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Mellin representation of the graviton bulk-to-bulk propagator in AdS.

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A Mellin-type representation of the graviton bulk-to-bulk propagator from Ref. [1] in terms of the integral over the product of bulk-to-boundary propagators is derived.

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The correlation functions of the conformal $\mathcal{N}=4$ SYM at large coupling constant are reduced via AdS/CFT correspondence [2–4] to Witten diagrams [4] in AdS space. A powerful method to calculate Witten diagrams in AdS space is to represent bulk-to-bulk propagators as Mellin integrals over the bulk-to-boundary propagators, calculate the tree-level "star" integrals with vertices over the AdS space and then convert the remaining integrals over the flat space into Mellin transforms of the conformal ratios using Symanzik's star formula [5] (see the discussion in Ref. [6]). For the scalar propagator with mass $m^2=(\Delta-d)\Delta$ the Mellin representation of bulk-to-bulk propagator has the form [6, 7]

$$\begin{aligned}&\Pi^\Delta_d(x,y) \qquad (1)\\ &= -\frac{i\Gamma\big(\frac{d}{2}\big)}{4\pi^{\frac{d}{2}+1}}\int_{-i\infty}^{i\infty}\frac{d\lambda}{\big(\Delta-\frac{d}{2}\big)^2-\lambda^2}\int\frac{d^dz}{\pi^{d/2}}\\ &\times\frac{(x^0)^{\frac{d}{2}+\lambda}\Gamma\big(\frac{d}{2}+\lambda\big)}{\Gamma(\lambda)[(x^0)^2+(\vec{x}-\vec{z})^2]^{\frac{d}{2}+\lambda}}\frac{(y^0)^{\frac{d}{2}-\lambda}\Gamma\big(\frac{d}{2}-\lambda\big)}{\Gamma(-\lambda)[(y^0)^2+(\vec{y}-\vec{z})^2]^{\frac{d}{2}-\lambda}}\end{aligned}$$

Here we used Poincare coordinates $x=(x^0,x^i)$ where $\vec{x}$ is a d-dimensional Euclidean vector (our metric is $dx^2=\frac{1}{(x^0)^2}[(dx^0)^2+d\vec{x}^2]$ with the size of AdS space $R=1$). The above equation looks like the integral of the product of two bulk-to-boundary propagators with unphysical complex masses $m=\pm i\sqrt{\frac{d^2}{4}-\lambda^2}$ over the usual flat space and over λ. The easiest way to prove this formula is to calculate explicitly the integral over z in the r.h.s. of Eq. (1). One obtains (cf. Ref. [8])

$$\begin{aligned}&\Gamma\Big(\frac{d}{2}\Big)\int\frac{d^dz}{\pi^{d/2}}\frac{(x^0)^{\frac{d}{2}+\lambda}\Gamma\big(\frac{d}{2}+\lambda\big)}{\Gamma(\lambda)(|x-z|^2)^{\frac{d}{2}+\lambda}}\frac{(y^0)^{\frac{d}{2}-\lambda}\Gamma\big(\frac{d}{2}-\lambda\big)}{\Gamma(-\lambda)(|y-z|^2)^{\frac{d}{2}-\lambda}}\\ &= f_\lambda(u)+f_{-\lambda}(u) \qquad (2)\end{aligned}$$

where $|x-z|^2\equiv(x^0)^2+(\vec{x}-\vec{z})^2$,

$$u(x,y)=\frac{(x^0-y^0)^2+(\vec{x}-\vec{y})^2}{2x^0y^0}$$

is the chordal distance between points x and y and

$$\begin{aligned}&f_\lambda(u) \qquad (3)\\ &= r^{\frac{d}{4}+\frac{\lambda}{2}}(1-r)^{-\frac{d}{2}}\frac{\Gamma\big(\frac{d}{2}+\lambda\big)}{\Gamma(\lambda)}F\Big(\frac{d}{2},1-\frac{d}{2},1+\lambda,\frac{-r}{1-r}\Big)\end{aligned}$$

Here F is the hypergeometric function ${}_2F_1$ and the variable $r(u)$ is defined as

$$r(u)\ \equiv\ \frac{1+u-\sqrt{u(2+u)}}{1+u+\sqrt{u(2+u)}} \qquad (4)$$

Substituting the integral (2) to Eq. (1) we get

$$\Pi^\Delta_d(u)=\ -\frac{i}{2\pi^{\frac{d}{2}+1}}\int_{-i\infty}^{i\infty}\frac{d\lambda}{\big(\Delta-\frac{d}{2}\big)^2-\lambda^2}f_\lambda(u) \qquad (5)$$

