THE GEOMETRY OF NONCOMMUTATIVE SINGULARITY RESOLUTIONS

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Abstract. We introduce a geometric realization of noncommutative singularity resolutions. To do this, we first present a new conjectural method of obtaining conventional resolutions using coordinate rings of matrix-valued functions. We verify this conjecture for all cyclic quotient surface singularities, the Kleinian D_n and E_6 surface singularities, the conifold singularity, and a non-isolated singularity, using appropriate quiver algebras. This conjecture provides a possible new generalization of the classical McKay correspondence. Then, using symplectic reduction within these rings, we obtain new, non-conventional resolutions that are hidden if only commutative functions are considered. Geometrically, these non-conventional resolutions result from shrinking exceptional loci to ramified (non-Azumaya) point-like spheres.

Contents

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1. Motivation: a geometric perspective

We aim to make progress towards answering the following question.

> Given a variety X with mild singularities, find a coordinate ring of matrix-valued functions on X that "sees" appropriate conventional resolutions in a new way. Using these matrix-valued functions, obtain new, non-conventional resolutions that are hidden if only commutative functions on X are considered.

The rings of matrix-valued functions that we will consider are quiver algebras. Stated concisely, a quiver algebra is a quotient of an algebra whose basis consists of all paths in a quiver (that is, directed graph), and the product of two paths is their concatenation if defined and zero otherwise. A representation of (or module over) a quiver algebra is obtained by associating a vector space to each vertex of the quiver, representing each arrow by a linear map from the vector space at its tail to the vector space at its head, and requiring these linear maps satisfy the relations of the algebra.

To motivate our approach to geometry, let R be a commutative noetherian $\mathbb{C}$-algebra. The points m of the affine variety $X = \operatorname{Max} R$ may always be identified with the simple modules $R/m \cong \mathbb{C}$ over the ring of polynomial functions R on X, and a point m in X is smooth (singular) if and only if the projective dimension of the corresponding simple module R/m equals the complex topological dimension of X at m,

$$\operatorname{pd}_R(R/m) = \dim (R_m)$$

(resp. is infinite). It is therefore natural to extend this idea to noncommutative coordinate rings: if a f.g. noncommutative $\mathbb{C}$-algebra A is a finitely generated module over its center Z (or "module-finite over its center"), then we deem a point $p \in \operatorname{Max} A$ (equivalently, simple A-module V whose annihilator is p [S, Corollary 4.2.3]) smooth if its projective dimension equals the topological dimension at $p \cap Z \in \operatorname{Max} Z$,

$$\operatorname{pd}_A(V) = \dim (Z_{p\cap Z}) . \tag{1}$$

Moreover, in the commutative case the evaluation of a function $f = f(x) \in R$ at the point $m = (x-a)$ is the corresponding representation of f, namely $f(a) = [f] \in R/m$, so we say the evaluation of a function $f \in A$ at the point p is the representation of f corresponding to V, and thus in general f will be a matrix-valued function.

The algebras A and Z are both noetherian by the Artin-Tate lemma [S, Theorem 4.2.1], and $\operatorname{Max} A$ admits the Zariski topology with closed sets

$$V(I) := \{p \in \operatorname{Max} A \mid I \subseteq p\}$$

with I any ideal (since maximal ideals are prime). If in addition A is prime then the map $\phi : \operatorname{Max} A \to \operatorname{Max} Z$ given by $p \mapsto p \cap Z$ is bijective and continuous over an open dense subset of $\operatorname{Max} Z$ called the Azumaya locus of A [S, Theorem 4.2.7], so $\operatorname{Max} A$ and $\operatorname{Max} Z$ may be regarded in some sense as birationally equivalent. We therefore call the map ϕ a noncommutative resolution of Z if A is smooth in the sense that (1) holds for each $p \in \operatorname{Max} A$. Such resolutions were first proposed by the physicists Berenstein, Douglas, and Leigh [Be, BD, BL] in the context of string theory (see also [DGM]), and formalized independently and more abstractly by Van den Bergh in his definition of a noncommutative crepant resolution [V, Definition 4.1]. In Van den Bergh's approach, birationality is extended to the noncommutative setting by replacing isomorphic function fields [H, Corollary 4.5] with Morita equivalent "noncommutative function fields" (see for example [B, section 5.2]).

We will propose a program to unify, in a geometric sense, the commutative resolutions of a singularity with its noncommutative resolutions. In so doing we will present a new conjectural method of obtaining commutative resolutions from a noncommutative coordinate ring in section 2.1. Using this ring we will then introduce, in section 2.2, a way of shrinking the irreducible components of the exceptional locus to smooth point-like spheres, where many such spheres may occupy the same point in space. From this we obtain new resolutions, unseen by the commutative functions, that are (possibly proper) subsets of the maximal ideal spectra of the noncommutative coordinate ring. The conjecture will be verified for a number of examples in section 4, including at least one where the singularity is not two-dimensional; not a quotient by a finite group; non-Gorenstein; non-toric; non-isolated.

It would be interesting to understand how our construction is related to Van den Bergh's construction, where a commutative resolution of $\operatorname{Spec} R$ is obtained from a noncommutative R-algebra A as an open subset of the fine moduli space $\mathcal{M}_d^{\theta}(A)$ of stable A-modules with a fixed dimension vector $d \in \mathbb{Z}_{\geq 0}^{|Q_0|}$ and generic stability parameter $\theta \in \mathbb{Z}^{|Q_0|}$ [V, Theorem 6.3.1], which is based on the methods of [BKR].

Conventions. A denotes a finitely generated (= f.g.) algebra (usually over $\mathbb{C}$). All modules are left modules, and all representations are complex unless specified otherwise. The A-module V corresponding to a representation $\rho : A \to \operatorname{End}_{\mathbb{C}}(V)$ is the module defined by $av := \rho(a)v$ for $a \in A$, $v \in V$. A module isoclass will often be referred to as just a module. Multiplication of paths in a quiver algebra is read right to left, following the composition of maps. Q_ℓ denotes the set of paths of length ℓ in a quiver Q, and $Q_{\geq 0}$ denotes the set of all paths in Q. Given a quiver algebra $A = \mathbb{C}Q/I$ and vertex $i \in Q_0$, we denote by S_i the "vertex simple module" corresponding to the representation of A with a single 1-dimensional vector space at vertex i, and with all arrows represented by zero.

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2. Almost large modules

2.1. **Definition and conjecture.** We call a simple module (and its corresponding representation) *large* if it is of maximal $\mathbb{C}$-dimension[1]

$$d = \max \{\dim_{\mathbb{C}} V \mid V \text{ a simple } A\text{-module}\}. \tag{2}$$

If A is a f.g. $\mathbb{C}$-algebra, module-finite over its center Z, then $d < \infty$ [S, Theorem 4.2.2]. If A is also prime then the Azumaya locus of A is the open dense set of points $m \in \operatorname{Max} Z$ such that $A/Am \cong \operatorname{Mat}_d(\mathbb{C})$ (characterizing the "noncommutative residue fields" of A). Furthermore, there is a bijection between $Am \in \operatorname{Max} A$ and the large modules V, given by $Am = \operatorname{ann}_A V$ [S, Theorem 4.2.7], and so the large modules are parameterized by the Azumaya locus. Under suitable conditions the Azumaya locus coincides with the smooth locus of Z, a fact first discovered by Le Bruyn when the algebra is graded [Le, Theorem 1], and by Brown and Goodearl when the algebra is not graded [BG, section 3].

Theorem 2.1. *(Le Bruyn, Brown-Goodearl* [BG, Theorem 3.8]*.) If an algebra is prime, noetherian, Auslander-regular, Cohen-Macaulay, and module-finite over its center Z, and if the compliment of the Azumaya locus has codimension at least 2 in* $\operatorname{Max} Z$*, then the Azumaya and smooth loci coincide.*[2]

We introduce the following definitions in hopes of extending this theorem to smooth resolutions of the center of A when A is an infinite dimensional basic algebra, module-finite over its center.

Recall that two idempotents e_i and e_j are orthogonal if $e_ie_j = e_je_i = \delta_{ij}e_i$; an idempotent is primitive if it cannot be expressed as the sum of two nontrivial orthogonal idempotents; and a set of idempotents is complete if their sum is $1 \in A$. If $\{e_1, \ldots, e_n\}$ is a complete set of primitive orthogonal idempotents then A decomposes into a direct sum of indecomposable A-modules $A = Ae_1 \oplus \cdots \oplus Ae_n$, which is unique up to isomorphism and permutation of the factors since each Ae_i is projective [L, Corollary 20.23]. A subset $\{e_{i_1}, \ldots, e_{i_m}\}$ of $\{e_1, \ldots, e_n\}$ is basic if $Ae_{i_1}, \ldots, Ae_{i_m}$ is a complete, non-redundant set of representatives of A-modules of the form Ae for some primitive idempotent e, and A is basic if $\{e_{i_1}, \ldots, e_{i_m}\} = \{e_1, \ldots, e_n\}$. Finally, if A

[1] When A is module-finite over its center, such modules are also tiny [S, Theorem 4.2.2]!

[2] A ring S is *Auslander-regular* if S has finite global dimension and satisfies the Auslander condition, namely, that if $p < q$ are non-negative integers and M is a finitely generated R-module, then $\operatorname{Ext}^p_S(N, S) = 0$ for every submodule N of $\operatorname{Ext}^q_S(M, S)$. S is *Cohen-Macaulay* if it has finite Gelfand-Kirillov dimension $\operatorname{GKdim}(S) < \infty$ and

$$\min \{r \mid \operatorname{Ext}^r_S(M, S) \neq 0\} + \operatorname{GKdim}(M) = \operatorname{GKdim}(S)$$

for every finitely generated S-module M.

is a basic k-algebra and $d \in (\mathbb{Z}_{\geq 0})^n$, then we denote by $\operatorname{Rep}_d A$ the set of A-modules V with dimension vector $d = (\dim_k(e_i V))$.

We introduce the following definition in order to capture the notion of a path in a quiver algebra without having to refer to one specific basis.

Definition 2.2. We say a subset $\mathcal{P}$ of a basic k-algebra A is a *path-like set* if $\mathcal{P} \setminus \{0\}$ is a k-basis for A, $\mathcal{P}$ contains a basic set of idempotents, and $a, b \in \mathcal{P}$ implies $ab \in \mathcal{P}$.

Remark 2.3. If $A = \mathbb{C}Q/I$ is a quiver algebra with vertex set $Q_0 = \{1, 2, \ldots, n\}$ and $a \in e_2 Q_1 e_1$, then the set $\{e_1 + a, e_2 - a, e_3, \ldots, e_n\}$ is a complete set of primitive orthogonal idempotents in A different from the vertex idempotents. Note that $e_1 + a$ and $e_2 - a$ are primitive since there are A-module isomorphisms $A(e_1 + a) \cong Ae_1$ and $A(e_2 - a) \cong Ae_2$, and Ae_1 and Ae_2 are indecomposable.[3]

Recall that in a noetherian integral domain R, the codimension of a prime ideal p is the length ℓ of a maximal chain $p_0 \subsetneq p_1 \subsetneq \cdots \subsetneq p_\ell = p$ of distinct prime ideals, and ℓ equals the codimension of the subvariety defined by p in $\operatorname{Max} R$.

Definition 2.4. Let A be a f.g. basic algebra, module-finite over its prime center Z. Suppose d is the dimension vector of a large A-module. For $1 \leq \ell \leq \dim Z$, we say a subset P of A has *codimension* ℓ if there is a path-like set $\mathcal{P}$ of A and a maximal chain of subsets

$$0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_\ell = P \tag{3}$$

such that each P_j is the $\mathcal{P}$-annihilator of a module in $\operatorname{Rep}_d A$. If $V \in \operatorname{Rep}_d A$ is non-simple and satisfies $\operatorname{ann}_{\mathcal{P}} V = P$ then we say V is an *almost large A-module.*

Note that P is a multiplicatively closed subset of A. Also, if $d \neq (1, \ldots, 1)$ then the ideal generated by P will in general not be prime. We will call V an $\ell_{\mathcal{P}} = \ell$ almost large module.

Recall that the top $\operatorname{Top} V$ of a module V is the largest semisimple quotient of V, while the socle $\operatorname{Soc} V$ ("bottom") is the largest semisimple submodule of V. If A is module-finite over its noetherian center Z, then we say A is *homologically smooth* if (1) holds for each $p \in \operatorname{Max} A$.

Conjecture 2.5. *Let A be as in Definition 2.4, and in addition homologically smooth with a singular center Z. Suppose a primitive idempotent $e \in A$ satisfies*

$$\max \{\dim_{\mathbb{C}}(eW) \mid W \text{ a large } A\text{-module}\} = 1. \tag{4}$$

If the large A-module isoclasses are parameterized by the smooth locus of $\operatorname{Max} Z$ *then the following hold:*

(1) The isoclasses of almost large A-modules V, with $\operatorname{Soc} V = e \operatorname{Soc} V$, *are parameterized by the exceptional locus E of a smooth resolution* $Y \to \operatorname{Max} Z$.

