^{ik_2 X} (z_2) :: \bar{\partial} \tilde{X}^T \bar{\partial} \tilde{X}^P \bar{\partial}^2 \tilde{X}^P e^{ik_2 \tilde{X}} (\bar{z}_2) : \Big\rangle \end{aligned} \tag{3.10}$$

$$\begin{aligned} &= i^6 \int d^2 z_1 d^2 z_2 \\ &\cdot \Big[\exp \Big\{ \Big\langle ik_1 X (z_1) \ \ ik_1 \tilde{X} (\bar{z}_1) \Big\rangle \\ &+ \Big\langle \Big(\varepsilon_T^{(1)} \partial X^T + \varepsilon_P^{(1)} \partial X^P + \varepsilon_P^{(2)} \partial^2 X^P + ik_2 X \Big) (z_2) \\ &\qquad \cdot \Big(\varepsilon_T'^{(1)} \bar{\partial} \tilde{X}^T + \varepsilon_P'^{(1)} \bar{\partial} \tilde{X}^P + \varepsilon_P'^{(2)} \bar{\partial}^2 \tilde{X}^P + ik_2 \tilde{X} \Big) (\bar{z}_2) \Big\rangle \\ &+ \Big\langle ik_1 X (z_1) \Big(\varepsilon_T^{(1)} \partial X^T + \varepsilon_P^{(1)} \partial X^P + \varepsilon_P^{(2)} \partial^2 X^P + ik_2 X \Big) (z_2) \Big\rangle \\ &+ \Big\langle ik_1 \tilde{X} (\bar{z}_1) \Big(\varepsilon_T'^{(1)} \bar{\partial} \tilde{X}^T + \varepsilon_P'^{(1)} \bar{\partial} \tilde{X}^P + \varepsilon_P'^{(2)} \bar{\partial}^2 \tilde{X}^P + ik_2 \tilde{X} \Big) (\bar{z}_2) \Big\rangle \\ &+ \Big\langle ik_1 X (z_1) \Big(\varepsilon_T'^{(1)} \bar{\partial} \tilde{X}^T + \varepsilon_P'^{(1)} \bar{\partial} \tilde{X}^P + \varepsilon_P'^{(2)} \bar{\partial}^2 \tilde{X}^P + ik_2 \tilde{X} \Big) (\bar{z}_2) \Big\rangle \\ &+ \Big\langle ik_1 \tilde{X} (\bar{z}_1) \Big(\varepsilon_T^{(1)} \partial X^T + \varepsilon_P^{(1)} \partial X^P + \varepsilon_P^{(2)} \partial^2 X^P + ik_2 X \Big) (z_2) \Big\rangle \Big\} \Big]_{\text{linear terms}} \end{aligned}$$

$$
\begin{aligned}
&= \int d^2z_1 d^2z_2 \left\langle : e^{ik_1X}\left(z_1\right) :: e^{ik_1\tilde{X}}\left(\bar{z}_1\right) :: e^{ik_2X}\left(z_2\right) :: e^{ik_2\tilde{X}}\left(\bar{z}_2\right) : \right\rangle \\
&\cdot \Big[\exp\Big\{ \\
&- \varepsilon_T^{(1)} \left[ie^T k_1 \partial_2 \left\langle X\left(z_1\right) X\left(z_2\right)\right\rangle + ie^T Dk_1 \partial_2 \left\langle \tilde{X}\left(\bar{z}_1\right) X\left(z_2\right)\right\rangle + ie^T Dk_2 \partial_2 \left\langle \tilde{X}\left(\bar{z}_2\right) X\left(z_2\right)\right\rangle \right] \\
&- \varepsilon_T'^{(1)} \left[ie^T Dk_1 \bar{\partial}_2 \left\langle X\left(z_1\right) \tilde{X}\left(\bar{z}_2\right)\right\rangle + ie^T k_1 \bar{\partial}_2 \left\langle \tilde{X}\left(\bar{z}_1\right) \tilde{X}\left(\bar{z}_2\right)\right\rangle + ie^T Dk_2 \bar{\partial}_2 \left\langle X\left(z_2\right) \tilde{X}\left(\bar{z}_2\right)\right\rangle \right] \\
&- \varepsilon_P^{(1)} \left[ie^P k_1 \partial_2 \left\langle X\left(z_1\right) X\left(z_2\right)\right\rangle + ie^P Dk_1 \partial_2 \left\langle \tilde{X}\left(\bar{z}_1\right) X\left(z_2\right)\right\rangle + ie^P Dk_2 \partial_2 \left\langle \tilde{X}\left(\bar{z}_2\right) X\left(z_2\right)\right\rangle \right] \\
&- \varepsilon_P^{(2)} \left[ie^P k_1 \partial_2^2 \left\langle X\left(z_1\right) X\left(z_2\right)\right\rangle + ie^P Dk_1 \partial_2^2 \left\langle \tilde{X}\left(\bar{z}_1\right) X\left(z_2\right)\right\rangle + ie^P Dk_2 \partial_2^2 \left\langle \tilde{X}\left(\bar{z}_2\right) X\left(z_2\right)\right\rangle \right] \\
&- \varepsilon_P'^{(1)} \left[ie^P Dk_1 \bar{\partial}_2 \left\langle X\left(z_1\right) \tilde{X}\left(\bar{z}_2\right)\right\rangle + ie^P k_1 \bar{\partial}_2 \left\langle \tilde{X}\left(\bar{z}_1\right) \tilde{X}\left(\bar{z}_2\right)\right\rangle + ie^P Dk_2 \bar{\partial}_2 \left\langle X\left(z_2\right) \tilde{X}\left(\bar{z}_2\right)\right\rangle \right] \\
&- \varepsilon_P'^{(2)} \left[ie^P Dk_1 \bar{\partial}_2^2 \left\langle X\left(z_1\right) \tilde{X}\left(\bar{z}_2\right)\right\rangle + ie^P k_1 \bar{\partial}_2^2 \left\langle \tilde{X}\left(\bar{z}_1\right) \tilde{X}\left(\bar{z}_2\right)\right\rangle + ie^P Dk_2 \bar{\partial}_2^2 \left\langle X\left(z_2\right) \tilde{X}\left(\bar{z}_2\right)\right\rangle \right] \\
&- \varepsilon_T^{(1)} \varepsilon_T'^{(1)} \left[e^T De^T \partial\bar{\partial} \left\langle X\left(z_2\right) \tilde{X}\left(\bar{z}_2\right)\right\rangle \right] \\
&- \varepsilon_P^{(1)} \varepsilon_P'^{(1)} \left[e^P De^P \partial\bar{\partial} \left\langle X\left(z_2\right) \tilde{X}\left(\bar{z}_2\right)\right\rangle \right] - \varepsilon_P^{(1)} \varepsilon_P'^{(2)} \left[e^P De^P \partial\bar{\partial}^2 \left\langle X\left(z_2\right) \tilde{X}\left(\bar{z}_2\right)\right\rangle \right] \\
&- \varepsilon_P^{(2)} \varepsilon_P'^{(1)} \left[e^P De^P \partial^2\bar{\partial} \left\langle X\left(z_2\right) \tilde{X}\left(\bar{z}_2\right)\right\rangle \right] - \varepsilon_P^{(2)} \varepsilon_P'^{(2)} \left[e^P De^P \partial^2\bar{\partial}^2 \left\langle X\left(z_2\right) \tilde{X}\left(\bar{z}_2\right)\right\rangle \right] \\
&- \varepsilon_T^{(1)} \varepsilon_P'^{(1)} \left[e^T De^P \partial\bar{\partial} \left\langle X\left(z_2\right) \tilde{X}\left(\bar{z}_2\right)\right\rangle \right] - \varepsilon_T^{(1)} \varepsilon_P'^{(2)} \left[e^T De^P \partial\bar{\partial}^2 \left\langle X\left(z_2\right) \tilde{X}\left(\bar{z}_2\right)\right\rangle \right] \\
&- \varepsilon_P^{(1)} \varepsilon_T'^{\,(1)} \left[e^P De^T \partial\bar{\partial} \left\langle X\left(z_2\right) \tilde{X}\left(\bar{z}_2\right)\right\rangle \right] - \varepsilon_P^{(2)} \varepsilon_T'^{\,(1)} \left[e^P De^T \partial^2\bar{\partial} \left\langle X\left(z_2\right) \tilde{X}\left(\bar{z}_2\right)\right\rangle \right] \Big\} \Big]_{\text{linear terms}}
\end{aligned}
$$