Since $r<1$ and the function

$$\begin{aligned}&F\Big(\frac{d}{2},1-\frac{d}{2},1+\lambda,\frac{-r}{1-r}\Big) \qquad (6)\\ &= \frac{\Gamma(1+\lambda)\Gamma^{-1}\big(\frac{d}{2}\big)}{\Gamma\big(1+\lambda-\frac{d}{2}\big)}\int_0^1 dt\ (1-t)^{\lambda-\frac{d}{2}}\Big[t+\frac{t^2r}{1-r}\Big]^{\frac{d}{2}-1}\end{aligned}$$

is regular in the right half-plane and behaves like $\lambda^{\frac{d}{2}-1}$ as $\Re\lambda\to\infty$ one can close the contour over λ in Eq. (5) in the right semi-plane and get the result as a residue at $\lambda=\Delta-\frac{d}{2}$

$$\begin{aligned}\Pi^d_\Delta(u)\ &=\ \frac{f_{\Delta-\frac{d}{2}}(u)}{\pi^{d/2}(2\Delta-d)}\ =\ \frac{\pi^{-\frac{d}{2}}\Gamma(\Delta)}{2\Gamma\big(\Delta-\frac{d}{2}+1\big)}\\ &\times r^{\frac{\Delta}{2}}(1-r)^{-\frac{d}{2}}F\Big(\frac{d}{2},1-\frac{d}{2},\Delta-\frac{d}{2}+1,\frac{-r}{1-r}\Big) \qquad (7)\end{aligned}$$

It is easy to see that the r.h.s. of Eq. (7) is equal to the bulk-to-bulk scalar propagator [9].

As we mentioned above, the formula (1) is extremely convenient for the calculation of Witten diagrams in the Mellin representation so it would be advantageous to get similar expression for the bulk-to-bulk graviton propagator. This propagator can be represented as [1]

$$\begin{aligned}&G^{\alpha\beta;\mu\nu}(x,y)\\ &= (\partial^\alpha\partial^\mu u\partial^\beta\partial^\nu u+\alpha\leftrightarrow\beta)\Pi^d_d(u)+g_{\alpha\beta}g_{\mu\nu}H(u)\\ &+\{(D^\alpha[\partial^\beta\partial^\mu u\partial^\nu uX(u)]+D^\alpha[\partial^\beta u\partial^\mu u\partial^\nu uY(u)]\\ &+\alpha\leftrightarrow\beta)\ +\ D^\alpha[\partial^\beta Z(u)]g^{\mu\nu}+(\alpha\leftrightarrow\mu,\beta\leftrightarrow\nu)\} \qquad (8)\end{aligned}$$

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where D_μ is a covariant derivative and

$$H(u) = -\frac{2}{d-1}\Big[(1+u)^2\Pi_d^d(u) - (d-2)(1+u)\int_u^\infty du'\ \Pi_d^d(u')\Big] \quad (9)$$

The remaining three functions $X(u)$, $Y(u)$ and $Z(u)$ are gauge artifacts. Hereafter the Greek indices from the first half of alphabet refer to the point x and from the second to y.

The Mellin representation of the graviton propagator has the form

$$G^{\alpha\beta;\mu\nu}(x,y) \quad (10)$$
$$= \frac{i\Gamma(d/2)}{2(d-1)\pi^{\frac{d}{2}+1}}\int_{-i\infty}^{i\infty}\frac{d\lambda}{(d/2)^2-\lambda^2}\frac{\left(\frac{d}{2}+1\right)^2-\lambda^2}{\Gamma(\lambda)\Gamma(-\lambda)}$$
$$\int\frac{d^dz}{\pi^{d/2}}\frac{(x_0)^{\frac{d}{2}+\lambda+2}\Gamma\left(\frac{d}{2}+\lambda\right)}{(|x-z|^2)^{\frac{d}{2}+\lambda}}\frac{(y^0)^{\frac{d}{2}-\lambda+2}\Gamma\left(\frac{d}{2}-\lambda\right)}{(|y-z|^2)^{\frac{d}{2}-\lambda}}$$
$$\times\ J^{\alpha i}(x-z)J^{\beta j}(x-z)\mathcal{E}_{ij;kl}J^{k\mu}(z-y)J^{l\nu}(z-y)$$

where

$$J^{\mu i}(x-z) = \delta^{\mu i} - 2\frac{(x-z)^\mu(x-z)^i}{|x-z|^2} \quad (11)$$

(and similarly for other J's) while $\mathcal{E}_{ij;kl}$ is a traceless symmetric projector

$$\mathcal{E}_{ij;kl} = \frac{1}{2}(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}) - \frac{\delta_{ij}\delta_{kl}}{d} \quad (12)$$

The d-dimensional Latin indices of this projector are raised and lowered with the flat metric.

