

Conformal Aspects of Spinor-Vector Duality

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Abstract

We present a detailed study of various aspects of Spinor-Vector duality in Heterotic string compactifications and expose its origin in terms of the internal conformal field theory. In particular, we illustrate the main features of the duality map by using simple toroidal orbifolds preserving $\mathcal{N}_4 = 1$ and $\mathcal{N}_4 = 2$ spacetime supersymmetries in four dimensions. We explain how the duality map arises in this context by turning on special values of the Wilson lines around the compact cycles of the manifold. We argue that in models with $\mathcal{N}_4 = 2$ spacetime supersymmetry, the interpolation between the Spinor-Vector dual vacua can be continuously realized. We trace the origin of the Spinor-Vector duality map to the presence of underlying $N = (2, 2)$ and $N = (4, 4)$ SCFTs, and explicitly show that the induced spectral-flow in the twisted sectors is responsible for the observed duality. The isomorphism between current algebra representations gives rise to a number of chiral character identities, reminiscent of the recently-discovered MSDS symmetry.

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1 Introduction

String theory provides a detailed framework to explore the unification of the gauge and gravitational interactions. Progress in this endeavour mandates both a deeper understanding of the various mathematical structures underlying the theory as well as the development of phenomenological models that aspire to make contact with observation. It is clear that a deeper understanding of the structure and various dualities underlying such models may further elucidate their basic properties.

Over the last few years a novel ‘Spinor-Vector’ duality map has been observed in the massless spectra of Heterotic $\mathbb{Z}_2 \times \mathbb{Z}_2$ compactifications under the exchange of the vectorial and spinorial representations of the $SO(10)$ GUT gauge group. The initial observation of Spinor-Vector duality [1–3] in $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetric orbifold vacua was made by using the powerful and systematic [4, 5] classification methods of the free fermionic formulation, by means of numerical and analytical techniques. A special property of \mathbb{Z}_2 -type orbifolds is that $\mathcal{N}_4 = 1$ twisted sectors inherit the structure of the $\mathcal{N}_4 = 2$ ones. In particular, this implies that, as the duality map is realised internally in each twisted sector, it can be seen to hold in T^4/\mathbb{Z}_2 vacua [3] as well. Analytic proof of the Spinor-Vector duality within the framework of the fermionic construction was given in refs. [2, 3, 6] for the \mathbb{Z}_2 - as well as the $\mathbb{Z}_2 \times \mathbb{Z}_2$ - case.

In ref. [7] the $E_8 \times E_8$ heterotic string compactified on a symmetric, non-freely-acting $T^2 \times (T^4/\mathbb{Z}_2)$ orbifold was considered. This was then followed by two additional freely-acting $\mathbb{Z}'_2 \times \mathbb{Z}''_2$ orbifolds [8], each correlating the charges of an E_8 -factor to a half-shift along a compact cycle of the untwisted T^2 . The resulting vacuum was characterized by $\mathcal{N}_4 = 2$ spacetime supersymmetry and its genus-1 partition function included 8 independent orbits and, therefore, 7 discrete torsions. Within that framework, Spinor-Vector duality was seen to arise from different choices for the values of these discrete torsions.

At a deeper level, Spinor-Vector duality in $\mathcal{N}_4 = 1$ theories can be seen to be a remnant of the spontaneous breaking of $N = (2, 2)$ worldsheet supersymmetry to $N = (0, 2)$. The $N = (2, 2)$ -constructions [9], correspond to compactifications on Calabi-Yau surfaces that extend the gauge symmetry from $SO(10) \times U(1)$ to E_6 . The preservation of the global left-moving $N = 2$ superconformal algebra in this setting, corresponds to the self-dual case under the duality map, in the sense that both vectorial and spinorial representations of the $SO(10) \subset E_6$ are then massless. This reflects the fact that the matter representations of $SO(10)$, namely the **16** (spinorial) and **10** (vectorial), together with the singlet **1**, do fit nicely into the **27** representation of the enhanced symmetry group E_6 . Then one may give non-vanishing mass to either the vectorial or the spinorial representations (or both) by turning on suitable discrete Wilson lines. The resulting vacua will be dual to each other through the above Spinor-Vector duality map.

In the case of T^4/\mathbb{Z}_2 Heterotic compactifications with $\mathcal{N}_4 = 2$ supersymmetry, the $\hat{c} = 6$ internal CFT breaks into a $\hat{c} = 2$, $N = 2$ superconformal system in terms of 2

free compact (super-)coordinates while the 4 remaining internal coordinates form an $\hat{c} = 4$, $N = 4$ system [10]. At the enhanced symmetry point, where $SO(12) \times SU(2) \rightarrow E_7$, the global left-moving internal CFT becomes enhanced into $N = (4, 4)$. The result of this enhancement is twofold. First of all it guarantees the presence of $SO(12)$ spinorials and vectorials (always accompanied by singlets) in the massless spectrum, marking this as the self-dual point under the duality map. Secondly, it ensures the existence of an $N = 4$ spectral-flow operator, transforming the (spinorial) **32** representation of $SO(12)$ into the vectorial **12** plus the singlet **1**. This spectral-flow is responsible for the fact that the number of massless degrees of freedom in the spinorial representation is the same as that in the vectorial and singlets. As before, by turning on suitable Wilson lines, either the spinorial or the vectorial (plus singlet) representations of $SO(12)$ may acquire non-vanishing mass, which manifests itself as the observed Spinor-Vector duality. However, the matching of the number of massless degrees of freedom between these representations at the points of symmetry enhancement, ensures that these numbers will continue to be equal as the theory is deformed away from these critical points.

The initial observation and study of the Spinor-Vector duality map in [1–3, 6] was made within the framework of the fermionic construction [11]. However, even though such formulations at the fermionic point are very effective for scanning the space of phenomenologically attractive vacua [12, 13], they are typically limited only to particular points in moduli space, where the compactification radii and other background fields take specific values. This limited description may sometimes obscure the underlying physics and may mask the true CFT structure and origin of various maps, such as the Spinor-Vector duality map. For this purpose, it is important to deform these theories away from the ‘special’ fermionic points, or to directly develop constructions where the duality is manifested at generic points in moduli space. This will be achieved partially in the present paper, where our arguments will be valid at a generic point in moduli space.

The purpose of this paper is to further investigate the CFT nature of Spinor-Vector duality and explicitly demonstrate how the duality results from the spectral flow of global $N = 2$ or $N = 4$ SCFTs, that arise from the embedding of the spin connection of Type II theories into the gauge connection of Heterotic ones.

An interesting discovery is that the relevant spectral-flow operator in the twisted sector is identical to the operator generating the recently discovered *Massive Spectral boson-fermion Degeneracy Symmetry* (MSDS) [14], [15]. The MSDS structure typically arises chirally in the worldsheet supersymmetric sector of exotic 2d string constructions living at special extended symmetry points in the moduli space. In those constructions, it stems from a special breaking of the global $N = 2$ SCFT generating spacetime supersymmetry, to a novel enhanced current algebra, thus, implying a very specific (spontaneous) breaking of spacetime supersymmetry. The trademark of these constructions is that all massive bosonic and fermionic modes are matched, similarly to the case of conventional supersymmetry. However, massless bosonic and

fermionic modes remain unpaired:

$$n_b - n_f \begin{cases} = 0 & \text{for } m > 0 \\ \neq 0 & \text{for } m = 0 \end{cases} . \quad (1.1)$$

The paper is organized as follows:

In Section 2 we present an overview of Spinor-Vector duality in terms of a specific $\mathcal{N}_4 = 2$ model in which the duality is exhibited in a clear and simple way. In particular, we start with an $\mathcal{N}_4 = 2$ Heterotic compactification on $S^1 \times \tilde{S}^1 \times (T^4/\mathbb{Z}_2)$ and then consider an additional freely-acting \mathbb{Z}'_2 -orbifold, correlating the Cartan charges of the full $E_8 \times E_8$ with a half-shift along the compact S^1 -circle. We demonstrate how the duality map arises within this description for different choices of the discrete torsion.

In Section 2.3, we proceed to demonstrate how the freely-acting \mathbb{Z}'_2 can be reformulated as a Wilson line background around the S^1 -circle, where now the choice of discrete torsion is translated into the specific choices for the value of the Wilson line. We show that within the moduli space of $\mathcal{N}_4 = 2$ vacua, the interpolations between vacua with massless $SO(12)$ -spinorials and those with massless vectorials (plus singlets) can be continuously performed. This situation differs substantially from the $\mathcal{N}_4 = 1$ compactifications on $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$, where the analogous Wilson lines do not correspond to invariant marginal operators that may be used to perturb the σ -model and, hence, can only take specific discrete values.

In Section 3, we analyze the superconformal properties of $N = (2, 2)$ and $N = (4, 4)$ internal CFTs and illuminate the true source and structure of the Spinor-Vector duality map. In particular, we show how embedding the $N = 2$ and $N = 4$ SCFTs of Type II theories into the left-moving bosonic side of Heterotic string theory, gives rise to a spectral-flow which is responsible for transforming the spinorial representations of $SO(10)$ and $SO(12)$ into the vectorial and singlet representations. We explicitly construct the spectral-flow operator in each case and we demonstrate the induced isomorphism between the representations of the current algebra. In particular, this explains why the number of massless degrees of freedom remains unchanged under the duality map.

In Section 4, we give a complementary discussion from the Hamiltonian perspective, using the underlying Narain lattice. In this formulation the interpolation between models, and the patterns of symmetry breaking and symmetry enhancement is particularly transparent. In Section 5 we briefly review how the orbifold models considered in this paper are related to generic K3 compactifications of the $E_8 \times E_8$ heterotic string.

Finally, in Section 6 we present our conclusions and directions for future research.

2 Review of Spinor-Vector Duality

In this Section we provide an overview of Spinor-Vector duality. After introducing our conventions we directly proceed with the definition of a very specific model, which will serve as a working example in which the structure of the duality map will be illustrated.

2.1 Generalities and Conventions

Throughout the paper we set $\alpha' = 2$, in order to avoid additional $\sqrt{2}$ -factors in the exponents of vertex operators. In particular, this implies that a free complex fermion $\Psi(z)$ is bosonized in terms of a real compact boson $\Phi(z)$ as:

$$\left. \begin{aligned} \Psi(z) &= e^{i\Phi} \\ \Psi^\dagger(z) &= e^{-i\Phi} \end{aligned} \right\} \leftrightarrow \Psi\Psi^\dagger(z) = i\partial\Phi(z). \quad (2.1)$$

In these conventions, the above equivalence can be realized at a bosonic radius $R = 1$, commonly referred to as the ‘fermionic point’. More generally, a vertex operator $e^{iq\Phi}$ has conformal weight $\Delta = q^2/2$.

For Heterotic theories, we adopt the usual convention in which spacetime fermions arise from spin fields of the right-moving (anti-holomorphic) sector. Hence, the right-moving sector is characterized by a local $N = 1$ superconformal algebra, which results from gauge fixing the (super-)reparametrization invariance, whereas the left-moving (holomorphic) sector is similar to the bosonic string and contains the gauge degrees of freedom.

In the bosonic formulation [16] of the $E_8 \times E_8$ Heterotic string¹, the conformal anomaly in the left-moving sector is canceled by introducing 16 additional bosons compactified on the $E_8 \times E_8$ chiral root lattice.

Here we will rather use the fermionic formulation of the Heterotic string, where the left-moving conformal anomaly is instead canceled by the insertion of 16 free (complex) worldsheet fermions Ψ^A, λ^A , with $A = 1, \dots, 8$. If all 16 complex fermions are assigned the same (real) boundary conditions, the sum over the spin structures yields the $Spin(32)/\mathbb{Z}_2$ lattice:

$$\Gamma_{16} = \frac{1}{2} \sum_{\gamma, \delta=0,1} \frac{\theta_{[\delta]}^{[\gamma]16}}{\eta^{16}}. \quad (2.2)$$

On the other hand, by grouping the complex fermions into two groups of eight, $\{\Psi^A\}$, $\{\lambda^A\}$, such that the fermions in each group share common boundary conditions and summing over the (independent) possible boundary conditions of each group, one

¹For some recent developments on the phenomenology of Heterotic orbifolds see, for example, refs [17], [18] and references therein.

obtains the representation of the $E_8 \times E_8$ lattice in terms of Jacobi θ -functions:

$$\Gamma_{E_8 \times E_8}(\tau) = \left[\frac{1}{2} \sum_{k,\ell=0,1} \frac{\theta_{[\ell]}^{[k]8}}{\eta^8} \right] \left[\frac{1}{2} \sum_{\rho,\sigma=0,1} \frac{\theta_{[\sigma]}^{[\rho]8}}{\eta^8} \right]. \quad (2.3)$$

The modular invariant partition function of the ten-dimensional $E_8 \times E_8$ Heterotic string then becomes:

$$Z_{E_8 \times E_8} = \frac{1}{\tau_2^4 \eta^8 \bar{\eta}^8} \left[\frac{1}{2} \sum_{\bar{a}, \bar{b}=0,1} (-)^{\bar{a}+\bar{b}+\bar{a}\bar{b}} \frac{\bar{\theta}_{[\bar{b}]}^{[\bar{a}]4}}{\bar{\eta}^4} \right] \Gamma_{E_8 \times E_8}(\tau). \quad (2.4)$$

It is convenient to decompose the spectrum into characters of the global the $SO(2n)$ worldsheet current algebra realized in terms of worldsheet fermions:

$$Z_{E_8 \times E_8} = \frac{1}{\tau_2^4 \eta^8 \bar{\eta}^8} (\bar{V}_8 - \bar{S}_8) (O_{16} + S_{16}) (O_{16} + S_{16}), \quad (2.5)$$

where:

$$\begin{aligned} O_{2n} &= \frac{1}{2} \left(\frac{\theta_3^n}{\eta^n} + \frac{\theta_4^n}{\eta^n} \right), \\ V_{2n} &= \frac{1}{2} \left(\frac{\theta_3^n}{\eta^n} - \frac{\theta_4^n}{\eta^n} \right), \\ S_{2n} &= \frac{1}{2} \left(\frac{\theta_2^n}{\eta^n} + e^{-i\pi n/2} \frac{\theta_1^n}{\eta^n} \right), \\ C_{2n} &= \frac{1}{2} \left(\frac{\theta_2^n}{\eta^n} - e^{-i\pi n/2} \frac{\theta_1^n}{\eta^n} \right), \end{aligned} \quad (2.6)$$

and $\theta_1 \equiv \theta_{[1]}^1$, $\theta_2 \equiv \theta_{[0]}^1$, $\theta_3 \equiv \theta_{[0]}^0$, $\theta_4 \equiv \theta_{[1]}^0$.

Finally, Appendix A contains a detailed calculation of the partition function for the simple $\mathcal{N}_4 = 2$ model, that is used to illustrate the Spinor-Vector duality map. Moreover, in Appendix B, we summarize useful OPEs involving spin-fields of the $SO(N)$ current algebra.

2.2 Spinor-Vector duality in a T^4/\mathbb{Z}_2 orbifold

We are now ready to present the structure of Spinor-Vector duality in terms of an explicit example, realized in a simple compactification of the $E_8 \times E_8$ heterotic string on a $T^2 \times T^4/\mathbb{Z}_2$ orbifold. Because twisted sectors of $\mathcal{N}_4 = 1$, $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ constructions inherit the $\mathcal{N}_4 = 2$ structure of twisted sectors in T^4/\mathbb{Z}_2 vacua, it will be sufficient for our purposes to analyze the latter case in some detail. For simplicity,

we will consider the case where the T^2 -torus factorizes² into two independent circles $S^1(R) \times \tilde{S}^1(\tilde{R})$, parametrized by the X^8 and X^9 internal coordinates, respectively.

The orbifold group acts on the internal coordinates of T^4 and on their fermionic superpartners as:

$$g : \begin{cases} \bar{\psi}^I(\bar{z}) \rightarrow -\bar{\psi}^I(\bar{z}) \\ X^I(z, \bar{z}) \rightarrow -X^I(z, \bar{z}) \end{cases} , \quad \text{for } I = 4, 5, 6, 7 \quad (2.7)$$

whereas the remaining two internal coordinates X^8, X^9 , parametrizing $S^1 \times \tilde{S}^1$, are invariant.

The standard embedding of the point group in the gauge sector is realized as a twist in the boundary conditions of two complex fermions associated to the gauge degrees of freedom:

$$g : \Psi^A \rightarrow -\Psi^A , \quad \text{for } A = 1, 2, \quad (2.8)$$

while the remaining 14 left-moving fermions remain untwisted. Note that in the bosonic formulation, where the complex fermions Ψ are bosonized as $\Psi(z) = e^{iH(z)}$, the orbifold action becomes realized as a half-shift in the compact bosonic coordinate $H(z) \rightarrow H(z) + \pi$.

It is convenient to decompose the $SO(2n)$ characters into characters of lower-dimensional subgroups, in which the \mathbb{Z}_2 -orbifold action is diagonal:

$$\bar{V}_8 - \bar{S}_8 = \bar{V}_4\bar{O}_4 + \bar{O}_4\bar{V}_4 - \bar{S}_4\bar{S}_4 - \bar{C}_4\bar{C}_4 \quad (2.9)$$

This is the standard decomposition of the $SO(8)$ -little group into representations of $SO(4) \times SO(4)$, where the first $SO(4)$ -factor corresponds to the 2 transverse worldsheet fermions ψ^μ , with $\mu = 2, 3$ and the fermionic superpartners of the two untwisted toroidal coordinates $X^{8,9}$ parametrizing $S^1 \times \tilde{S}^1$. The second $SO(4)$ -factor will correspond to the supercoordinates $(X^I, \psi^I, I = 4, 5, 6, 7)$, that are twisted under the \mathbb{Z}_2 -action.

Similarly, the global $SO(16)$ -characters in the left-moving sector can be decomposed into characters of $SO(4) \times SO(12)$:

$$O_{16} + S_{16} = O_4O_{12} + V_4V_{12} + S_4S_{12} + C_4C_{12} , \quad (2.10)$$

where the $SO(4)$ -subgroup is realized by the 2 complex fermions $\Psi^{1,2}$ that transform under the orbifold action, while the $SO(12)$ -subgroup is associated to the remaining 6 fermions $\Psi^{3,\dots,8}$ on which the orbifold embedding is trivial. Since the orbifold action does not twist the remaining fermions λ^A , associated to E'_8 , the relevant contribution will still be in terms of $SO(16)$ -characters.

²It should be noted that this factorizability of T^2 is not necessary for the general results of this section. It can be shown that Spinor-Vector duality continues to persist for generic values of the T^2 -moduli.

The generic action of \mathbb{Z}_2 on the $SO(2n)$ -characters associated to the twisted fermions:

$$\begin{aligned}
O_{2n} &\rightarrow +O_{2n}, \\
V_{2n} &\rightarrow -V_{2n}, \\
S_{2n} &\rightarrow +e^{-i\pi n/2}S_{2n}, \\
C_{2n} &\rightarrow -e^{-i\pi n/2}C_{2n}.
\end{aligned} \tag{2.11}$$

This, of course, reflects the fact that the O_{2n} representation corresponds to the vacuum state (and the adjoint in the first excited level), which stays invariant under a twist in the boundary conditions of the $SO(2n)$ -fermions. On the other hand, the vectorial representation in the $SO(2n)$ -current algebra is linear in the worldsheet fermions and, thus, V_{2n} changes sign under the fermion twist.