$$
\begin{aligned}
&= \int d^2z_1 d^2z_2 \left(1 - z_1\bar{z}_1\right)^{a_0} \left(1 - z_2\bar{z}_2\right)^{a_0'} \left|z_1 - z_2\right|^{b_0-1} \left|1 - z_1\bar{z}_2\right|^{c_0-1} \\
&\cdot \Big[\exp\Big\{ \\
&\varepsilon_T^{(1)} \left[\frac{ie^T k_1}{\left(z_1 - z_2\right)} + \frac{ie^T Dk_1 \bar{z}_1}{\left(1 - \bar{z}_1 z_2\right)} + \frac{ie^T Dk_2 \bar{z}_2}{\left(1 - \bar{z}_2 z_2\right)} \right] + \varepsilon_T'^{(1)} \left[\frac{ie^T Dk_1 z_1}{\left(1 - z_1\bar{z}_2\right)} + \frac{ie^T k_1}{\left(\bar{z}_1 - \bar{z}_2\right)} + \frac{ie^T Dk_2 z_2}{\left(1 - z_2\bar{z}_2\right)} \right] \\
&+ \varepsilon_P^{(1)} \left[\frac{ie^P k_1}{\left(z_1 - z_2\right)} + \frac{ie^P Dk_1 \bar{z}_1}{\left(1 - \bar{z}_1 z_2\right)} + \frac{ie^P Dk_2 \bar{z}_2}{\left(1 - \bar{z}_2 z_2\right)} \right] + \varepsilon_P^{(2)} \left[\frac{ie^P k_1}{\left(z_1 - z_2\right)^2} + \frac{ie^P Dk_1 \bar{z}_1^2}{\left(1 - \bar{z}_1 z_2\right)^2} + \frac{ie^P Dk_2 \bar{z}_2^2}{\left(1 - \bar{z}_2 z_2\right)^2} \right] \\
&+ \varepsilon_P'^{(1)} \left[\frac{ie^P Dk_1 z_1}{\left(1 - z_1\bar{z}_2\right)} + \frac{ie^P k_1}{\left(\bar{z}_1 - \bar{z}_2\right)} + \frac{ie^P Dk_2 z_2}{\left(1 - z_2\bar{z}_2\right)} \right] + \varepsilon_P'^{(2)} \left[\frac{ie^P Dk_1 z_1^2}{\left(1 - z_1\bar{z}_2\right)^2} + \frac{ie^P k_1}{\left(\bar{z}_1 - \bar{z}_2\right)^2} + \frac{ie^P Dk_2 z_2^2}{\left(1 - z_2\bar{z}_2\right)^2} \right] \\
&+ \varepsilon_T^{(1)} \varepsilon_T'^{(1)} \frac{e^T De^T}{\left(1 - z_2\bar{z}_2\right)^2} \\
&+ \varepsilon_P^{(1)} \varepsilon_P'^{(1)} \frac{e^P De^P}{\left(1 - z_2\bar{z}_2\right)^2} + 2\varepsilon_P^{(1)} \varepsilon_P'^{(2)} \frac{e^P De^P z_2}{\left(1 - z_2\bar{z}_2\right)^3} + 2\varepsilon_P^{(2)} \varepsilon_P'^{(1)} \frac{e^P De^P \bar{z}_2}{\left(1 - z_2\bar{z}_2\right)^3} + 2\varepsilon_P^{(2)} \varepsilon_P'^{(2)} \frac{e^P De^P \left(1 + 2z_2\bar{z}_2\right)}{\left(1 - z_2\bar{z}_2\right)^4} \\
&+ \varepsilon_T^{(1)} \varepsilon_P'^{(1)} \frac{e^T De^P}{\left(1 - z_2\bar{z}_2\right)^2} + 2\varepsilon_T^{(1)} \varepsilon_P'^{(2)} \frac{e^T De^P z_2}{\left(1 - z_2\bar{z}_2\right)^3} + \varepsilon_P^{(1)} \varepsilon_T'^{(1)} \frac{e^P De^T}{\left(1 - z_2\bar{z}_2\right)^2} + 2\varepsilon_P^{(2)} \varepsilon_T'^{(1)} \frac{e^P De^T \bar{z}_2}{\left(1 - z_2\bar{z}_2\right)^3} \\
&\Big\} \Big]_{\text{linear terms}}
\end{aligned}
\tag{3.11}
$$

To fix the $SL(2,R)$ modulus group on the disk, we set $z_1 = 0$ and $z_2 = r$, then $d^2z_1 d^2z_2 =$

$d\left(r^{2}\right)$. By using Eq.(2.17), the amplitude can then be reduced to

$$A=\int_{0}^{1} d\left(r^{2}\right)\left(1-r^{2}\right)^{a_{0}^{\prime}} r^{b_{0}-1}$$
$$\cdot\Bigg[$$
$$\exp\left\{\begin{array}{l}
\varepsilon_{T}^{(1)}\left[-\frac{i\sqrt{\tilde{b}_{0}}}{-r}\right]+\varepsilon_{T}^{\prime(1)}\left[-\frac{i\sqrt{\tilde{b}_{0}}}{-r}\right] \\
+\varepsilon_{P}^{(1)}\left[\frac{i\frac{b_{0}-1}{2M_{2}}}{-r}+\frac{i\frac{a_{0}}{M_{2}}}{\left(1-r^{2}\right)/r}\right]+\varepsilon_{P}^{(2)}\left[\frac{i\frac{b_{0}-1}{2M_{2}}}{(-r)^{2}}+\frac{i\frac{a_{0}}{M_{2}}}{\left[\left(1-r^{2}\right)/r\right]^{2}}\right] \\
+\varepsilon_{P}^{\prime(1)}\left[\frac{i\frac{b_{0}-1}{2M_{2}}}{-r}+\frac{i\frac{a_{0}}{M_{2}}}{\left(1-r^{2}\right)/r}\right]+\varepsilon_{P}^{\prime(2)}\left[\frac{i\frac{b_{0}-1}{2M_{2}}}{(-r)^{2}}+\frac{i\frac{a_{0}}{M_{2}}}{\left[1-r^{2}/r\right]^{2}}\right] \\
-\varepsilon_{T}^{(1)}\varepsilon_{T}^{\prime(1)}\frac{1}{\left(1-r^{2}\right)^{2}} \\
+\varepsilon_{P}^{(1)}\varepsilon_{P}^{\prime(1)}\frac{\frac{a_{0}}{M_{2}^{2}}}{\left(1-r^{2}\right)^{2}}+2\varepsilon_{P}^{(1)}\varepsilon_{P}^{\prime(2)}\frac{\frac{a_{0}}{M_{2}^{2}}r}{\left(1-r^{2}\right)^{3}}+2\varepsilon_{P}^{(2)}\varepsilon_{P}^{\prime(1)}\frac{\frac{a_{0}}{M_{2}^{2}}r}{\left(1-r^{2}\right)^{3}}+2\varepsilon_{P}^{(2)}\varepsilon_{P}^{\prime(2)}\frac{\frac{a_{0}}{M_{2}^{2}}\left(1+2r^{2}\right)}{\left(1-r^{2}\right)^{4}}
\end{array}\right\}$$
$$\Bigg]_{\text{linear terms}} \tag{3.12}$$

Although in Eq.(3.12) we have dropped several subleading terms by using the kinematic relations Eq.(2.17), Eq.(3.12) still has subleading terms. We can see that by performing the integration of a generic term in Eq.(3.12) and looking at its behavior in the Regge limit explicitly.

$$\begin{aligned}
\int_{0}^{1} d\left(r^{2}\right)\left(1-r^{2}\right)^{a_{0}^{\prime}+n_{a}} r^{b_{0}-1-N+n_{b}} &= B\left(a_{0}^{\prime}+1+n_{a}, \frac{b_{0}-N+1}{2}+\frac{n_{b}}{2}\right) \\
&= B\left(a_{0}^{\prime}+1, \frac{b_{0}-N+1}{2}\right) \frac{\left(a_{0}^{\prime}+1\right)_{n_{a}}\left(\frac{b_{0}-N+1}{2}\right)_{\frac{n_{b}}{2}}}{\left(a_{0}^{\prime}+1+\frac{b_{0}-N+1}{2}\right)_{n_{a}+\frac{n_{b}}{2}}} \\
&\sim B\left(a_{0}+1, \frac{b_{0}-N+1}{2}\right)\left(\frac{b_{0}-N+1}{2}\right)_{\frac{n_{b}}{2}}\left(a_{0}\right)^{-\frac{n_{b}}{2}}
\end{aligned} \tag{3.13}$$