Note that the covariant derivative and the trace of the graviton propagator (10) vanish:

$$g_{\alpha\beta}G^{\alpha\beta;\mu\nu}(x,y)=0, \qquad D_\alpha G^{\alpha\beta;\mu\nu}(x,y)=0 \quad (13)$$

Let us compare the integrand in the formula (10) to bulk-to-boundary propagator of the graviton. The general solution of the Dirichlet problem with the boundary data $\hat{h}_{ab}$ has the form [10]:

$$h^\alpha_\beta(x) = \frac{(d+1)\Gamma(d)}{(d-1)\Gamma(d/2)} \quad (14)$$
$$\times\int\frac{d^dz}{\pi^{d/2}}\frac{(x^0)^d}{(|x-z|^2)^d}J^{\alpha i}(x-z)J^{\beta j}(x-z)\mathcal{E}_{ij;ab}\hat{h}_{ab}$$

We see that similarly to the scalar case, the Mellin representation (10) looks like an integral of the product of two bulk-to-boundary propagators with unphysical complex graviton masses $m = \pm i\sqrt{\frac{d^2}{4}-\lambda^2}$ over the usual flat space and over λ.

Now let us prove the Eq. (10). The central point of the proof is the calculation of the following integral

$$I^{\alpha\beta;\mu\nu}(x,y;\lambda) = 2\left[\left(\frac{d}{2}+1\right)^2-\lambda^2\right]\Gamma(d/2) \quad (15)$$
$$\times\int\frac{d^dz}{\pi^{d/2}}\frac{(x^0)^{\frac{d}{2}+\lambda-2}\Gamma\left(\frac{d}{2}+\lambda\right)}{\Gamma(\lambda)(|x-z|^2)^{\frac{d}{2}+\lambda}}\frac{(y^0)^{\frac{d}{2}-\lambda-2}\Gamma\left(\frac{d}{2}-\lambda\right)}{\Gamma(-\lambda)(|y-z|^2)^{\frac{d}{2}-\lambda}}$$
$$\times\ J^{\alpha i}(x-z)J^{\beta j}(x-z)\mathcal{E}_{ij;kl}J^{k\mu}(z-y)J^{l\nu}(z-y)$$

It can be decomposed in the same set of structures as the propagator (8)

$$I^{\alpha\beta;\mu\nu}(x,y;\lambda)$$
$$= (\partial^\alpha\partial^\mu u\partial^\beta\partial^\nu u + \alpha\leftrightarrow\beta)G_\lambda(u) + g^{\alpha\beta}g^{\mu\nu}H_\lambda(u)$$
$$+\{(D^\alpha[\partial^\beta\partial^\mu u\partial^\nu u X_\lambda(u)] + D^\alpha[\partial^\beta u\partial^\mu u\partial^\nu u Y_\lambda(u)] + \alpha\leftrightarrow\beta) + (\alpha\leftrightarrow\mu,\beta\leftrightarrow\mu)\}$$
$$+ D^\alpha[\partial^\beta Z_\lambda(u)]g^{\mu\nu} + D^\mu[\partial^\nu Z_{-\lambda}(u)]g^{\alpha\beta} \quad (16)$$