This implies the following action of the \mathbb{Z}_2 -orbifold on the right-moving $SO(4) \times SO(4)$ fermion characters:

$$\bar{V}_8 - \bar{S}_8 \xrightarrow{\mathbb{Z}_2} \bar{V}_4\bar{O}_4 - \bar{O}_4\bar{V}_4 + \bar{S}_4\bar{S}_4 - \bar{C}_4\bar{C}_4. \tag{2.12}$$

Already it becomes visible that the action of the \mathbb{Z}_2 -orbifold projects out half of the gravitini, so that the compactification on T^4/\mathbb{Z}_2 will describe an $\mathcal{N}_4 = 2$ supersymmetric vacuum.

Similarly, the orbifold action on the left-moving gauge sector transforms the $SO(4) \times SO(12)$ fermion characters of the first E_8 factor as:

$$O_{16} + S_{16} \xrightarrow{\mathbb{Z}_2} O_4O_{12} - V_4V_{12} - S_4S_{12} + C_4C_{12}. \tag{2.13}$$

Under the action of the \mathbb{Z}_2 -orbifold, the (untwisted) moduli space of the $\mathcal{N}_4 = 2$ theory is reduced down to:

$$\frac{SO(16+6,6)}{SO(16+6) \times SO(6)} \xrightarrow{\mathbb{Z}_2} \frac{SO(4,4)}{SO(4) \times SO(4)} \times \frac{SO(16+2,2)}{SO(16+2) \times SO(2)}. \tag{2.14}$$

The $\frac{SO(4,4)}{SO(4) \times SO(4)}$ -factor corresponds to Lorentz boosts of the $\Gamma_{(4,4)}$ lattice associated with the T^4 or, equivalently, to marginal deformations with respect to the 16 moduli G_{IJ}, B_{IJ} .

Similarly, the $\frac{SO(16+2,2)}{SO(16+2) \times SO(2)}$ -factor contains the Lorentz boosts in the $\Gamma_{(18,2)}$ -lattice, which can be equivalently obtained from any particular point in moduli space³ by marginally deforming with respect to the 4 moduli in T^2 , G_{ij} and B_{ij} , as well as by turning on Wilson lines A_i^a , with $a = 1, \dots, 16$ taking values along the 16 Cartan generators of $E_8 \times E_8$ and $i, j = 8, 9$.

³For example, one could define the theory at the so-called ‘‘fermionic’’ point in moduli space, where all internal bosonic coordinates X^I can be consistently fermionized in terms of free (complex) worldsheet fermions $i\partial X_{L,R}^I = i\bar{\psi}_{L,R}^I\psi_{L,R}^I$.

Furthermore, we introduce an additional freely-acting \mathbb{Z}'_2 -orbifold :

$$g' = e^{2\pi i(Q_8 + Q'_8)} \delta, \quad (2.15)$$

where δ is a half-shift along the $S^1(R)$ compact direction:

$$X^8 \rightarrow X^8 + \pi R, \quad (2.16)$$

and Q_8 and Q'_8 are the $U(1)$ gauge charges with respect to the generators in the Cartan subalgebra of E_8 and E'_8 , respectively. The spinorial (or anti-spinorial) representations carry half-integer charges $Q \in \mathbb{Z} + \frac{1}{2}$, whereas the adjoint and vectorial representations have integer charges $Q \in \mathbb{Z}$. In the fermionic formulation of E_8 (resp. E'_8), the parity operator $e^{2\pi i Q_8}$ (resp. $e^{2\pi i Q'_8}$) becomes associated to the spin structure of the corresponding set $\{\Psi^A\}$ (resp. $\{\lambda^A\}$) of the complex worldsheet fermions, similarly to the spacetime fermion number $(-)^F$ for the right-moving worldsheet fermions.

The effect of the freely acting \mathbb{Z}'_2 is to correlate the gauge charges Q_8, Q'_8 with a half-shift along the circle $S^1(R)$ parametrized by X^8 . As will be shown in the next section, this freely acting orbifold corresponds to a particular choice of the Wilson line along the X^8 circle and, as such, it can be equivalently described as a Lorentz boost of the full (untwisted) $\Gamma_{(18,2)}$ -lattice. In terms of the $\Gamma_{(1,1)}$ lattice associated to the X^8 -circle, the action of the freely-acting \mathbb{Z}'_2 is that of a momentum shift. Indeed, in the Hamiltonian representation, the $\Gamma_{(1,1)}(R)$ -lattice takes the form:

$$\Gamma_{(1,1)}(R) = \sum_{m,n \in \mathbb{Z}} \Lambda_{m,n}(R) = \frac{1}{\eta\bar{\eta}} \sum_{m,n \in \mathbb{Z}} q^{\frac{1}{2}P_L^2} \bar{q}^{\frac{1}{2}P_R^2}, \quad (2.17)$$

where:

$$P_{L,R} = \frac{m}{R} \pm \frac{nR}{2}, \quad (2.18)$$

and m, n are the momentum and winding numbers around the X^8 -cicle, respectively. Then the action of the freely-acting orbifold \mathbb{Z}'_2 on the S^1 lattice is simply a momentum shift:

$$\Lambda_{m,n}(R) \xrightarrow{\delta} (-)^m \Lambda_{m,n}(R). \quad (2.19)$$

The modular invariant partition function of the model can be decomposed into:

$$Z = \frac{1}{(\sqrt{\tau_2} \eta \bar{\eta})^2} [Z_{(0,0)} + Z_{(1,0)} + Z_{(0,1)} + Z_{(1,1)}], \quad (2.20)$$

where

$$Z_{(h,h')} = \frac{1}{2^2} \sum_{g,g'=0,1} Z_{[g,g']}^{[h,h']}. \quad (2.21)$$

Here, $Z_{(h,h')}$ denote the sectors twisted by the $\mathbb{Z}_2 \times \mathbb{Z}'_2$ orbifold and the summation over g, g' incorporates the projection to invariant states.

It is convenient to define the twisted characters:

$$Q_o = \overline{O}_4 \overline{V}_4 - \overline{S}_4 \overline{S}_4 \quad , \quad Q_v = \overline{V}_4 \overline{O}_4 - \overline{C}_4 \overline{C}_4 \quad (2.22)$$

$$P_o = \overline{O}_4 \overline{C}_4 - \overline{S}_4 \overline{O}_4 \quad , \quad P_v = \overline{V}_4 \overline{S}_4 - \overline{C}_4 \overline{V}_4 \quad , \quad (2.23)$$

which are the linear combinations of standard $SO(4) \times SO(4)$ characters that are eigenvectors with respect to the orbifold action.

The relative sign between the two orbits is arbitrary and is parametrized by the discrete torsion coefficient $\epsilon = \pm 1$. In terms of the discrete torsion parameter, the modular invariant partition function can be written as:

$$Z = \frac{1}{(\sqrt{\tau_2 \eta \bar{\eta}})^2} \frac{1}{2^2} \sum_{h,g=0,1} \sum_{h',g'=0,1} (-)^{\frac{1-\epsilon}{2}(hg'-gh')} Z_{[g,g']}^{[h,h']}, \quad (2.24)$$

where the inclusion or not of the modular invariant cocycle $(-)^{hg'-h'g}$ alternates (or not) the sign of the second orbit. Furthermore, in the interest of simplicity, and throughout this paper, the contribution of the spectator $\Gamma_{(1,1)}(\tilde{R})$ -lattice, associated to the X^9 -circle will be suppressed.

The explicit calculation of the partition function is presented in considerable detail in Appendix A. Here we will directly discuss the spectrum of this model and comment on the appearance of the Spinor-Vector duality, which will now become transparent. Massless states can be seen to arise from the fully untwisted sector $Z_{(0,0)}$ and from the sector $Z_{(1,0)}$ twisted under the non freely-acting \mathbb{Z}_2 . On the other hand, both sectors $Z_{(0,1)}$ and $Z_{(1,1)}$, which are twisted by the freely-acting \mathbb{Z}'_2 , are characterized by non-trivial winding, hence, rendering all states within these two sectors massive.

The untwisted sector contains the representation

$$Q_v \Lambda_{2m,n} \Gamma_{(+)}^{h=0} O_{12} O_4 O_{16},$$

which gives rise both to the gravity multiplet as well as the space-time vector bosons generating the $SO(12) \times SO(4) \times SO(16)$ gauge symmetry. In addition, it contains

$$Q_o \Lambda_{2m,n} \Gamma_{(+)}^{h=0} V_{12} V_4 O_{16},$$

giving rise to scalar multiplets transforming in the bi-vector representation of $SO(12) \times SO(4)$.

Let us now examine the \mathbb{Z}_2 -twisted sector $Z_{(1,0)}$ and see how the Spinor-Vector duality operates. We first note that massless states can only arise for vanishing momentum and winding quantum numbers $m = n = 0$ along the S^1 -circle and only from the P_o -sector. The representations that can become massless are then :

$$\begin{aligned} P_o \Lambda_{2m+\frac{1-\epsilon}{2},n} \left(\Gamma_{(+)}^{h=1} V_{12} C_4 O_{16} + \Gamma_{(-)}^{h=1} O_{12} S_4 O_{16} \right) \\ P_o \Lambda_{2m+\frac{1+\epsilon}{2},n} \Gamma_{(+)}^{h=1} S_{12} O_4 O_{16}. \end{aligned} \quad (2.25)$$

It is then clear that, distinct choices of discrete torsion ($\epsilon = \pm 1$) give mass either to the spinorial representation of $SO(12)$, while keeping the the vectorial and scalar representations massless, or render the spinorial massless and give mass to the vectorial and scalar representations, instead.

Indeed, in the case with $\epsilon = +1$ the zero lattice modes attach to the $P_o V_{12} C_4 O_{16}$ -representation, that produces 8 massless $\mathcal{N}_4 = 2$ hypermultiplets in the vectorial representation $(\mathbf{12}, \mathbf{2})$ of $SO(12) \times SO(4)$, whereas in the case with $\epsilon = -1$ the zero lattice modes attach to $P_o S_{12} O_4 O_{16}$, which produces 8 massless $N = 2$ hypermultiplets in the $(\mathbf{32}, \mathbf{1})$ spinorial representation. Furthermore, in the case with $\epsilon = +1$ the first excited twisted lattice modes produce $8 \times 2 \times 4$ massless $SO(12)$ -singlets $(\mathbf{1}, \mathbf{2})$ from the term $P_o O_{12} S_4 O_{16}$. A very interesting observation is that the total number of massless states (equal to $2 \times 8 \times 32$) is the same in both cases $\epsilon = \pm 1$.

2.3 Spinor-Vector Duality map via non-trivial Wilson line backgrounds

In the previous section we explicitly analysed the spectrum of a particular Heterotic model, compactified on $S^1 \times \tilde{S}^1 \times T^4/\mathbb{Z}_2$. There, an additional freely-acting \mathbb{Z}'_2 orbifold, correlating the gauge charges with a translation along the S^1 -circle, was introduced and the change in the choice of the discrete torsion ϵ , associated to the two independent modular orbits, was shown to produce the Spinor-Vector dual theory.

However, this formulation of the duality in terms of the choice of discrete torsion is not suitable to reveal its underlying structure. In order to display this structure it will be convenient to rewrite the partition function in a representation where modular covariance will be manifest. Indeed, it is straightforward to obtain the following covariant expression for the $\mathbb{Z}_2 \times \mathbb{Z}'_2$ orbifold blocks:

$$\begin{aligned}
Z_{[g, g']}^{[h, h']} &= \left[\frac{1}{2} \sum_{\bar{a}, \bar{b}=0,1} (-)^{\bar{a}+\bar{b}+\bar{a}\bar{b}} C_1 \frac{\bar{\theta}_{[\bar{b}]^2}^{\bar{a}+h} \bar{\theta}_{[\bar{b}+g]}^{\bar{a}+h} \bar{\theta}_{[\bar{b}-g]}^{\bar{a}-h}}{\bar{\eta}^4} \right] \Gamma_{(4,4)}[h] \\
&\times \Gamma_{(1,1)}[g'] (-)^{h'(\ell+\sigma)+g'(k+\rho)} \left[\frac{1}{2} \sum_{k, \ell=0,1} C_2 \frac{\theta_{[\ell]}^{[k]6} \theta_{[\ell+g]}^{[k+h]} \theta_{[\ell-g]}^{[k-h]}}{\eta^8} \right] \left[\frac{1}{2} \sum_{\rho, \sigma=0,1} \frac{\theta_{[\sigma]}^{[\rho]8}}{\eta^8} \right],
\end{aligned} \tag{2.26}$$

where C_1 and C_2 are modular invariant phases fixing the chiralities of the spinorial current algebra representations. In order to have agreement with the chirality conventions that appear in the definition of the model in the previous section, we choose:

$$\begin{aligned}
C_1 &= (-)^{\bar{a}\bar{b}} (-)^{(\bar{a}+h)(\bar{b}+g)}, \\
C_2 &= (-)^{k\ell} (-)^{(k+h)(\ell+g)}.
\end{aligned} \tag{2.27}$$

Also,

$$\Gamma_{(1,1)}[g'](R) = \frac{R}{\sqrt{2\tau_2}} \sum_{\tilde{m}, n \in \mathbb{Z}} e^{-\frac{\pi R^2}{2\tau_2} |\tilde{m} + \frac{g'}{2} + \tau(n + \frac{h'}{2})|^2} \quad (2.28)$$

is the $(1, 1)$ -lattice in the Lagrangian representation with half-shifted windings. This can be easily verified by noting that in the $[g']$ -twisted sector the boundary conditions of the compact scalar X^8 can also be satisfied as:

$$\begin{aligned} X^8(\sigma^1 + 2\pi, \sigma^2) &\sim X^8(\sigma^1, \sigma^2) + 2\pi nR + h'\pi R, \\ X^8(\sigma^1, \sigma^2 + 2\pi) &\sim X^8(\sigma^1, \sigma^2) + 2\pi \tilde{m}R + g'\pi R, \end{aligned} \quad (2.29)$$

where \tilde{m}, n are the two winding numbers along the X^8 -circle. Then, the insertion of the modular invariant cocycle $(-)^{h'(\ell+\sigma)+g'(k+\rho)}$ has exactly the effect of alternating the sign depending on the gauge charges $e^{2\pi i(Q_8+Q'_8)}$, as is required by the freely-acting \mathbb{Z}'_2 action.

In what follows we will illustrate that the freely acting \mathbb{Z}'_2 -orbifold is equivalent to a very specific choice of Wilson line along the S^1 -circle. From the latter perspective, the arbitrariness in the choice of discrete torsion will be seen to correspond to a particular freedom in the choice of the Wilson lines.

We begin by performing a double Poisson resummation to bring the $\Gamma_{(1,1)}$ lattice to its dual form

$$\Gamma_{(1,1)}[g'] = \frac{(1/R)}{\sqrt{2\tau_2}} \sum_{\tilde{m}, n \in \mathbb{Z}} e^{-\frac{\pi}{2\tau_2} (\frac{1}{R})^2 |n + \tau \tilde{m}|^2} (-)^{\tilde{m}g' + nh'}, \quad (2.30)$$

so that h', g' only appear through the phase. It is now possible to completely perform the summation over h', g' :

$$Z = \frac{1}{(\sqrt{\tau_2} \eta \bar{\eta})^2} \frac{1}{2} \sum_{h, g=0,1} Z[h], \quad (2.31)$$

and reduce the partition function down to the sum of the orbifold blocks of the non-freely acting \mathbb{Z}_2 :

$$\begin{aligned} Z[h] &= \frac{1}{2} \sum_{h', g'=0,1} (-)^{\frac{1-\epsilon}{2}(hg' - gh')} Z[h, h'] \\ &= \left[\frac{1}{2} \sum_{\bar{a}, \bar{b}=0,1} (-)^{\bar{a} + \bar{b} + \bar{a}\bar{b}} C_1 \frac{\bar{\theta}[\bar{a}]^2 \bar{\theta}[\bar{a}+h] \bar{\theta}[\bar{a}-h]}{\bar{\eta}^4} \right] \Gamma_{(4,4)}[h] \\ &\times \tilde{\Gamma}_{(1,1)}[X] (R/2) \left[\frac{1}{2} \sum_{k, \ell=0,1} C_2 \frac{\theta[\ell]^{16} \theta[\ell+h] \theta[\ell-h]}{\eta^8} \right] \left[\frac{1}{2} \sum_{\rho, \sigma=0,1} \frac{\theta[\rho]^8}{\eta^8} \right]. \end{aligned} \quad (2.32)$$

Here,

$$\tilde{\Gamma}_{(1,1)[Y^X]}(R/2) = \frac{R/2}{\sqrt{2\tau_2}} \sum_{\tilde{m}, n \in \mathbb{Z}} e^{-\frac{\pi}{2\tau_2}(R/2)^2 |\tilde{m} + \tau n|^2} (-)^{\tilde{m}X + nY}, \quad (2.33)$$

and

$$X \equiv k + \rho + \left(\frac{1 - \epsilon}{2}\right) h, \quad (2.34)$$

$$Y \equiv \ell + \sigma + \left(\frac{1 + \epsilon}{2}\right) g. \quad (2.35)$$

Note that the ‘shift’ parameters X, Y now depend on the ϵ -parameter that was previously introduced as a discrete torsion and which will be shortly reinterpreted as a parameter controlling the choice of Wilson line along the S^1 .

In order to illustrate the effect that the X, Y -coupling of the $(1, 1)$ -lattice to the gauge charges has on the spectrum, we Poisson resum the $\tilde{\Gamma}_{(1,1)[Y^X]}$ -lattice and cast it in Hamiltonian form:

$$\tilde{\Gamma}_{(1,1)[Y^X]}(R/2) = \sum_{m, n \in \mathbb{Z}} (-)^{nY} \Lambda_{2m+X, \frac{n}{2}}(R). \quad (2.36)$$

From this form, it becomes clear that the \mathbb{Z}'_2 -freely acting orbifold has the effect of shifting the momentum quantum number by $\frac{1}{2}X$ and also modifying the generalized GSO- or \mathbb{Z}_2 -orbifold projections depending on the winding of the states.

Let us now comment on how Spinor-Vector duality arises in this framework. We focus only on the twisted sector $h = 1$ and notice that only states with $\rho = 0$, i.e. states which are ‘uncharged’ under $SO(16)$, can contribute to the massless spectrum.

Let us pick directly the vectorial of $SO(12)$, by noticing that if it exists in the massless spectrum it must necessarily come from the sector:

$$P_o \Gamma_{(+)}^{h=1} \times \Lambda_{2m+\frac{1-\epsilon}{2}, n}(R) \times V_{12} \times \{S_4 \oplus C_4\} \times \{O_{16} \oplus V_{16}\}. \quad (2.37)$$

A few comments are in order here. First of all, after performing the σ -projection, one finds that only the vacuum representation O_{16} of $SO(16)$ survives. Of course, massless states can only occur in the sector of unshifted momentum/winding quantum numbers, which permits us to restrict our attention only to states with $X \in 2\mathbb{Z}$. For the vectorial representation, $X = (1 - \epsilon)/2$, so that the vectorial can become massless only for the choice $\epsilon = +1$, as found in the previous section. In addition, the P_o representation carries conformal weight $(0, \frac{1}{4})$, while the low-lying modes in P_v start from conformal weight $(0, \frac{3}{4})$ and are already anti-chirally massive. Therefore, since the vectorial representation V_{12} has conformal weight $(\frac{1}{2}, 0)$, only states from the $P_o \Gamma_{(+)}$ -sector can be massless, because the contribution of the twisted lattices to the conformal weights are $(\frac{1}{4}, \frac{1}{4})$ for $\Gamma_{(+)}^{h=1}$ and $(\frac{3}{4}, \frac{1}{4})$ for $\Gamma_{(-)}^{h=1}$. Furthermore, the

generalized GSO-projection realized by the ℓ -summation selects the C_4 representation of $SO(4)$, whereas the orbifold projection g' projects onto states where the twisted $\{\Gamma_{(\pm)}^{h=1}\}$ -lattices and the twisted $\{P_o, P_v\}$ characters are correlated with the same \mathbb{Z}_2 -parity, so that the surviving representation is :

$$P_o \Gamma_{(+)}^{h=1} \times \Lambda_{2m+\frac{1-\epsilon}{2},n}(R) \times V_{12} C_4 O_{16}. \quad (2.38)$$

In the same spirit, we can construct the massless states in the vacuum representation of $SO(12)$:

$$P_o \Gamma_{(-)}^{h=1} \times \Lambda_{2m+\frac{1-\epsilon}{2},n}(R) \times O_{12} S_4 O_{16}. \quad (2.39)$$

This time the ℓ -projection will pick the S_4 representation of $SO(4)$, while the balance of conformal weights now indicates that the twisted lattice $\Gamma_{(-)}^{h=1}$ with odd \mathbb{Z}_2 -parity has to be used in order for these states to become massless. As this representation comes from the same sector $k = \rho = 0$ as the vectorial, the conditions for it to be massless are again $\epsilon = +1$, so that this singlet representation of $SO(12)$ is always present in the massless spectrum whenever the vectorial one is.