Here the Pochhammer symbol is defined by $(x)_{y}=\frac{\Gamma(x+y)}{\Gamma(x)}$, which, if y is a positive integer, is reduced to $(x)_{y}=x(x+1)(x+2)\cdots(x+y-1)$. From the Regge behavior Eq.(3.13), we see that increasing one power of $1/r$ in the integrand results in increasing one-half power of a_{0}. Thus we obtain the following rules to determine which terms in the exponent of Eq.(3.12) contribute to the leading behavior of the amplitude:

$$1/r \rightarrow E, \quad a_{0} \rightarrow E^{2}. \tag{3.14}$$

We can now drop the subleading terms in energy to get

$$A = \int_0^1 d\left(r^2\right)\left(1-r^2\right)^{a_0'} r^{b_0-1}$$
$$\cdot \left[\exp\left\{\varepsilon_T^{(1)}\left[-\frac{i\sqrt{\tilde{b}_0}}{-r}\right] + \varepsilon_T^{\prime(1)}\left[-\frac{i\sqrt{\tilde{b}_0}}{-r}\right] + \varepsilon_P^{(2)}\left[\frac{i\frac{b_0-1}{2M_2}}{(-r)^2}\right] + \varepsilon_P^{\prime(2)}\left[\frac{i\frac{b_0-1}{2M_2}}{(-r)^2}\right]\right\}\right]_{\epsilon_{TPTP}}$$
$$\cdot \left[\exp\left\{\varepsilon_P^{(1)}\left[\frac{i\frac{b_0-1}{2M_2}}{-r} + \frac{i\frac{a_0}{M_2}}{\left(1-r^2\right)/r}\right] + \varepsilon_P^{\prime(1)}\left[\frac{i\frac{b_0-1}{2M_2}}{-r} + \frac{i\frac{a_0}{M_2}}{\left(1-r^2\right)/r}\right] + \varepsilon_P^{(1)}\varepsilon_P^{\prime(1)}\frac{\frac{a_0}{M_2^2}}{\left(1-r^2\right)^2}\right\}\right]_{\epsilon_{PP}} \tag{3.15}$$

where $[\cdots]_{\epsilon_{TPTP}}$ in the second line and $[\cdots]_{\epsilon_{PP}}$ in the third line indicate that we take the coefficients of $\varepsilon_T^{(1)}\varepsilon_T^{\prime(1)}\varepsilon_P^{(2)}\varepsilon_P^{\prime(2)}$ and $\varepsilon_P^{(1)}\varepsilon_P^{\prime(1)}$ respectively. Because of the difference in the powers of $1/r$ and a_0 in the exponent of Eq.(3.12), Eq.(3.15) has much more structure for $\varepsilon_P^{(1)}$ and $\varepsilon_P^{\prime(1)}$ than for $\varepsilon_T^{(1)}$, $\varepsilon_T^{\prime(1)}$, $\varepsilon_P^{(2)}$, and $\varepsilon_P^{\prime(2)}$, and fits into the aforementioned rules (3.5)(3.6)(3.7)(3.8). It is also worth noting that the appearance of the last term in the second exponent of Eq.(3.15) originates from the contraction between $\partial X\left(z_2\right)$ and $\bar{\partial}\tilde{X}\left(\bar{z}_2\right)$ in Eq.(3.10), which is a characteristic of string D-brane scattering.

The explicit form of the amplitude for the current example is

$$A = \int_0^1 d\left(r^2\right)\left(1-r^2\right)^{a_0'} r^{b_0-1}\left(-\frac{i\sqrt{\tilde{b}_0}}{-r}\right)\left(-\frac{i\sqrt{\tilde{b}_0}}{-r}\right)\left(\frac{i\frac{b_0-1}{2M_2}}{(-r)^2}\right)\left(\frac{i\frac{b_0-1}{2M_2}}{(-r)^2}\right)$$
$$\cdot\left[\left(\frac{i\frac{b_0-1}{2M_2}}{-r} + \frac{i\frac{a_0}{M_2}}{\left(1-r^2\right)/r}\right)\left(\frac{i\frac{b_0-1}{2M_2}}{-r} + \frac{i\frac{a_0}{M_2}}{\left(1-r^2\right)/r}\right) + \frac{\frac{a_0}{M_2^2}}{\left(1-r^2\right)^2}\right] \tag{3.16}$$
$$= -\left(\sqrt{\tilde{b}_0}\right)^2\left(\frac{b_0-1}{2M_2}\right)^4\int_0^1 d\left(r^2\right)\left(1-r^2\right)^{a_0'} r^{b_0-9}$$
$$\cdot\left[\left(\sum_{l=0}^{2}\binom{2}{l}\left(\frac{-r^2}{\left(1-r^2\right)}\frac{2a_0}{b_0-1}\right)^l\right) - \frac{r^2}{\left(1-r^2\right)^2}\frac{4a_0}{\left(b_0-1\right)^2}\right] \tag{3.17}$$
$$\sim -\left(\sqrt{\tilde{b}_0}\right)^2\left(\frac{b_0-1}{2M_2}\right)^4 B\left(a_0+1, \frac{b_0-7}{2}\right)$$
$$\cdot\left[\left(\sum_{l=0}^{2}\binom{2}{l}\left(-\frac{2}{b_0-1}\right)^l\left(\frac{b_0-7}{2}\right)_l\right) - \frac{4}{\left(b_0-1\right)^2}\left(\frac{b_0-7}{2}\right)\right] \tag{3.18}$$
$$= -\left(\sqrt{\tilde{b}_0}\right)^2\left(\frac{b_0-1}{2M_2}\right)^4 B\left(a_0+1, \frac{b_0-7}{2}\right)$$
$$\cdot\left[{}_2F_0\left(-2, \frac{b_0-7}{2}, \frac{2}{b_0-1}\right) - \frac{4}{\left(b_0-1\right)^2}\left(\frac{b_0-7}{2}\right)\right] \tag{3.19}$$

where we have used Eq.(3.13).

B. General cases

Now we move on to general cases. The vertex operator corresponding to a general massive state with d left-modes and d' right-modes is of the following form.

$$V = i^{d+d'}\varepsilon_{\mu_1\cdots\mu_{d+d'}} :\partial^{n_1}X^{\mu_1}\cdots\partial^{n_d}X^{\mu_d}e^{ik_2X}\,(z): :\bar{\partial}^{n_{d+1}}\tilde{X}^{\mu_{d+1}}\cdots\bar{\partial}^{n_{d+d'}}\tilde{X}^{\mu_{d+d'}}e^{ik_2\tilde{X}}\,(\bar{z}): \tag{3.20}$$

The vertex operators corresponding to the states Eq.(3.1) are expressed in this covariant form by

$$d = \sum_{n>0} p_n + q_n, \quad d' = \sum_{n>0} p'_n + q'_n$$

$$(n_1, n_2, \cdots, n_{d+d'}) = \left(\cdots, \underbrace{m, \cdots, m}_{p_m}, \cdots, \underbrace{n, \cdots, n}_{q_n}, \cdots, \underbrace{m', \cdots, m'}_{p'_{m'}}, \cdots, \underbrace{n', \cdots, n'}_{q'_{n'}}, \cdots\right)$$

$$\varepsilon_{\cdots\underbrace{T\cdots T}_{p_m}\cdots\underbrace{P\cdots P}_{q_n}\cdots\underbrace{T\cdots T}_{p'_{m'}}\cdots\underbrace{P\cdots P}_{q'_{n'}}\cdots} = 1.$$

For the calculation of the correlator involving the operator Eq.(3.20), we introduce parameters associated with the polarization tensor and exponentiate the kinematic factors.

$$\varepsilon_{TTT\cdots PPP\cdots TTT\cdots PPP\cdots} \to \prod_{n>0}\prod_{i=1}^{p_n}\prod_{j=1}^{q_n}\prod_{i'=1}^{p'_n}\prod_{j'=1}^{q'_n}\varepsilon^{(n)}_{T_i}\varepsilon^{(n)}_{P_j}\varepsilon'^{(n)}_{T_{i'}}\varepsilon'^{(n)}_{P_{j'}}$$

$$\begin{aligned} V = (i)^{\sum_{n>0} p_n + p'_n + q_n + q'_n} \Bigg[&: \exp\left\{ ik_2X(z) + \sum_{n>0}\sum_{i=1}^{p_n}\varepsilon^{(n)}_{T_i}\partial^n X^T(z) + \sum_{m>0}\sum_{j=1}^{q_m}\varepsilon^{(m)}_{P_j}\partial^m X^P(z)\right\}: \\ &\times : \exp\left\{ ik_2\tilde{X}(\bar{z}) + \sum_{n>0}\sum_{i=1}^{p'_n}\varepsilon'^{(n)}_{T_i}\partial^n \tilde{X}^T(\bar{z}) + \sum_{m>0}\sum_{j=1}^{q'_m}\varepsilon'^{(m)}_{P_j}\partial^m \tilde{X}^P(\bar{z})\right\}: \Bigg]_{\text{linear terms}} \end{aligned} \tag{3.21}$$

where "linear terms" means the terms linear in all of $\varepsilon^{(n)}_{T_i}, \varepsilon^{(m)}_{P_j}, \varepsilon'^{(n)}_{T_i}$, and $\varepsilon'^{(m)}_{P_j}$. Below we use symbols like

$$\varepsilon_{T^3P^2TP^3} \equiv \varepsilon^{(1)}_{T_1}\varepsilon^{(3)}_{T_1}\varepsilon^{(3)}_{T_2}\varepsilon^{(2)}_{P_1}\varepsilon^{(5)}_{P_1}\varepsilon'^{(1)}_{T_1}\varepsilon'^{(1)}_{P_1}\varepsilon'^{(1)}_{P_2}\varepsilon'^{(2)}_{P_1}, \qquad \varepsilon_T\sum_n p_n \equiv \sum_{n>0}\sum_{i=1}^{p_n}\varepsilon^{(n)}_{T_i}$$

(the meanings of these symbols are not unique.) and do not write the normal ordering symbol : : to avoid messy expressions.