A straightforward but somewhat lenghtly calculation yields (cf. Ref. [8])

$$G_\lambda(u) = \left[\left(\frac{d}{2}-1\right)^2-\lambda^2\right]f_\lambda(u) + (\lambda\leftrightarrow-\lambda)$$
$$H_\lambda(u) = 2(1+u)^2f_\lambda(u) - \frac{2}{d}\left(\frac{d^2}{4}-\lambda^2\right)f_\lambda(u) + 2(d-2)(1+u)F_\lambda(u) + (\lambda\leftrightarrow-\lambda) \quad (17)$$

for the two physical structures and

$$\left[\frac{d^2}{4}-\lambda^2\right]X_\lambda(u) = \left[(1+u)^2-\frac{1}{d}\right]f''_\lambda(u) + \left[\left(\frac{d}{2}+1\right)^2 - \lambda^2\right](1+u)f'_\lambda(u) + d\left(\frac{d^2}{4}-\lambda^2\right)f_\lambda(u + (\lambda\leftrightarrow-\lambda)),$$
$$\left[\frac{d^2}{4}-\lambda^2\right]Y_\lambda(u) = \left[(1+u)^2-\frac{1}{d}\right]f'''_\lambda(u) + (d+1)(1+u)f''_\lambda(u) + \frac{d(d+1)}{2}f'_\lambda(u) + (\lambda\leftrightarrow-\lambda),$$
$$\left[\frac{d^2}{4}-\lambda^2\right]Z_\lambda(u) = \left[(1+u)^3-\frac{1}{d}\right][f''_\lambda(u)+f''_{-\lambda}(u)] + \left[(1+d-2\lambda)(1+u)+\left(\frac{d}{2}+\frac{2}{d}\lambda^2-\frac{1}{d}\right)\right]$$
$$\times\ [f'_\lambda(u)+f'_{-\lambda}(u)] + 2(d-1)\lambda(1+u)[f_\lambda(u)+f_{-\lambda}(u)]$$
$$+\left[2(d-1)\lambda+(2-d-2\lambda)\left(\frac{d^2}{4}-\lambda^2\right)\right][F_\lambda(u)+F_{-\lambda}(u)] \quad (18)$$

for three gauge-dependent ones. Here

$$F_\lambda(u) = -\int_u^\infty f_\lambda(v)dv = -\frac{\Gamma\left(\frac{d}{2}+\lambda\right)}{\Gamma(d/2)} \quad (19)$$
$$\times\ \frac{r^{\frac{\lambda}{2}+\frac{d-2}{4}}(1-r)^{1-\frac{d}{2}}}{(d-2+2\lambda)}F\left(\frac{d}{2}-1, 2-\frac{d}{2}, 1+\lambda, \frac{-r}{1-r}\right)$$

One can easily see that the function $F\left(\frac{d}{2}-1, 2-\frac{d}{2}, 1+\lambda, \frac{-r}{1-r}\right)$ is also regular at the right half-plane and behaves like $\lambda^{\frac{d}{2}-1}$ as $\Re\lambda\to\infty$, cf. Eq. (6).

Let us now return to the proof of Eq. (10) which can be rewritten as

$$G^{\alpha\beta;\mu\nu}(x,y) \quad (20)$$
$$= \frac{i(d-1)^{-1}}{4\pi^{\frac{d}{2}+1}}\int_{-i\infty}^{i\infty}\frac{d\lambda}{(d/2)^2-\lambda^2}I^{\alpha\beta;\mu\nu}(x,y,\lambda)$$

Let us discuss the two gauge-invariant structures $G(u)$ and $H(u)$. The corresponding terms in the r.h.s of Eq.

(20) are

$$(\partial^\alpha\partial^\mu u\partial^\beta\partial^\nu u + \alpha \leftrightarrow \beta)\frac{i(d-1)^{-1}}{4\pi^{\frac{d}{2}+1}}\int_{-i\infty}^{i\infty}\frac{d\lambda}{\frac{d^2}{4}-\lambda^2}G_\lambda(u)$$
$$+ g^{\alpha\beta}g^{\mu\nu}\frac{i(d-1)^{-1}}{4\pi^{\frac{d}{2}+1}}\int_{-i\infty}^{i\infty}\frac{d\lambda}{\frac{d^2}{4}-\lambda^2}H_\lambda(u)$$
$$= (\partial^\alpha\partial^\mu u\partial^\beta\partial^\nu u + \alpha \leftrightarrow \beta)\frac{i(d-1)^{-1}}{2\pi^{\frac{d}{2}+1}}\int_{-i\infty}^{i\infty}\frac{d\lambda}{\frac{d^2}{4}-\lambda^2}$$
$$\times \left[\left(\frac{d}{2}-1\right)^2-\lambda^2\right]f_\lambda(u)$$
$$+ g^{\alpha\beta}g^{\mu\nu}\frac{i(d-1)^{-1}}{2\pi^{\frac{d}{2}+1}}\int_{-i\infty}^{i\infty}\frac{d\lambda}{\frac{d^2}{4}-\lambda^2}[2(1+u)^2f_\lambda(u)$$
$$- \frac{2}{d}\left(\frac{d^2}{4}-\lambda^2\right)f_\lambda(u) + 2(d-2)(1+u)F_\lambda(u)] \quad (21)$$