Similarly, we can construct the spinorial of $SO(12)$ as:

$$P_o \Gamma_{(+)}^{h=1} \times \Lambda_{2m+\frac{1+\epsilon}{2},n}(R) \times S_{12} \{O_4 + V_4\} O_{16}. \quad (2.40)$$

Again, the ℓ -projection picks the vacuum representation of $SO(4)$ and the balance of conformal weights straightforwardly picks the $\Gamma_{(+)}^{h=1}$ twisted lattice. Then, the \mathbb{Z}_2 -projection can be carried out to show that this representation is indeed invariant. The condition for it to be massless is again, $X \in 2\mathbb{Z}$, which is now satisfied for $\epsilon = -1$.

This reproduces exactly the same conditions for the Spinor-Vector duality of the previous section, in terms of the choice of the discrete torsion. This simple and direct check will be generalized in the next section to provide the conditions for the presence of the Spinor-Vector duality, in terms of the value of the Wilson line along S^1 .

2.4 Turning on a general Wilson line background

Since the effect of the freely-acting \mathbb{Z}'_2 is simply to correlate half-shifts along the S^1 -circle with the charges of the gauge sector, it must have a natural interpretation as a special Wilson line. We will illustrate this point and identify the particular choices of Wilson lines that correspond to the two possible values of the ϵ -parameter. To this end, we will start with the initial $\mathcal{N}_4 = 2$ theory compactified on $S^1 \times \tilde{S}^1 \times T^4/\mathbb{Z}_2$, before the freely-acting \mathbb{Z}'_2 orbifold is introduced. The partition function is again

simply the sum of the \mathbb{Z}_2 orbifold blocks as in (2.31), where now:

$$Z[g] = \left[\frac{1}{2} \sum_{\bar{a}, \bar{b}=0,1} (-)^{\bar{a}+\bar{b}+\bar{a}\bar{b}} C_1 \frac{\bar{\theta}[\bar{a}]^2 \bar{\theta}[\bar{a}+h] \bar{\theta}[\bar{a}-h]}{\bar{\eta}^4} \right] \Gamma_{(4,4)}[g] \\ \times \Gamma_{(1,1)}(R/2) \left[\frac{1}{2} \sum_{k, \ell=0,1} C_2 \frac{\theta[\ell]^6 \theta[\ell+h] \theta[\ell-h]}{\eta^8} \right] \left[\frac{1}{2} \sum_{\rho, \sigma=0,1} \frac{\theta[\rho] \theta[\sigma]^8}{\eta^8} \right], \quad (2.41)$$

and here the $\Gamma_{(1,1)}$ -lattice is a spectator:

$$\Gamma_{(1,1)}(R/2) = \frac{R/2}{\sqrt{2\tau_2}} \sum_{\tilde{m}, n \in \mathbb{Z}} e^{-\frac{\pi}{2\tau_2} (R/2)^2 |\tilde{m} + \tau n|^2}.$$

We will now turn on a general Wilson line along the X^8 compact direction. This amounts to a perturbation of the σ -model by the injection of the following marginal $(1, 1)$ -operator:

$$\delta S = \int d^2 z A_8^a \bar{\partial} X^8(\bar{z}) J^a(z), \quad (2.42)$$

$J^a(z)$, with $a = 1, \dots, 16$, are the currents in the Cartan subalgebra of $E_8 \times E_8$. In the fermionic formulation they correspond to left-moving complex fermion bilinears $J^a(z) = i\bar{\Psi}^a \Psi^a(z)$. This deformation corresponds to turning on a non-trivial Wilson line A_8^a around the S^1 -circle. In fact, it is more convenient to factor out the S^1 radius and define $y^a \equiv A_8^a/R$. The effect of the insertion of this marginal operator is to deform the torus amplitude. By modifying the boundary conditions of the left-moving fermions, or by directly carrying out the path integral one finds:

$$Z[g] = \left[\frac{1}{2} \sum_{\bar{a}, \bar{b}=0,1} (-)^{\bar{a}+\bar{b}+\bar{a}\bar{b}} C_1 \frac{\bar{\theta}[\bar{a}]^2 \bar{\theta}[\bar{a}+h] \bar{\theta}[\bar{a}-h]}{\bar{\eta}^4} \right] \Gamma_{(4,4)}[g] \left[\sum_{\tilde{m}, n \in \mathbb{Z}} e^{-\frac{\pi}{\tau_2} (R/2)^2 |\tilde{m} + \tau n|^2} \right. \\ \left. \times \frac{1}{2} \sum_{k, \ell=0,1} C_2 \left(\frac{\theta[\ell+h-2y^1 n] \theta[\ell-h-2y^2 n]}{\eta^2} \prod_{B=3}^8 \frac{\theta[\ell-2y^B n]}{\eta} \right) \frac{1}{2} \sum_{\rho, \sigma=0,1} \prod_{C=9}^{16} \frac{\theta[\rho-2y^C n]}{\eta} e^{-i\pi \Xi_{\tilde{m}, n}(y)} \right], \quad (2.43)$$

where the phase:

$$\Xi_{\tilde{m}, n}(y) = \tilde{m} n \sum_{a=1}^{16} y^a y^a - n \left((\ell + g)y^1 + (\ell - g)y^2 + \ell \sum_{B=3}^8 y^B + \sigma \sum_{C=9}^{16} y^C \right), \quad (2.44)$$

ensures that the deformation preserves modular invariance.

So far, a particularly simple $\mathbb{Z}_2 \times \mathbb{Z}'_2$ -model was chosen in order to exhibit the basic structure of Spinor-Vector duality. Initially, it was defined in terms of the freely-acting \mathbb{Z}'_2 -orbifold correlating the gauge charges with half-shifts along a compact S^1 , performed on top of the non-freely acting \mathbb{Z}_2 -rotation orbifold. In the previous section, it was shown that the effect of the freely-acting component could be resummed, in order to provide a realization of the same model solely in terms of the orbifold blocks of the non-freely acting \mathbb{Z}_2 , eqs. (2.32)-(2.36).

Now we are in the position to demonstrate that the effect of the freely acting \mathbb{Z}'_2 is equivalent to turning on a particular choice of Wilson line, along S^1 . In fact, it will be instructive to determine the general conditions for the choice of the Wilson line which result in Spinor-Vector duality.

In general, non-rational values for the Wilson line will typically break the enhanced gauge group down to its Cartan factors. However, rational values of the Wilson line may preserve the enhancement. For example, turning on a Wilson line with rational values $y^a = p/q$, with $p < q$ being relatively prime integers, is equivalent to a freely-acting $\mathbb{Z}_{(1+p \bmod 2)q}$ orbifold. For simplicity, and for the purposes of our discussion, it will be sufficient to restrict our attention to the \mathbb{Z}_2 -case, namely to specific discrete points $y^a \in \mathbb{Z}$ along the -otherwise continuous- Wilson line.

Using the periodicity properties of θ -functions, it is straightforward to show that the partition function reduces exactly to the form (2.32), (2.33) of the $Z_g^{[h]}$ -orbifold blocks, where now the $\tilde{\Gamma}_{(1,1)}[X_Y]$ -lattice of (2.34) is shifted by:

$$X = (k+h)y^1 + (k-h)y^2 + k \sum_{B=3}^8 y^B + \rho \sum_{C=9}^{16} y^C + n \sum_{a=1}^{16} y^a y^a, \quad (2.45)$$

$$Y = (\ell+g)y^1 + (\ell-g)y^2 + \ell \sum_{B=3}^8 y^B + \sigma \sum_{C=9}^{16} y^C. \quad (2.46)$$

Comparison of the above equation with (2.34) explicitly illustrates the correspondence. As a particular example, consider turning on the Wilson line:

$$y^a = (0, 0|1, 0, 0, 0, 0, 0|1, 0, 0, 0, 0, 0, 0, 0). \quad (2.47)$$

This choice corresponds precisely to the $\mathbb{Z}_2 \times \mathbb{Z}'_2$ orbifold with the particular choice $\epsilon = +1$ for the discrete torsion. For the opposite discrete torsion $\epsilon = -1$, one may take, instead:

$$y^a = (1, 0|0, 0, 0, 0, 0, 0|1, 0, 0, 0, 0, 0, 0, 0). \quad (2.48)$$

This clearly illustrates that the freely-acting \mathbb{Z}'_2 introduced in the original formulation of the model is nothing but a particular choice of the Wilson line around the compact X^8 -circle.

Furthermore, it is easy to obtain the general conditions for the Wilson line $y^a \in \mathbb{Z}$ that lead to manifestations of Spinor-Vector duality. In particular, it has been already argued that the vectorial, vacuum and spinorial representations of $SO(12)$ arise from the sectors exhibited in eqs. (2.38), (2.39) and (2.40), respectively. Of course, since massless states come from the even winding sector, the Y -shift introduces no modification to the projections.

The conditions for the low-lying states in these sectors to be massless can be found by imposing $X \in 2\mathbb{Z}$. Noting that, for these particular sectors, $h = 1$ and $\rho = 0$ we can distinguish between two conditions controlling the presence of massless states:

- If $\sum_{A=1}^2 y^A \in 2\mathbb{Z}$, then both $k = 0$ -sectors V_{12} and O_{12} are massless.
- If $\sum_{B=3}^8 y^B \in 2\mathbb{Z}$, then the spinorial sector S_{12} is massless.

Clearly, for certain choices of the Wilson line both conditions can be simultaneously satisfied, in which case one recovers the Spinor-Vector self-dual models with enhanced E_7 -gauge symmetry. Similarly, it is possible to choose the Wilson line such that none of the above conditions are satisfied, in which case all charged hypermultiplets become massive. Finally, by choosing to violate only one out of the two conditions, the Wilson line higgses either the vectorial V_{12} (always followed by the vacuum O_{12}) representations while keeping the spinorial S_{12} massless, or vice-versa.

Let us note here an important property, present in this class of models, where $\mathcal{N}_4 = 2$ supersymmetry is preserved by the orbifold action. As shown in eq. (2.42), the Wilson line deformation arises from the injection into the σ -model of a marginal operator which survives the orbifold projection. It is clear that this $(1, 1)$ -operator is associated to a scalar in the physical massless spectrum of the theory. In particular, it corresponds to a modulus within the $\frac{SO(16+2,2)}{SO(16+2) \times SO(2)}$ -factor of the full moduli space and can, thus, take continuous values. This provides the basis for the continuous connection of all the above vacua, in view of the fact that they can all be recovered for specific choices of the Wilson line y^a around S^1 .

In fact, this can be used to illustrate the fact that Spinor-Vector duality is directly interrelated with the enhancement of the gauge symmetry at the Spinor-Vector self-dual point. Indeed, by the very structure of the massless representations and the above conditions, it is clear that at the self-dual points $X \in 2\mathbb{Z}$, independently of the values of k or h . This guarantees that the mapping

$$\left\{ \begin{array}{l} P_o \Gamma_{(-)}^{h=1} O_{12} S_4 O_{16} \\ P_o \Gamma_{(+)}^{h=1} V_{12} C_4 O_{16} \\ P_o \Gamma_{(+)}^{h=1} S_{12} O_4 O_{16} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{l} Q_v \Gamma_{(+)}^{h=0} O_{12} O_4 O_{16} \\ Q_v \Gamma_{(+)}^{h=0} O_{12} O_4 S_{16} \\ Q_v \Gamma_{(+)}^{h=0} C_{12} C_4 O_{16} \end{array} \right\}, \quad (2.49)$$

preserves conformal weights and is one to one. This illustrates how gauge symmetry enhancement translates into Spinor-Vector self-duality in the twisted massless spectrum.

In this particular example, by continuously deforming away from the critical self-dual point along one of the flat directions y^1 , $E_7 \times SU(2)$ spontaneously breaks down to $SO(12) \times U(1) \times U(1)$. When one reaches $y^1 = 1$, the gauge symmetry becomes enhanced back to $SO(12) \times SO(4)$ with the states transforming with the vectorial representation of $SO(12)$ having acquired a mass, while keeping the spinorial massless. An alternative way to give mass to the vectorial of $SO(12)$ is by deforming only one of the two $SU(2)$ -factors. This corresponds to deforming along the trajectory $y^1 = y^2 = \lambda$. As we move continuously away from $\lambda = 0$, the $E_7 \times SU(2)$ breaks down to $SO(12) \times U(1) \times SU(2)$ and V_{12} becomes massive. At point $\lambda = 1/2$, the $U(1)$ gets enhanced so that one again recovers $SO(12) \times SO(4)$.

One may try to deform along a different flat direction in order to give mass to the spinorial while keeping the vectorial massless. However, since the massless $SO(12)$ -spinorials in the twisted sector are always attached to the vacuum representations of $SO(4) \times E'_8$, i.e. $S_{12}O_4O_{16}$, the only way to give them mass (while keeping the vectorial massless) is via non-vanishing expectation values for the Wilson line around S^1 , associated to the $SO(12)$ factor. This inevitably breaks $SO(12)$ spontaneously to one of its subgroups. As an example, consider deforming along the y^3 flat direction, where $SO(12)$ spontaneously breaks down to $SO(10) \times U(1)$, until the point $y^3 = 1$ is reached. There, the $SO(10) \times U(1)$ gets enhanced back to $SO(12)$ but, now, the spinorial representation has become massive whereas the vectorial is kept massless. Of course, deforming along some generic flat direction may render both vectorial and spinorial representations massive.

The possibility of continuous interpolation between vacua with massless vectorials and massless spinorials of $SO(12)$ is special to the $\mathcal{N}_4 = 2$ case. In the case of $\mathbb{Z}_2 \times \mathbb{Z}_2$ models, which preserve $\mathcal{N}_4 = 1$ supersymmetry, and where both \mathbb{Z}_2 's act as rotations on the full T^6 -torus, the situation is substantially different. There, Wilson lines no longer correspond to marginal operators surviving the orbifold projections and, as a result, the associated deformation parameters are no longer continuous. Instead, the only allowed possibility would be to turn on discrete Wilson lines, as in [6]. In fact, a special property of $\mathbb{Z}_2 \times \mathbb{Z}_2$ models with $\mathcal{N}_4 = 1$ supersymmetry is that their twisted sectors actually inherit the structure of the $\mathcal{N}_4 = 2$ theories. However, whereas in the $\mathcal{N}_4 = 2$ case that we are considering here the Wilson line deformation parameters y^a can be continuously varied, in the $\mathcal{N}_4 = 1$ case they can only take discrete values and are, essentially, discrete remnants of the Wilson line deformations of the $\mathcal{N}_4 = 2$ theory.

Therefore, the interpretation of Spinor-Vector duality as the result of turning on specific Wilson lines that give masses either to vectorial or spinorial representations, with the initial (undeformed $y^a = 0$) theory containing both representations in its massless spectrum, has two important consequences. First of all, it unifies a class of models, including the model considered in this paper as well as those presented in ref. [7], and exhibits their common origin. Secondly and most importantly, it sheds light into the origin and nature of the duality and exhibits its close relation to the

inherently stringy phenomenon of symmetry enhancement. This will be discussed in more detail in the next section, where it will be shown how Spinor-Vector duality is, in fact, a discrete remnant of the spectral flow of a spontaneously broken, left-moving, (global) extended superconformal algebra.

3 Spinor-Vector duality from $N = 2$ and $N = 4$ Spectral-Flow

In the previous sections we illustrated how Spinor-Vector duality arises by turning on particular Wilson lines in a ‘parent’ theory with enhanced gauge symmetry. In the particular example considered there, different choices for the Wilson line around the S^1 -circle resulted in massive vectorial or spinorial representations of the $SO(12)$ -gauge group. Let us recall that all vacua containing massless vectorial representations also contained massless hypermultiplets, which were singlets under the $SO(12)$ -group. What is more, the total number of massless states in the twisted sector, vectorials V_{12} and singlets O_{12} on the one hand, and spinorials S_{12} on the other, was found to be the same. This equality is not a numerical coincidence and the reason behind this matching lies in the presence of (at least) an unbroken global $N = 4$ worldsheet superconformal symmetry, in the left-moving sector, associated to the gauge symmetry enhancement in the Spinor-Vector self-dual case.

The presence of this unbroken $N = 4$ algebra can be seen as an embedding⁴ of the $N = 4$ worldsheet superconformal algebra of Type II theories into the bosonic (left-moving) sector of the Heterotic string. The presence of an unbroken $N = 4$ SCFT introduces a spectral flow, which can be seen to transform the spinorial representations of $SO(12)$ into the vectorial (always followed by the scalar) representations and vice-versa. In the following subsections, we will display the way the mapping arises in the $\mathcal{N}_4 = 1$ and $\mathcal{N}_4 = 2$ cases⁵, by explicitly constructing the spectral-flow operator and exhibiting its action on the vertex operators.

3.1 Spinor -Vextor Duality and $N = 2$ Spectral Flow in $\mathcal{N}_4 = 1$ Vacua

We will first start with the simpler case of unbroken $\mathcal{N}_4 = 1$ spacetime supersymmetry and expand upon the analysis of ref. [6]. Spacetime supersymmetry requires the local right-moving $\hat{c} = 6$, $N_R = 1$ internal SCFT to become enhanced to $N_R = 2$. Now consider the case where the left-moving internal CFT also becomes enhanced to a global $N_L = 2$ SCFT (see, for example, [10], [19] and references therein).

This enhancement arises naturally via the Gepner map, as follows. Consider first the left-moving worldsheet degrees of freedom of a Type II theory with an enhanced

⁴This embedding of the spin connection of Type II into the gauge connection of Heterotic theories is known as the Gepner map [9].

⁵As will be discussed in detail in this section, in the $\mathcal{N}_4 = 1$ case, the enhancement arises from the presence of an $N = 2$ SCFT, whose spectral flow induces the spinor-to-vector map.