The string D-particle scattering amplitudes of these string states can be calculated to be

$$A = \int d^2 z_1 d^2 z_2 \cdot \varepsilon_{T\sum p_n P\sum q_n T\sum p'_n P\sum q'_n} \tag{3.22}$$

$$\cdot \left\langle \begin{array}{c} e^{ik_1 X}(z_1)\, e^{ik_1\tilde{X}}(\bar{z}_1) \cdot \prod_{n>0} \left(i\partial^n X^T\right)^{p_n} \prod_{m>0} \left(i\partial^m X^P\right)^{q_m} e^{ik_2 X}(z_2) \\ \cdot \prod_{n>0} \left(i\bar{\partial}^n \tilde{X}^T\right)^{p'_n} \prod_{m>0} \left(i\bar{\partial}^m \tilde{X}^P\right)^{q'_m} e^{ik_2\tilde{X}}(\bar{z}_2) \end{array} \right\rangle$$

$$\equiv (i)^{\sum_{n>0} p_n + p'_n + q_n + q'_n} A' \tag{3.23}$$

$$= (i)^{\sum_{n>0} p_n + p'_n + q_n + q'_n} \int d^2 z_1 d^2 z_2$$

$$\cdot \exp \left\{ \begin{array}{c} \left\langle (ik_1 X)(z_1) \left(ik_1 \tilde{X}\right)(\bar{z}_1) \right\rangle \\ + \left\langle \begin{array}{c} \left(\varepsilon_T \sum_{n>0} p_n \partial^n X^T + \varepsilon_P \sum_{m>0} q_m \partial^m X^P + ik_2 X\right)(z_2) \\ \left(\varepsilon'_T \sum_{n>0} p'_n \bar{\partial}^n \tilde{X}^T + \varepsilon'_P \sum_{m>0} q'_m \bar{\partial}^m \tilde{X}^P + ik_2 \tilde{X}\right)(\bar{z}_2) \end{array} \right\rangle \\ + \left\langle (ik_1 X)(z_1) \left(\varepsilon_T \sum_{n>0} p_n \partial^n X^T + \varepsilon_P \sum_{m>0} q_m \partial^m X^P + ik_2 X\right)(z_2) \right\rangle \\ + \left\langle \left(ik_1 \tilde{X}\right)(\bar{z}_1) \left(\varepsilon'_T \sum_{n>0} p'_n \bar{\partial}^n \tilde{X}^T + \varepsilon'_P \sum_{m>0} q'_m \bar{\partial}^m \tilde{X}^P + ik_2 \tilde{X}\right)(\bar{z}_2) \right\rangle \\ + \left\langle (ik_1 X)(z_1) \left(\varepsilon'_T \sum_{n>0} p'_n \bar{\partial}^n \tilde{X}^T + \varepsilon'_P \sum_{m>0} q'_m \bar{\partial}^m \tilde{X}^P + ik_2 \tilde{X}\right)(\bar{z}_2) \right\rangle \\ + \left\langle \left(ik_1 \tilde{X}\right)(\bar{z}_1) \left(\varepsilon_T \sum_{n>0} p_n \partial^n X^T + \varepsilon_P \sum_{m>0} q_m \partial^m X^P + ik_2 X\right)(z_2) \right\rangle \end{array} \right\} \tag{3.24}$$

where only linear terms are taken in the expansion of the exponential (in the sense of Eq.(3.21)). In Eq.(3.24), we have used the simplified notation $\varepsilon_{T_j}^{(n)} \equiv \varepsilon_T$, $j = 1, 2, ...p_n$, $n \in Z_+$ for the spin polarizations, and similarly for the other polarizations. The exact meanings of the summations in the exponent are the ones like Eq.(3.21). Note that there will be terms corresponding to quadratic in the spin polarization. The amplitude A' can be reduced to

$$
\begin{aligned}
A' = \int d^2 z_1 d^2 z_2 & \left\langle e^{ik_1 X}\left(z_1\right) e^{ik_1 \tilde{X}}\left(\bar{z}_1\right) e^{ik_2 X}\left(z_2\right) e^{ik_2 \tilde{X}}\left(\bar{z}_2\right)\right\rangle \\
\cdot \exp & \left\{
\begin{array}{l}
-\varepsilon_T \sum\limits_{n>0} p_n \left[\begin{array}{c} ie^T \cdot k_1 \partial_2^n \left\langle X\left(z_1\right) X\left(z_2\right)\right\rangle + ie^T \cdot D \cdot k_1 \partial_2^n \left\langle \tilde{X}\left(\bar{z}_1\right) X\left(z_2\right)\right\rangle \\ + ie^T \cdot D \cdot k_2 \partial_2^n \left\langle \tilde{X}\left(\bar{z}_2\right) X\left(z_2\right)\right\rangle \end{array}\right] \\
-\varepsilon_T' \sum\limits_{n'>0} p_{n'}' \left[\begin{array}{c} ie^T \cdot D \cdot k_1 \bar{\partial}_2^{n'} \left\langle X\left(z_1\right) \tilde{X}\left(\bar{z}_2\right)\right\rangle + ie^T \cdot k_1 \bar{\partial}_2^{n'} \left\langle \tilde{X}\left(\bar{z}_1\right) \tilde{X}\left(\bar{z}_2\right)\right\rangle \\ + ie^T \cdot D \cdot k_2 \bar{\partial}_2^{n'} \left\langle X\left(z_2\right) \tilde{X}\left(\bar{z}_2\right)\right\rangle \end{array}\right] \\
-\varepsilon_P \sum\limits_{m>0} q_m \left[\begin{array}{c} ie^P \cdot k_1 \partial_2^m \left\langle X\left(z_1\right) X\left(z_2\right)\right\rangle + ie^P \cdot D \cdot k_1 \partial_2^m \left\langle \tilde{X}\left(\bar{z}_1\right) X\left(z_2\right)\right\rangle \\ + ie^P \cdot D \cdot k_2 \partial_2^m \left\langle \tilde{X}\left(\bar{z}_2\right) X\left(z_2\right)\right\rangle \end{array}\right] \\
-\varepsilon_P' \sum\limits_{m'>0} q_{m'}' \left[\begin{array}{c} ie^P \cdot D \cdot k_1 \bar{\partial}_2^{m'} \left\langle X\left(z_1\right) \tilde{X}\left(\bar{z}_2\right)\right\rangle + ie^P \cdot k_1 \bar{\partial}_2^{m'} \left\langle \tilde{X}\left(\bar{z}_1\right) \tilde{X}\left(\bar{z}_2\right)\right\rangle \\ + ie^P \cdot D \cdot k_2 \bar{\partial}_2^{m'} \left\langle X\left(z_2\right) \tilde{X}\left(\bar{z}_2\right)\right\rangle \end{array}\right] \\
-\varepsilon_T \varepsilon_T' \sum\limits_{n,n'>0} p_n p_{n'}' \left(e^T \cdot D \cdot e^T\right) \partial^n \bar{\partial}^{n'} \left\langle X\left(z_2\right) \tilde{X}\left(\bar{z}_2\right)\right\rangle \\
-\varepsilon_P \varepsilon_P' \sum\limits_{m,m'>0} q_m q_{m'}' \left(e^P \cdot D \cdot e^P\right) \partial^m \bar{\partial}^{m'} \left\langle X\left(z_2\right) \tilde{X}\left(\bar{z}_2\right)\right\rangle \\
-\varepsilon_T \varepsilon_P' \sum\limits_{n,m'>0} p_n q_{m'}' \left(e^T \cdot D \cdot e^P\right) \partial^n \bar{\partial}^{m'} \left\langle X\left(z_2\right) \tilde{X}\left(\bar{z}_2\right)\right\rangle \\
-\varepsilon_P \varepsilon_T' \sum\limits_{n',m>0} q_m p_{n'}' \left(e^P \cdot D \cdot e^T\right) \partial^m \bar{\partial}^{n'} \left\langle X\left(z_2\right) \tilde{X}\left(\bar{z}_2\right)\right\rangle
\end{array}
\right\}
\end{aligned}
\tag{3.25}
$$