[3]There are A-module monomorphisms $A(e_1 + a) \xrightarrow{\text{id}} Ae_1$; $Ae_1 \xrightarrow{\text{id}} A(e_1 + a)$; $A(e_2 - a) \xrightarrow{\cdot e_2} Ae_2$; and $Ae_2 \xrightarrow{\cdot (e_2 - a)} A(e_2 - a)$.

(2) For any fixed path-like set $\mathcal{P}$ of A, there is a natural bijection between the irreducible components E_i of E and the distinct subsets P with the properties that P is the $\mathcal{P}$-annihilator of an almost large module V with $\operatorname{Soc} V = e \operatorname{Soc} V$, and if $P = P_\ell$ occurs in a maximal chain (3) then the proceeding term $P_{\ell-1}$ is the $\mathcal{P}$-annihilator of a large module.

(3) If there exists a sequence of $\mathcal{P}$-annihilators

$$0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_j \subsetneq \cdots \subsetneq P_\ell,$$

where P_j corresponds to the irreducible component E_i by the natural bijection, then the isoclasses of almost large modules V, with $\operatorname{Soc} V = e \operatorname{Soc} V$ and $\mathcal{P}$-annihilator P_ℓ, are parameterized by a codimension ℓ (in Y) quasi-projective subvariety of E_i.

We will verify this conjecture for a number of examples in section 4. The underlying idea is then

smooth locus of an affine variety	$\longleftrightarrow$	large module isoclasses
exceptional locus of a smooth resolution	$\longleftrightarrow$	almost large module isoclasses with isomorphic 1-dim'l socles
exceptional locus shrunk to zero size	$\longleftrightarrow$	tops of these almost large module isoclasses

where the correspondence is given by parameterization. The last item will be introduced in the next section. The guiding principle is that if V and W are two non-isomorphic large modules and the points $\operatorname{ann}_Z V$ and $\operatorname{ann}_Z W$ lie on the same line that passes through a singular point of $\operatorname{Max} Z$, then V and W become isomorphic, and hence $\operatorname{ann}_Z V$ and $\operatorname{ann}_Z W$ become identified, when a minimal number of elements in A are set equal to zero.

Remark 2.6. We will only verify (2) in Conjecture 2.5 for the path-like set $\mathcal{P} = Q_{\geq 0} \cup \{0\}$, though it will easily follow for any path-like set containing the vertex idempotents, since such a set is multiplicatively generated by the vertex idempotents and a basis for $\mathbb{C}Q_1$ consisting of elements of the form $\sum_{a \in e_j Q_1 e_i} \gamma_a a$, with $\gamma_a \in \mathbb{C}$, $i, j \in Q_0$.

Remark 2.7. In physics terms, a path-like set $\mathcal{P}$ may be viewed as the set of dibaryon operators in a quiver gauge theory, and the $\mathcal{P}$-annihilator of a point in the vacuum moduli space would then be the set of all dibaryons with zero vev at that point (in some sense, since a non-cyclic path will not be gauge invariant, and vev's are gauge invariant).

Remark 2.8. Let $A = \mathbb{C}Q/I$ be a quiver algebra satisfying the hypothesis of Conjecture 2.5, and let $i \in Q_0$ be such that e_i satisfies (4). We ask the question: does the set of almost large A-modules whose socles are isomorphic to the vertex simple S_i always equal the entire set of non-simple modules whose socles are isomorphic to S_i and whose dimension vector d equals that of a large module? Similarly, if a resolution of the center of A is an open subset of the θ-stable moduli space $\mathcal{M}_d^\theta(A)$ with generic stability parameter $\theta = \left(-1 + \sum_{j \in Q_0} d_j, -1, \ldots, -1\right) \in \mathbb{Z}^{|Q_0|}$, where the first component is θ_i, then is the resolution necessarily the entire moduli space?

2.2. **Shrinking families of almost large modules.** In most cases we consider, isoclasses of almost large modules are parameterized by collections of $\mathbb{P}^n$'s. To make precise the notion of a $\mathbb{P}^n$-family of module isoclasses, we introduce the following definition; note the similarity with Definition 3.3 given below.

Definition 2.9. Let A be a $\mathbb{C}$-algebra, set $\mathbb{C}[t] := \mathbb{C}[t_1, \ldots, t_{n+1}]$, and suppose that there exists an algebra monomorphism

$$\sigma : A \longrightarrow \operatorname{End}_{\mathbb{C}[t]}(\mathbb{C}[t]^{\oplus d}). \tag{5}$$

Then for each $z \in \mathbb{C}^{n+1}$ the composition of σ with the evaluation map at z,

$$A \xrightarrow{\sigma} \operatorname{End}_{\mathbb{C}[t]}(\mathbb{C}[t]^{\oplus d}) \xrightarrow{\epsilon_z} \operatorname{End}_{\mathbb{C}[t]}((\mathbb{C}[t]/(t-z))^{\oplus d}) \cong \operatorname{End}_{\mathbb{C}}(\mathbb{C}^d),$$

is a representation of A, and $V_z := \mathbb{C}^d$ is an A-module with $av := \epsilon_z \sigma(a) v$. We say that the set of module isoclasses

$$\left\{ [V_z] \mid z \in \mathbb{C}^{n+1} \setminus 0 \right\}$$

is a $\mathbb{P}^n$-*family* if it has the property that $V_z \cong V_{z'}$ if and only if there exists a $\lambda \in \mathbb{C}^*$ such that $(z'_1, \ldots, z'_{n+1}) = (\lambda z_1, \ldots, \lambda z_{n+1})$.

In section 2.3 we will recall how $|\lambda|$ may be realized as the radius of $\mathbb{P}^n$ when viewed as an n-dimensional sphere using symplectic geometry. Let $A = kQ/I$ be a quiver algebra admitting a $\mathbb{P}^n$-family $\{[V_z]\}$ of A-modules. For $i \in Q_0$ set $d_i := \dim_{\mathbb{C}}(e_i V_x)$ and $d := \sum_i d_i$. Denote by λ an indeterminate and λ_* an arbitrary element of $\mathbb{C}^*$. Let $V_t := \mathbb{C}[t]^{\oplus d}$ be the A-module defined by $av := \sigma(a)v$. Suppose there exists an isomorphism

$$\phi_\lambda : V_t \xrightarrow{\cong} V_{\lambda t} \tag{6}$$

where

$$\phi_\lambda \in \bigoplus_{i \in Q_0} \operatorname{GL}_{d_i}\left(\mathbb{C}(\lambda)\right).$$

Then for each $z \in \mathbb{C}^{n+1} \setminus 0$ and $\lambda_* \in \mathbb{C}^*$ there is an isomorphism

$$\phi_{\lambda_*} : V_z \xrightarrow{\cong} V_{\lambda_* z}.$$

For each $i \in Q_0$ we will denote by $\phi_{\lambda,i}$ the restriction of ϕ_λ to the factor $\operatorname{GL}_{d_i}(\mathbb{C})$.

Suppose the least power of λ that appears in all the matrix entries of ϕ_λ is $m \in \mathbb{Z}$. Since there is a trivial diagonal $\mathbb{C}^*$-action on the isomorphism parameters, there is also an isomorphism $\lambda_*^{-m}\phi_{\lambda_*} : V_z \xrightarrow{\cong} V_{\lambda_* z}$. With this choice of rescaling, the limit

$$\phi_0 := \lim_{\lambda \to 0} \lambda^{-m}\phi_\lambda \in \bigoplus_{i \in Q_0} \operatorname{Mat}_{d_i}(\mathbb{C})$$

is nonzero and finite. We will write ϕ_λ as ϕ_λ^z when we need to specify the module V_z on which ϕ_λ is acting.

Lemma 2.10. *$V_z / \ker \phi_0^z \cong V_{z'} / \ker \phi_0^{z'}$ for each $z, z' \in \mathbb{C}^{n+1} \setminus 0$.*

Proof. Let $\sigma_{\lambda t} : A \to \operatorname{End}_{\mathbb{C}[t]}(\mathbb{C}[t]^{\oplus d})$ be the $\mathbb{C}[t]$-representation corresponding to $V_{\lambda t}$, so in particular $\sigma_t := \sigma$, and without loss of generality suppose the least power of λ that appears in the matrix entries of ϕ_λ is zero. For each arrow $a \in Q_1$, each t_i that appears in the matrix entries of $\sigma(a) = \sigma_t(a)$ is mapped to λt_i in the matrix $\sigma_{\lambda t}(a)$ under the transformation given by

$$\phi_{\lambda, \mathrm{h}(a)} \sigma_t(a) = \sigma_{\lambda t}(a) \phi_{\lambda, \mathrm{t}(a)}.$$

In particular t_i is mapped to 0 in the matrix $\sigma_{0t}(a)$ under the transformation given by

$$\phi_{0, \mathrm{h}(a)} \sigma_t(a) = \sigma_{0t}(a) \phi_{0, \mathrm{t}(a)},$$

so $\sigma_0(a) = \sigma_{0t}(a)$ does not depend on the t_i, and thus the matrix $\epsilon_z \sigma_0(a)$ does not depend on the choice of z. Now a acts on V_{0z} by $\epsilon_z \sigma_0(a)$, so $V_{0z} = V_{0z'}$ for each $z, z' \in \mathbb{C}^{n+1} \setminus 0$, and under this identification, $\operatorname{im} \phi_0^z = \operatorname{im} \phi_0^{z'}$.

The module epimorphisms

$$\phi_0^z : V_z \to \operatorname{im} \phi_0^z \quad \text{and} \quad \phi_0^{z'} : V_{z'} \to \operatorname{im} \phi_0^{z'}$$

then imply $V_z / \ker \phi_0^z \cong \operatorname{im} \phi_0^z = \operatorname{im} \phi_0^{z'} \cong V_{z'} / \ker \phi_0^{z'}$. □

Set $V_0 := V_z / \ker \phi_0^z$. By Lemma 2.10, V_0 does not depend on the choice of $z \in \mathbb{C}^{n+1} \setminus 0$ up to isomorphism.

Lemma 2.11. *If there is a $z \in \mathbb{C}^{n+1} \setminus 0$ such that the socle of V_z is 1-dimensional, then V_0 does not depend on the choice of ϕ_λ satisfying (6).*

Proof. Let $z \in \mathbb{C}^{n+1} \setminus 0$ be such that $\operatorname{Soc} V_z$ is 1-dimensional, say at $0 \in Q_0$. Since z is fixed we will write $\ker \phi_0^z$ as just $\ker \phi_0$.

Let ϕ_λ and ϕ'_λ be two isomorphisms $V_t \xrightarrow{\cong} V_{\lambda t}$, so they are also isomorphisms $V_z \xrightarrow{\cong} V_{\lambda z}$. We claim that $\ker \phi_0 = \ker \phi'_0 \subset V_z$. Denote by ρ and ρ_{λ_*} the representations $A \to \operatorname{Mat}_d(\mathbb{C})$ corresponding to V_z and $V_{\lambda_* z}$ respectively.

Fix $i \in Q_0$. Then for each path $p \in e_0 Q_{\geq 0} e_i$,

$$c\rho(p)\phi_{\lambda,i}^{-1} = \phi_{\lambda,0}\rho(p)\phi_{\lambda,i}^{-1} = \rho_\lambda(p) = \phi'_{\lambda,0}\rho(p)\phi'^{-1}_{\lambda,i} = c'\rho(p)\phi'^{-1}_{\lambda,i}, \tag{7}$$

where $\phi_{\lambda,0} = c \in \mathbb{C}^*$ and $\phi'_{\lambda,0} = c' \in \mathbb{C}^*$. Choose $d_i = \dim_{\mathbb{C}} e_iV_z$ paths $\{p_1, \ldots, p_{d_i}\}$ from i to $0 \in Q_0$ inductively as follows. Choose $v_1 \in e_iV_z \cong \mathbb{C}^{d_i}$. Since $\operatorname{Soc} V_z \cong \mathbb{C}$ is at $0 \in Q_0$, there exists a path $p_1 \in e_0Q_{\geq 0}e_i$ such that $\rho(p_1)v_1 \neq 0$. Now suppose the paths $\{p_1, \ldots, p_{j-1}\}$ have been chosen. Choose $v_j \in \ker \rho(p_1) \cap \ldots \cap \ker \rho(p_{j-1}) \cap e_iV_z$. Again since $\operatorname{Soc} V_z \cong \mathbb{C}$ is at $0 \in Q_0$ there exists a path $p_j \in e_0Q_{\geq 0}e_i$ such that $\rho(p_j)v_j \neq 0$. View each $\rho(p_k)$ as an element of $\operatorname{Mat}_{1\times d_i}(\mathbb{C})$ and recall $\phi_{\lambda,i} \in \operatorname{Mat}_{d_i\times d_i}(\mathbb{C})$. Then

$$\dim \ker \begin{bmatrix} \rho(p_1) \\ \vdots \\ \rho(p_{j-1}) \\ \rho(p_j) \end{bmatrix} < \dim \ker \begin{bmatrix} \rho(p_1) \\ \vdots \\ \rho(p_{j-1}) \end{bmatrix} < \dim \ker \begin{bmatrix} \rho(p_1) \end{bmatrix} = d_i - 1.$$

Thus setting

$$B := \begin{bmatrix} \rho(p_1) \\ \rho(p_2) \\ \vdots \\ \rho(p_{d_i}) \end{bmatrix} \in \operatorname{Mat}_{d_i\times d_i}(\mathbb{C})$$

we have $\dim \ker B = 0$ so B is injective. But from (7),

$$B\phi_{\lambda,i}^{-1}\phi'_{\lambda,i} = c^{-1}c'B,$$

and since B is injective $\phi_{\lambda,i}^{-1}\phi'_{\lambda,i} = c^{-1}c'\mathbf{1}_{d_i}$, so $\phi_{\lambda,i} = cc'^{-1}\phi'_{\lambda,i}$, so $w \in \ker \phi_{0,i} \cap e_iV_z$ if and only if $w \in \ker \phi'_{0,i} \cap e_iV_z$, and thus $\ker \phi_0 = \ker \phi'_0$, proving our claim.