$N_L = 2$ global superconformal algebra. The vertex operators are generically proportional to:

$$e^{q\phi + is_0 H_0 + is_1 H_1 + i\frac{Q}{\sqrt{3}}H}, \quad (3.1)$$

where q is the superghost charge (picture), s_0, s_1 are the $SO(1, 3)$ helicity charges and Q is the charge with respect to the $U(1)$ current $J(z) = i\sqrt{3}\partial H(z)$ of the internal $N_L = 2$ SCFT. The currents generating spacetime supersymmetry are constructed in terms of the free boson as:

$$\begin{aligned} e^{-\phi/2} S_\alpha \Sigma(z) &= e^{-\frac{1}{2}\phi \pm \frac{i}{2}(H_0 + H_1) + i\frac{\sqrt{3}}{2}H}, \\ e^{-\phi/2} C_{\dot{\alpha}} \Sigma^\dagger(z) &= e^{-\frac{1}{2}\phi \pm \frac{i}{2}(H_0 - H_1) - i\frac{\sqrt{3}}{2}H}. \end{aligned} \quad (3.2)$$

Here Σ, Σ^\dagger are the maximal charge ground states of the R-sector with conformal weight $(\frac{3}{8}, 0)$. Imposing a good action of the supersymmetry currents on the vertex operators of the spectrum requires:

$$q + s_0 + s_1 + Q \in 2\mathbb{Z}. \quad (3.3)$$

The $N_L = 2$ spectral flow arises from shifting the $U(1)$ charges so that one may obtain a continuous interpolation between the NS ($\alpha = 0$) and R ($\alpha = \pm\frac{1}{2}$) sectors:

$$\begin{aligned} J_n &\rightarrow J_n - 3\alpha\delta_{n,0} \\ L_n &\rightarrow L_n - \alpha J_n + \frac{3}{2}\alpha^2\delta_{n,0}. \end{aligned} \quad (3.4)$$

Now consider embedding the $N_L = 2$ SCFT into the left-moving ('bosonic') sector of the $E_8 \times E_8$ Heterotic string. The analogue of the supersymmetry current (3.2) is now built as:

$$\mathcal{I}(z)\Sigma(z), \quad (3.5)$$

where $\mathcal{I}(z)$ is a $(\frac{5}{8}, 0)$ -operator that dresses the internal $N_L = 2$ ground state by replacing the superghost and spacetime fermion contributions $e^{-\phi/2 \pm \frac{i}{2}H_0 \pm \frac{i}{2}H_1}$ of the Type II case. In the Heterotic side this operator will arise from the gauge degrees of freedom. The spin connection may be naturally embedded into the gauge connection by setting:

$$\mathcal{I}(z) = e^{i\lambda \cdot Z(z)}, \quad (3.6)$$

with $\lambda^A = \pm\frac{1}{2}$ and $A = 1, \dots, 5$. This is simply the bosonization of the R-sector ground state for the 5 complex current algebra fermions Ψ^A in E_8 . The GSO projection is then naturally generalized by requiring that the spectral-flow operator (3.5) has a well-defined action on the states:

$$\sum_{A=1}^5 \oint \frac{dz}{2\pi i} \partial Z^A(z) + Q \in 2\mathbb{Z}. \quad (3.7)$$

This constrains the sum of the number operator for the 5 current algebra fermions Ψ^A and the charge Q of the $N_L = 2$ SCFT to be even.

The $(1, 0)$ -currents surviving the GSO projection are then:

$$\Psi^A \Psi^B(z) \ , \ S_{10} \Sigma(z) \ , \ C_{10} \Sigma^\dagger(z) \ , \ J(z). \quad (3.8)$$

Here $A, B = 1, \dots, 5$ run over the 5 complex current algebra fermions so that the fermion bilinears $\Psi^A \Psi^B$ transform as the adjoint representation **45** of $SO(10)$. Similarly, S_{10} and C_{10} are the R-sector vertex operators for the spinorial **16** and conjugate spinorial $\overline{\mathbf{16}}$ representations of $SO(10)$. Together with the singlet generated by $J(z)$, the currents form the adjoint representation **78** of E_6 .

As a concrete example, consider the $N = (2, 2)$ compactification on $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$, which preserves $\mathcal{N}_4 = 1$ left-moving spacetime supersymmetry. In terms of modular covariant conformal blocks, the above Gepner map is realized as:

$$\frac{1}{2} \sum_{a,b} (-)^{a+b+ab} \theta_{[b]}^a \theta_{[b+g_1]}^{[a+h_1]} \theta_{[b+g_2]}^{[a+h_2]} \theta_{[b-g_1-g_2]}^{[a-h_1-h_2]} \rightarrow \frac{1}{2} \sum_{k,\ell} \theta_{[\ell]}^{[k]} \theta_{[\ell+g_1]}^{[k+h_1]} \theta_{[\ell+g_2]}^{[k+h_2]} \theta_{[\ell-g_1-g_2]}^{[k-h_1-h_2]}. \quad (3.9)$$

The full modular invariant partition function of the theory can be organized into orbifold blocks, as before:

$$\begin{aligned} Z_{[g_1, g_2]}^{[h_1, h_2]} &= \left[\frac{1}{2} \sum_{\bar{a}, \bar{b}=0,1} (-)^{\bar{a}+\bar{b}+\bar{a}\bar{b}} C_1 \frac{\bar{\theta}_{[\bar{b}]}^{\bar{a}} \bar{\theta}_{[\bar{b}+g_1]}^{\bar{a}+h_1} \bar{\theta}_{[\bar{b}+g_2]}^{\bar{a}+h_2} \bar{\theta}_{[\bar{b}-g_1-g_2]}^{\bar{a}-h_1-h_2}}{\bar{\eta}^4} \right] \Gamma_{(6,6)}^{[h_1, h_2]}_{[g_1, g_2]} \\ &\times \left[\frac{1}{2} \sum_{k, \ell=0,1} C_2 \frac{\theta_{[\ell]}^{[k]} \theta_{[\ell+g_1]}^{[k+h_1]} \theta_{[\ell+g_2]}^{[k+h_2]} \theta_{[\ell-g_1-g_2]}^{[k-h_1-h_2]}}{\eta^8} \right] \left[\frac{1}{2} \sum_{\rho, \sigma=0,1} \frac{\theta_{[\rho]}^{[\sigma]8}}{\eta^8} \right], \end{aligned} \quad (3.10)$$

where, for concreteness, we make the following choice of chiralities:

$$\begin{aligned} C_1 &= (-)^{\bar{a}\bar{b}}, \\ C_2 &= 1. \end{aligned} \quad (3.11)$$

We will consider here the simple case where the twisted $\Gamma_{(6,6)}^{[h_1, h_2]}_{[g_1, g_2]}$ lattice is factorized into the product of three $\Gamma_{(2,2)}^{[h]}_{[g]}$ lattices as follows:

$$\Gamma_{(6,6)}^{[h_1, h_2]}_{[g_1, g_2]} = \Gamma_{(2,2)}^{[h_1]}_{[g_2]} \Gamma_{(2,2)}^{[h_2]}_{[g_2]} \Gamma_{(2,2)}^{[h_1+h_2]}_{[g_1+g_2]}, \quad (3.12)$$

where :

$$\Gamma_{(2,2)}^{[h]}_{[g]} = \begin{cases} \Gamma_{(2,2)} & , \text{ for } (h, g) = (0, 0) \\ \left| \frac{2\eta}{\theta_{[1-g]}^{[1-h]}} \right|^2 & , \text{ for } (h, g) \neq (0, 0) \end{cases} \quad (3.13)$$

In terms of the fermionic formulation of the Heterotic string, the $\hat{c} = 6$ system realizing the $N_L = 2$ SCFT, is built out of the 3 complex fermions $\Psi^{6,7,8}$ which are bosonized as $e^{\pm iH^j(z)}$, with $j = 6, 7, 8$. The spectral-flow currents are then constructed out of free fields as:

$$\begin{aligned} C_{10}(z)e^{\frac{i}{2}H^6(z)+\frac{i}{2}H^7(z)+\frac{i}{2}H^8(z)} &\propto e^{i\frac{\sqrt{3}}{2}H(z)}, \\ S_{10}(z)e^{-\frac{i}{2}H^6(z)-\frac{i}{2}H^7(z)-\frac{i}{2}H^8(z)} &\propto e^{-i\frac{\sqrt{3}}{2}H(z)}. \end{aligned} \quad (3.14)$$

The presence of an $N_L = 2$ SCFT manifests itself in the ability to factor out the free $U(1)$ current $J(z)$. To see this, consider the following linear field redefinition for the free scalars:

$$\begin{aligned} H(z) &= (H^6 + H^7 + H^8) / \sqrt{3} \\ X(z) &= (2H^6 - H^7 - H^8) / \sqrt{6} \\ Y(z) &= (-H^7 + H^8) / \sqrt{2}. \end{aligned} \quad (3.15)$$

The bosons H, X, Y are still free and, in particular, $J(z) = i\sqrt{3}\partial H(z)$ is identified with the conserved $U(1)$ current of the $N_L = 2$ SCFT, in accordance with (3.5) and (3.14).

Consider now the action of the spectral flow in the untwisted sector. The scalar spectrum contains states in the NS-sector of the $N = (2, 2)$ SCFT saturating the BPS bound $(\Delta, \bar{\Delta}) = (\frac{1}{2}|Q|, \frac{1}{2}|\bar{Q}|)$. Their vertex operators can be written in terms of the $N = (2, 2)$ chiral primaries \mathcal{F} with charge $Q = \bar{Q} = 1$ as:

$$\Psi^A(z)\mathcal{F}(z, \bar{z}) = \Psi^A(z)e^{iH^j(z)}e^{-\bar{\phi}(\bar{z})+i\bar{H}_k(\bar{z})} \propto e^{i\frac{1}{\sqrt{3}}H(z)}, \quad (3.16)$$

where $A = 1, \dots, 5$, $j = 6, 7, 8$ and $k = 2, 3, 4$. This transforms as the vectorial of $SO(10)$. Under the spectral flow (3.4), the left-moving $U(1)$ charge is shifted by $-\frac{3}{2}$ units to yield $Q = 1 \rightarrow -\frac{1}{2}$. Taking, for example, $j = 6$:

$$e^{\pm iH^6} \rightarrow e^{\pm\frac{i}{2}H^6 \mp \frac{i}{2}H^7 \mp \frac{i}{2}H^8}, \quad (3.17)$$

we see that the spectral flow transforms the $V_{10}V_2O_2O_2$ representation (3.16) into the $C_{10}C_2S_2S_2$ and $S_{10}S_2C_2C_2$ representations in the R-sector of the $N_L = 2$ SCFT. It is described by the vertex operator:

$$\begin{aligned} C_{10}(z)e^{\frac{i}{2}H^6(z)-\frac{i}{2}H^7(z)-\frac{i}{2}H^8(z)}e^{-\bar{\phi}(\bar{z})+i\bar{H}_2(\bar{z})} &\propto e^{-i\frac{1}{2\sqrt{3}}H(z)}, \\ S_{10}(z)e^{-\frac{i}{2}H^6(z)+\frac{i}{2}H^7(z)+\frac{i}{2}H^8(z)}e^{-\bar{\phi}(\bar{z})+i\bar{H}_2(\bar{z})} &\propto e^{+i\frac{1}{2\sqrt{3}}H(z)}, \end{aligned} \quad (3.18)$$

transforming in the spinorial representation of $SO(10)$ and its conjugate, respectively. Note that the spectral flow between the NS and R sectors arises explicitly through the action of the $Q = \pm\frac{3}{2}$ operators in (3.14) on the states, in complete analogy to the case of spacetime supersymmetry.

Shifting the $U(1)$ charge of (3.18) once more, $Q = -\frac{1}{2} \rightarrow -2$, one finds the singlet representation $O_{10}O_2V_2V_2$ in the NS-sector of the $N = (2, 2)$ SCFT. Its vertex operator is written in terms of the chiral primary \mathcal{G} with charge $(Q, \bar{Q}) = (-2, 1)$:

$$\mathcal{G}(z, \bar{z}) = e^{-iH^7(z) - iH^8(z)} e^{-\bar{\phi}(\bar{z}) + i\bar{H}_2(\bar{z})} \propto e^{-i\frac{2}{\sqrt{3}}H(z)}. \quad (3.19)$$

The same analysis can be carried out in the twisted sectors. For concreteness, consider the $(h_1, h_2) = (1, 0)$ sector. The massless matter spectrum contains again fermionic states transforming in the vectorial representation of $SO(10)$. They are built out of twisted $Q = 1$ chiral primaries:

$$\Psi^A(z) e^{\frac{i}{2}H^6(z) + \frac{i}{2}H^8(z)} \Gamma_{(+,+,+)}^{(1,0)}(z, \bar{z}) e^{-\frac{1}{2}\bar{\phi}(\bar{z}) + \frac{i}{2}\bar{H}_0(\bar{z}) + \frac{i}{2}\bar{H}_1(\bar{z}) + \frac{i}{2}\bar{H}_3(\bar{z})} \propto e^{i\frac{1}{\sqrt{3}}H(z)}. \quad (3.20)$$

Here $\Gamma_{(+,+,+)}^{(1,0)}(z, \bar{z})$ is the weight- $(\frac{1}{4}, \frac{1}{4})$ invariant twist-field, associated to the $(h_1, h_2) = (1, 0)$ -twisted $\Gamma_{(6,6)}$ -lattice

$$\Gamma_{(s_1 s_3, s_2 s_3)}^{(h_1, h_2)} = \Gamma_{(s_1)}^{h_1} \Gamma_{(s_2)}^{h_2} \Gamma_{(s_3)}^{h_1+h_2}, \quad (3.21)$$

with $s_i = \pm$ being the definite \mathbb{Z}_2 -parities of the three $(2, 2)$ -sublattices, defined by analogy to (A.2). Note that the untwisted $\Gamma_{(-)}$ lattice with negative parity projects out the low-lying states, whereas $\Gamma_{(+)}$ preserves these states but starts with $(0, 0)$ -conformal dimension. On the other hand, each $\Gamma_{(2,2)}$ -twisted lattice $\Gamma_{(+)}^{h=1}$ of positive parity has conformal weight $(\frac{1}{8}, \frac{1}{8})$, while $\Gamma_{(-)}^{h=1}$ contains sectors with weights $(\frac{5}{8}, \frac{1}{8})$ and $(\frac{1}{8}, \frac{5}{8})$.

In terms of characters, (3.20) corresponds to the $V_{10}C_2O_2C_2$ representation. Under the spectral flow⁶, it will be mapped into the conjugate spinorial representation $C_{10}O_2S_2O_2$ with charge $Q = -\frac{1}{2}$:

$$C_{10}(z) e^{-\frac{i}{2}H^7(z)} \Gamma_{(+,+,+)}^{(1,0)}(z, \bar{z}) e^{-\frac{1}{2}\bar{\phi}(\bar{z}) + \frac{i}{2}\bar{H}_0(\bar{z}) + \frac{i}{2}\bar{H}_1(\bar{z}) + \frac{i}{2}\bar{H}_3(\bar{z})} \propto e^{-i\frac{1}{2\sqrt{3}}H(z)}. \quad (3.22)$$

To see this, we will explicitly construct the twisted spectral-flow operator with charge $Q = -\frac{3}{2}$ and consider its action on the vertex operator (3.20). Here, because the twist is only \mathbb{Z}_2 , it is possible to represent the twist-field vertex operators $\Gamma_{(\pm)}^{h_i}(z, \bar{z})$ associated to the twisted lattice in terms of level-one free-fermion characters. To this end, note that the topological contribution $\Gamma_{(\pm)}^h(z, \bar{z})$ of the twisted lattice (3.12),(3.13) can be represented in terms of free-fermions as:

$$\Gamma_{(s)}^{h=1} = \frac{1}{2^2 \eta^2 \bar{\eta}^2} \sum_{g=0,1} \sum_{\gamma, \delta=0,1} (-)^{\binom{1-s}{2}} g \theta_{[\delta]}^{\gamma} \theta_{[\delta+g]}^{\gamma+1} \times \bar{\theta}_{[\delta]}^{\gamma} \bar{\theta}_{[\delta+g]}^{\gamma+1}. \quad (3.23)$$

⁶Of course, by shifting the $U(1)$ charge by $+\frac{3}{2}$ units, the vectorial representation would be mapped into the massive spinorial, $S_{10}V_2C_2V_2$.

Performing the γ -summations and imposing the δ, g -projections we find the explicit form of the free-fermion representations of the twisted vertex operators:

$$\Gamma_{(+)}^{h=1}(z, \bar{z}) = \left\{ \begin{aligned} &O_2 S_2 \bar{O}_2 \bar{S}_2 \oplus O_2 C_2 \bar{O}_2 \bar{C}_2 \oplus V_2 S_2 \bar{V}_2 \bar{S}_2 \oplus V_2 C_2 \bar{V}_2 \bar{C}_2 \\ &\oplus S_2 O_2 \bar{S}_2 \bar{O}_2 \oplus S_2 V_2 \bar{S}_2 \bar{V}_2 \oplus C_2 O_2 \bar{C}_2 \bar{O}_2 \oplus C_2 V_2 \bar{C}_2 \bar{V}_2 \end{aligned} \right\}. \quad (3.24)$$

$$\Gamma_{(-)}^{h=1}(z, \bar{z}) = \left\{ \begin{aligned} &O_2 S_2 \bar{V}_2 \bar{C}_2 \oplus O_2 C_2 \bar{V}_2 \bar{S}_2 \oplus V_2 S_2 \bar{O}_2 \bar{C}_2 \oplus V_2 C_2 \bar{O}_2 \bar{S}_2 \\ &\oplus S_2 O_2 \bar{C}_2 \bar{V}_2 \oplus S_2 V_2 \bar{C}_2 \bar{O}_2 \oplus C_2 O_2 \bar{S}_2 \bar{V}_2 \oplus C_2 V_2 \bar{S}_2 \bar{O}_2 \end{aligned} \right\}. \quad (3.25)$$

For simplicity, we suppress the indices and directly label each vertex operator by its $SO(2n)$ -representation so that, for example, the O_2 -representation contains the identity operator $\mathbf{1}_2(z)$ as its ground state and the adjoint representation (realized as a bifermion) at the first excited level.