where only linear terms are taken in the expansion of the exponential. We can now put in the propagators in Eq.(2.4) to Eq.(2.6) to get

$$A' = \int d^2z_1 d^2z_2 \left(1-z_1\bar{z}_1\right)^{a_0}\left(1-z_2\bar{z}_2\right)^{a_0'}\left|z_1-z_2\right|^{b_0-1}\left|1-z_1\bar{z}_2\right|^{c_0-1}$$

$$\exp\left\{\begin{array}{l} \varepsilon_T\sum_{n>0}p_n\left[\frac{i\left(n-1\right)!e^T\cdot k_1}{\left(z_1-z_2\right)^n}+\frac{i\left(n-1\right)!e^T\cdot D\cdot k_1}{\left(1-\bar{z}_1z_2\right)^n}\bar{z}_1^n+\frac{i\left(n-1\right)!e^T\cdot D\cdot k_2}{\left(1-\bar{z}_2z_2\right)^n}\bar{z}_2^n\right] \\ +\varepsilon_T'\sum_{n'>0}p_{n'}'\left[\frac{i\left(n'-1\right)!e^T\cdot D\cdot k_1}{\left(1-z_1\bar{z}_2\right)^{n'}}z_1^{n'}+\frac{i\left(n'-1\right)!e^T\cdot k_1}{\left(\bar{z}_1-\bar{z}_2\right)^{n'}}+\frac{i\left(n'-1\right)!e^T\cdot D\cdot k_2}{\left(1-z_2\bar{z}_2\right)^{n'}}z_2^{n'}\right] \\ +\varepsilon_P\sum_{m>0}q_m\left[\frac{i\left(m-1\right)!e^P\cdot k_1}{\left(z_1-z_2\right)^m}+\frac{i\left(m-1\right)!e^P\cdot D\cdot k_1}{\left(1-\bar{z}_1z_2\right)^m}\bar{z}_1^m+\frac{i\left(m-1\right)!e^P\cdot D\cdot k_2}{\left(1-\bar{z}_2z_2\right)^m}\bar{z}_2^m\right] \\ +\varepsilon_P'\sum_{m'>0}q_{m'}'\left[\begin{array}{c}\frac{i\left(m'-1\right)!e^P\cdot D\cdot k_1}{\left(1-z_1\bar{z}_2\right)^{m'}}z_1^{m'}+\frac{i\left(m'-1\right)!e^P\cdot k_1}{\left(\bar{z}_1-\bar{z}_2\right)^{m'}} \\ +\frac{i\left(m'-1\right)!e^P\cdot D\cdot k_2}{\left(1-z_2\bar{z}_2\right)^{m'}}z_2^{m'}\end{array}\right] \\ -\varepsilon_T\varepsilon_T'\sum_{n,n'>0}p_np_{n'}'\left(e^T\cdot D\cdot e^T\right)\partial^n\bar{\partial}^{n'}\ln\left(1-z_2\bar{z}_2\right) \\ -\varepsilon_P\varepsilon_P'\sum_{m,m'>0}q_mq_{m'}'\left(e^P\cdot D\cdot e^P\right)\partial^m\bar{\partial}^{m'}\ln\left(1-z_2\bar{z}_2\right) \\ -\varepsilon_T\varepsilon_P'\sum_{n,m'>0}p_nq_{m'}'\left(e^T\cdot D\cdot e^P\right)\partial^n\bar{\partial}^{m'}\ln\left(1-z_2\bar{z}_2\right) \\ -\varepsilon_P\varepsilon_T'\sum_{n',m>0}q_mp_{n'}'\left(e^P\cdot D\cdot e^T\right)\partial^m\bar{\partial}^{n'}\ln\left(1-z_2\bar{z}_2\right) \end{array}\right\} \tag{3.26}$$

where only linear terms are taken in the expansion of the exponential. To fix the $SL(2,R)$ modulus group on the disk, we set $z_1=0$ and $z_2=r$, then $d^2z_1d^2z_2=d\left(r^2\right)$. By using Eq.(2.17), the amplitude can then be reduced to

$$A' = \int d\left(r^2\right)\left(1-r^2\right)^{a_0'}r^{b_0-1}$$

$$\exp\left\{\begin{array}{l} \varepsilon_T\sum_{n>0}p_n\left[-\frac{i\left(n-1\right)!\sqrt{\tilde{b}_0}}{\left(-r\right)^n}\right]+\varepsilon_T'\sum_{n'>0}p_{n'}'\left[-\frac{i\left(n'-1\right)!\sqrt{\tilde{b}_0}}{\left(-r\right)^{n'}}\right] \\ +\varepsilon_P\sum_{m>0}q_m\left[\frac{i\left(m-1\right)!\frac{b_0-1}{2M_2}}{\left(-r\right)^m}+\frac{i\left(m-1\right)!\frac{a_0}{M_2}}{\left[\left(1-r^2\right)/r\right]^m}\right] \\ +\varepsilon_P'\sum_{m'>0}q_{m'}'\left[\frac{i\left(m'-1\right)!\frac{b_0-1}{2M_2}}{\left(-r\right)^{m'}}+\frac{i\left(m'-1\right)!\frac{a_0}{M_2}}{\left[\left(1-r^2\right)/r\right]^{m'}}\right] \\ -\varepsilon_T\varepsilon_T'\sum_{n,n'>0}p_np_{n'}'\partial^n\bar{\partial}^{n'}\ln\left(1-z_2\bar{z}_2\right)\big|_{z_2=\bar{z}_2=r} \\ -\varepsilon_P\varepsilon_P'\sum_{m,m'>0}q_mq_{m'}'\partial^m\bar{\partial}^{m'}\ln\left(1-z_2\bar{z}_2\right)\big|_{z_2=\bar{z}_2=r}\frac{a_0}{M_2^2} \end{array}\right\} \tag{3.27}$$

where only linear terms are taken in the expansion of the exponential.

Now we use the energy counting (3.14) and show how we reach the rules (3.5)(3.6)(3.7)(3.8). We can see immediately that in the exponent of Eq.(3.27), the terms linear in $\varepsilon_{P_i}^{(n)}$ or $\varepsilon_{P_i}^{\prime(n)}$ are dominated by their first terms if $m \geq 2$ or $m' \geq 2$. We can see also that most of the terms in the forth and fifth lines of the exponent are discarded as subleading. If we start with the terms consisting of only the factors coming from the first three lines, the other terms are obtained by series of replacements of two factors in them with one factors coming from the forth and fifth lines, and for each of the replacements we can see how it changes the power of energy. We do not need to calculate the infinite number of derivatives. For each differentiation the increase of the power of $1/r$ is less than or equal to 1, while the powers of $1/r$ in the first three lines increase with n, n', m or m', which implies that if one term in the forth or fifth line is discarded, the terms with higher n, n', m, m' in the same line are also discarded. The sequences of those discarded terms start at $(n, n') = (1, 1)$, $(m, m') = (1, 2)$, and $(m, m') = (2, 1)$. In this way, we can see that only the terms with $m = m' = 1$ in the fifth line contribute to the leading behavior. Thus we obtain the generalization of Eq.(3.15)

$$
\begin{aligned}
A' = & \int d\left(r^2\right)\left(1-r^2\right)^{a_0'} r^{b_0-1} \\
& \exp\left\{\begin{array}{l} \varepsilon_T \sum\limits_{n>0} p_n \left[-\dfrac{i\,(n-1)!\sqrt{\tilde{b}_0}}{(-r)^n}\right] + \varepsilon_T' \sum\limits_{n'>0} p'_{n'} \left[-\dfrac{i\,(n'-1)!\sqrt{\tilde{b}_0}}{(-r)^{n'}}\right] \\ +\varepsilon_P \sum\limits_{m>1} q_m \left[\dfrac{i\,(m-1)!\frac{b_0-1}{2M_2}}{(-r)^m}\right] + \varepsilon_P' \sum\limits_{m'>1} q'_{m'} \left[\dfrac{i\,(m'-1)!\frac{b_0-1}{2M_2}}{(-r)^{m'}}\right] \end{array}\right\}_{\varepsilon_{T\sum p_n\, P\sum' q_n\, T\sum p'_n\, P\sum' q'_n}} \\
& \exp\left\{\varepsilon_P q_1 \left[\frac{i\frac{b_0-1}{2M_2}}{-r} + \frac{i\frac{a_0}{M_2}r}{1-r^2}\right] + \varepsilon_P' q_1' \left[\frac{i\frac{b_0-1}{2M_2}}{-r} + \frac{i\frac{a_0}{M_2}r}{1-r^2}\right] + \varepsilon_P \varepsilon_P' q_1 q_1' \frac{\frac{a_0}{M_2^2}}{\left(1-r^2\right)^2}\right\}_{\varepsilon_{P q_1\, P q_1'}}
\end{aligned}
\tag{3.28}
$$

where the symbols $\varepsilon_{\dots}$ are similar to the ones in Eq.(3.15) and indicate that we take the coefficients of the products of the dummy variables in the exponents. ($\varepsilon_{P_i}^{(1)}$ and $\varepsilon_{P_i}^{\prime(1)}$ are excluded in the "sums" $\sum'$.) Note that the last term in the last line of Eq.(3.28) is quadratic in the polarization. This term is a characteristic of string D-brane scattering and has no analog in any of the previous works. It will play a crucial role in the following calculation in this paper.