It follows that $V_z/\ker \phi_0 = V_z/\ker \phi'_0$, and so by Lemma 2.10,

$$V_0(\phi_\lambda) \cong V_z/\ker \phi_0 = V_z/\ker \phi'_0 \cong V_0(\phi'_\lambda).$$

□

Definition 2.12. Suppose A is module-finite over its noetherian center Z, and let $\{[V_z]\}$ be a $\mathbb{P}^n$-family where each member has a 1-dimensional socle. If $V_0 = \bigoplus W_i$ is semisimple with simple summands W_i then we say that the $\mathbb{P}^n$ parameterizing this family shrinks to the points $\operatorname{ann}_A W_i \in \operatorname{Max} A$, and sits over the points $\operatorname{ann}_Z W_i \in \operatorname{Max} Z$.

Remark 2.13. In all the examples we will encounter, V_0 is the top of each member of its corresponding $\mathbb{P}^n$-family, though in general V_0 need not be semisimple.

2.3. A first example: the blowup of $\mathbb{C}^n$.

We now introduce a new noncommutative perspective on the tautological line bundle

$$\pi : L := \left\{(x, v) \in \mathbb{P}^{n-1} \times \mathbb{C}^n \mid v \in x\right\} \to \mathbb{C}^n, \quad (x, v) \mapsto v,$$

whose total space is $\mathbb{C}^n$ blownup at the origin. Consider the quiver algebra

$$A := \mathbb{C}Q/\langle [c, c'] \mid c, c' \text{ cycles}\rangle \tag{8}$$

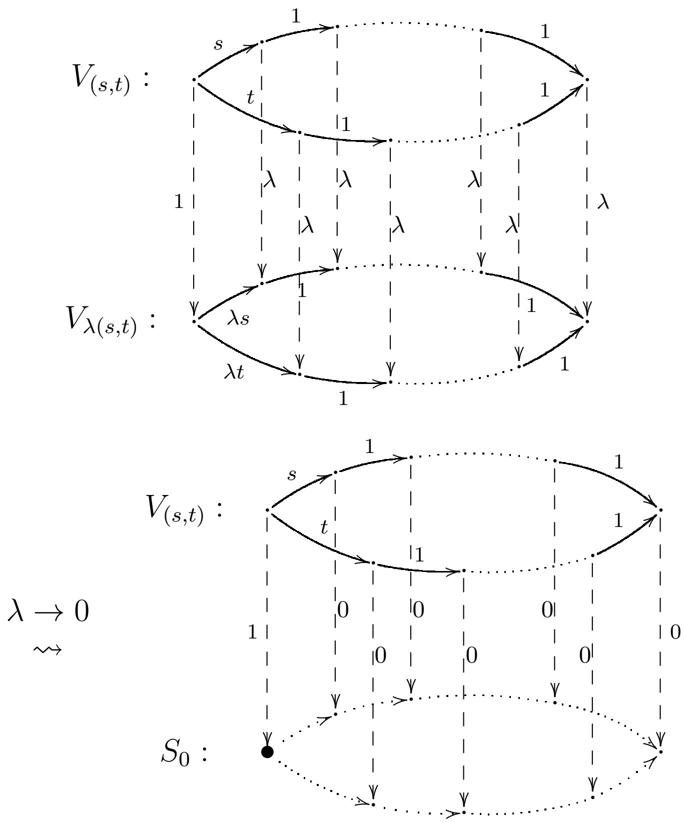


FIGURE 1. The $\mathbb{P}^1$-family $\{[V_{(s,t)}]\}$ shrunk to the vertex simple $[S_0]$ at the bold vertex. A dotted arrow denotes an arrow represented by zero and a dotted edge denotes some number of arrows.

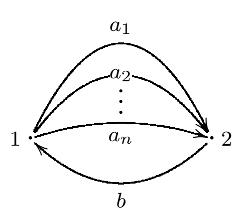


FIGURE 2. The tautological line bundle quiver.

with quiver given in figure 2. Recall that S_i denotes the vertex simple at $i \in Q_0$.

Proposition 2.14. *Let A be the quiver algebra (8). The isoclasses of large modules, and almost large modules with socle S_2 (resp. S_1), are parameterized by $\mathbb{C}^n$ blownup at the origin (resp. $\mathbb{C}^n$). Specifically,*

- *the large modules are parameterized by $\mathbb{C}^n \setminus \{0\}$, while*

- *the almost large modules with socle* S_2 *(resp.* S_1*) are parameterized by the exceptional divisor* $\pi^{-1}(0) = \mathbb{P}^{n-1}$ *(resp. the single point* 0*).*

Proof. Denote by Z the center of A. The ideal of relations of A is defined so that the corner rings $e_1Ae_1 = Ze_1$, $e_2Ae_2 = Ze_2$ are commutative, and so the algebra homomorphism

$$\tau : A \to \operatorname{End}_A(\mathbb{C}[z_1, \ldots, z_n])$$

defined by

$$\tau(a_i) = \begin{bmatrix} 0 & 0 \\ z_i & 0 \end{bmatrix}, \quad \tau(b) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \tau(e_1) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \tau(e_2) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

is a monomorphism. It then follows from [B, Proposition 2.9] that the large modules have dimension vector $(1,1)$. A module V with this dimension vector is simple if and only if there is some i such that a_i and b are represented by nonzero scalars, say z_i and y. However, if $y \neq 0$ then we may assume $y = 1$, as shown by the isomorphism (i) in figure 3 (the dashed lines denote the isomorphism parameters between A-modules W and V, where the resulting "squares commute"). Moreover, if two modules V and V' satisfy $y = y' = 1$, then $V \cong V'$ if and only if $z_i = z_i'$ for each i, and so the large module isoclasses are parameterized by $\mathbb{C}^n \setminus 0$.

Now consider the module isomorphisms (ii) and (iii) in figure 3, where the dotted arrows denote arrows represented by zero. Denote by $\mathcal{P}$ the path-like set $Q_{\geq 0} \cup \{0\}$. For $w_1, \ldots, w_j \in \{y, z_1, \ldots, z_n\}$ let $P(w_1 = \cdots = w_j = 0)$ denote the $\mathcal{P}$-annihilator of a module in $\operatorname{Rep}_{(1,1)} A$ with $w_1 = \cdots = w_j = 0$ and all other arrows represented by nonzero scalars. Note that $\dim Z = n$ since $Z \cong \mathbb{C}[z_1, \ldots, z_n]$. Then for $1 \leq \ell \leq n$ there is a maximal chain of subsets as in Definition 2.4,

$$0 \subsetneq P_1(y = 0) \subsetneq P_2(y = z_{i_1} = 0) \subsetneq P_3(y = z_{i_1} = z_{i_2} = 0) \subsetneq$$
$$\cdots \subsetneq P_\ell := P_\ell(y = z_{i_1} = z_{i_2} = \cdots = z_{i_{\ell-1}} = 0),$$

so any module whose $\mathcal{P}$-annihilator is P_ℓ is almost large. Similarly

$$0 \subsetneq P_1(z_1 = 0) \subsetneq P_2(z_1 = z_2 = 0) \subsetneq \cdots \subsetneq P' := P_n(z_1 = z_2 = \cdots = z_n = 0),$$

so any module whose $\mathcal{P}$-annihilator is P' is also almost large. Any module whose $\mathcal{P}$-annihilator is P_ℓ has socle S_2 (since $\ell \neq n+1$), and the isoclasses of all such modules forms a $\mathbb{P}^{n-1}$-family since $\lambda \in \mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^*$, which is shown by the module isomorphism (ii) in figure 3. Any module whose $\mathcal{P}$-annihilator is P' has socle S_1, and there is only one such module up to isomorphism, shown by the module isomorphism (iii) in figure 3. In this case $y \in \mathbb{C}^*$, and the Z-annihilator of this single isoclass is the maximal ideal m at the origin of $\mathbb{C}^n$. Note that any module whose $\mathcal{P}$-annihilator is $P(z_1 = \cdots = z_\ell = 0)$, where $1 \leq \ell \leq n-1$, is large and thus not almost large.

The path-like set $\mathcal{P} = Q_{\geq 0} \cup \{0\}$ is sufficient for determining all almost large modules since the almost large modules with socle S_1 or S_2 obtained from $Q_{\geq 0} \cup \{0\}$ exhaust the set of all modules in $\operatorname{Rep}_{(1,1)} A$ with socle S_1 or S_2. □

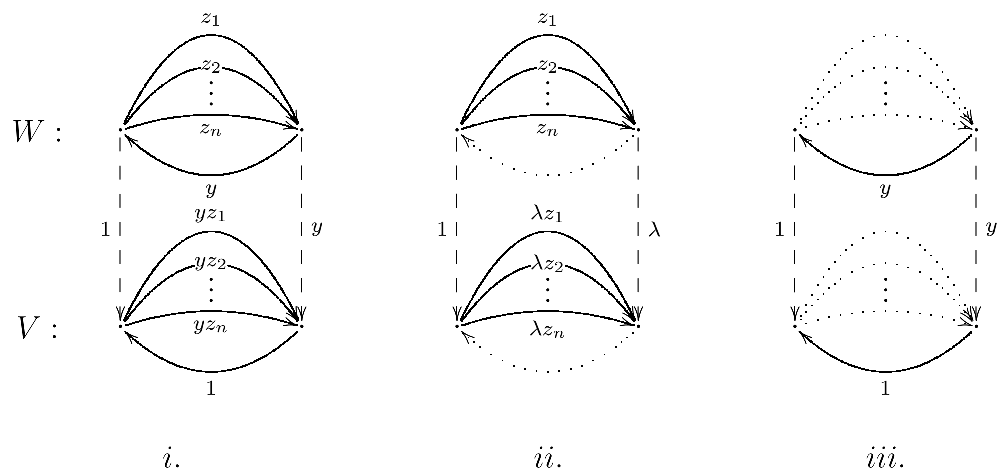


FIGURE 3. Some isomorphic A-modules. Dotted arrows denote arrows represented by zero, and dashed arrows denote isomorphism parameters between A-modules.

We now describe how to shrink the $\mathbb{P}^{n-1}$ to zero size using the noncommutative algebra A. Let $M = \mathbb{C}^n \setminus \{0\}$, $\mathbb{T} = U(1) \subset \mathbb{C}^*$, and consider the moment map

$$\mu : M \to g^* = \mathbb{R}$$

defined by

$$\mu(z_1, \ldots, z_n) = \frac{1}{2}\left(|z_1|^2 + \cdots + |z_n|^2\right).$$

Then

$$\begin{aligned} \mu^{-1}(1/2)/\mathbb{T} &= \left\{(z_1, \ldots, z_n) \in M \mid |z_1|^2 + \cdots + |z_n|^2 = 1\right\}/\mathbb{T} \\ &= \left\{\mathbb{P}^{n-1} \text{ with radius } 1\right\}, \end{aligned}$$

and more generally

$$\begin{aligned} \mu^{-1}(|\lambda|^2/2)/\mathbb{T} &= \left\{(\lambda z_1, \ldots, \lambda z_n) \in M \mid |z_1|^2 + \cdots + |z_n|^2 = 1\right\}/\mathbb{T} \\ &= \left\{\mathbb{P}^{n-1} \text{ with radius } |\lambda|\right\}. \end{aligned}$$

Varying λ is equivalent to varying the radius of $\mathbb{P}^{n-1}$. In particular, $\lambda \to 0$ is equivalent to the radius vanishing, and in our case of interest, the isomorphism (ii) of figure 3 becomes a module epimorphism, given in figure 4. The vertex simple S_1, which is not an almost large module, may therefore be viewed as the $\mathbb{P}^{n-1}$ shrunk to zero size. Note that S_1 is the top of every module in the $\mathbb{P}^{n-1}$-family. Moreover, even though this module corresponds to a point at the origin of $\mathbb{C}^n$, it is *not* the module (isoclass) corresponding to the actual origin of $\mathbb{C}^n$, namely the isoclass given in (iii) of figure 3.