In the twisted sectors, the spectral-flow currents (3.14) become extended by a *chiral* operator $\Omega_{(\pm, \pm, \pm)}(z)$, acting on the twist-field contribution $\Gamma^{(1,0)}(z, \bar{z})$ associated to the twisted lattices. The total invariant spectral-flow current is then decomposed into the following contributions:

$$\begin{aligned} j_{s.f.}(z) = &(C_{10} C_2 C_2 C_2)(z) \Omega_{(+, +, +)}(z) \oplus (C_{10} C_2 S_2 S_2)(z) \Omega_{(-, +, +)}(z) \\ &\oplus (C_{10} S_2 C_2 S_2)(z) \Omega_{(-, +, -)}(z) \oplus (C_{10} S_2 S_2 C_2)(z) \Omega_{(-, +, -)}(z) \\ &\oplus (S_{10} S_2 S_2 S_2)(z) \Omega_{(+, +, +)}(z) \oplus (S_{10} S_2 C_2 C_2)(z) \Omega_{(-, +, +)}(z) \\ &\oplus (S_{10} C_2 S_2 C_2)(z) \Omega_{(-, +, -)}(z) \oplus (S_{10} C_2 C_2 S_2)(z) \Omega_{(+, +, -)}(z). \end{aligned} \quad (3.26)$$

The chiral dressing $\Omega_{(\alpha, \beta, \gamma)}(z)$ transforms as $(\alpha\gamma, \beta\gamma)$ under the action of the orbifold group $\mathbb{Z}_2 \times \mathbb{Z}_2$, where $\alpha, \beta, \gamma = \pm 1$. Its conformal weight is $(\frac{3-\alpha-\beta-\gamma}{2}, 0)$ and its action on the relevant twisted vertex operators of the form (3.21), (3.24), (3.25) follows the fusion rule:

$$\Omega_{(\alpha, +, \gamma)}(z) \cdot \Gamma_{(r, +, s)}^{(1,0)}(w, \bar{w}) = \frac{\Gamma_{(\alpha r)}^1 \Gamma_{(+)}^0 \Gamma_{(\gamma s)}^1(w, \bar{w})}{(z-w)^{\frac{1}{2} - \frac{1}{4}(\alpha + \gamma)}} + \dots, \quad (3.27)$$

where again, we use a compact notation where the representation indices as well as the associated Dirac matrices arising from the OPEs are suppressed. The ellipsis denotes less singular terms. In terms of free bosonic fields, $\Omega_{(\alpha, \beta, \gamma)}(z)$ can be represented as:

$$\Omega_{(\alpha, \beta, \gamma)}(z) = e^{i\frac{1-\alpha}{2}(\pm\Phi_1 \pm \Phi_2) + i\frac{1-\beta}{2}(\pm\Phi_3 \pm \Phi_4) + i\frac{1-\gamma}{2}(\pm\Phi_5 \pm \Phi_6)}. \quad (3.28)$$

We are now ready to explicitly calculate the action of the spectral-flow operator on the vertex operators. Let us start with $C_{10}O_2S_2O_2$ given in (3.22). The spectral-flow operator responsible for the mapping between the various representations is the zero mode of the current (3.26) :

$$\begin{aligned}
Q_{\text{s.f.}} = \oint \frac{dz}{2\pi i} \Big[& (C_{10}C_2C_2C_2)(z)\Omega_{(+,+,+)}(z) + z(C_{10}C_2S_2S_2)(z)\Omega_{(-,+,+)}(z) \\
& + z^2(C_{10}S_2C_2S_2)(z)\Omega_{(-,+, -)}(z) + z^2(C_{10}S_2S_2C_2)(z)\Omega_{(+,+, -)}(z) \\
& + (S_{10}S_2S_2S_2)(z)\Omega_{(+,+,+)}(z) + z(S_{10}S_2C_2C_2)(z)\Omega_{(-,+,+)}(z) \\
& + z^2(S_{10}C_2S_2C_2)(z)\Omega_{(-,+, -)}(z) + z(S_{10}C_2C_2S_2)(z)\Omega_{(+,+, -)}(z) \Big]. \quad (3.29)
\end{aligned}$$

Note that this operator could also be obtained as the invariant truncation of the untwisted spectral-flow current $C_{16}(z)$ of E_8 . The lattice dressing $\Omega(z)$ ensures the operator survives the orbifold projections. The spectral flow charge can now act upon the various states $\mathcal{V}(0,0)|0\rangle$ centered at $z = \bar{z} = 0$. In particular, its action on $C_{10}O_2S_2O_2$ generates the following (massless) representations:

$$\begin{aligned}
Q_{\text{s.f.}} \cdot \Big[& C_{10}O_2S_2O_2 \Gamma_{(+,+,+)}^{(1,0)} e^{-\frac{1}{2}\bar{\phi} + \frac{i}{2}\bar{H}_0} \bar{S}_2\bar{O}_2\bar{S}_2\bar{O}_2 \Big] (0,0)|0\rangle \\
= \Big[& V_{10}C_2O_2C_2\Gamma_{(+,+,+)}^{(1,0)} + O_{10}S_2O_2C_2\Gamma_{(-,+,+)}^{(1,0)} + O_{10}S_2V_2S_2\Gamma_{(+,+,+)}^{(1,0)} \\
& + O_{10}C_2O_2S_2\Gamma_{(+,+, -)}^{(1,0)} \Big] e^{-\frac{1}{2}\bar{\phi} + \frac{i}{2}\bar{H}_0} \bar{S}_2\bar{O}_2\bar{S}_2\bar{O}_2|0\rangle. \quad (3.30)
\end{aligned}$$

It is straightforward to verify the above mapping by using the OPEs between $SO(N)$ -spin fields, given in Appendix B.

As expected, the conjugate spinorial $SO(10)$ -representation has been mapped into the vectorial $V_{10}C_2O_2C_2$, accompanied by the singlets. This illustrates the exact map between these representations, as it is induced by the spectral-flow current (3.26). It is instructive to count the numbers of (massless) degrees of freedom :

Spinorial	$C_{10}O_2S_2O_2 \rightarrow 2^{5-1} \times (4 \times 4)$	(3.31)
Vectorial	$V_{10}C_2O_2C_2 \rightarrow 10 \times (4 \times 4)$	
Singlets	$O_{10}S_2O_2C_2 \rightarrow 1 \times (8 \times 4)$	
	$O_{10}S_2V_2S_2 \rightarrow 2 \times (4 \times 4)$	
	$O_{10}C_2O_2S_2 \rightarrow 1 \times (4 \times 8)$	

The spectral-flow displayed above gives rise to a number of supersymmetric-like identities, realized internally in the left-moving sector. Here, in contrast to the right-moving sector, which enjoys local worldsheet supersymmetry, the spinors and vectors transform under the gauge rotation group $SO(10)$ rather than the spacetime little group and, hence, there is no cancellation between the contributions of the vectorial and spinorial representations⁷. The analogues of ‘‘bosons’’ and ‘‘fermions’’ have

⁷Of course, this implementation of the correct spin-statistics arises automatically, from the requirements of higher-genus modular invariance and factorization.

again an equal contribution to the partition function, however, the characters are summed rather than subtracted. Nevertheless, these identities may lead to considerable simplifications in various calculations involving, for example, integration of the modular parameters over the fundamental domain. Such integrations arise frequently when one is calculating the one-loop vacuum amplitude in cases where the spacetime supersymmetry is spontaneously broken.

For example, one may pick the weight $(*, \frac{1}{4})$ -contribution of the internal CFT, which is relevant for creating massless states in the right-moving side. For concreteness, let us take the contribution that couples to the right-moving lattice piece $\bar{O}_2 \bar{S}_2 \bar{O}_2 \bar{O}_2 \bar{O}_2 \bar{S}_2 \in \Gamma_{(\pm, +, \pm)}^{(1,0)}$. Gathering together all relevant factors one may write this contribution as $\mathcal{U}(\epsilon) \bar{O}_2 \bar{S}_2 \bar{O}_2 \bar{O}_2 \bar{O}_2 \bar{S}_2$, where:

$$\mathcal{U}(\epsilon) \equiv O_2 O_2 \times [(O_2 S_2 O_2 S_2) A_1 + (V_2 C_2 O_2 S_2) A_2 + (O_2 S_2 V_2 C_2) A_3 + (V_2 C_2 V_2 C_2) A_4], \quad (3.32)$$

and the A_i are:

$$\begin{aligned} A_1 &= [O_{10} S_2 V_2 S_2 + V_{10} C_2 O_2 C_2 + \epsilon(C_{10} O_2 S_2 O_2 + S_{10} V_2 C_2 V_2)] O_{16} \\ A_2 &= [O_{10} S_2 O_2 C_2 + V_{10} C_2 V_2 S_2 + \epsilon(C_{10} O_2 C_2 V_2 + S_{10} V_2 S_2 O_2)] O_{16} \\ A_3 &= [O_{10} C_2 O_2 S_2 + V_{10} S_2 V_2 C_2 + \epsilon(C_{10} V_2 C_2 O_2 + S_{10} O_2 S_2 V_2)] O_{16} \\ A_4 &= [O_{10} C_2 V_2 C_2 + V_{10} S_2 O_2 S_2 + \epsilon(C_{10} V_2 S_2 V_2 + S_{10} O_2 C_2 O_2)] O_{16} \end{aligned} \quad (3.33)$$

Here, $\epsilon = \pm 1$ denotes the spin-statistics sign. The value $\epsilon = -1$ would arise in the presence of worldsheet super-reparametrization invariance, hence, requiring a *local* $N = 1$ worldsheet SCFT. Its further enhancement to a global $N = 2$ would introduce the spectral-flow responsible for $\mathcal{N}_4 = 1$ spacetime supersymmetry. The spacetime supersymmetric structure manifests itself in terms of chiral identities between current algebra characters, such as $\mathcal{U}(-1) = 0$, which can be verified by using the Jacobi theta-function identities.

In our case, worldsheet supersymmetry is global in the left-moving sector and there are no pictures, which corresponds to the ‘bosonic’ case $\epsilon = +1$. Even though $\mathcal{U}(+1)$ is non-vanishing, the previous identity $\mathcal{U}(-1) = 0$ may still be used to illustrate the spectral-flow and to algebraically simplify the characters. Examples of such identities will be presented in more detail in the next section, where the $\mathcal{N}_4 = 2$ case will be considered. However, they are still present in the $\mathcal{N}_4 = 1$ case as well, even though they are somewhat more tedious to display explicitly.

A very important observation can be made already at this point. It turns out that the spectral-flow operator in the twisted sector is none other than a deformed version of the operator inducing the *Massive Spectral boson-fermion Degeneracy Symmetry* (MSDS) of [14], [15]. The action of the MSDS-operator on states is only well-defined provided a set of conditions is satisfied, [15], and these severely constrain the compactification. Whenever these are met, the chiral character identities emanating from

the MSDS spectral-flow can be utilised to relate ‘vectorial’ representations to ‘spinorials’, with the exception of weight $\Delta = \frac{1}{2}$ ground states⁸ which remain untransformed. In these cases, the spectral-flow operator precisely coincides with the MSDS charge, which is the zero mode of an invariant truncation of the $SO(24)$ spin-field $C_{24}(z)$.

In the particular example (3.10), (3.12) we considered in this section, the $\Gamma_{(6,6)}$ -lattice in the $(h_1, h_2) = (1, 0)$ plane was factorizable into three $\Gamma_{(2,2)}$ sublattices $\Gamma[\frac{1}{g_1}] \Gamma[\frac{0}{g_2}] \Gamma[\frac{1}{g_1+g_2}]$. This particular compactification does not satisfy⁹ the conditions for the MSDS spectral-flow and, as a result, identities such as those mentioned above can generically only arise in certain sub-sectors in which the spectral-flow current has a well-defined action. Of course, for the purposes of phenomenology, only the subsectors contributing to the massless spectrum are relevant. This is the case for the identity $\mathcal{U}(-1) = 0$ considered above. Similar identities can be obtained by considering other contributions in $\Gamma_{(\pm,+, \pm)}^{(1,0)}$, such as those coupling to $\bar{O}_2 \bar{S}_2 \bar{O}_2 \bar{O}_2 \bar{O}_2 \bar{C}_2$, or $\bar{O}_2 \bar{C}_2 \bar{O}_2 \bar{O}_2 \bar{O}_2 \bar{C}_2$ and so on, provided that the right-moving lattice contribution has conformal weight $(0, \frac{1}{4})$ in order to produce anti-chirally massless states.

Finally, by turning on discrete torsions¹⁰, as in [6], one may deform the theory and give masses to the vectorial representation (always accompanied by the singlets) or to the conjugate spinorial. Of course, this (discrete) deformation away from the enhanced symmetry point will have the effect of breaking the enhanced $N = (2, 2)$ SCFT down to $N = (0, 2)$, as is required for the preservation of the $\mathcal{N}_4 = 1$ spacetime supersymmetry.

It is then clear that the observed duality map between the two theories, one with massless (conjugate) spinorials and one with massless vectorials, is the direct result of the spectral-flow in the twisted sectors of the enhanced $N = (2, 2)$ compactification. In particular, the spectral-flow at the enhanced point guarantees that the number of massless degrees of freedom in the two -seemingly disconnected- theories always remains the same.

3.2 Spinor-Vector Duality and $N = 4$ Spectral Flow in $\mathcal{N}_4 = 2$ Vacua

In this section, we will briefly extend the analysis of the previous section to the $\mathcal{N}_4 = 2$ level. This time, spacetime supersymmetry requires the extension of the local right-moving $N_R = 1$, $\hat{c} = 6$ superconformal system into a free $N_R = 2$, $\hat{c} = 2$ SCFT system and an $N_R = 4$ SCFT with $\hat{c} = 4$ [10]:

$$\{ N = 1, \hat{c} = 6 \} \longrightarrow \{ N = 2, \hat{c} = 2 \} \oplus \{ N = 4, \hat{c} = 4 \}. \quad (3.34)$$

⁸States of conformal weight $(\frac{1}{2}, *)$ are (chirally) massless in Type II theories. These are precisely the states that remain invariant under the MSDS spectral-flow, in contrast to the case of conventional supersymmetry. The principle governing the MSDS spectral-flow is preserved intact when the spin connection of Type II theories is embedded in the gauge connection of the Heterotic string.

⁹It is possible to consider a discrete shift in the toroidal background parameters, compatible with the orbifold, such that the MSDS spectral-flow conditions would be satisfied in the $(1, 0)$ -plane.

¹⁰These can be seen, alternatively, as discrete Wilson lines.

As before, we are interested in the case where the left-moving internal CFT also becomes enhanced to a direct sum of global $\{N_L = 4, \hat{c} = 4\} \oplus \{N_L = 2, \hat{c} = 2\}$ SCFTs. In particular, the free $\hat{c} = 2$ system will give rise to a compactification on T^2 .

The starting point is, again, the Type II theory with $N_4 = 2$ supersymmetries arising from the left-moving side. The vertex operators of the states are now proportional to:

$$e^{q\phi + is_0 H_0 + is_1 H_1 + irY + iQ\sqrt{2}H}, \quad (3.35)$$

where the spacetime part is defined as in the previous section, r is the $U(1)$ charge of the free $j(z) = i\partial Y(z)$ boson and Q is the ‘isospin’ charge with respect to the diagonal $SU(2)_{k=1}$ current $J^3(z) = \frac{i}{\sqrt{2}}\partial H(z)$ of the internal $N_L = 4$ SCFT. The two spacetime supersymmetry currents are then of the form (3.2), the difference now is the presence of two weight- $(\frac{3}{8}, 0)$ R-ground states $\Sigma^1(z), \Sigma^2(z)$. In terms of the $Y(z), H(z)$ -scalars, they can be written as:

$$\begin{aligned} \Sigma^1(z) &= e^{\frac{i}{2}Y(z) + i\frac{1}{\sqrt{2}}H(z)}, \\ \Sigma^2(z) &= e^{\frac{i}{2}Y(z) - i\frac{1}{\sqrt{2}}H(z)}. \end{aligned} \quad (3.36)$$

The fermionization of $Y(z)$ provides the 2 real fermions of the free $\hat{c} = 2$ system. The generalization of the GSO projection ensuring the well-defined action of both supersymmetry currents on the states (3.35) requires:

$$q + s_0 + s_1 + r + 2Q \in 2\mathbb{Z} \quad , \quad 2Q \in \mathbb{Z}. \quad (3.37)$$

The $N_L = 4$ spectral-flow is similarly [19]:

$$\begin{aligned} J_n^3 &\rightarrow J_n^3 - \alpha\delta_{n,0}, \\ L_n &\rightarrow L_n - 2\alpha J_n^3 + \alpha^2\delta_{n,0}. \end{aligned} \quad (3.38)$$

We consider now the embedding of the left-moving spin connection of Type II into the ‘bosonic’ sector of the Heterotic string. The analysis is straightforward and parallel to the $N_L = 2$ case of the previous section. To this end, we build the spectral flow current as in (3.5). We will extend the $SO(10)$ current algebra of complex fermions Ψ^A (where $A = 1, \dots, 5$) with the free complex fermion $\Psi^6(z) \equiv e^{iY(z)}$ of the $\hat{c} = 2$ system so that Ψ^A will be, henceforth, taken to generate an $SO(12)_{k=1}$ current algebra. Hence, the GSO projection of (3.7) is carried through to the present case without modification. The invariant $(1, 0)$ -currents are then:

$$\Psi^A \Psi^B(z) \quad , \quad C_{12} e^{\pm i\frac{1}{\sqrt{2}}H}(z) \quad , \quad J^3(z) \quad , \quad J^\pm(z), \quad (3.39)$$

where $J^\pm(z) = e^{\pm i\sqrt{2}H(z)}$ and $I = 1, \dots, 6$. The fermion bilinears transform as the adjoint **66** of $SO(12)$. Similarly, C_{12} is charged under the conjugate spinorial **32**.

Together with the three $SU(2)_{k=1}$ currents J^3, J^\pm , which are $SO(12)$ -singlets, the above currents form the adjoint representation **133** of E_7 .

As before, it is convenient to display the spectrum and the spectral flow explicitly in a concrete example. To this end, we consider the $\mathcal{N}_4 = 2$ model (2.41) with enhanced $E_7 \times SU(2) \times E_8$ gauge symmetry, presented in Section 2.4. This model arises via a Gepner map from an $\mathcal{N}_4 = 4$ Type II compactification on $T^2 \times T^4/\mathbb{Z}_2$, in which 2 spacetime supersymmetries arise from each of the left- and right-moving sectors. It corresponds precisely to an $N = (4, 4) \oplus (2, 2)$ compactification. In terms of covariant conformal blocks, the Gepner map is realized as:

$$\frac{1}{2} \sum_{a,b} (-)^{a+b+ab} \theta_{[b]}^{[a]} \theta_{[b+g]}^{[a+h]} \theta_{[b-g]}^{[a-h]} \rightarrow \frac{1}{2} \sum_{k,\ell} \theta_{[\ell]}^{[k]} \theta_{[\ell+g]}^{[k+h]} \theta_{[\ell-g]}^{[k-h]}. \quad (3.40)$$

In the free-field description, the $\hat{c} = 4$ system realizing the $N_L = 4$ SCFT, is built out of 2 complex fermions $\Psi^{7,8}$ which are bosonized as $e^{\pm iH^j(z)}$, with $j = 7, 8$. The spectral-flow currents are then constructed out of the free fields as:

$$C_{12}(z) e^{\pm \frac{i}{2}(H^7(z) - H^8(z))} \propto e^{\pm i \frac{1}{\sqrt{2}} H(z)}, \quad (3.41)$$

where we emphasize above that, by the properties of $N_L = 4$ SCFT, the spectral-flow current has to carry $\pm \frac{1}{2}$ units of J^3 -charge.

As before, we perform the linear field redefinition:

$$\begin{aligned} Y(z) &= H^6(z) \\ X(z) &= (H^7(z) + H^8(z))/\sqrt{2} \\ H(z) &= (H^7(z) - H^8(z))/\sqrt{2}, \end{aligned} \quad (3.42)$$

where the scalars Y, X, H are still free and, in particular, $i\partial Y(z)$ is identified with the $U(1)$ -charge of the $\hat{c} = 2$ system, while $J(z) = \frac{i}{\sqrt{2}} \partial H(z)$ is the Cartan charge of the $SU(2)_{k=1}$ current algebra of the $\hat{c} = 4, N_L = 4$ system.

We are now in the position to consider the action of the spectral-flow in the untwisted scalar spectrum. The starting point is the vertex operator in the vectorial representation of $SO(12)$:

$$\Psi^A(z) e^{\pm iH^j(z)} e^{-\bar{\phi}(\bar{z}) \pm i\bar{H}_k(\bar{z})} \propto e^{\pm i\epsilon_j \frac{1}{\sqrt{2}} H(z)}, \quad (3.43)$$

where $j = 7, 8$ and $k = 3, 4$. The ϵ_j are defined as $\epsilon_7 = 1$ and $\epsilon_8 = -1$. We now shift the $SU(2)_{k=1}$ charge Q by $\pm \frac{1}{2}$ units in order to make it vanish¹¹. For concreteness, take $j = 7$:

$$e^{\pm iH^7(z)} \rightarrow e^{\pm \frac{i}{2}(H^7(z) + H^8(z))}. \quad (3.44)$$

¹¹Note that, for an $SU(2)_k$ affine algebra, only the integrable representations $|Q| \leq k/2$ are unitary.