For further calculation, we first note that

$$\exp\left\{\varepsilon_P q_1\left[\frac{i\frac{b_0-1}{2M_2}}{-r}+\frac{i\frac{a_0}{M_2}r}{1-r^2}\right]+\varepsilon'_P q'_1\left[\frac{i\frac{b_0-1}{2M_2}}{-r}+\frac{i\frac{a_0}{M_2}r}{1-r^2}\right]+\varepsilon_P\varepsilon'_P q_1 q'_1\frac{\frac{a_0}{M_2^2}}{\left(1-r^2\right)^2}\right\}_{\varepsilon_{P^{q_1}P'^{q'_1}}}$$

$$=\varepsilon_{P^{q_1}P'^{q'_1}}\sum_{j=0}^{\min\left\{q_1,q'_1\right\}}\binom{q_1}{j}\binom{q'_1}{j}j!\left(\frac{i\frac{b_0-1}{2M_2}}{-r}+\frac{i\frac{a_0}{M_2}r}{1-r^2}\right)^{q_1+q'_1-2j}\left(\frac{\frac{a_0}{M_2^2}}{\left(1-r^2\right)^2}\right)^j. \qquad (3.29)$$

Thus the amplitude can be further reduced to

$$\begin{aligned}A'=&\int d\left(r^2\right)\left(1-r^2\right)^{a'_0}r^{b_0-1}\\&\cdot\prod_{n>0}\left[-\frac{i\left(n-1\right)!\sqrt{\tilde{b}_0}}{\left(-r\right)^n}\right]^{p_n}\prod_{n'>0}\left[-\frac{i\left(n'-1\right)!\sqrt{\tilde{b}_0}}{\left(-r\right)^{n'}}\right]^{p'_{n'}}\\&\cdot\prod_{m>1}\left[\frac{i\left(m-1\right)!\frac{b_0-1}{2M_2}}{\left(-r\right)^m}\right]^{q_m}\prod_{m'>1}\left[\frac{i\left(m'-1\right)!\frac{b_0-1}{2M_2}}{\left(-r\right)^{m'}}\right]^{q_{m'}}\\&\cdot\sum_{j=0}^{\min\left\{q_1,q'_1\right\}}\sum_{l=0}^{q_1+q'_1-2j}j!\binom{q_1}{j}\binom{q'_1}{j}\binom{q_1+q'_1-2j}{l}\\&\cdot\left(\frac{i\frac{b_0-1}{2M_2}}{-r}\right)^{q_1+q'_1-2j-l}\left(\frac{i\frac{a_0}{M_2}r}{1-r^2}\right)^l\left(\frac{\frac{a_0}{M_2^2}}{\left(1-r^2\right)^2}\right)^j,\end{aligned} \qquad (3.30)$$

which, in the case of the state (3.4), is reduced to Eq.(3.17). We can now do the integration to get

$$\begin{aligned}A'=&\left(i\frac{b_0-1}{2M_2}\right)^{q_1+q'_1}\cdot\prod_{n>0}\left(\left[-i\left(n-1\right)!\sqrt{\tilde{b}_0}\right]^{p_n}\left[-i\left(n-1\right)!\sqrt{\tilde{b}_0}\right]^{p'_n}\right)\\&\cdot\prod_{m>1}\left(\left[i\left(m-1\right)!\frac{b_0-1}{2M_2}\right]^{q_m}\left[i\left(m-1\right)!\frac{b_0-1}{2M_2}\right]^{q_m}\right)\\&\cdot\sum_{j=0}^{\min\left\{q_1,q'_1\right\}}\sum_{l=0}^{q_1+q'_1-2j}j!\binom{q_1}{j}\binom{q'_1}{j}\binom{q_1+q'_1-2j}{l}\left(\frac{-2}{b_0-1}\right)^l\left(\frac{-4}{\left(b_0-1\right)^2}\right)^j\\&\cdot B\left(a_0+1,\frac{b_0+1-N}{2}\right)\left(\frac{b_0+1-N}{2}\right)_j\left(\frac{b_0+1-N}{2}+j\right)_l\end{aligned} \qquad (3.31)$$

where we have done the expansion of the beta function in the RR as following

$$\begin{aligned}&B\left(a'_0+1-l-2j,\frac{b_0+1-N}{2}+l+j\right)\\&\approx B\left(a_0+1,\frac{b_0+1-N}{2}\right)\frac{\left(\frac{b_0+1-N}{2}\right)_{l+j}}{a_0^{l+j}}\\&=B\left(a_0+1,\frac{b_0+1-N}{2}\right)\frac{\left(\frac{b_0+1-N}{2}\right)_j\left(\frac{b_0+1-N}{2}+j\right)_l}{a_0^{l+j}}.\end{aligned} \qquad (3.32)$$

Note that in the case of the state (3.4), Eq.(3.31) is reduced to Eq.(3.18). Performing the summation over n, we obtain

$$
\begin{aligned}
A' = & \left(i\frac{b_0-1}{2M_2}\right)^{q_1+q_1} \cdot \prod_{n>0}\left(\left[-i\,(n-1)!\sqrt{\tilde{b}_0}\right]^{p_n+p'_n}\right)\prod_{m>1}\left(\left[i\,(m-1)!\frac{b_0-1}{2M_2}\right]^{q_m+q'_m}\right) \\
& \cdot B\left(a_0+1,\frac{b_0+1-N}{2}\right)\sum_{j=0}^{\min\left\{q_1,q'_1\right\}}(-1)^j j!\binom{q_1}{j}\binom{q'_1}{j}\left(\frac{b_0+1-N}{2}\right)_j\left(\frac{2}{b_0-1}\right)^{2j} \\
& \cdot{}_2F_0\left(-q_1-q'_1+2j,\frac{b_0+1-N}{2}+j,\frac{2}{b_0-1}\right),
\end{aligned}
\tag{3.33}
$$

which, in the case of the state (3.4), is reduced to Eq.(3.19). Finally we can use the identity of the Kummer function

$$
\begin{aligned}
& 2^{2m}\,\tilde{t}^{-2m}U\left(-2m,\frac{t}{2}+2-2m,\frac{\tilde{t}}{2}\right) \\
= &\ {}_2F_0\left(-2m,-1-\frac{t}{2},-\frac{2}{\tilde{t}}\right) \\
\equiv & \sum_{j=0}^{2m}(-2m)_j\left(-1-\frac{t}{2}\right)_j\frac{\left(-\frac{2}{\tilde{t}}\right)^j}{j!} \\
= & \sum_{j=0}^{2m}\binom{2m}{j}\left(-1-\frac{t}{2}\right)_j\left(\frac{2}{\tilde{t}}\right)^j
\end{aligned}
\tag{3.34}
$$

to get the final form of the amplitude

$$
\begin{aligned}
A' = & \prod_{n>0}\left(\left[-i\,(n-1)!\sqrt{\tilde{b}_0}\right]^{p_n+p'_n}\right)\prod_{m>1}\left(\left[i\,(m-1)!\frac{b_0-1}{2M_2}\right]^{q_m+q'_m}\right)\left(-\frac{i}{M_2}\right)^{q_1+q'_1} \\
& \cdot B\left(a_0+1,\frac{b_0+1-N}{2}\right)\sum_{j=0}^{\min\left\{q_1,q'_1\right\}}(-1)^j j!\binom{q_1}{j}\binom{q'_1}{j}\left(\frac{b_0+1-N}{2}\right)_j \\
& \cdot U\left(-q_1-q'_1+2j,\frac{-b_0+N+1}{2}-q_1-q'_1+j,-\frac{b_0-1}{2}\right).
\end{aligned}
\tag{3.35}
$$

Note that the amplitudes in Eq.(3.35) can not be factorized into two open string D-particle scattering amplitudes as in the case of closed string-string scattering amplitudes [11, 32]. In Eq.(3.35) U is the Kummer function of the second kind and is defined to be

$$
U(a,c,x)=\frac{\pi}{\sin\pi c}\left[\frac{M(a,c,x)}{(a-c)!(c-1)!}-\frac{x^{1-c}M(a+1-c,2-c,x)}{(a-1)!(1-c)!}\right]\quad(c\neq 2,3,4...)
\tag{3.36}
$$

where $M(a,c,x)=\sum_{j=0}^{\infty}\frac{(a)_j}{(c)_j}\frac{x^j}{j!}$ is the Kummer function of the first kind. Note that the second argument of Kummer function $c=c(b_0)$, and is not a constant as in the usual case. As a result, U as a function of b_0 is not a solution of the Kummer equation.