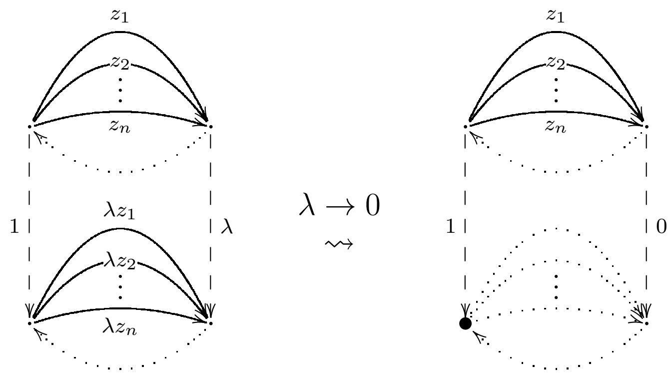


FIGURE 4. Shrinking the exceptional locus to zero size.

2.3.1. *Socle vs. top.* In Conjecture 2.5 we made a choice of restricting our attention to almost large modules with isomorphic 1-dimensional socles rather than isomorphic 1-dimensional tops. These two choices–either fixing the socle or fixing the top–appear equally suitable for the examples we will encounter in section 4, but they are not equal in regards to the noncommutative tautological line bundle algebra A defined in (8). For consider the geometric interpretation of projective dimension: if R is the (commutative) coordinate ring for an algebraic variety and $p \in \operatorname{Spec} R$ is smooth, then the projective dimension of R_p/p_p equals the codimension of p (that is, the codimension of the irreducible subvariety defined by p). Therefore since $\operatorname{pd}_A(S_1) = n$ and $\operatorname{pd}_A(S_2) = 1$, S_1 should be viewed as a zero-dimensional point in $\operatorname{Max} A$ while S_2 should be viewed as an $(n-1)$-dimensional "point".[4] It follows that if the $\mathbb{P}^{n-1}$ shrinks to a zero-dimensional point, then it should shrink to S_1 and not S_2.

[4]Given any almost large module W with socle S_2, there exists minimal projective resolutions of W and the vertex simple S_1 that are identical except for a factor of b that "switches sides" in the first two connecting maps. For an explicit example, consider $n = 3$. The homomorphism $Ae_1 \otimes e_1 V \stackrel{\delta}{\longrightarrow} V$, $\delta(c \otimes v) = cv$, is a projective cover for both $V = W$ and $V = S_1$. Let $I \subset Ae_1$ be the left ideal such that $\ker \delta_0 = I \otimes e_1 V$; then if $V = W$ (resp. $V = S_1$),

$$I = \langle c_i := x_i a_{i+1} - x_{i+1} a_i,\ ba_i \mid i = 1, 2, 3 \rangle = \langle c_1, c_2, ba_1 \rangle$$
$$(\text{resp. } I = \langle a_1, a_2, a_3 \rangle = \langle c_1, c_2, a_1 \rangle).$$

The sequence

$$0 \to Ae_2 \otimes V \xrightarrow{\cdot \left[\begin{array}{ccc} a_1 b & c_2 b & c_1 b \end{array}\right] \otimes 1} (Ae_1)^{\oplus 3} \otimes V \xrightarrow{\cdot \left[\begin{array}{ccc} c_2 b & -c_1 b & 0 \\ -a_1 b & 0 & c_1 \beta_2 \\ 0 & a_1 b & -c_2 \beta_2 \end{array}\right] \otimes 1}$$
$$\Big((Ae_2)^{\oplus 2} \oplus Ae_1\Big) \otimes V \xrightarrow{\cdot \left[\begin{array}{c} c_1 \\ c_2 \\ \beta_1 a_1 \end{array}\right] \otimes 1} Ae_1 \otimes V \stackrel{\delta}{\longrightarrow} V \to 0$$

3. $\mathbb{P}^n$-FAMILIES

3.1. **Determining $\mathbb{P}^n$-families.** We now give an explicit method for determining a $\mathbb{P}^n$-family of module isoclasses over a quiver algebra $A = kQ/I$. Recall the notation of Definition 2.9.

1. *Fix the support of σ.* This may be done efficiently by fixing a *pulled-apart* supporting subquiver $\widetilde{Q}$ of Q; given a representation $\rho : A \to \operatorname{Mat}_d(\mathbb{C})$, or the corresponding A-module $\mathbb{C}^d$, the quiver $\widetilde{Q}$ is defined by

$$\begin{aligned} \widetilde{Q}_0 &= \{1, \ldots, \operatorname{rank} \rho(1)\}, \\ \widetilde{Q}_1 &= \textstyle\bigsqcup_{a \in Q_1} \{ i \to j \mid (\rho(a))_{ji} \neq 0 \}, \end{aligned}$$

where $(\rho(a))_{ji}$ is the ji-th entry of the matrix $\rho(a)$. Note that this quiver depends on a choice of basis for $\mathbb{C}^d$. If ρ has dimension vector $(1, \ldots, 1)$, then $\widetilde{Q}_0 = Q_0$.

For fixed $\widetilde{Q}$, define the ideal $J_0 \subset \mathbb{C}[x_a] := \mathbb{C}[x_a \mid a \in \widetilde{Q}_1]$ so that the map

$$\sigma_0 : A \to \operatorname{Mat}_d(\mathbb{C}[x_a]/J_0), \qquad \sigma_0(a) := \begin{cases} x_a E_a & \text{if } a \in \widetilde{Q}_1, \\ E_a & \text{if } a \in \widetilde{Q}_0, \end{cases} \tag{9}$$

is an algebra monomorphism, where for a path a in $\widetilde{Q}$, E_a denotes the matrix with a 1 in the $(\mathrm{h}(a), \mathrm{t}(a))$-th slot and zeros elsewhere.

2. *Trivialize the ideal J_0.* Suppose $\widetilde{Q}$ is a pulled-apart subquiver of Q (with respect to some basis) that contains a sink at $0 \in \widetilde{Q}_0$. We apply the following iterative procedure on n to trivialize the ideal J_0 in (9). For $n \geq 1$, define

$$\sigma_n : A \longrightarrow \operatorname{Mat}_d(\mathbb{C}[x_a]/J_n) \tag{10}$$

as follows:

If $n = 1$, let $i = 0$.

Suppose $b \in \widetilde{Q}_1 e_i$. If for each $a \in \widetilde{Q}_1 e_i$ there is some $\alpha_a \in \mathbb{C}$ such that $x_b = \alpha_a x_a$ (modulo J_{n-1}) (in particular, if $\widetilde{Q}_1 e_i = \{b\}$), then set

$$\begin{aligned} \sigma_n(a) &:= \begin{cases} \alpha_a E_a & \text{if } a \in \widetilde{Q}_1 e_i \\ x_a E_a & \text{otherwise} \end{cases} \\ J_n &:= \left\langle I_n, x_a \mid a \in \widetilde{Q}_1 e_i \right\rangle, \end{aligned} \tag{11}$$

---

is a minimal projective resolution of $V = W$ (resp. $V = S_1$) when

$$(\beta_2, \beta_1) = \begin{cases} (1, b) & \text{if } V = W \\ (b, 1) & \text{if } V = S_1 \end{cases}.$$

However, for any n the projective dimension of the vertex simple S_2 is only 1,

$$0 \to Ae_1 \otimes e_2 S_2 \xrightarrow{\cdot c \otimes 1} Ae_2 \otimes e_2 S_2 \xrightarrow{\delta_0} S_2 \to 0.$$

where the ideal I_n is defined so that (10) is an algebra monomorphism. Otherwise do nothing.

Next, if $e_i\widetilde{Q}_1$ is non-empty, choose $a \in e_i\widetilde{Q}_1$ and set $j := \mathrm{t}(a) \in \widetilde{Q}_0$. Otherwise choose any vertex j where there exists an $a \in \widetilde{Q}_1 e_j$ such that $\sigma_n(a) = x_a E_a$ and $\sigma_n(b) \propto E_b$ for all $b \in \widetilde{Q}_1 e_{\mathrm{h}(a)}$ (the latter condition is trivially satisfied if j is a sink).

Repeat this process with $i = j$ until there does not exist such a j, and denote the final representation by

$$\sigma : A \longrightarrow \mathrm{Mat}_d(\mathbb{C}[x_a]/J).$$

In the examples we will consider, we will find that $\mathbb{C}[x_a]/J \cong \mathbb{C}[t_1, \dots, t_m]$ for some m. The following lemma says that when this is the case, it possible that the family of all modules supported on $\widetilde{Q}$ forms a $\mathbb{P}^{m-1}$-family. Denote by ϵ_z : $\mathrm{Mat}_d(\mathbb{C}[x_a]/J_n) \longrightarrow \mathrm{Mat}_d((\mathbb{C}[x_a]/J_n)/(x_a - z_a)) \cong \mathrm{Mat}_d(\mathbb{C})$ the evaluation map at the point $z = (z_a)_{a\in\widetilde{Q}_1} \in \mathbb{C}^{|\widetilde{Q}_1|}$.

Lemma 3.1. *If ρ is a representation of A with pulled-apart supporting subquiver $\widetilde{Q}$, then there exists a point $z \in (\mathbb{C}^*)^{|\widetilde{Q}_1|}$ such that*

$$\rho \cong \epsilon_z \cdot \sigma.$$

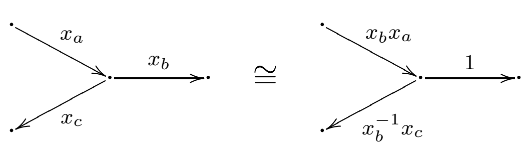
Proof. Clearly there exists some $z \in (\mathbb{C}^*)^{|\widetilde{Q}_1|}$ such that $\rho = \epsilon_z \cdot \sigma_0$. We claim that given any point $z \in (\mathbb{C}^*)^{|\widetilde{Q}_1|}$ there exists a point $z' \in (\mathbb{C}^*)^{|\widetilde{Q}_1|}$ such that

$$\epsilon_z \cdot \sigma_{n-1} \cong \epsilon_{z'} \cdot \sigma_n. \tag{12}$$

Let $b \in \widetilde{Q}_1$ be such that $\sigma_{n-1}(b) = bE_b$ and $\sigma_n(b) = E_b$, and set

$$z'_a := \begin{cases} z_b z_a & \text{if } \; a \in \widetilde{Q}_1 e_{\mathrm{h}(b)} \\ z_b^{-1} z_a & \text{if } \; a \in e_{\mathrm{h}(b)}\widetilde{Q}_1 \\ z_a & \text{otherwise} \end{cases}$$

In particular, $z'_b = 1$. The isomorphism (12) then follows from the definition of σ_n (11), explicitly given by $\mathrm{diag}(1, \dots, 1, z_a^{-1}, 1, \dots, 1) \in \mathrm{GL}_d(\mathbb{C})$, with z_a^{-1} in the $\mathrm{h}(a)$-th slot. Schematically, there is an isomorphism of representations:



Consequently there is some $z^0, z^1, \dots, z^N \in \mathbb{C}^{|\widetilde{Q}_1|}$ such that

$$\rho = \epsilon_{z^0} \cdot \sigma_0 \cong \epsilon_{z^1} \cdot \sigma_1 \cong \quad \cdots \quad \cong \epsilon_{z^N} \cdot \sigma.$$

$\square$

3. *Solve the isomorphism parameters.* Suppose that $\mathbb{C}[x_\alpha]/J \cong \mathbb{C}[t_1, \ldots, t_m]$. Set $\phi : V_{(t_1,\ldots,t_m)} \xrightarrow{\cong} V_{(\lambda_1 t_1,\ldots,\lambda_m t_m)}$, so that $(t_1, \ldots, t_m) \sim (\lambda_1 t_1, \ldots, \lambda_m t_m)$, and solve the relations among the λ_i.

In the following example we demonstrate how to "solve the isomorphism parameters" to show that a family of modules is a $\mathbb{P}^1$-family.