The flow then takes the $V_{12}V_4$ representation (3.43) into the $S_{12}S_4$ representation in the R-sector of the $N_L = 4$ SCFT, with vertex operator:

$$S_{12}(z)e^{\pm\frac{i}{2}(H^7(z)+H^8(z))}e^{-\bar{\phi}(\bar{z})\pm i\bar{H}_k(\bar{z})}. \quad (3.45)$$

Again, the transformation can be verified straightforwardly by considering the action of the spectral-flow current (3.41) on the vertex operator (3.43).

We now focus our attention to the twisted fermionic massless spectrum, which is relevant for the Spinor-Vector duality map. As in the $\mathcal{N}_4 = 1$ case, we start from the states transforming in the vectorial representation of $SO(12)$. The vertex operator is:

$$\Psi^A e^{\pm\frac{i}{2}(H^7(z)-H^8(z))}\Gamma_{(+)}^1(z, \bar{z})e^{-\frac{1}{2}\bar{\phi}(\bar{z})+\frac{i}{2}\bar{H}_0(\bar{z})\pm\frac{i}{2}(\bar{H}_1(\bar{z})+\bar{H}_2(\bar{z}))}, \quad (3.46)$$

where again $A = 1, \dots, 6$. It involves the invariant twist-field contribution $\Gamma_{(+)}^{h=1}(z, \bar{z})$, which starts with conformal weight $(\frac{1}{4}, \frac{1}{4})$ and is associated to the topological contribution of the $h = 1$ -twisted $\Gamma_{(4,4)}$ -lattice with definite (positive) \mathbb{Z}_2 -parity:

$$\Gamma_{(s)}^{h=1} = \frac{1}{2^2\eta^4\bar{\eta}^4} \sum_{g=0,1} \sum_{\gamma,\delta=0,1} (-)^{\binom{1-s}{2}g} \theta_{[\delta]}^{[\gamma]^2} \theta_{[\delta+g]}^{[\gamma+1]^2} \times \bar{\theta}_{[\delta]}^{[\gamma]^2} \bar{\theta}_{[\delta+g]}^{[\gamma+1]^2}. \quad (3.47)$$

Under the spectral flow, the $V_{12}C_4O_{16}$ representation (3.46) will be mapped into the spinorial $S_{12}O_4O_{16}$. This can be seen by shifting the $SU(2)_{k=1}$ charge by $\delta Q = \mp\frac{1}{4}$ units so that it vanishes:

$$e^{\pm\frac{i}{2}(H^7(z)-H^8(z))} \rightarrow \mathbf{1}(z), \quad (3.48)$$

where by the identity operator $\mathbf{1}(z)$, we imply not only the vacuum representation but also its higher excitations (with even 2d fermion parity). Together, they build up the fermionic O_4 -representation and one recovers the vertex operator in the spinorial representation of $SO(12)$:

$$S_{12}(z)\mathbf{1}(z)\Gamma_{(+)}^1(z, \bar{z})e^{-\frac{1}{2}\bar{\phi}(\bar{z})+\frac{i}{2}\bar{H}_0(\bar{z})\pm\frac{i}{2}(\bar{H}_1(\bar{z})+\bar{H}_2(\bar{z}))}. \quad (3.49)$$

We will now carry out the analysis explicitly by constructing the spectral-flow currents in the twisted sector and applying them on the vertex operators of the states. As before, the special \mathbb{Z}_2 -nature of the twist permits us to avoid the twist-field formalism and represent the relevant $\Gamma_{(\pm)}^{h=1}(z, \bar{z})$ contributions entirely via free fermion characters. Performing the summation and projections in (3.47), we find the explicit form of this representation of the twisted vertex operators :

$$\Gamma_{(+)}^{h=1}(z, \bar{z}) = \left\{ \begin{aligned} &O_4S_4\bar{O}_4\bar{S}_4 \oplus O_4C_4\bar{O}_4\bar{C}_4 \oplus V_4S_4\bar{V}_4\bar{S}_4 \oplus V_4C_4\bar{V}_4\bar{C}_4 \\ &\oplus S_4O_4\bar{S}_4\bar{O}_4 \oplus S_4V_4\bar{S}_4\bar{V}_4 \oplus C_4O_4\bar{C}_4\bar{O}_4 \oplus C_4V_4\bar{C}_4\bar{V}_4 \end{aligned} \right\}. \quad (3.50)$$

$$\Gamma_{(-)}^{h=1}(z, \bar{z}) = \left\{ \begin{aligned} &O_4 S_4 \bar{V}_4 \bar{C}_4 \oplus O_4 C_4 \bar{V}_4 \bar{S}_4 \oplus V_4 S_4 \bar{O}_4 \bar{C}_4 \oplus V_4 C_4 \bar{O}_4 \bar{S}_4 \\ &\oplus S_4 O_4 \bar{C}_4 \bar{V}_4 \oplus S_4 V_4 \bar{C}_4 \bar{O}_4 \oplus C_4 O_4 \bar{S}_4 \bar{V}_4 \oplus C_4 V_4 \bar{S}_4 \bar{O}_4 \end{aligned} \right\}. \quad (3.51)$$

Out of these, only the twisted ‘ground states’ with conformal weight $(*, \frac{1}{4})$ will be considered in the fusion rules, since only they can give rise to massless states.

We are now ready to construct the spectral-flow currents in the twisted sector. As argued in the previous section, the untwisted spectral-flow operators (3.41) become extended by a *chiral* dressing $\Omega_{(\pm)}^I(z)$, with $I = 1, 2$, of conformal weight $\Delta_{1,(\pm)} = (\frac{1}{2}, 0)$, $\Delta_{2,(\pm)} = (1 \mp 1, 0)$, acting on the twist-field contribution $\Gamma_{(\pm)}^{h=1}(z, \bar{z})$. The operator $\Omega_{(\pm)}^I(z)$ transforms as $\Omega_{(\pm)}^I \rightarrow \pm \Omega_{(\pm)}^I$ under the \mathbb{Z}_2 -orbifold. Its action on the twisted ground-state vertex operators relevant for the massless spectrum follows the fusion rule:

$$\begin{aligned} \Omega_{(\pm)}^1(z) \cdot \Gamma_{(r)}^1(w, \bar{w}) &= \frac{\Gamma_{(+)}^1(w, \bar{w})}{(z-w)^{\frac{1}{2} + \frac{1-r}{4}}} + \frac{\Gamma_{(-)}^1(w, \bar{w})}{(z-w)^{\frac{1-r}{4}}} + \dots \\ \Omega_{(+)}^2(z) \cdot \Gamma_{(r)}^1(w, \bar{w}) &= \Gamma_{(r)}^1(w, \bar{w}) + \dots \\ \Omega_{(-)}^2(z) \cdot \Gamma_{(r)}^1(w, \bar{w}) &= \frac{\Gamma_{(-r)}^1(w, \bar{w})}{(z-w)^{1-\frac{r}{2}}} + \dots \end{aligned} \quad (3.52)$$

Again, in the interest of notational simplicity, we suppress the representation indices and the Dirac matrices of the transformation. The ellipsis denotes, as usual, less singular terms. In terms of free bosonic fields, $\Omega_{(\pm)}^I(z)$ can be represented as:

$$\begin{aligned} \Omega_{(\alpha)}^1(z) &= e^{\pm \frac{i}{2}(\Phi_1 - \alpha \Phi_2) \pm \frac{i}{2}(\Phi_3 - \alpha \Phi_4)}, \\ \Omega_{(+)}^2(z) &= \mathbf{1}(z), \\ \Omega_{(-)}^2(z) &= e^{\pm i(\frac{1+r}{2})\Phi_1 \pm i(\frac{1-r}{2})\Phi_2 \pm i(\frac{1+s}{2})\Phi_3 \pm i(\frac{1-s}{2})\Phi_4} \end{aligned} \quad (3.53)$$

where the \pm -signs are arbitrary and independent, $\alpha = \pm 1$ is the \mathbb{Z}_2 -parity of the operator and $r, s = \pm 1$. There are two invariant spectral-flow operators responsible for the mapping between the various representations. They are given as the zero mode of the invariant current:

$$Q_{\text{s.f.}}^I = \oint \frac{dz}{2\pi i} \left[z^{1-I/2} (C_{12} C_4)(z) \Omega_{(+)}^I(z) + z^{I/2} (S_{12} S_4)(z) \Omega_{(-)}^I(z) \right], \quad (3.54)$$

or explicitly:

$$\begin{aligned} Q_{\text{s.f.}}^1 &= \oint \frac{dz}{2\pi i} z^{1/2} (C_{12} C_4 C_4 C_4 + S_{12} S_4 S_4 S_4)(z), \\ Q_{\text{s.f.}}^2 &= \oint \frac{dz}{2\pi i} (C_{12} C_4 \mathbf{1}_4 \mathbf{1}_4 + z S_{12} S_4 V_4 V_4)(z). \end{aligned} \quad (3.55)$$

where in the above we made use of the explicit spin-field representations (3.53).

It is now straightforward to consider the action of the spectral-flow charge on the massless $S_{12}O_4O_{16}$ -spinorial representation:

$$Q_{\text{s.f.}} \cdot \left[S_{12}O_4 \Gamma_{(+)}^1 e^{-\frac{1}{2}\bar{\phi} + \frac{i}{2}\bar{H}_0} \bar{S}_4 \bar{O}_4 \right] |0\rangle = \left[V_{12}C_4 \Gamma_{(+)}^1 + O_{12}S_4 \Gamma_{(-)}^1 \right] e^{-\frac{1}{2}\bar{\phi} + \frac{i}{2}\bar{H}_0} \bar{S}_4 \bar{O}_4 |0\rangle. \quad (3.56)$$

We, thus, see that the spinorial representation of $SO(12)$ is precisely mapped into the vectorial $V_{12}C_4O_{16}$, together with the accompanying singlet $O_{12}S_4O_{16}$. The exact map between the representations is, again, seen to arise from the spectral-flow of the twisted $N_L = 4$ SCFT. The matching of the numbers of massless degrees of freedom is, hence, a byproduct of the spectral-flow map:

Spinorial	$S_{12}O_4 \rightarrow 2^{6-1} \times (4 \times 4)$	(3.57)
Vectorial	$V_{12}C_4 \rightarrow 12 \times 2 \times (4 \times 4)$	
Singlet	$O_{12}S_4 \rightarrow 1 \times 2 \times (8 \times 2 \times 4)$	

As was the case in the previous section, here as well the spectral-flow is responsible for a number of supersymmetric-like identities, realized internally in the left-moving sector. In the present case, however, the analogous identities are not those of ‘conventional’ supersymmetry, but rather exhibit the precise degeneracy structure of MSDS constructions [14], [15]. The reason for this is that the relevant orbifold block:

$$Z_{[g]}^{[1]} = \frac{1}{2^2 \eta^{12} \bar{\eta}^4} \left[\sum_{\ell=0,1} (-)^\ell \theta_{[\ell]}^{[k]6} \theta_{[\ell+g]}^{[k+1]2} \right] \left[\sum_{\gamma,\delta=0,1} \theta_{[\delta]}^{[\gamma]2} \theta_{[\delta+g]}^{[\gamma+1]2} \bar{\theta}_{[\delta]}^{[\gamma]2} \bar{\theta}_{[\delta+g]}^{[\gamma+1]2} \right] \quad (3.58)$$

corresponds to boundary conditions for the free fields that satisfy the conditions [15] for the MSDS spectral-flow operator to have a well-defined action on the spectrum. Indeed, the operator (3.55) is exactly the invariant $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ -truncation of the maximal MSDS charge, whose vertex operator is proportional to the spin-field $C_{24}(z)$ of $SO(24)$. The spectral-flow is, hence, identified with the spectral-flow of constructions with MSDS structure, the only difference being that the ‘equal’ characters between ‘vectors’ and ‘spinors’ are, again, summed rather than subtracted.

Let us exhibit these identities in a systematic way. Let us pick the contribution from the $Z_{(+)}^{h=1}$ -orbifold block of positive \mathbb{Z}_2 -parity, which couples to the P_0 twisted right-moving characters and which is the only $h = 1$ sector which contains massless fermions:

$$Z_{(+)}^{h=1}(\epsilon) = (O_{12}S_4 + \epsilon C_{12}V_4) \Gamma_{(-)}^{h=1} + (V_{12}C_4 + \epsilon S_{12}O_4) \Gamma_{(+)}^{h=1}. \quad (3.59)$$

Here we explicitly kept the dependence on $\epsilon = \pm 1$, which distinguishes the cases where local worldsheet supersymmetry is present ($\epsilon = -1$) or absent ($\epsilon = +1$). As was argued in the previous section, the two cases are characterized by the fact that only

from the sector with local worldsheet supersymmetry can spacetime fermions arise. In our case, where $N_L = 4$ is embedded inside the ‘bosonic’ sector of the Heterotic string, there are no cancellations between the vectorial and spinorial contributions to the partition function. However, it will be instructive to explicitly display the identities in the $\epsilon = -1$ case, which illustrate the spectral-flow of the representations in a particularly clear way and, at the same time, may give rise to considerable algebraic simplifications, as will be shown below.

The direct evidence of an MSDS spectral-flow at work can be obtained straightforwardly by calculating the $\epsilon = -1$ contributions that couple to the various right-moving lattice pieces $\bar{V}_4\bar{C}_4, \bar{V}_4\bar{S}_4, \dots \in \Gamma_{(+)}^{h=1}$:

$$\begin{aligned}
\bar{V}_4\bar{C}_4 \cdot \left[(O_{12}S_4 - C_{12}V_4) O_4S_4 + (V_{12}C_4 - S_{12}O_4) V_4C_4 \right] &= 4\bar{V}_4\bar{C}_4 , \\
\bar{V}_4\bar{S}_4 \cdot \left[(O_{12}S_4 - C_{12}V_4) O_4C_4 + (V_{12}C_4 - S_{12}O_4) V_4S_4 \right] &= 4\bar{V}_4\bar{S}_4 , \\
\bar{O}_4\bar{C}_4 \cdot \left[(O_{12}S_4 - C_{12}V_4) V_4S_4 + (V_{12}C_4 - S_{12}O_4) O_4C_4 \right] &= 0 , \\
\bar{O}_4\bar{S}_4 \cdot \left[(O_{12}S_4 - C_{12}V_4) V_4C_4 + (V_{12}C_4 - S_{12}O_4) O_4S_4 \right] &= 0 , \\
\bar{C}_4\bar{V}_4 \cdot \left[(O_{12}S_4 - C_{12}V_4) S_4O_4 + (V_{12}C_4 - S_{12}O_4) C_4V_4 \right] &= 4\bar{C}_4\bar{V}_4 , \\
\bar{S}_4\bar{V}_4 \cdot \left[(O_{12}S_4 - C_{12}V_4) C_4O_4 + (V_{12}C_4 - S_{12}O_4) S_4V_4 \right] &= 4\bar{S}_4\bar{V}_4 , \\
\bar{S}_4\bar{O}_4 \cdot \left[(O_{12}S_4 - C_{12}V_4) C_4O_4 + (V_{12}C_4 - S_{12}O_4) S_4O_4 \right] &= 0 , \\
\bar{C}_4\bar{O}_4 \cdot \left[(O_{12}S_4 - C_{12}V_4) S_4V_4 + (V_{12}C_4 - S_{12}O_4) C_4O_4 \right] &= 0 . \tag{3.60}
\end{aligned}$$

Of course, if we restrict our attention to the massless spectrum, only the weight- $(*, \frac{1}{4})$ contributions in $Z_{(+)}^{h=1}$ are relevant and the corresponding identities are the ‘supersymmetric-like’ ones, with a vanishing r.h.s. as shown above. It is, nevertheless, possible to utilize the above identities in the form:

$$V_{12}C_4\Gamma_{(+)}^{h=1} + O_{12}S_4\Gamma_{(-)}^{h=1} = S_{12}O_4\Gamma_{(+)}^{h=1} + C_{12}V_4\Gamma_{(-)}^{h=1} + 4\bar{\mathcal{Z}}(\bar{q}), \tag{3.61}$$

which permits algebraic simplifications in the partition function. Here, the right-moving contribution $\bar{\mathcal{Z}}(\bar{q})$ corresponds to effectively spurious modes and is eliminated by imposing level matching, or by integration of the Teichmüller parameter.

This concludes our analysis of the twisted $N_L = 4$ case. It becomes clear how the duality between $SO(12)$ spinors and vectors is a direct consequence of the spectral-flow of the $N_L = 4$ SCFT. A number of chiral identities between current algebra characters illustrate this spectral-flow and can, in some cases, be utilized to algebraically simplify the partition function.

4 Narain lattices

Partition functions offer a very compact way to encode the spectrum of a conformal field theory or string compactification. Some of the phenomena discussed in the previous sections become more intuitive if we use a more explicit description provided by the relation between states and the Narain lattice [21]. For toroidal compactifications the Narain lattice encodes the possible winding, momentum and gauge charges as a function of the background fields [22]. While twists which can be realized as pure shifts of the Narain lattice just relate one toroidal compactification to another, more general twists lead to orbifold models with reduced supersymmetry and reduced gauge groups. As reviewed in Section 2 the orbifold partition function depends on lattices obtained from the original Narain lattices by shifting and projection onto invariant states. However, as long as all gauge symmetries come from the untwisted sector, the twist invariant part of the Narain lattice contains the full information about symmetry enhancement and symmetry breaking. This allows to discuss the continuous interpolation between models in a very explicit way.

4.1 Review of Narain lattices

Modular invariance implies that the Narain lattice $\Gamma = \Gamma_{(22,6)}$ underlying a toroidal compactification of the heterotic string to four dimensions must be an even self-dual lattice with respect to the quadratic form of type $(+)^{22}(-)^6$. All such lattices form a single continuous family and can be deformed into one another by $SO(22,6)$ transformations. String vacua related by $SO(22) \times SO(6)$ transformations are equivalent, so that the moduli space locally takes the form

$$\frac{SO(22,6)}{SO(22) \times SO(6)} .$$

The following standard basis of the Narain lattice provides an explicit parametrization of this moduli space in terms of the background fields $G_{IJ}, B_{IJ}, A_I = (A_I^a)$, $I = 1, \dots, 6$, $a = 1, \dots, 16$ [23]:

$$\begin{aligned} \bar{k}^I &= \left(0, \frac{1}{2}e^{*I}; \frac{1}{2}e^{*I} \right) , \\ k_I &= \left(A_I, e_I + B_{IJ}e^{*J} - \left(\frac{1}{4}A_I \cdot A_J\right)e^{*J}; -e_I + B_{IJ}e^{*J} - \left(\frac{1}{4}A_I \cdot A_J\right)e^{*J} \right) , \\ l_a &= \left(\alpha_a, -(\alpha_a \cdot A_K)\frac{1}{2}e^{*K}; -(\alpha_a \cdot A_K)\frac{1}{2}e^{*K} \right) . \end{aligned} \quad (4.1)$$

Here $\{e_I\}$ is a basis of the compactification lattice Λ , $\{e^{*I}\}$ is the dual basis of the dual lattice Λ^* , and α_a , $a = 1, \dots, 16$ are a set of simple roots of $E_8 \times E_8$. The basis vectors have the following mutual scalar products:

$$\bar{k}^I \cdot k_J = \delta_J^I , \quad l_a \cdot l_b = C_{ab} ,$$

where C_{ab} is the Cartan matrix of $E_8 \times E_8$, and where all scalar products which are not displayed are zero. The background fields parametrizing the moduli space are the lattice metric $G_{IJ} = e_I \cdot e_J$ of Λ , the antisymmetric tensor field B_{IJ} and the Wilson lines $A_I = (A_I^a) \in \mathbb{R}^{16}$. We will take the lattice Λ to be generic throughout, and for simplicity we will only consider models with vanishing B -field, $B_{IJ} = 0$. The integer expansion coefficients n_I, m^I, Q^a of a Narain vector $v \in \Gamma_{(22,6)}$ with respect to this basis

$$v = n_I \bar{k}^I + m^I k_I + Q^a l_a ,$$

are the momentum, winding and gauge quantum numbers of the corresponding states.