An interesting application of Eq.(3.35) is the universal power law behavior of the amplitudes. We first define the Mandelstam variables as $s = 2E^2$ and $t = -(k_1 + k_2)^2$. The second argument of the beta function in Eq.(3.35) can be calculated to be

$$\frac{b_0 + 1 - N}{2} = \frac{2k_1 \cdot k_2 + 1 + 1 - N}{2} = \frac{(k_1 + k_2)^2 - k_1^2 - k_2^2 + 2 - N}{2} = \frac{-t - 2}{2} \tag{3.37}$$

where we have used Eq.(2.9) and $M_2^2 = (N - 2)$. The amplitudes thus give the universal power-law behavior for string states at *all* mass levels

$$A \sim s^{\alpha(t)} \quad \text{(in the RR)} \tag{3.38}$$

where

$$\alpha(t) = a(0) + \alpha' t, \;\; a(0) = 1 \text{ and } \alpha' = \frac{1}{2}. \tag{3.39}$$

IV. RATIOS ON THE FIXED ANGLE REGIME

We begin with a brief review of high-energy open string-string scattering in the fixed angle regime, namely

$$s, -t \to \infty, t/s \approx -\sin^2 \frac{\phi}{2} = \text{fixed (but } \phi \neq 0) \tag{4.1}$$

where s, t and u are the Mandelstam variables and ϕ is the CM scattering angle. It was shown that for the 26D open bosonic string the only states that will survive the high-energy limit at mass level $M_2^2 = 2(N - 1)$ are of the form [7, 8]

$$|N, 2m, q\rangle \equiv (\alpha_{-1}^T)^{N-2m-2q} (\alpha_{-1}^L)^{2m} (\alpha_{-2}^L)^q |0, k\rangle \tag{4.2}$$

where N, m and q are non-negative integers and $N \geq 2m + 2q$. It can be shown that the high-energy vertex in Eq.(4.2) are conformal invariants up to a subleading order term in the high-energy expansion. Note that e^P approaches to e^L in the fixed angle regime [4][5]. For simplicity, one chooses k_1, k_3 and k_4 to be tachyons. It turns out that the high-energy fixed angle scattering amplitudes can be calculated by using the saddle-point method. The complete ratios among the amplitudes at each fixed mass level can be calculated to be [7, 8]

$$\frac{T^{(N,2m,q)}}{T^{(N,0,0)}} = \left(-\frac{1}{M_2}\right)^{2m+q} \left(\frac{1}{2}\right)^{m+q} (2m - 1)!!. \tag{4.3}$$

A calculation based on the decoupling of high-energy ZNS gave the same result as in Eq.(4.3).

To compare the RR amplitudes Eq.(3.35) with the fixed angle amplitudes corresponding to states in Eq.(4.2), we consider the RR amplitudes of the following closed string states

$$\begin{aligned} &|N;2m,2m';q,q'\rangle \\ &= \left(\alpha_{-1}^{T}\right)^{N/2-2m-2q}\left(\alpha_{-1}^{P}\right)^{2m}\left(\alpha_{-2}^{P}\right)^{q}\otimes\left(\tilde{\alpha}_{-1}^{T}\right)^{N/2-2m'-2q'}\left(\tilde{\alpha}_{-1}^{P}\right)^{2m'}\left(\tilde{\alpha}_{-2}^{P}\right)^{q'}|0,k\rangle. \end{aligned} \tag{4.4}$$

where m, m', q and q' are non-negative integers. We can take the following values

$$p_1 = N/2-2m-2q, p_1' = N/2-2m'-2q', \tag{4.5}$$

$$q_1 = 2m, q_1' = 2m', \tag{4.6}$$

$$q_2 = q, q_2' = q' \tag{4.7}$$

in Eq.(3.35), and include the phase factor in Eq.(3.23) to get

$$\begin{aligned} A^{(N;2m,2m';q,q')} &= (i)^{N-q-q'}\left(-i\sqrt{\tilde{b}_0}\right)^{N-2(m+m')-2(q+q')}\left(i\frac{b_0-1}{2M_2}\right)^{q+q'}\left(-\frac{i}{M_2}\right)^{2m+2m''} \\ &\cdot B\left(a_0+1,\frac{b_0+1-N}{2}\right)\sum_{j=0}^{\min\{2m,2m'\}}(-1)^j j!\binom{2m}{j}\binom{2m'}{j}\left(\frac{b_0+1-N}{2}\right)_j \\ &\cdot U\left(-2m-2m'+2j,\frac{-b_0+N+1}{2}-2m-2m'+j,-\frac{b_0-1}{2}\right). \end{aligned} \tag{4.8}$$

It is now easy to calculate the RR ratios for each fixed mass level

$$\begin{aligned} \frac{A^{(N;2m,2m';q,q')}}{A^{(N,0,0,0,0)}} &= (i)^{-q-q'}\left(-i\frac{b_0-1}{2\tilde{b}_0 M_2}\right)^{q+q'}\left(\frac{1}{\tilde{b}_0 M_2^2}\right)^{m+m'} \\ &\cdot\sum_{j=0}^{\min\{2m,2m'\}}(-1)^j j!\binom{2m}{j}\binom{2m'}{j}\left(\frac{b_0+1-N}{2}\right)_j \\ &\cdot U\left(-2m-2m'+2j,\frac{-b_0+N+1}{2}-2m-2m'+j,-\frac{b_0-1}{2}\right) \end{aligned} \tag{4.9}$$

which is a b_0-dependent function.

Before studying the fixed angle ratios for string D-particle scatterings, we first make a pause to review previous results on *string-string* scatterings.

A. String-String Scatterings

1. *Open String*

For open string-string scatterings, either the saddle-point method ($t-u$ channel only) or the decoupling of high-energy zero-norm states (ZNS) can be used to calculate the fixed angle ratios [4–9]. It was discovered that there was an interesting link between high-energy fixed angle amplitudes T and RR amplitudes A. To the leading order in energy, the ratios among fixed angle amplitudes are ϕ-independent numbers, whereas the ratios among RR amplitudes are t-dependent functions. However, It was discovered [31] that the coefficients of the high-energy RR ratios in the leading power of t can be identified with the fixed angle ratios, namely [31]

$$\lim_{\tilde{t}'\to\infty}\frac{A^{(N,2m,q)}}{A^{(N,0,0,)}}=\left(-\frac{1}{M_2}\right)^{2m+q}\left(\frac{1}{2}\right)^{m+q}(2m-1)!!=\frac{T^{(N,2m,q)}}{T^{(N,0,0)}}. \tag{4.10}$$

To ensure this identification, one needs the following identity [31–34]

$$\begin{aligned}&\sum_{j=0}^{2m}(-2m)_j\left(-L-\frac{\tilde{t}'}{2}\right)_j\frac{(-2/\tilde{t}')^j}{j!}\\&=0(-\tilde{t}')^0+0(-\tilde{t}')^{-1}+...+0(-\tilde{t}')^{-m+1}+\frac{(2m)!}{m!}(-\tilde{t}')^{-m}+O\left\{\left(\frac{1}{\tilde{t}'}\right)^{m+1}\right\}\end{aligned} \tag{4.11}$$

where $L=1-N$ and is an integer. Note that L effects only the subleading terms in $O\left\{\left(\frac{1}{\tilde{t}'}\right)^{m+1}\right\}$. Mathematically, the complete proof of Eq.(4.11) for *arbitrary real values* L was recently worked out in [34] by using an identity of signless Stirling number of the first kind in combinatorial theory.

2. *Open Superstring*

For all four classes [10] of high-energy fixed angle open superstring scattering amplitudes, both the corresponding RR amplitudes and the complete ratios of the leading (in t) RR amplitudes can be calculated [33]. For the fixed angle regime [10], the complete ratios can be calculated by the decoupling of high-energy zero norm states. It turns out that the identification in Eq.(4.10) continues to work, and L is an integer again for this case [33].