As we discussed above (see Eqs. (3), (6), and (19)), the functions $f_\lambda(u)$ and $F_\lambda(u)$ are regular in the right half-plane and decrease as $\lambda^{\frac{d}{2}-1}e^{-\frac{\lambda}{2}|\ln r|}$ when $\Re\lambda \to \infty$ so one can close the contour over λ and take the residue at $\lambda = \frac{d}{2}$. One obtains

$$G^{\alpha\beta;\mu\nu}(x,y) = \frac{f_{d/2}(u)}{d\pi^{d/2}}(\partial^\alpha\partial^\mu u\partial^\beta\partial^\nu u + \alpha \leftrightarrow \beta)$$
$$- \frac{2}{(d-1)\pi^{d/2}}\left[(1+u)^2 f_{\frac{d}{2}}(u) + (d-2)(1+u)F_{\frac{d}{2}}(u)\right]$$
$$\times g^{\alpha\beta}g^{\mu\nu} + \text{gauge} - \text{dependent structures} \quad (22)$$

which coincides with Eq. (8) and Eq. (9) since $\Pi_d^d(u) = \frac{1}{d\pi^{d/2}}f_{\frac{d}{2}}(u)$. Thus, we proved that the integral (10) can serve as a gravition bulk-to-bulk propagator in the gauge $D_\alpha G^{\alpha\beta;\mu\nu} = 0$. It should be mentioned that similar but somewhat more complicated representation of the graviton propagator was obtained in Ref. [7]). It has a function $\frac{(\frac{d}{2}+1)^2-\lambda^2}{(\frac{d}{2}-1)^2-\lambda^2}$ in place of $\frac{(\frac{d}{2}+1)^2-\lambda^2}{1-d}$ in Eq. (10) as well as additional terms proportional to the tensor structure obtained from that of Eq. (10) by replacement $\mathcal{E}_{ij;kl} \to \delta_{ij}\delta_{kl}$ and to the $g^{\mu\nu}g^{\alpha\beta}$ structure.

For completeness, let us briefly discuss the gauge boson propagator [11]

$$G_{\alpha;\mu}(x,y) = -\frac{f_{\frac{d}{2}-1}(u)}{2\pi^{d/2}\left(\frac{d}{2}-1\right)}\,\partial^\mu\partial^\alpha u + \partial^\mu\partial^\nu S(u) \quad (23)$$

where the second structure depends on the choice of gauge. The Mellin representaton of this propagator has the form [7]

$$G^{\alpha;\mu}(x,y) \quad (24)$$
$$= \frac{i\Gamma(d/2)}{4\pi^{\frac{d}{2}+1}}\int_{-i\infty}^{i\infty}d\lambda\,\frac{(d/2)^2-\lambda^2}{\left[\left(\frac{d}{2}+1\right)^2-\lambda^2\right]^2}$$
$$\int\frac{d^dz}{\pi^{d/2}}\frac{(x_0)^{\frac{d}{2}+\lambda+2}\Gamma\left(\frac{d}{2}+\lambda\right)}{\Gamma(\lambda)(|x-z|^2)^{\frac{d}{2}+\lambda}}\frac{(y^0)^{\frac{d}{2}-\lambda+2}\Gamma\left(\frac{d}{2}-\lambda\right)}{\Gamma(-\lambda)(|y-z|^2)^{\frac{d}{2}-\lambda}}$$
$$\times J^{\alpha i}(x-z)\delta_{ik}J^{k\mu}(z-y)$$

The explicit calculation of the integral in the r.h.s. of this equation confirms this expression obtained in Ref. [7] by solution of Einstein equations. Again, the gauge condition for the propagator (24) is $D_\alpha G^{\alpha;\mu}(x,y) = 0$.

We have represented the graviton bulk-to-bulk propagator in the form of the Mellin integral of the product of bulk-to-boundary propagators (with nonphysical masses). This formula permits us to apply the Mellin-transformation method of Ref. [6] to Witten diagrams with graviton (and gauge boson) propagators.

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