If we switch off the Wilson lines, $A_I = 0$, the lattice basis takes the simple form

$$\begin{aligned} \bar{k}^I &= \left(0, \frac{1}{2} e^{*I}; \frac{1}{2} e^{*I} \right) , \\ k_I &= (0, e_I; -e_I) , \\ l_a &= (\alpha_a, 0_6; 0_6) . \end{aligned} \tag{4.2}$$

On the special locus $A_I = 0$ the Narain lattice factorizes, $\Gamma = \Gamma_{16} \Gamma_{(6,6)}$ and the generic gauge symmetry $U(1)^{28}$ is enhanced to $E_8 \times E_8 \times U(1)^{12}$. Non-abelian gauge symmetries are identified by looking for Narain vectors of the purely left-moving form $(p_L; 0_6)$, with $p_L^2 = 2$. Such vectors automatically form the root system of a semi-simple ADE-type Lie algebra.

Orbifold twists are often defined with respect to a particular subspace of the Narain moduli space. One of the standard constructions, which we use for the \mathbb{Z}_2 orbifold, is to combine an automorphism (rotation or reflection) $\theta_{(6)}$ of the compactification lattice Λ with a shift $\delta_{(16)}$ in the $E_8 \times E_8$ root lattice. Modular invariance imposes constraints on the allowed pairs $(\theta_{(6)}, \delta_{(16)})$. The induced action on the Narain lattice is obvious as long as Wilson lines are switched off, $A_I = 0$, so that the lattice takes the factorized form $\Gamma = \Gamma_{16} \Gamma_{(6,6)}$. In particular, the ‘gauge twist’ $\delta_{(16)}$ acts on Γ by the trivially extended shift vector

$$\delta = (\delta_{(16)}, 0_6; 0_6) . \tag{4.3}$$

It is clear that this vector is no longer an admissible shift vector for deformed Narain lattices obtained by switching on Wilson lines. The reason is that a shift vector of order N must have the property that $N\delta \in \Gamma$, while $k\delta \notin \Gamma$ for $0 < k < N$. To check whether any given vector is in Γ , we only need to check whether its scalar product with the basis vectors is integer, because Γ is self-dual. Assuming that $N\delta$, with δ given by (4.3) is in the lattice generated by (4.2) it is clear that it cannot be in the lattice generated by the deformed basis (4.1), except possibly for special values of the A_i . However, it is easy to see that the gauge twist $\delta_{(16)}$ is consistent with the most general continuous Wilson lines. We just have to modify the extended shift vector (4.3) by applying to it the same $SO(22, 6)$ boost which relates the two bases (4.2)

and (4.1):

$$\delta = \left(\delta_{(16)}, -(\delta \cdot A_K) \frac{1}{2} e^{*K}; -(\delta \cdot A_K) \frac{1}{2} e^{*K} \right). \quad (4.4)$$

To check that (4.4) is an admissible shift vectors for all values A_I of the Wilson lines, we note that since $\delta_{(16)}$ is an admissible shift of Γ_{16} ,

$$N\delta_{(16)} = \sum_{a=1}^{16} Q^a \alpha_a,$$

where $Q^a \in \mathbb{Z}$, and where the simple roots α_a of $E_8 \times E_8$ form a lattice basis of Γ_{16} . For $A_I = 0$, it follows that

$$N\delta = (\delta_{(16)}, 0_6; 0_6) = \sum_{a=1}^{16} Q^a l_a.$$

If we switch on the most general Wilson lines, the basis vectors are deformed according to (4.1), and the resulting deformed shift vector is indeed (4.4). Thus a gauge twist acting as a pure shift does not restrict the allowed values of the Wilson lines, which in particular remain continuous.

The \mathbb{Z}_2 orbifold considered in Section 2 combines an order 2 automorphism $\theta_{(6)}$ of the compactification lattice Λ , with an order 2 shift of the gauge lattice Γ_{16} , which corresponds to the standard embedding (of the spin into the gauge connection). Since $\theta_{(6)}$ acts as identity on two directions and as a reflection on the other four, the underlying torus must factorize as $T^2 \times T^4$. The action of the twist on the gauge lattices is by a pure shift, and therefore we can switch on Wilson lines along the T^2 , which can have arbitrary continuous values, as discussed above. In other words the \mathbb{Z}_2 twist acts consistently on all Narain lattices of the form

$$\Gamma = \Gamma_{(16+2,2)} \Gamma_{(4,4)},$$

where $\Gamma_{(16+2,2)}$ combines the T^2 and gauge degrees of freedom, whereas $\Gamma_{(4,4)}$ is the momentum/winding lattice of the T^4 . It is manifest that the (untwisted) moduli space of this family of orbifold models is given by (2.14). This demonstrates how the same conclusion can be reached using either the partition function or the Narain lattice, and illustrates how a simpler, more geometric description arises by using the Narain lattice.

4.2 Gauge symmetries and continuous Wilson lines

Another advantage of the Narain lattice is that it is very easy to trace patterns of symmetry enhancement and symmetry breaking. For toroidal models one has complete control of the possible non-abelian gauge symmetries, and for orbifold models

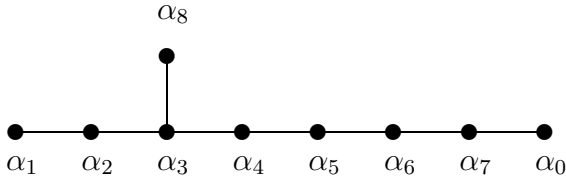


Figure 1: The extended Dynkin diagram of E_8 .

the same is true at least for the untwisted sector which is obtained from the underlying toroidal model by projection onto invariant states. While in general the contributions of twisted sectors need to be investigated explicitly, one can verify, using the partition function, for the orbifolds considered in this article that no gauge symmetry enhancements can arise from the twisted sectors. Therefore we focus on the projected, untwisted sector and on the underlying toroidal model in the following.

Let us first review how the breaking of $E_8 \times E_8$ by Wilson lines can be controlled and parametrized, following [25]. For simplicity we only consider one E_8 factor, with simple roots α_a , $a = 1 \dots, 8$. There is a well known algorithm for constructing successively the maximal (regular) subalgebras of a simple Lie algebra, which works by successively removing dots from the extended Dynkin diagram [24]. The extended Dynkin diagram of E_8 is obtained (as for any simple Lie algebra) by adding the lowest root

$$\alpha_0 = - \sum_{a=1}^8 k^a \alpha_a ,$$

to the Dynkin diagram. The coefficients $(k^a) = (2, 4, 6, 5, 4, 3, 2, 3)$, together with $k^0 = 1$ are known as the Kac labels. The extended Dynkin diagram of E_8 is displayed in Figure 1. For concreteness, we specify an explicit choice of simple roots for E_8 together with the resulting lowest root:

$$\begin{aligned}
 \alpha_1 &= \left(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} \right) , \\
 \alpha_2 &= (0, 0, 0, 0, 0, 1, 0, 1) , \\
 \alpha_3 &= (0, 0, 0, 0, 1, -1, 0, 0) , \\
 \alpha_4 &= (0, 0, 0, 0, -1, 0, 1, 0) , \\
 \alpha_5 &= (0, 0, 0, 1, 0, 0, -1, 0) , \\
 \alpha_6 &= (0, 0, 1, -1, 0, 0, 0, 0) , \\
 \alpha_7 &= (0, 1, -1, 0, 0, 0, 0, 0) , \\
 \alpha_8 &= (0, 0, 0, 0, 0, 1, 0, -1) , \\
 \alpha_0 &= (1, -1, 0, 0, 0, 0, 0, 0) .
 \end{aligned} \tag{4.5}$$

The breaking of any of the two E_8 groups in Narain models can be controlled by

monitoring which of the Narain vectors corresponding to simple roots are ‘projected out.’ Since

$$(A_I, *_{6}; *_{6}) \cdot (\alpha_a, 0_6; 0_6) = A_I \cdot \alpha_a ,$$

it is clear that a massless boson with charges corresponding to α_a is present in the spectrum if and only if $A_I \cdot \alpha_a \in \mathbb{Z}$. If this condition is violated by changing the Wilson lines continuously, then this state acquires a mass, controlled by the Wilson lines, through the Higgs mechanism.

As an example we consider the breaking of E_8 to $E_7 \times U(1)$ and $E_7 \times SU(2)$. By inspection of Figure 1, the removal of the dot corresponding to α_7 results in the Dynkin diagram of $E_7 \times SU(2)$, while removing in addition the dot corresponding to α_0 gives the Dynkin diagram of E_7 . ‘Removing dots’ can be implemented by choosing Wilson lines which have non-integer scalar products with the dots one wants to remove but integer scalar products with the dots one wants to keep [25]. For concreteness, consider switching on a Wilson line of the form $A_1 = \lambda \alpha_7^*$, where λ is a continuous parameter, and where α_7^* is the dual of the seventh root (= the seventh fundamental weight),

$$\alpha_a^* \cdot \alpha_b = \delta_{ab} .$$

In the representation we have chosen

$$\alpha_7^* = (-1, 1, 0, 0, 0, 0, 0, 0) .$$

Depending on the value of λ , there are the following three cases:

1. If $\lambda \in \mathbb{Z}$, then the Wilson line A_1 has integer scalar product with all simple roots of E_8 , and the gauge symmetry is not reduced. For $\lambda \neq 0$ this corresponds to a T-duality transformation, where the Narain lattice is mapped to itself.¹²
2. If $\lambda = \frac{1}{2} \bmod \mathbb{Z}$, then the Wilson line has an integer scalar product with all simple E_8 roots, except with α_7 , and the scalar product with the lowest root α_0 is integer. The resulting root system corresponds to $E_7 \times SU(2)$, which shows that these discrete Wilson lines break E_8 to this maximal subgroup.
3. If λ is not an integer multiple of $\frac{1}{2}$, then the Wilson line has integer scalar product with all simple roots of E_8 except α_7 , and the scalar product with the lowest root α_0 is not integer. Since the number of Cartan generators is not reduced, the unbroken subgroup is $E_7 \times U(1)$.

Thus the continuous Wilson line $A_1 = \lambda \alpha_7^*$ generically breaks E_8 to $E_7 \times U(1)$, but at special values this is re-enhanced to the maximal subgroup $E_7 \times SU(2)$. Moreover $\lambda \simeq \lambda + 1$ by T-duality.

¹²The T-duality group $SO(22, 6, \mathbb{Z})$ consists precisely of those isometries that act automorphically on the lattice, and thus can be ‘undone’ by a basis transformation.

Discrete Wilson lines, where a multiple of the Wilson line lies in Γ_{16} , can be re-interpreted as orbifolds acting by pure shifts. For the case at hand, note that the discrete Wilson line

$$A_1 = -\frac{1}{2}\alpha_7^* = \left(\frac{1}{2}, -\frac{1}{2}, 0, 0, 0, 0, 0, 0 \right),$$

which breaks E_8 to $E_7 \times SU(2)$, can be re-interpreted as a shift vector. Indeed, $\delta_{(16)} = -\frac{1}{2}\alpha_7^*$ is an admissible shift vector of order 2, because $2\delta_{(16)} \in \Gamma_{16}$. Note that $\alpha_7^* \in \Gamma_{16}$, because the E_8 root lattice is selfdual. This is an explicit example where a pure shift can be re-interpreted as a discrete change of background fields. Incidentally, the shift $\delta_{(16)} = -\frac{1}{2}\alpha_7^*$ is the bosonic version of the standard embedding gauge twist (2.8) of the \mathbb{Z}_2 orbifold. Note, however, that if an orbifold acts by a combination of an automorphism of Λ with a gauge shift, it need not be true any more that the gauge shift can be replaced by a discrete background field. In particular, once the shift $\delta_{(16)} = -\frac{1}{2}\alpha_7^*$ becomes part of the definition of the \mathbb{Z}_2 orbifold it cannot be replaced by a discrete Wilson line any more. However, the \mathbb{Z}_2 orbifold can be re-interpreted in terms of background fields, because it acts by a pure shift.

Let us next investigate the symmetry breaking patterns of the \mathbb{Z}_2 orbifold model. As far as gauge symmetry breaking is concerned, the shift

$$\delta_{(16)} = -\frac{1}{2}\alpha_7^* = \left(\frac{1}{2}, -\frac{1}{2}, 0, 0, 0, 0, 0, 0, 0, \dots, 0 \right) \quad (4.6)$$

has the same effect as the Wilson line discussed above and breaks $E_8 \times E_8$ to $E_7 \times SU(2) \times E_8$. This is clear because at the $E_8 \times E_8$ point the invariant states of the untwisted sector correspond to Narain lattice vectors which have integer scalar products with $(\delta_{(16)}, 0, \dots, 0)$. We will now investigate what happens to this sector if we switch on a Wilson line, which we parametrize as

$$A_1 = (a_1, a_2 | b_1, b_2, b_3, b_4, b_5, b_6 | c_1, c_2, \dots, c_8).$$

The symmetry breaking of the second E_8 by the c_1, \dots, c_8 follows the same pattern as in toroidal models. Therefore we will focus on the first E_8 and frequently suppress the vector components corresponding to the second E_8 . The explicit basis (4.5) is not adapted to the subgroup $E_7 \times SU(2)$ unbroken by the twist, but instead to the maximal subgroup $SO(16)$, since all roots are chosen to be either adjoint weights of $SO(16)$, $(\alpha_0, \alpha_2, \dots, \alpha_8)$ or conjugate spinor weights (α_1) . It is convenient to further decompose the E_8 roots with respect to the subgroup

$$SO(4) \times SO(12) \subset SO(16) \subset E_8.$$

As we will see explicitly below, using $SO(4) \simeq SU(2) \times SU(2)$, this allows us to fit states into representations of $E_7 \times SU(2)$ through the chain of subgroups

$$SU(2) \times SU(2) \times SO(12) \subset SU(2) \times E_7 \subset E_8.$$

In terms of $SO(16)$ weights, the 240 roots of E_8 are obtained by combining the 112 roots of $SO(16)$

$$(0 \cdots \pm 1 \cdots \pm 1 \cdots 0)$$

(where precisely two entries are non-vanishing and take values ± 1) and the 128 conjugate spinor weights

$$\underbrace{\left(\pm \frac{1}{2}, \pm \frac{1}{2}, \cdots, \pm \frac{1}{2}\right)}_{\text{even}}.$$

(where the number of $(-)$ -signs is even). In the untwisted sector of the orbifold, all vectors which do not have integer scalar products with the shift vector (4.6) are projected out. Out of the 240 E_8 root vectors the following 128 vectors survive the projection,

$$\begin{aligned} & (\pm 1, \pm 1 | 0, 0, 0, 0, 0, 0), \\ & (0, 0, | \cdots \pm 1 \cdots \pm 1 \cdots), \\ & \left(\underbrace{\pm \frac{1}{2}, \pm \frac{1}{2}}_{\text{even}} \mid \underbrace{\pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}}_{\text{even}} \right), \end{aligned} \quad (4.7)$$

and it is easy to see that they span the $(0, 0)$ and (c, c) conjugacy classes of $SO(4) \times SO(12)$, where (0) denotes the class of the adjoint and (c) denotes the class of the conjugate spinor representation. Taking into account that none of the E_8 Cartan generators is projected out, we have precisely the right number of states to fill the adjoint representation $(3, 1) \oplus (1, 133)$ of $SU(2) \times E_7$. The weight vectors of this representation can be seen explicitly when performing a rotation by 45 degree in the first two entries, which corresponds to the isomorphism $SO(4) \rightarrow SU(2) \times SU(2)$:

Weights	$SU(2) \times SU(2) \times SO(12)$	$SU(2) \times E_7$
$(\pm\sqrt{2}, 0, 0, 0, 0, 0, 0, 0)$	$(3, 1, 1)$	$(3, 1)$
$(0, \pm\sqrt{2}, 0, 0, 0, 0, 0, 0)$	$(1, 3, 1)$	$(1, 133)$
$(0, 0, \cdots \pm 1 \cdots \pm 1)$	$(1, 1, 66)$	
$(0, \pm\frac{1}{2}\sqrt{2}, \underbrace{\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}}_{\text{even}})$	$(1, 2, 32)$	

With this completely explicit description of the gauge group at the point of maximal symmetry, we can now easily study the effect of continuous Wilson lines. We parametrize the Wilson line in the rotated basis as

$$\tilde{A}_1 = (\tilde{a}_1, \tilde{a}_2, b_1, b_2, b_3, b_4, b_5, b_6),$$

where

$$\tilde{a}_1 = \frac{1}{2}\sqrt{2}(a_1 - a_2), \quad \tilde{a}_2 = \frac{1}{2}\sqrt{2}(a_1 + a_2).$$

Now we use that a vector corresponds to a massless gauge boson in the untwisted sector of the orbifold if and only if it has an integer scalar product with the Wilson line.¹³ The following cases can occur.

1. If $\sqrt{2}\tilde{a}_1$ is integer, then the first $SU(2)$ factor of $SU(2) \times SU(2) \times SO(12)$ is unbroken. Otherwise it is broken.
2. If $\sqrt{2}\tilde{a}_2$ is integer, then the second $SU(2)$ factor of $SU(2) \times SU(2) \times SO(12)$ is unbroken. Otherwise it is broken. If the second $SU(2)$ is broken there cannot be an unbroken E_7 .
3. If $b = (b_1, b_2, b_3, b_4, b_5, b_6)$ is a weight of $SO(12)$, then the $SO(12)$ subgroup of $SU(2) \times SU(2) \times SO(12)$ is unbroken. Otherwise it is broken and then in particular the E_7 symmetry is broken.
4. If $b = (b_1, b_2, b_3, b_4, b_5, b_6)$ is a weight in the adjoint conjugacy class of $SO(12)$, and if $\sqrt{2}\tilde{a}_2$ is an integer, then the E_7 is unbroken. Note that if we only require b to be a weight of $SO(12)$, this only implies an unbroken $SU(2) \times SO(12)$ subgroup. But if b is an adjoint weight, then the additional weights required to extend the root system to from $SU(2) \times SO(12)$ to E_7 are present, because scalar products between adjoint weights and weights in any conjugacy class are integer valued for ADE-type Lie algebras.
5. We remark that by tuning the Wilson line, the $SO(12)$ factor can be broken to any regular subgroup, in the way described in [25, 26].