3. Compactified Open String

For compactified open string scatterings, both the amplitudes and the complete ratios of leading (in t) RR can be calculated [35]. For the fixed angle regime, the complete ratios can be calculated by the decoupling of high-energy zero norm states. The identification in Eq.(4.10) continues to work. However, only scattering amplitudes corresponding to the cases $m = 0$ were calculated. The difficulties has been as following. First, it seems that the saddle-point method is not applicable here. On the other hand, it was shown that [4–6] the leading order amplitudes containing $(\alpha_{-1}^{L})^{2m}$ component will drop from energy order E^{4m} to E^{2m}, and one needs to calculate the complicated naive subleading order terms in order to get the real leading order amplitude. One encounters this difficulty even for some cases in the non-compactified string calculation. In these cases, the method of decoupling of high-energy ZNS was adapted.

It was important to discover [35] that the identity in Eq.(4.11) for arbitrary real values L can only be realized in high-energy *compactified* string scatterings. This is due to the dependence of the value L on winding momenta K_i^{25} [35]

$$L = 1 - N - (K_2^{25})^2 + K_2^{25} K_3^{25}. \tag{4.12}$$

All other high-energy string scatterings calculated previously [31–33] correspond to integer value of L only.

4. Closed String

For closed string scatterings [32], one can use the KLT formula [36], which expresses the relation between tree amplitudes of closed and two channels of open string ($\alpha'_{\text{closed}} = 4\alpha'_{\text{open}} = 2$), to simplify the calculations. Both ratios of leading (in t) RR and fixed angle amplitudes were found to be the tensor product of two ratios in Eq.(4.10), namely [32]

$$\begin{aligned}\lim_{\tilde{t}'\to\infty} \frac{A_{\text{closed}}^{\left(N;2m,2m';q,q'\right)}}{A_{\text{closed}}^{(N;0,0;0,0)}} &= \left(-\frac{1}{M_2}\right)^{2(m+m')+q+q'} \left(\frac{1}{2}\right)^{m+m'+q+q'} (2m-1)!!(2m'-1)!! \\ &= \frac{T_{\text{closed}}^{\left(N;2m,2m';q,q'\right)}}{T_{\text{closed}}^{(N;0,0;0,0)}}.\end{aligned} \tag{4.13}$$

We now begin to discuss the RR *closed string, D-particle* scatterings considered in this paper.

B. Closed String D-particle Scatterings

1. $m = m' = 0$ Case

In [19], the high-energy scattering amplitudes and ratios of fixed angle closed string D-particle scatterings were calculated only for the case $m = m' = 0$. For nonzero m or m' cases, one encounters similar difficulties stated in the paragraph before Eq.(4.12) to calculate the complete fixed angle amplitudes. A subset of ratios was found to be [19]

$$\frac{T_{SD}^{(N,0,0,q,q')}}{T_{SD}^{(N,0,0,0,0)}} = \left(-\frac{1}{2M_2}\right)^{q+q'}. \tag{4.14}$$

In view of the non-factorizability of Regge string D-particle scattering amplitudes calculated in Eq.(3.35), one is tempted to conjecture that the complete ratios of fixed angle closed string D-particle scatterings may not be factorized. On the other hand, the decoupling of high-energy ZNS implies the factorizability of the fixed angle ratios.

2. General Case

We can show explicitly that the leading behaviors of the inner products in Eq.(3.26) involving k_1, k_2, e^T, e^P and D are not affected by the replacement of e^P with e^L if we take the limit $b_0 \to \infty$ after taking the Regge limit. Therefore we proceed as in the previous works on Regge scattering. The calculation for the complete ratios of leading (in b_0) RR *closed string, D-particle* scatterings from Eq.(4.9) gives

$$\lim_{b_0\to\infty}\frac{A_{SD}^{(N;2m,2m';q,q')}}{A_{SD}^{(N,0,0,0,0)}}$$
$$=(i)^{-q-q'}\left(-i\frac{b_0}{2b_0M_2}\right)^{q+q'}\left(\frac{1}{b_0M_2^2}\right)^{m+m'}$$
$$\cdot\sum_{j=0}^{\min\{2m,2m'\}}(-1)^j j!\binom{2m}{j}\binom{2m'}{j}\left(\frac{b_0}{2}\right)^j\frac{(2m+2m'-2j)!}{(m+m'-j)!}2^{-2m-2m'+2j}b_0^{m+m'-j}$$
$$=(i)^{-q-q'}\left(-i\frac{1}{2M_2}\right)^{q+q'}\left(\frac{1}{2M_2}\right)^{2m+2m'}$$
$$\cdot\sum_{j=0}^{\min\{2m,2m'\}}j!\binom{2m}{j}\binom{2m'}{j}(-2)^j\frac{(2m+2m'-2j)!}{(m+m'-j)!}. \tag{4.15}$$

In deriving Eq.(4.15), we have made use of Eq.(3.34) and Eq.(4.11). Note that each term in the summation of Eq.(4.15) is not factorized. Surprisingly, the summation in Eq.(4.15) can be performed, and the ratios can be calculated to be

$$\lim_{b_0\to\infty}\frac{A_{SD}^{(N;2m,2m';q,q')}}{A_{SD}^{(N,0,0,0,0)}}$$
$$=(-)^{q+q'}\left(\frac{1}{2}\right)^{q+q'+2m+2m'}\left(\frac{1}{M_2}\right)^{2m+2m'+q+q'}$$
$$\cdot\frac{2^{2m+2m'}\pi\sec\left[\frac{\pi}{2}(2m+2m')\right]}{\Gamma\left(\frac{1-2m}{2}\right)\Gamma\left(\frac{1-2m'}{2}\right)}$$
$$=\left(-\frac{1}{M_2}\right)^{2m+q}\left(\frac{1}{2}\right)^{m+q}(2m-1)!!\left(-\frac{1}{M_2}\right)^{2m'+q'}\left(\frac{1}{2}\right)^{m'+q'}(2m^{'}-1)!! \tag{4.16}$$

which are *factorized*. They are exactly the same with the ratios of the high-energy, fixed angle closed string-string scattering amplitudes calculated in Eq.(4.13) and again consistent with the decoupling of high-energy zero norm states [4–11]. *We thus conclude that the identification in Eq.(4.10) continues to work for string D-particle scatterings.* So the complete ratios of fixed angle closed string D-particle scatterings are

$$\frac{T_{SD}^{\left(N;2m,2m^{'};q,q^{'}\right)}}{T_{SD}^{(N;0,0;0,0)}}=\left(-\frac{1}{M_2}\right)^{2(m+m^{'})+q+q^{'}}\left(\frac{1}{2}\right)^{m+m^{'}+q+q^{'}}(2m-1)!!(2m'-1)!!$$
$$=\lim_{b_0\to\infty}\frac{A_{SD}^{(N;2m,2m';q,q')}}{A_{SD}^{(N,0,0,0,0)}} \tag{4.17}$$

where the first equality can be deduced from the decoupling of high-energy ZNS. Note that, for $m=m^{'}=0$, Eq.(4.17) reduces to Eq.(4.14) calculated previously [19].

It is well known that the closed string-string scattering amplitudes can be factorized into two open string-string scattering amplitudes due to the existence of the KLT formula [36]. On the contrary, there is no physical picture for open string D-particle tree scattering amplitudes and thus no factorizaion for closed string D-particle scatterings into two channels of open string D-particle scatterings, and hence no KLT-like formula there. Here what we really mean is: two string, two D-particle scattering in the limit of infinite D-particle mass. This can also be seen from the nontrivial string D-particle propagator in Eq.(2.6), which vanishes for the case of closed string-string scattering. Thus the factorized ratios in high-energy fixed angle regime calculated in the RR in Eq.(4.16) and Eq.(4.17) came as a surprise. However, these ratios are consistent with the decoupling of high-energy zero norm states calculated previously [4–11]. It will be interesting if one can calculate the complete fixed angle amplitudes directly and see how the non-factorized amplitudes can give the result of factorized ratios. We hope to pursue this issue in the future.

V. CONCLUSION

In this paper, we study scatterings of higher spin massive closed string states from D-particle in the Regge regime. We extract the complete infinite ratios among high-energy scattering amplitudes of different string states in the fixed angle regime from these Regge string scattering amplitudes. The ratios calculated by this indirect method include a subset of ratios calculated previously by direct fixed angle calculation [19]. Moreover, we discover that in spite of the non-factorizability of the closed string D-particle scattering amplitudes, the complete ratios derived for the fixed angle regime are found to be *factorized*. The ratios for string D-particle scattering amplitudes are consistent with the decoupling of high-energy zero norm states calculated previously. [4–11].

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