Example 3.2. Consider the family of modules over the path algebra given in the second column of figure 5.iii. To show that this is a $\mathbb{P}^1$-family we need to show that $\lambda = \mu$. Denote the isomorphism parameters by

$$1, f, g \in \mathrm{GL}_1(\mathbb{C}), \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2(\mathbb{C}),$$

at the respective vertices $1, 2, 3, 4 \in Q_0$; we then solve for these parameters by requiring that the relevant "squares commute":

$$\begin{bmatrix} 1 & 0 \end{bmatrix} f = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \Rightarrow \quad b = 0 \text{ and } f = a$$

$$\begin{bmatrix} a & 0 \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} f \quad \Rightarrow \quad d = f (= a)$$

$$\begin{bmatrix} 0 & 1 \end{bmatrix} g = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} a & 0 \\ c & a \end{bmatrix} \quad \Rightarrow \quad c = 0 \text{ and } g = a$$

$$\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} = 1 \begin{bmatrix} \lambda s \\ \mu t \end{bmatrix} \quad \Rightarrow \quad \lambda = a = \mu$$

3.2. **Coordinates on resolved singularities via impressions.** In this section we recall the definition of an impression, a notion the author introduced in [B, section 2.1]. An impression may be thought of as a way of placing (commutative) coordinates within an algebra that is module-finite over its center.

Definition/Lemma 3.3. [B, Definition 2.1] *Let k be an algebraically closed field, and let A be a f.g. k-algebra, module-finite over its center Z. Suppose that there exists a commutative noetherian reduced k-algebra B, an open dense subset $U \subseteq \operatorname{Max} B$, and an algebra momomorphism $\tau : A \to \operatorname{End}_B(B^d)$ such that the composition*

$$\tau_m : A \xrightarrow{\tau} \operatorname{End}_B\left(B^d\right) \xrightarrow{\epsilon_m} \operatorname{End}_B\left((B/m)^d\right) \cong \operatorname{End}_k\left(k^d\right)$$

is a large representation of A for each $m \in U$. Then

$$Z \cong \{f \in B \mid f 1_d \in \operatorname{im} \tau\} \subset B. \tag{13}$$

If the induced morphism of varieties

$$\operatorname{Max} B \stackrel{\phi}{\to} \operatorname{Max} Z \tag{14}$$

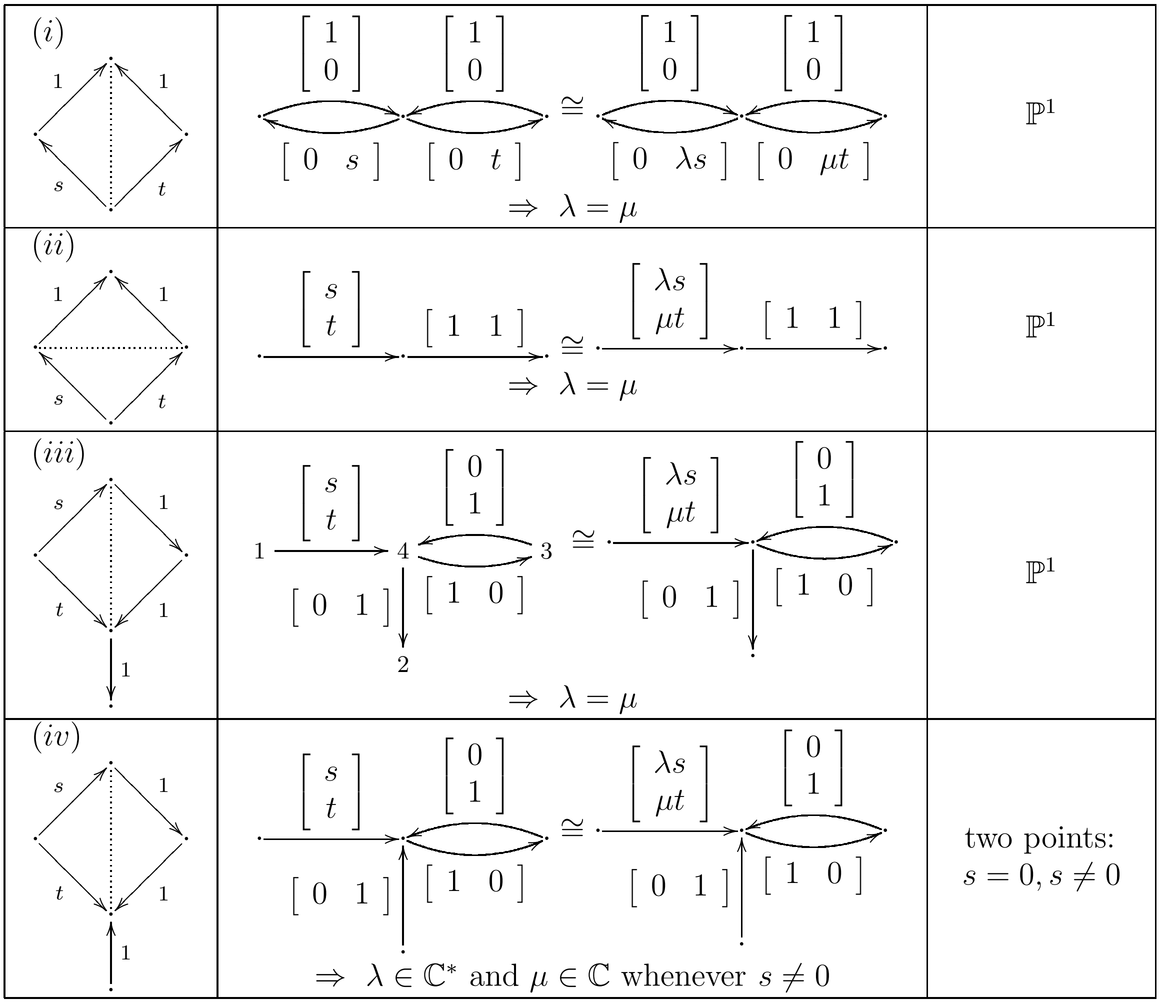


FIGURE 5. Examples of modules over path algebras, with $s, t \in \mathbb{C}$ not both zero, are given in the middle column, and their corresponding pulled-apart quivers (with respect to the standard basis) are given in the left column. Vertices in the pulled-apart quiver connected by a dotted edge correspond to the same vertex in the quiver itself. In (i) - (iii), the coordinates $(s : t)$ parameterize $\mathbb{P}^1$-families of almost large modules that will appear in section 4.3, and (iv) is cautionary.

is surjective, then we call (τ, B) *an* impression *of* A.

The following demonstrates the utility of an impression.

Proposition 3.4. [B, Proposition 2.5] *Let* (τ, B) *be an impression of a prime algebra* A. *If* V *is a large* A*-module, then there is some* $r \in \operatorname{Max} B$ *such that* $V \cong (B/r)^d$.

Now let $A = kQ/I$ be a quiver algebra. For $i \in Q_0$, set $d_i := \operatorname{rank} \tau(e_i)$. If $a \in e_j A e_i$ for some $i, j \in Q_0$, then we denote by $\bar{\tau}(a)$ the restriction of $\tau(a)$ to

$$B^{d_i} \cong \tau(e_i) B^d \to B^{d_j} \cong \tau(e_j) B^d. \tag{15}$$

For example, if the large A-modules have dimension vector $(1, \ldots, 1)$, then $\bar{\tau}(a) \in B$ whenever $a \in e_j A e_i$. In sections 4.1, 4.2, and 4.4, we will consider quiver algebras that admit impressions (τ, B) satisfying[5]

$$\bar{\tau}(e_i A e_i) = \bar{\tau}(e_j A e_j) \subset B \quad \text{for each} \quad i, j \in Q_0. \tag{16}$$

In each of these examples, (τ, B) determines a structure sheaf $\mathcal{O}_X$ on the parameterizing space X of isoclasses of large modules and almost large modules with fixed vertex simple socle, that coincides precisely with the structure sheaf obtained by blowing up the singularity. The construction of $\mathcal{O}_X$ from (τ, B) is as follows.

For each $x \in X$, let $Q(x)$ denote the supporting subquiver of x, and for each Zariski-open affine subset $U \subset X$, set

$$Q(U) := \bigcap_{x \in U} Q(x) \subseteq Q.$$

Define the new quiver

$$Q'(U) := \left\{ \begin{array}{lcl} Q'_0(U) & = & Q_0, \\ Q'_1(U) & = & Q_1 \cup \left\{ \mathrm{h}(a) \stackrel{a^*}{\longrightarrow} \mathrm{t}(a) \mid a \in Q_1(U) \right\}, \end{array} \right.$$

which contains Q as a subquiver, and set

$$A(U) := kQ'(U) / \left\langle I, \ aa^* - e_{\mathrm{h}(a)}, \ a^* a - e_{\mathrm{t}(a)} \mid a \in Q_1(U) \right\rangle,$$

which contains A as a subalgebra. Extend $\tau : A \to \operatorname{Mat}_d(B)$ to an algebra monomorphism

$$\tau : A(U) \longrightarrow \operatorname{Mat}_d(\operatorname{Frac}(B))$$

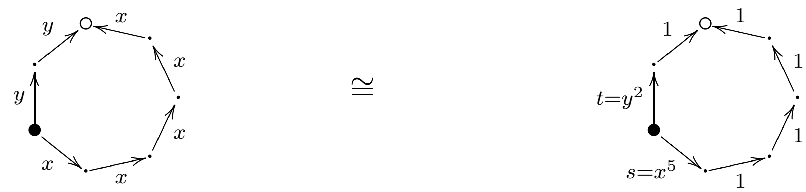
defined by

$$\begin{array}{rclcll} \tau(a) & := & \tau(a) & = & \bar{\tau}(a) E_{\mathrm{h}(a),\mathrm{t}(a)} & \text{for } a \in Q_1, \\ \tau(a^*) & := & & & \bar{\tau}(a)^{-1} E_{\mathrm{t}(a),\mathrm{h}(a)} & \text{for } a \in Q_1(U), \end{array} \tag{17}$$

[5] Let A be a quiver algebra and suppose that B, τ, and U are as in Definition 3.3 with $d < \infty$, but without requiring A be module-finite over its center or that ϕ exists. It was shown [B, Theorem 2.7] that if (16) holds, then A and its center Z are both noetherian rings, A is a finitely generated Z-module, and

$$Z = k \left[\sum_{i \in Q_0} \gamma_i \in \bigoplus_{i \in Q_0} e_i A e_i \mid \bar{\tau}(\gamma_i) = \bar{\tau}(\gamma_j) \text{ for each } i, j \in Q_0 \right].$$

Moreover, if we only assume that there is an algebra monomorphism $\tau : A \to \operatorname{End}_B(B^d)$ such that (16) holds, then the dimension vector d of any large A-module is bounded by $d \leq (1, \ldots, 1)$ [B, Proposition 2.9].



(i) coordinates $(x^5 : y^2)$
$\mathcal{O}_X(U) := \bar{\tau}\,(e_i A(U) e_i)$

(ii) coordinates $(s : t)$
$\sigma : A \to \mathrm{End}_{\mathbb{C}[s,t]}\,(\mathbb{C}[s,t]^{\oplus 7})$

FIGURE 6. Isomorphic labeling of arrows for the supporting subquiver $\widetilde{Q}^5$ of the $\mathbb{P}^1$-family of modules over the A_7 preprojective algebra given in figure 9 below. (i) determines coordinates of the $\mathbb{P}^1$-family from an impression of the A_7 preprojective algebra, and hence coordinates related to the singularity $\mathbb{C}^2/\rho(\mu_7)$, while (ii) specifies the $\mathbb{P}^1$-family (Definition 2.9) and is necessary for the intersections of the different $\mathbb{P}^1$-families to be parameterized by the intersections of the corresponding $\mathbb{P}^1$'s in the minimal resolution of $\mathbb{C}^2/\rho(\mu_7)$.

where E_{ij} denotes the matrix whose ijth entry is 1, and zeros elsewhere. We may then define the structure sheaf $\mathcal{O}_X$ induced by the impression (τ, B) to be

$$\mathcal{O}_X(U) := \bar{\tau}\,(e_i A(U) e_i)\,. \tag{18}$$

Remark 3.5. If the dimension vector of the large modules over a quiver algebra is not $(1, \ldots, 1)$ then it is not immediately clear how to generalize this construction, specifically (17), since in general $\bar{\tau}(a)$ may not be invertible.

4. Resolving singularities

In this section we verify Conjecture 2.5 in a number of examples. In these examples the noncommutative algebra is the path algebra of a McKay quiver, modulo relations. The McKay quiver Q of a group G and representation $\rho : G \to \mathrm{GL}_n(\mathbb{C})$ is defined to have a vertex for each irreducible representation $\phi_0, \phi_1, \ldots, \phi_m$ of G, and an arrow from j to i for each direct summand of ϕ_j in $\rho \otimes_{\mathbb{C}} \phi_i$. In the special cases $\rho : G \to \mathrm{SL}_2(\mathbb{C})$, Q is the double of any quiver whose underlying graph is the extended Dynkin graph of G, and McKay observed that this is the dual graph of the exceptional locus of the minimal resolution of $\mathbb{C}^2/\rho(G)$. Our program extends this correspondence by realizing the vertex simples at the vertices of the McKay quiver as the respective irreducible components of the exceptional locus shrunk to (smooth) point-like spheres.