As an explicit example, the one-parameter family of Wilson lines

$$A_1(\lambda) = (\lambda, 0|1 - \lambda, 0, \dots, 0|1, 0, \dots, 0), \quad 0 \leq \lambda \leq 1,$$

interpolating between the two discrete Wilson lines (2.47) and (2.48), which correspond to the $\mathbb{Z}_2 \times \mathbb{Z}'_2$ orbifold with discrete torsion $\epsilon = +1$ and $\epsilon = -1$, respectively. Note that we now include the second E_8 . Comparing the last 8 entries of the Wilson line to (4.5) and Figure 1, we see that the second E_8 is broken to $SO(16)$. The gauge shift already breaks the first E_8 to $SU(2) \times E_7$, and to investigate the further breaking by the Wilson line, we rewrite it in the E_7 -adapted basis

$$\tilde{A}_1(\lambda) = \left(\frac{\sqrt{2}}{2}\lambda, \frac{\sqrt{2}}{2}\lambda|1 - \lambda, 0, \dots, 0|1, 0, \dots, 0 \right), \quad 0 \leq \lambda \leq 1.$$

Using the above analysis, we see that for $0 < \lambda < 1$ the subgroup $SU(2) \times SU(2)$ is broken to $U(1) \times U(1)$, while $SO(12)$ is broken to $U(1) \times SO(10)$. For $\lambda = 0, 1$ the group $SU(2) \times SU(2) \times SO(12)$ is unbroken, but the additional weights needed

¹³This was developed systematically in [26].

for enhancement to $SU(2) \times E_7$ are projected out. Thus we have a family of models with gauge group

$$U(1)^3 \times SO(10) \times SO(16), \quad 0 < \delta < 1,$$

which is enhanced to

$$SU(2) \times SU(2) \times SO(12) \times SO(16) \quad \text{for } \delta = 0, 1,$$

which are the two models related by spinor-vector duality.

5 Heterotic $K3 \times T^2$ compactifications

It is well known that the singularities of the orbifold T^4/\mathbb{Z}_2 can be deformed to obtain a smooth K3 surface [20]. To put our results into perspective, we will now review some results about K3 compactifications of the heterotic string [27, 29, 30]. Anomaly cancellation requires that non-trivial gauge fields are switched on along the K3. The precise condition is that the Euler number 24 of K3 is cancelled by the instanton number of the gauge field configuration. Geometrically, this corresponds to the choice of an $E_8 \times E_8$ vector bundle V over K3, with second Chern class $c_2(V) = 24$. Since the gauge group has two simple factors, the gauge bundle is a sum $V_1 \oplus V_2$, and one is free to distribute the total instanton number between the two bundles,

$$c_2(V_1) + c_2(V_2) = 24 = \chi_{K3}.$$

The resulting family of models is parametrized by an integer k ,

$$(c_2(V_1), c_2(V_2)) = (12 + k, 12 - k) \quad k = 0, 1, 2, \dots, 12.$$

By heterotic-type IIA duality, these models are equivalent to compactifications of type-IIA string theory on a family of Calabi-Yau three-folds which are elliptic fibrations over the Hirzebruch surfaces F_k . Moreover the members of the family are related by going through loci of enhanced symmetry, which on the type-II side correspond to singularities of the F_k basis.

One example is the heterotic K3 compactification with standard embedding. Here the spin connection on K3 is identified with an $SU(2)$ subgroup of one of the E_8 , and the unbroken gauge group is the commutant $E_7 \times E_8$ of this subgroup. Together with the abelian factors from further reduction on T^2 this results in a gauge group $E_7 \times E_8 \times U(1)^4$. Since all instantons are valued in the same E_8 -factor, this corresponds to taking $k = 12$ above. Taking all 24 instantons to be valued in the same $SU(2)$ subgroup is a very special choice which leaves the gauge group as large as possible. A generic distribution of the instantons within an E_8 factor breaks it completely.

The gauge group can also be reduced in another way. As in any $N = 2$ gauge theory, one can ‘go to the Coulomb branch’ by turning on generic vacuum expectation

values for the scalars in the four-dimensional vector multiplets. This breaks the gauge group to the maximal abelian subgroup $U(1)^{7+8+4} = U(1)^{19}$, and only neutral hypermultiplets remain massless. The scalars in these hypermultiplets are the moduli of the K3 surface and of the gauge bundle. The (quaternionic) dimensions of these moduli spaces are known to be 20 and 45, respectively, so that this model has a rank 19 abelian gauge group and 65 hypermultiplets on the Coulomb branch.

In contrast, the \mathbb{Z}_2 orbifold model considered in this article is the compactification of the heterotic string on the orbifold limit T^4/\mathbb{Z}_2 of K3 with standard embedding, which was first studied in [28]. At the point of maximal symmetry this results in a model with gauge group $E_7 \times SU(2) \times E_8 \times U(1)^4$ together with charged and neutral hypermultiplets. For orbifolds going to the Coulomb branch corresponds to switching on generic Wilson lines, which breaks the gauge group to $U(1)^{20}$ and makes all charged hypermultiplets massive. The only remaining massless states are the $\mathcal{N}_4 = 2$ gravity multiplet, the dilaton vector multiplet and 18 vector multiplets and 4 hypermultiplets corresponding to the untwisted moduli space (2.14). To relate this model to a smooth K3 compactification, one must first go to an enhancement locus where at least one $SU(2)$ factor is present, because a smooth K3 compactification requires a non-vanishing instanton number. Switching on instantons breaks (at least) one $SU(2)$ factor and the resulting gauge group has (at most) rank 19. One specific route for going from an heterotic orbifold to a smooth heterotic K3 compactification was described in [27]. In the orbifold model one first goes to the $E_7 \times SU(2)$ locus, and then moves to the Higgs branch. This means to give vacuum expectation values to scalars in hypermultiplets. Each such hypermultiplet combines with a vector multiplet into a massive, long (non-BPS) vector multiplet. This mechanism is able to give mass to neutral vector multiplets, and therefore it reduces the rank of the gauge group. In the example described in [27] 3 hypermultiplets are used to Higgs the $SU(2)$. After going to the Coulomb branch they obtain a model with gauge group $U(1)^{19}$ and 65 hypermultiplets, which is the generic spectrum of the heterotic string on a smooth K3 surface with standard embedding.

6 Conclusions

In this paper we studied some of the conformal properties of Spinor-Vector duality, aspiring to trace back its CFT origin. In particular, after reviewing the duality map, we demonstrated how (discrete or even continuous) Wilson lines may be turned on to give masses to the vectorial or spinorial representations of the GUT gauge group. This provided a realization of the duality map as arising from different deformations of the same initial ‘parent’ theory (the S-V self-dual point), which corresponds precisely to points of exceptional gauge symmetry enhancement.

The fact that the number of massless degrees of freedom in the theory with massless vectorials (accompanied by the singlets) was found to precisely coincide with the number of massless states in the SV-dual theory with massless spinorials was

the most serious indication that there is an underlying spectral-flow at work. The enhancement points are marked by the appearance of global $N_L = 2$ and $N_L = 4$ superconformal algebras, which can be regarded as embeddings of the $N = 2$, $N = 4$ SCFTs of Type II theories into the ‘bosonic’ sector of the Heterotic string, via the Gepner map.

The superconformal properties and spectral-flow of the $N_L = 2$ and $N_L = 4$ SCFTs, which are relevant for $\mathcal{N}_4 = 1$ and $\mathcal{N}_4 = 2$ spacetime supersymmetry, respectively, were analyzed in some detail. In particular, the invariant spectral-flow operator was constructed explicitly in the twisted sectors and its action was shown to give rise to a specific isomorphism between the current algebra representations. A byproduct of this exact map is the presence of a series of identities, which may be employed both to illustrate the flow between the representations, as well as to algebraically simplify the partition function.

The unexpected and non-trivial result has been the identification of the spectral-flow operator in the $\mathcal{N}_4 = 2$ case, with the MSDS spectral-flow operator, which otherwise arises as a target-space symmetry in very special 2d string constructions. The MSDS constructions have been recently employed as candidate models in order to probe the early non-geometrical hot temperature phase of the universe. Their thermal interpretation and marginal deformations were discussed in [31], in relation to the construction of Hagedorn- and tachyon-free theories. Quite recently, their special symmetric structure was utilized in [32], in order to construct a stringy thermal model whose induced cosmological evolution is simultaneously free of initial gravitational-type singularities as well as Hagedorn-type instabilities of high temperature string theory, while remaining within the perturbative domain at every stage of the evolution.

The appearance of the MSDS structure in the context of Heterotic $N = (4, 4)$ -compactifications is highly non-trivial and merits a careful study on its own. What is more, through the Gepner map, the current algebra of conventional supersymmetry is precisely mapped into the MSDS current algebra, manifested as the algebra of the twisted $N_L = 4$ spectral-flow operators. The implications of this spectacular result remain to be investigated at greater depth in future work. However, the present observation already sheds some light into the realizations of the MSDS-structure and the nature of its algebra.

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A Partition Function in the specific $\mathcal{N}_4 = 2$ example

We describe here the procedure for constructing the partition function of the Heterotic orbifold compactification on $T^2 \times T^4/\mathbb{Z}_2$, which was defined in Section 2.2. The full partition function is the sum of two (disconnected) orbits of the modular group. One first starts with the (unprojected) partition function $Z_{[0,0]}^{[0,0]}$ in the (fully) untwisted sector:

$$Z_{[0,0]}^{[0,0]} = (\bar{V}_8 - \bar{S}_8) \Gamma_{(4,4)}(G_{IJ}, B_{IJ}) \Gamma_{(1,1)}(R) (O_{16} + S_{16}) (O_{16} + S_{16}), \quad (\text{A.1})$$

One then acts on $Z_{[0,0]}^{[0,0]}$ by the non-trivial elements $\{\alpha, \beta, \alpha\beta\}$ of the full orbifold group $\mathbb{Z}_2 \times \mathbb{Z}'_2$ to obtain:

$$Z_{[1,0]}^{[0,0]} = \alpha Z_{[0,0]}^{[0,0]} \quad , \quad Z_{[0,1]}^{[0,0]} = \beta Z_{[0,0]}^{[0,0]} \quad , \quad Z_{[1,1]}^{[0,0]} = \alpha\beta Z_{[0,0]}^{[0,0]}.$$

The modular orbit is then completed by acting on these elements with the modular group generators S, T (see, for example [33]):

$$Z_{[0,0]}^{[1,0]} = S\alpha Z_{[0,0]}^{[0,0]} \quad , \quad Z_{[0,0]}^{[0,1]} = S\beta Z_{[0,0]}^{[0,0]} \quad , \quad Z_{[0,0]}^{[1,1]} = S\alpha\beta Z_{[0,0]}^{[0,0]}.$$

$$Z_{[1,0]}^{[1,0]} = TS\alpha Z_{[0,0]}^{[0,0]} \quad , \quad Z_{[0,1]}^{[0,1]} = TS\beta Z_{[0,0]}^{[0,0]} \quad , \quad Z_{[1,1]}^{[1,1]} = TS\alpha\beta Z_{[0,0]}^{[0,0]}.$$

The second orbit can be easily constructed by picking an S -transformed element in the first orbit, such as $Z_{[0,0]}^{[1,0]}$, and acting on it with the β -group element so that we obtain an element outside the first orbit:

$$Z_{[0,1]}^{[1,0]} = \beta S\alpha Z_{[0,0]}^{[0,0]}.$$

The remaining elements in the second orbit can be constructed by the repeated action of the S, T -generators on $Z_{[0,1]}^{[1,0]}$:

$$Z_{[1,0]}^{[0,1]} = SZ_{[0,1]}^{[1,0]} \quad , \quad Z_{[1,1]}^{[1,0]} = TZ_{[0,1]}^{[1,0]} \quad , \quad Z_{[1,1]}^{[0,1]} = TSZ_{[0,1]}^{[1,0]},$$

$$Z_{[1,0]}^{[1,1]} = STZ_{[0,1]}^{[1,0]} \quad , \quad Z_{[0,1]}^{[1,1]} = TSTZ_{[0,1]}^{[1,0]}.$$

After taking into account the g, g' -projections, the untwisted sector can be written as:

$$\begin{aligned} Z_{(0,0)} &= Q_o \Lambda_{2m,n} \left[\Gamma_{(+)}^{h=0} (V_{12}V_4O_{16} + S_{12}S_4S_{16}) + \Gamma_{(-)}^{h=0} (O_{12}O_4O_{16} + C_{12}C_4S_{16}) \right] \\ &+ Q_o \Lambda_{2m+1,n} \left[\Gamma_{(+)}^{h=0} (S_{12}S_4O_{16} + V_{12}V_4S_{16}) + \Gamma_{(-)}^{h=0} (C_{12}C_4O_{16} + O_{12}O_4S_{16}) \right] \\ &+ Q_v \Lambda_{2m,n} \left[\Gamma_{(-)}^{h=0} (V_{12}V_4O_{16} + S_{12}S_4S_{16}) + \Gamma_{(+)}^{h=0} (O_{12}O_4O_{16} + C_{12}C_4S_{16}) \right] \\ &+ Q_v \Lambda_{2m+1,n} \left[\Gamma_{(-)}^{h=0} (S_{12}S_4O_{16} + V_{12}V_4S_{16}) + \Gamma_{(+)}^{h=0} (C_{12}C_4O_{16} + O_{12}O_4S_{16}) \right]. \end{aligned}$$

Similarly, for the sector twisted under the freely acting \mathbb{Z}'_2 :

$$\begin{aligned}
Z_{(0,1)} = & Q_o \Lambda_{2m,n+\frac{1}{2}} \left[\Gamma_{(+\epsilon)}^{h=0} (O_{12}V_4V_{16} + C_{12}S_4C_{16}) + \Gamma_{(-\epsilon)}^{h=0} (V_{12}O_4V_{16} + S_{12}C_4C_{16}) \right] \\
& + Q_o \Lambda_{2m+1,n+\frac{1}{2}} \left[\Gamma_{(+\epsilon)}^{h=0} (C_{12}S_4V_{16} + O_{12}V_4C_{16}) + \Gamma_{(-\epsilon)}^{h=0} (S_{12}C_4V_{16} + V_{12}O_4C_{16}) \right] \\
& + Q_v \Lambda_{2m,n+\frac{1}{2}} \left[\Gamma_{(-\epsilon)}^{h=0} (O_{12}V_4V_{16} + C_{12}S_4C_{16}) + \Gamma_{(+\epsilon)}^{h=0} (V_{12}O_4V_{16} + S_{12}C_4C_{16}) \right] \\
& + Q_v \Lambda_{2m+1,n+\frac{1}{2}} \left[\Gamma_{(-\epsilon)}^{h=0} (C_{12}S_4V_{16} + O_{12}V_4C_{16}) + \Gamma_{(+\epsilon)}^{h=0} (S_{12}C_4V_{16} + V_{12}O_4C_{16}) \right].
\end{aligned}$$

For the sector twisted under the non-freely acting \mathbb{Z}_2 :

$$\begin{aligned}
Z_{(1,0)} = & P_o \Lambda_{2m+\frac{1-\epsilon}{2},n} \left[\Gamma_{(+)}^{h=1} (V_{12}C_4O_{16} + S_{12}O_4S_{16}) + \Gamma_{(-)}^{h=1} (O_{12}S_4O_{16} + C_{12}V_4S_{16}) \right] \\
& + P_o \Lambda_{2m+\frac{1+\epsilon}{2},n} \left[\Gamma_{(+)}^{h=1} (S_{12}O_4O_{16} + V_{12}C_4S_{16}) + \Gamma_{(-)}^{h=1} (C_{12}V_4O_{16} + O_{12}S_4S_{16}) \right] \\
& + P_v \Lambda_{2m+\frac{1-\epsilon}{2},n} \left[\Gamma_{(-)}^{h=1} (V_{12}C_4O_{16} + S_{12}O_4S_{16}) + \Gamma_{(+)}^{h=1} (O_{12}S_4O_{16} + C_{12}V_4S_{16}) \right] \\
& + P_v \Lambda_{2m+\frac{1+\epsilon}{2},n} \left[\Gamma_{(-)}^{h=1} (S_{12}O_4O_{16} + V_{12}C_4S_{16}) + \Gamma_{(+)}^{h=1} (C_{12}V_4O_{16} + O_{12}S_4S_{16}) \right].
\end{aligned}$$

Finally, the sector twisted under both $\mathbb{Z}_2 \times \mathbb{Z}'_2$ is:

$$\begin{aligned}
Z_{(1,1)} = & P_o \Lambda_{2m+\frac{1-\epsilon}{2},n+\frac{1}{2}} \left[\Gamma_{(+\epsilon)}^{h=1} (O_{12}C_4V_{16} + C_{12}O_4C_{16}) + \Gamma_{(-\epsilon)}^{h=1} (V_{12}S_4V_{16} + S_{12}V_4C_{16}) \right] \\
& + P_o \Lambda_{2m+\frac{1+\epsilon}{2},n+\frac{1}{2}} \left[\Gamma_{(+\epsilon)}^{h=1} (C_{12}O_4V_{16} + O_{12}C_4C_{16}) + \Gamma_{(-\epsilon)}^{h=1} (S_{12}V_4V_{16} + V_{12}S_4C_{16}) \right] \\
& + P_v \Lambda_{2m+\frac{1-\epsilon}{2},n+\frac{1}{2}} \left[\Gamma_{(-\epsilon)}^{h=1} (O_{12}C_4V_{16} + C_{12}O_4C_{16}) + \Gamma_{(+\epsilon)}^{h=1} (V_{12}S_4V_{16} + S_{12}V_4C_{16}) \right] \\
& + P_v \Lambda_{2m+\frac{1+\epsilon}{2},n+\frac{1}{2}} \left[\Gamma_{(-\epsilon)}^{h=1} (C_{12}O_4V_{16} + O_{12}C_4C_{16}) + \Gamma_{(+\epsilon)}^{h=1} (S_{12}V_4V_{16} + V_{12}S_4C_{16}) \right].
\end{aligned}$$

In the above expressions we make use of the decomposition:

$$\Gamma_{(\pm)}^h \equiv \frac{1}{2} \left(\Gamma_{(4,4)[0]}^{[h]} \pm \Gamma_{(4,4)[1]}^{[h]} \right), \quad (\text{A.2})$$

of the symmetrically twisted (4, 4)-lattice:

$$\Gamma_{(4,4)}[h] = \begin{cases} \Gamma_{(4,4)}(G, B) & , \text{ for } (h, g) = (0, 0) \\ \left| \frac{2n}{\theta_{[1-g]}^{[1-h]}} \right|^4 & , \text{ for } (h, g) \neq (0, 0) \end{cases} \quad (\text{A.3})$$

into linear combinations with a definite \mathbb{Z}_2 -parity.

B Operator Products of $SO(N)$ Spin Fields

The fusion rules in the text can be verified straightforwardly by repeated use of the very useful OPEs involving the spin-fields of $SO(N)$ (see, for example, [34]):

$$\psi^a(z)S_\alpha(w) = \frac{\gamma_{\alpha\beta}^a}{\sqrt{2}} \frac{S_\beta(w)}{(z-w)^{1/2}} + \dots \quad (\text{B.1})$$

$$\begin{aligned} S_\alpha(z)S_\beta(w) &= \frac{c_{\alpha\beta}}{(z-w)^{N/8}} + \frac{\gamma_{\alpha\beta}^a}{\sqrt{2}} \frac{\psi^a(w)}{(z-w)^{N/8-1/2}} \\ &+ \frac{\gamma_{\alpha\beta}^{ab}}{2} \frac{\psi^a\psi^b(w)}{(z-w)^{N/8-1}} + \frac{1}{2\sqrt{2}} \frac{\gamma_{\alpha\beta}^{abc} \psi^a\psi^b\psi^c(w) + \gamma_{\alpha\beta}^a \partial\psi^a(w)}{(z-w)^{N/8-3/2}} \\ &+ \frac{1}{4} \frac{\gamma_{\alpha\beta}^{abcd} \psi^a\psi^b\psi^c\psi^d(w) + \gamma_{\alpha\beta}^{ab} \partial(\psi^a\psi^b)(w) + \frac{1}{2}c_{\alpha\beta}(\partial\psi^a)\psi^a(w)}{(z-w)^{N/8-2}} + \dots, \end{aligned} \quad (\text{B.2})$$

where $c_{\alpha\beta}$ is the charge conjugation matrix in the Dirac representation and the ellipses denote less singular terms.

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