4.1. **The conifold.** The well-known quiver algebra for the conifold (quadric cone) $R := \mathbb{C}\,[xz, xw, yz, yw] \cong \mathbb{C}\,[s,t,u,v]\,/(sv - tu)$ is

$$A := \mathbb{C}Q/\left\langle a_i b_j a_k - a_k b_j a_i,\, b_i a_j b_k - b_k a_j b_i \mid i,j,k = 1,2 \right\rangle$$

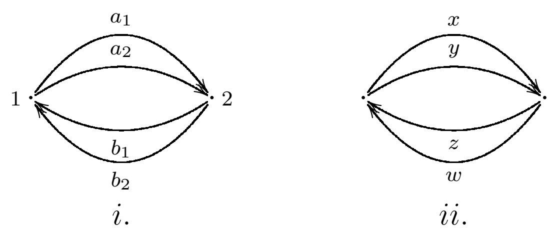


FIGURE 7. The conifold quiver and its impression.

with quiver given in figure 7.i. Since A is a square superpotential algebra, by [B, Theorem 3.7] A admits an impression $(\tau, \mathbb{C}[x,y,z,w])$, where τ is defined by the labeling of arrows in figure 7.ii, namely,

(19)

$$\tau(a_1) = \begin{bmatrix} 0 & 0 \\ x & 0 \end{bmatrix}, \quad \tau(a_2) = \begin{bmatrix} 0 & 0 \\ y & 0 \end{bmatrix}, \quad \tau(b_1) = \begin{bmatrix} 0 & z \\ 0 & 0 \end{bmatrix}, \quad \tau(b_2) = \begin{bmatrix} 0 & w \\ 0 & 0 \end{bmatrix},$$

$$\tau(e_1) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \tau(e_2) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

The center Z of A is isomorphic to R, and the non-Azumaya locus of A is the unique singular point $0 \in \operatorname{Max} R$ [B, Theorem 6.5]. $\operatorname{Max} R$ admits two crepant resolutions $\pi_\pm : Y^\pm \to \operatorname{Max} R$ given by the two birational transforms (with $s' = 1$),

$$sv - tu = s(v - t'u), \qquad sv - tu = s(v - tu').$$

The exceptional locus $\pi^{-1}(0)$ is given by $v - t'u = 0$ (resp. $v - tu' = 0$) with $s = t = u = v = 0$, so since $s'(xw) = s't = st' = (xz)t'$ (resp. $s'(yz) = s'u = su' = (xz)u'$), the ratios $t'/s' = w/z$ (resp. $u'/s' = y/x$) are free to vary. Thus in terms of the original coordinates x, y, z, w, $\pi_+^{-1}(0) = \mathbb{P}^1$ has coordinates $(z : w)$, while $\pi_-^{-1}(0) = \mathbb{P}^1$ has coordinates $(x : y)$. We now show that these coordinates agree with those obtained from the almost large A-modules.

Proposition 4.1. *Let A be the conifold quiver algebra. Then the large A-module isoclasses are parameterized by the smooth locus of* $\operatorname{Max} R$*, while the almost large module isoclasses with socle S_2 (resp. S_1) are parameterized by the exceptional locus $\pi_-^{-1}(0) = \mathbb{P}^1$ (resp. $\pi_+^{-1}(0)$), having coordinates $(x : y)$ (resp. $(z : w)$). Moreover, the coordinates on $Y^\pm$ obtained from the impression $(\tau, \mathbb{C}[x,y,z,w])$, namely (18), agree with those obtained by blowing up.*

Proof. The fact that the large modules are parameterized by the smooth locus follows from [B, Theorem 6.5]. By [B, Theorem 3.7] the large modules have dimension vector $(1,1)$. Denote by $\mathcal{P}$ the path-like set $Q_{\geq 0} \cup \{0\}$. As in the proof of Proposition 2.14, for $w_1, \ldots, w_j \in \{y, z_1, \ldots, z_n\}$ let $P(w_1 = \cdots = w_j = 0)$ denote the $\mathcal{P}$-annihilator of a module in $\operatorname{Rep}_{(1,1)} A$ with $w_1 = \cdots = w_j = 0$ and all other arrows represented

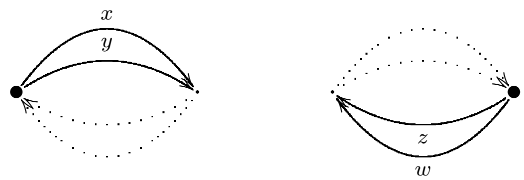


Figure 8. The exceptional loci of the two crepant resolutions of the conifold. Each $\mathbb{P}^1$ shrinks to the vertex simple at the bold vertex.

by nonzero scalars. Noting that $\dim Z = 3$, there is then a maximal chain as in Definition 2.4,

$$0 \subsetneq P_1(z=0) \subsetneq P_2(z=w=0) \subsetneq P_3(z=w=x=0),$$

so any module with $\mathcal{P}$-annihilator $P(z=w=0)$, $P(z=w=x=0)$, or $P(z=w=y=0)$ is almost large with socle S_2, and similarly any module with $\mathcal{P}$-annihilator $P(x=y=0)$, $P(x=y=z=0)$, or $P(x=y=w=0)$ is almost large with socle S_1. These two families of almost large modules form $\mathbb{P}^1$-families (recall the module isomorphism (ii) in figure 3), with respective coordinates $(x:y)$ and $(z:w)$ determined from (18) and the impression of A given by (19); see figure 8. The path-like set $\mathcal{P} = Q_{\geq 0} \cup \{0\}$ is sufficient for determining all almost large modules since the almost large modules with socle S_1 or S_2 obtained from $Q_{\geq 0} \cup \{0\}$ exhaust the set of all modules in $\operatorname{Rep}_{(1,1)} A$ with socle S_1 or S_2. □

4.2. **Cyclic quotient surface singularities.** Consider the linear action of the finite abelian group $\mu_r = \langle g \rangle$ of order r on $\mathbb{C}[x,y]$ by the representation

$$\rho(g) = \begin{bmatrix} e^{2\pi i/r} & 0 \\ 0 & e^{2\pi i b/r} \end{bmatrix},$$

that is, $g \cdot (x,y) = \left(e^{2\pi i/r}x, e^{2\pi i b/r}y\right)$. The ring of invariants $R := \mathbb{C}[x,y]^{\rho(\mu_r)}$ is the coordinate ring for the cyclic quotient surface singularity $\mathbb{C}^2/\rho(\mu_r) := \operatorname{Max} R$ of type $\frac{1}{r}(1,b)$. We suppose μ_r acts freely on $\mathbb{C}^2 \setminus 0$, and so we take $\gcd(r,b) = 1$, thus neglecting quasi-reflections. We will find that the minimal resolution $Y \to \mathbb{C}^2/\rho(\mu_r)$ of such a singularity (the total number of irreducible components of the exceptional locus and the coordinates on each component) can be read off directly from the associated McKay quiver: this information is simply hidden within the quiver, and is extracted by determining the supporting subquivers of the almost large modules over the McKay quiver algebra.

Lemma 4.2. *Let Q be the McKay quiver of (μ_r, ρ), so for each $i \in Q_0 = \{1, \ldots, r\}$ there are arrows*

$$e_i \xrightarrow{a_i} e_{i+1}, \qquad e_i \xrightarrow{b_i} e_{i+b}.$$

Then the associated McKay quiver algebra

$$A := \mathbb{C}Q / \langle b_{i+1}a_i - a_{i+r}b_i \mid i \in Q_0 \rangle$$

admits an impression $(\tau, \mathbb{C}[x,y])$*, where* τ *is defined by the labeling*

$$\bar{\tau}(a_i) = x, \qquad \bar{\tau}(b_i) = y, \tag{20}$$

for each $i \in Q_0$*.*

Proof. Since the corner rings e_iAe_i are commutative, the algebra homomorphism $\tau : A \to \operatorname{End}_{\mathbb{C}[x,y]}(\mathbb{C}[x,y])$ defined by (20) is a monomorphism. Thus the large A-modules have dimension vector $(1,\ldots,1)$ by [B, Proposition 2.9]. Take $U = \mathbb{C}^2 \setminus 0$. Since r, b are coprime, V_{τ_m} will be a large module for each $m \in U$. Since $Z \cong \mathbb{C}[x,y]^{\mu_r}$, the canonical morphism $\phi : \operatorname{Max} B \to \operatorname{Max} Z$ is a surjection. □

The following theorem extends the fact that the large A-modules are parameterized by the smooth locus of $\mathbb{C}^2/\rho(\mu_r)$.

Theorem 4.3. *Let* $\mathbb{C}^2/\rho(\mu_r)$ *be a cyclic quotient surface singularity,* $Y \to \mathbb{C}^2/\rho(\mu_r)$ *its minimal (Hirzebruch-Jung) resolution, and* A *the associated McKay quiver algebra. Then for each* $i \in Q_0$*, the set of almost large modules with socle* S_i *are parameterized by the exceptional locus of* Y*. Moreover, the coordinates on* Y *obtained from the impression* $(\tau, \mathbb{C}[x,y])$*, namely (18), agree with those obtained from the Hirzebruch-Jung resolution.*

Proof. As noted above, the large modules have dimension vector $(1,\ldots,1)$, so we may fix any vertex $0 \in Q_0 = \{0,\ldots,r-1\}$ and consider the isoclasses of almost large modules with socle isomorphic to the vertex simple S_0.

Let L denote the lattice $\mathbb{Z}^2 + \mathbb{Z}\cdot\frac{1}{r}(1,b) \subset \mathbb{R}^2$. For $m \in \{1,\ldots,r-1\}$, let $p \in e_0Q_{\geq 1}$ the unique path satisfying $\bar{\tau}(p) = x^m$, that is, $p = a_1a_2\cdots a_m$, and let $q \in e_0Q_{\geq 1}e_{\operatorname{t}(p)}$ be the unique path satisfying $\bar{\tau}(q) = y^n$ for some $n \in \{1,\ldots r-1\}$. Then $m = \overline{nb}$, so $\frac{1}{r}(n,m) = \frac{1}{r}\left(n, \overline{nb}\right) \in L$ is in the unit square of $\mathbb{R}^2$.

Let $Q^m \subset Q$ be the subquiver defined by

$$Q_i^m = \{a \in Q_i \mid a \text{ is a subpath of } p \text{ or } q\}, \quad i = 0,1.$$

Note that $0 \in Q_0^m$ is a sink for Q^m and

$$j := \operatorname{t}(p) = \operatorname{t}(q) \in Q_0^m$$

is a source (denoted by the bold vertices in figure 9).

Consider two subquivers Q^m and $Q^{m'}$ of Q where $m = \overline{nb}$ and $m' = \overline{n'b}$ with $1 \leq n, n' \leq r-1$. If $n < n'$ and $m' < m$ then clearly $Q^m \not\subset Q^{m'}$ and $Q^{m'} \not\subset Q^m$. Now the boundary lattice points of the convex hull of $L \subset \mathbb{R}^2$ in the positive quadrant, excluding the origin, are precisely the points $\frac{1}{r}(n', m')$ for which $n < n'$ implies $m' < m$ (and these points are in 1-1 correspondence with the irreducible components of the exceptional locus in Y). There is thus a 1-1 correspondence between the maximal chains of subquivers

$$Q^{m_1} \subsetneq Q^{m_2} \subsetneq \cdots \subsetneq Q^{m_\ell} \tag{21}$$

and the boundary lattice points.

Now let Q^m be the minimal term in a maximal chain (21), and construct the subquiver $\widetilde{Q}^m \supset Q^m$ of Q by adding the arrows a_i and b_i to Q^m for each $i \notin Q^m$ (these are denoted by the dotted arrows in figure 9).

Since $\dim Z = 2$, we must determine all maximal chains $0 \subsetneq P_1 \subsetneq P_2$ as in Definition 2.4. Denote by $\mathcal{P}$ the path-like set $Q_{\geq 0} \cup \{0\}$.

(i) *If Q^m is the minimal term in a maximal chain (21), then p and q cannot have a common vertex subpath e_k different from the sink and source of Q^m, namely e_0 and e_j.* Suppose otherwise; let p_1 and q_1 be the (unique) subpaths of p and q respectively, satisfying $p_1, q_1 \in e_k Q_{\geq 1} e_j$. Then there are subpaths p_2 and q_2 of p and q such that $p_2, q_2 \in e_1 Q_{\geq 1} e_{\mathrm{t}(p_2)}$. The subquiver corresponding to p_2 and q_2 is then a subquiver of Q^m, contracting the minimality of Q^m in a maximal chain. It follows that p and q have no common vertex subpaths other than the source and sink of Q^m. □

(ii) *$\widetilde{Q}^m$ supports an A-module with dimension vector $(1, \ldots, 1)$.* It suffices to show that if $a_i, b_{\mathrm{h}(a_i)} \in \widetilde{Q}^m_1$ then $b_i, a_{\mathrm{h}(b_i)} \in \widetilde{Q}^m_1$ as well, since the relation $b_{\mathrm{h}(a_i)} a_i = a_{\mathrm{h}(b_i)} b_i$ must hold.

If $i \in Q^m_0$ and $a_i \in \widetilde{Q}^m_1$ then $b_{\mathrm{h}(a_i)} \notin \widetilde{Q}^m_1$ by (i), so it must be that $i \notin Q^m_0$. But then $b_i \in \widetilde{Q}^m_1$ by construction of $\widetilde{Q}^m$, so we just need to show that $a_{\mathrm{h}(b_i)} \in \widetilde{Q}^m_1$ as well. If $\mathrm{h}(b_i) \notin Q^m_0$ then $a_{\mathrm{h}(b_i)} \in \widetilde{Q}^m_1$, again by construction. Otherwise suppose $\mathrm{h}(b_i) \in Q^m_0$. Since $i \notin Q^m_0$, $b_i \notin Q^m_1$, so $e_{\mathrm{h}(b_i)}$ cannot be a subpath of q different from e_j since there is only one b arrow whose head is at any given vertex, and thus $e_{\mathrm{h}(b_i)}$ must be a subpath of p. Moreover, $\mathrm{h}(b_i) \neq 0$ since q contains the b arrow whose head is at 0. But then $a_{\mathrm{h}(b_i)} \in Q^m_1 \subseteq \widetilde{Q}^m_1$, proving our claim. □

(iii) *Any module $V \in \mathrm{Rep}_{(1,\ldots,1)} A$ supported on $\widetilde{Q}^m$ has socle S_0 and therefore is not simple.* Since $0 \in \widetilde{Q}^m_0$, a_0 and b_0 will not be added to Q^m to form $\widetilde{Q}^m_0$, and so 0 is a sink in $\widetilde{Q}^m$. It therefore suffices to show that for each $i \in \widetilde{Q}^m_0 = Q_0$ there is a path s in Q from i to 0 that is contained in $\widetilde{Q}^m$ (that is, s does not annihilate V) since the dimension vector of V is $(1, \ldots, 1)$. We claim there exists a path $s = r a_{k_t} \cdots a_{k_2} a_{k_1}$, where r is a subpath of p or q with head at 0.

For $1 \leq u \leq t$, if $\mathrm{h}(a_{k_u}) \in Q^m_0$ then $u = t$; otherwise $\mathrm{h}(a_{k_u}) \notin Q^m_0$, in which case there exists arrows $a_{\mathrm{h}(a_{k_u})}$ and $b_{\mathrm{h}(a_{k_u})}$ in $\widetilde{Q}^m_1$ by construction, so $a_{k_{u+1}} a_{k_u}$ is a path in $\widetilde{Q}^m$. Now $a_{k_{u+1}}$ cannot be a subpath of $a_{k_u} \cdots a_{k_1}$ since r and b are coprime and 0 is a sink, and it follows that t must exist since the number of vertices is finite. □

(iv) *If Q' supports an A-module with dimension vector* $(1,\ldots,1)$ *and* $\widetilde{Q}^m \subsetneq Q' \subseteq Q$ *then* $Q' = Q$. Suppose $b_i \in Q'_1 \setminus \widetilde{Q}^m_1$. Then $i \in \widetilde{Q}^m_0 \setminus \{j\}$, specifically i is a vertex subpath of $p = a_1 a_2 \cdots a_m$.

Now if $\mathrm{h}(b_i) \notin Q^m_0$ then $a_{\mathrm{h}(b_i)} \in \widetilde{Q}^m_1 \subset Q'_1$, while if $\mathrm{h}(b_i) \in Q^m_0$ then $a_{\mathrm{h}(b_i)} \in Q^m_1 \subseteq Q'_1$ (otherwise the head of b_i would coincide with the head of a b arrow in Q^m_1, and since there is precisely one b arrow with head at any given vertex then b_i would be in Q^m_1, contrary to our original assumption). Therefore in either case $a_{\mathrm{h}(b_i)} \in Q'_1$. Since Q' supports an A-module and $a_{\mathrm{h}(b_i)}$ and b_i are both in Q'_1, the relation

$$a_{\mathrm{h}(b_i)} b_i = b_{\mathrm{h}(a_i)} a_i$$

implies $b_{\mathrm{h}(a_i)}$ is also in Q'_1. We can apply this argument iteratively (next with $b_{\mathrm{h}(a_i)}$ in place of b_i) to deduce that

$$b_{\mathrm{h}(a_1 a_2 \cdots a_{i-1} a_i)} = b_0$$

is in Q'_1. A similar argument with the a arrows then implies $Q'_1 = Q_1$, and hence $Q' = Q$. $\square$

(v) *Any module $V \in \mathrm{Rep}_{(1,\ldots,1)} A$ supported on $\widetilde{Q}^m$ is an $\ell_{\mathcal{P}} = 1$ almost large module.* By (ii) $\widetilde{Q}^m$ supports an A-module; by (iii) V is not simple; and by (iv) the chain $0 \subsetneq P_1$, where P_1 is the $\mathcal{P}$-annihilator of V, is maximal. $\square$

(vi) *$\widetilde{Q}^m$ supports a $\mathbb{P}^1$-family, minus the two points where one of the coordinates is zero.* By (i) p and q have no common vertex subpaths and so clearly Q^m supports a $\mathbb{P}^1$-family (minus two points); see the upper diagram in figure 1. By (iii) any module V supported on $\widetilde{Q}^m$ will have socle S_0, and so together with the "commutation" relations from I this implies that V is isomorphic to a module in which all the a arrows in $\widetilde{Q}^m_1$ are represented by the same scalar, and all the b arrows are represented by the same scalar. The claim then follows since the subquiver Q^m of $\widetilde{Q}^m$ supports a $\mathbb{P}^1$-family. $\square$

(vii) *If Q' supports an $\ell_{\mathcal{P}} = 1$ almost large module with socle S_0 then $Q' = \widetilde{Q}^m$ for some m.* By our assumptions on Q', Q' must contain as a subquiver a minimal term Q^m in some maximal chain (21), and by (iv) we may assume that Q' does not properly contain $\widetilde{Q}^m$, for otherwise it would equal Q. In addition, by assuming $\ell_{\mathcal{P}} = 1$, Q' cannot be properly contained in $\widetilde{Q}^m$. Suppose that $a_i \in Q'_1 \setminus Q^m_1$, where $i \neq j$ is a vertex subpath of q. By the argument in (iv), a_0 must then also be in Q'_1, and so the socle of any module supported on Q'_1 would not be S_0. Similarly $b_i \notin Q'_1$ if $i \neq j$ is a vertex subpath of p. Thus if a_i or b_i is in $Q'_1 \setminus Q^m_1$ then i must be not be in Q^m_0, so by the construction of $\widetilde{Q}^m$ we have $Q'_1 \subseteq \widetilde{Q}^m_1$ and hence $Q' = \widetilde{Q}^m$. $\square$

We have now characterized the $\ell_{\mathcal{P}} = 1$ almost large module isoclasses with socle S_0, and now we characterize the $\ell_{\mathcal{P}} = 2$ almost large modules.

Set

$$\alpha := \left\{ a_i \in \widetilde{Q}_1^m \mid i = j \text{ or } b_k \cdots b_{\mathrm{h}(b_i)} b_i \in e_j \widetilde{Q}_{\geq 1}^m \right\},$$

$$\beta := \left\{ b_i \in \widetilde{Q}_1^m \mid i = j \text{ or } a_k \cdots a_{\mathrm{h}(a_i)} a_i \in e_j \widetilde{Q}_{\geq 1}^m \right\}.$$

Consider the subquivers $\widetilde{Q}^{m,a}$ and $\widetilde{Q}^{m,b}$ of $\widetilde{Q}^m$ with vertex sets Q_0 and arrow sets $\widetilde{Q}_1^m \setminus \alpha$ and $\widetilde{Q}_1^m \setminus \beta$, respectively.

(viii) *The subquivers $\widetilde{Q}^{m,a}$ and $\widetilde{Q}^{m,b}$ support A-modules with dimension vector* $(1, \ldots, 1)$. Let $\rho \in \operatorname{Rep}_{(1,\ldots,1)} A$ be a representation supported on $\widetilde{Q}^{m,a}$, and suppose $a_i \in \alpha$, so $\rho(a_i) = 0$. It suffices to show that the relations

$$\rho(a_{\mathrm{h}(b_i)} b_i) = \rho(b_{\mathrm{h}(a_i)} a_i) = 0 \tag{22}$$

and

$$\rho(b_{\mathrm{h}(a_t)} a_t) = \rho(a_i b_t) = 0 \tag{23}$$

hold, where $\mathrm{h}(b_t) = i$.

In the first case, if $i = j$ then by (i) $a_{\mathrm{h}(b_i)} \notin Q_1^m$, hence $a_{\mathrm{h}(b_i)} \notin \widetilde{Q}_1^m$ since $\mathrm{h}(b_i) \in Q_0^m$, so (22) holds.

If $i \neq j$ then $a_i \in \alpha$ implies $b_k \cdots b_{\mathrm{h}(b_i)} b_i \in e_j \widetilde{Q}_{\geq 1}^m$. If the length of the path $b_k \cdots b_i$ is 1 (so the path is really just b_i), then $\mathrm{h}(b_i) = j$, so $a_{\mathrm{h}(b_i)} = a_j \in \alpha$, so (22) holds. Otherwise if the length of the path $b_k \cdots b_i$ is at least 2 then $b_k \cdots b_{\mathrm{h}(b_i)} \in e_j \widetilde{Q}_{\geq 1}^m$ as well, in which case $a_{\mathrm{h}(b_i)} \in \alpha$, hence (22) holds.

Now in the second case, first suppose $b_t \in \widetilde{Q}_1^m$. If $i = j$ then $b_t \in e_j \widetilde{Q}_1^m$, so $a_t \in \alpha$, hence (23) holds. If $i \neq j$ then $a_i \in \alpha$ implies $b_k \cdots b_i \in e_j \widetilde{Q}_{\geq 1}^m$, hence $b_k \cdots b_i b_t \in e_j \widetilde{Q}_{\geq 1}^m$ since $\mathrm{h}(b_t) = i$, and so $a_t \in \alpha$, hence (23) holds.

Otherwise suppose $b_t \notin \widetilde{Q}_1^m$. Then $t \notin Q_0^m$, so it must be that $a_t \in Q_1^m$, and by (i), $b_{\mathrm{h}(a_t)} \notin Q_1^m$, hence $b_{\mathrm{h}(a_t)} \notin \widetilde{Q}_1^m$ since $\mathrm{h}(a_t) \in Q_0^m$, and so (23) holds.

We have shown that in all cases the relations (22) and (23) are satisfied, so $Q^{m,a}$ supports an A-module, and similarly for $Q^{m,b}$. □

(ix) *Any module in* $\operatorname{Rep}_{(1,\ldots,1)} A$ *supported on $Q^{m,a}$ or $Q^{m,b}$ has socle S_0.* Since 0 is a sink in $\widetilde{Q}^m$ by (iii), it is also a sink in $\widetilde{Q}^{m,a}$, and so it suffices to show that for any vertex $k \in \widetilde{Q}_0^{m,a} = Q_0$ there exists a path s from k to 0 in $\widetilde{Q}^{m,a}$. No b arrows are removed from $\widetilde{Q}^m$ to form $\widetilde{Q}^{m,a}$, and so by (iii) we may take s to be $r b_{j_t} \cdots b_{j_2} b_{j_1}$, where r is a subpath of p or q with head at 0 (q if $\mathrm{h}(b_{j_t}) = j$). We therefore only need to show that if $a_i \in \alpha$ and $i \neq j$, then a_i is not a subpath of p. Suppose otherwise; since $a_i \in \alpha$ and $i \neq j$, $b_k \cdots b_i$ is a path in $\widetilde{Q}^m$, so b_i is a path in $\widetilde{Q}^m$, and hence a path in Q^m since $i \in Q_0^m$, which is a contradiction by (i). $\square$

(x) *If Q' supports an $\ell_{\mathcal{P}} = 2$ almost large module with socle S_0, then $Q' = \widetilde{Q}^{m,a}$ or $Q' = \widetilde{Q}^{m,b}$.* Suppose $b_i \notin Q_1'$; then we claim that b_j is also not in Q_1', hence $Q' \subseteq \widetilde{Q}^{m,b}$ since Q' supports an A-module, so $Q' = \widetilde{Q}^{m,b}$ since Q' supports an $\ell_{\mathcal{P}} = 2$ almost large module. First suppose $i \in Q_0^m \setminus \{j\}$. Since $Q' \subset \widetilde{Q}^m$ and (i) holds, the vertex i would be a sink of Q', and so S_i would be a direct summand of the socle of any module supported on Q', contrary to our assumption. So suppose $i \notin Q_0^m$. Since Q' supports a module with socle S_0, there exists a path r from $\mathrm{h}(b_i)$ to 0. Since $Q' \subset \widetilde{Q}^m$ and (i) holds, $r = r'p$ or $r = r'q$ for some path r' in Q' from $\mathrm{h}(b_i)$ to j. Thus for any $\rho \in \operatorname{Rep}_{(1,\ldots,1)} A$ supported on Q', $\rho(b_j r') = \rho(r'' b_i) = 0$ for some path r'' in Q. But $\rho(r') \neq 0$ since r' is a path in Q', so it must be that $\rho(b_j) = 0$, hence $b_j \notin Q_1'$, proving our claim. $\square$

(xi) *There is only one module in $\operatorname{Rep}_{(1,\ldots,1)} A$ supported respectively on $Q^{m,a}$ and $Q^{m,b}$, up to isomorphism.* Suppose to the contrary that the underlying graph of $\widetilde{Q}^{m,a}$ contains a cycle. Since $a_0, b_0 \notin \widetilde{Q}_1^m$, $\widetilde{Q}^{m,a}$ contains no oriented cycles, so there must be a vertex i for which both a_i and b_i are in $\widetilde{Q}_1^{m,a}$. Since $\widetilde{Q}^{m,a}$ supports a representation $\rho \in \operatorname{Rep}_{(1,\ldots,1)} A$ with socle S_0 by (ix), there exists a path r from $\mathrm{h}(b_i)$ to j in $\widetilde{Q}^{m,a}$ (recall the proof of (x)). But $a_j \notin \widetilde{Q}^{m,a}$ implies $\rho(b_j r a_i) = \rho(a_j r b_i) = 0$, so $\rho(a_i) = 0$ since $\rho(b_j r) \neq 0$, hence $a_i \notin \widetilde{Q}^{m,a}$, a contradiction. Similarly the underlying graph of $Q^{m,b}$ contains no cycles. The claim then follows since we are considering modules with dimension vector $(1, \ldots, 1)$. $\square$

(xii) *There is an equality of subquivers $Q^{m_i,b} = Q^{m_{i+1},a}$.* Denote by j_i, p_i, and q_i, the source j and respective paths p and q of Q^{m_i}. Suppose to the contrary that $b_{j_i} \in \widetilde{Q}_1^{m_{i+1}}$. Since p_i is a subpath of p_{i+1}, it follows that $a_{j_i} \in Q_1^{m_{i+1}}$, hence $a_{j_i} \in \widetilde{Q}_1^{m_{i+1}}$ as well, so it must be that $j_i \notin Q_0^{m_{i+1}}$ by (i) since then both a_{j_i} and b_{j_i} are in $Q_1^{m_{i+1}}$ and $j_i \neq j_{i+1}$. But $a_{j_i} \in Q_1^{m_{i+1}}$ implies $j_i \in Q_0^{m_{i+1}}$, a contradiction.

Similarly suppose to the contrary that $a_{j_{i+1}} \in \widetilde{Q}_1^{m_i}$. Since p_i is a subpath of p_{i+1}, by (i) it must be that q_{i+1} is a subpath of q_i. Therefore $b_{j_{i+1}} \in Q_1^{m_i}$, hence $b_{j_{i+1}} \in \widetilde{Q}_1^{m_i}$ as well, so it must be that $j_{i+1} \notin Q_0^{m_i}$ by (i) and $j_{i+1} \neq j_i$. But $b_{j_{i+1}} \in Q_1^{m_i}$ implies $j_{i+1} \in Q_0^{m_i}$, a contradiction.

Since $b_{j_i} \notin \widetilde{Q}_1^{m_{i+1}}$ and $\widetilde{Q}^{m_{i+1}}$ supports an A-module, we have $\widetilde{Q}^{m_{i+1}} \subseteq \widetilde{Q}^{m_i}$. Similarly since $a_{j_{i+1}} \notin \widetilde{Q}^{m_i}$ we have $\widetilde{Q}^{m_i} \subseteq \widetilde{Q}^{m_{i+1}}$, proving our claim. $\square$

(xiii) *If $V \in \operatorname{Rep}_{(1,\ldots,1)} A$ has socle S_0 then V is an almost large module.* Suppose Q' supports V. Since Q' supports a module with dimension vector $(1, \ldots, 1)$ and socle

S_0, for each $i \in Q_0$ there must be a path in Q' from i to 0. Therefore there is some m such that $Q_1^m \setminus \{a_j\}$ or $Q_1^m \setminus \{b_j\}$ is a subset of Q_1'.

First suppose Q^m is a subquiver of Q', and suppose $a_i \in Q_1' \setminus Q_1^m$. Then since Q' supports an A-module with socle S_0, there is a path p from i to j, say $p = a_i p'$ with p' a path. By the relations of A, $ap'b_j = b_i p''$ for some path p'', so $b_i \in Q_1'$. Since $i \notin Q_0^m$ was arbitrary, $Q' = \widetilde{Q}^m$ by the construction of $\widetilde{Q}^m$.

Now suppose $Q_1^m \setminus \{a_j\}$ is a subset of Q_1', but Q_1^m is not. By the proof of (x), $Q' \subseteq \widetilde{Q}^{m,a}$. Suppose to the contrary that this containment is proper, that is, there is some a_i or b_i in $\widetilde{Q}_1^{m,a} \setminus Q_1'$ with $i \notin Q_0^m$. If $a_i \notin Q_1'$ then there must exist a path consisting entirely of b arrows from i to 0, for if p is a path from i to 0 containing an a arrow then by the relations of A, $p = a_i p'$ for some path p'. But by the construction of $\widetilde{Q}^{m,a}$, there is no path consisting entirely of b arrows from i to 0 since $a_i \in \widetilde{Q}_1^{m,a}$. Similarly, if $b_i \notin Q_1'$ then there must exist a path consisting entirely of a arrows from i to 0 in Q'. But there is no such path in $\widetilde{Q}^{m,a}$ since such a path would necessarily contain a_j, and $a_j \notin \widetilde{Q}_1^{m,a}$, so $a_j \notin Q_1'$ as well. Thus the containment cannot be proper and so $Q' = \widetilde{Q}^{m,a}$. The case where $Q_1^m \setminus \{b_j\}$ is a subset of Q_1' is similar. □

The path-like set $\mathcal{P} = Q_{\geq 0} \cup \{0\}$ is sufficient for determining all almost large modules since the almost large modules with socle S_0 obtained from $Q_{\geq 0} \cup \{0\}$ exhaust the set of all modules in $\operatorname{Rep}_{(1,\ldots,1)} A$ with socle S_0 by (xiii). By (x) and (xi), the $\ell_{\mathcal{P}} = 2$ almost large module isoclasses with socle S_0 are parameterized by the points in the $\mathbb{P}^1$-families where one coordinate is zero, namely $(0:1)$ or $(1:0)$. Together with (vi) and (vii), it follows that there is a 1-1 correspondence between the supporting subquivers of almost large modules with socle S_0 (each of which supports a $\mathbb{P}^1$-family) and the boundary lattice points of L, and hence the irreducible components of the exceptional locus, each of which is a $\mathbb{P}^1$ [R2, Proposition 2.2, Theorem 3.2]. Furthermore, by (xii) the intersections of the irreducible components parameterize the intersections of the $\mathbb{P}^1$-families of almost large modules.

Finally, if $Q^{m,n}$ is the minimal term in the chain (21), then $Q^{m,n}$, and hence $\widetilde{Q}^{m,n}$, supports a $\mathbb{P}^1$-family with homogeneous coordinates $(x^n : y^m)$, obtained from (18) and the impression $(\tau, \mathbb{C}[x,y])$ of A given in lemma 4.2. But these are precisely the coordinates obtained from the Hirzebruch-Jung resolution; see for example [R2, Theorem 3.2] and references therein. □

Example 4.4. The supporting subquivers $\widetilde{Q}^m$ of the $\ell_{\mathcal{P}} = 1$ almost large modules over the $\frac{1}{7}(1,b)$ McKay quiver algebra A with $1 \leq b \leq 6$ are shown in figure 9.

Remark 4.5. Ishii showed that for small finite subgroups $G \subset \mathrm{GL}_2(\mathbb{C})$, the G-Hilbert scheme $\operatorname{Hilb}^G(\mathbb{C}^2)$ coincides with the minimal resolution of $\mathbb{C}^2/G$ using Wunram's special representations of G [I, Theorem 3.1]. It would therefore be interesting to understand how special representations are related to almost large modules.

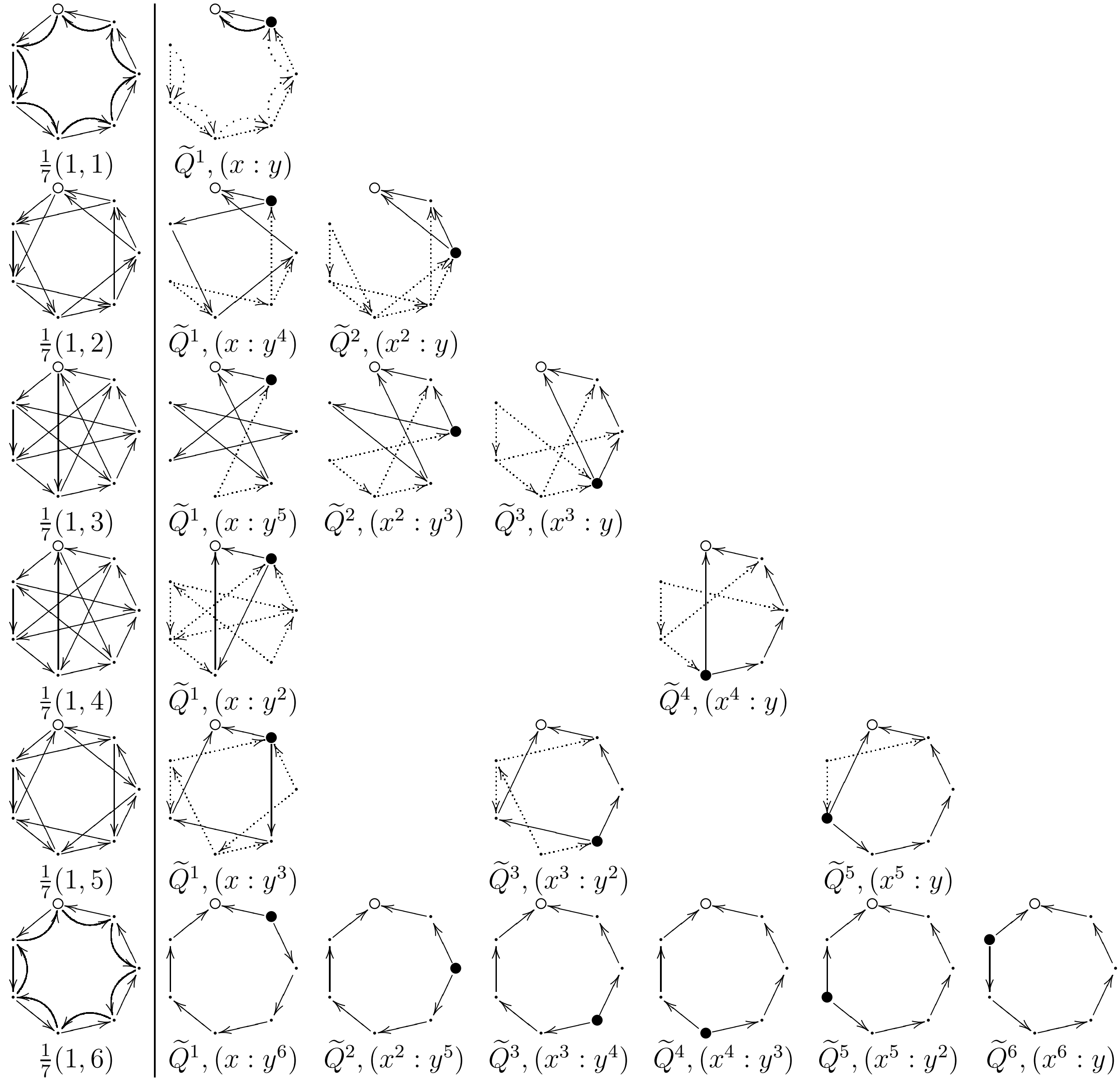


FIGURE 9. The supporting subquivers of the $\ell_{\mathcal{P}} = 1$ almost large modules over the McKay quiver algebras of type $\frac{1}{7}(1, b)$.

A cyclic quotient surface singularity is Gorenstein if and only if it is of type $\frac{1}{n}(1, -1)$, in which case the McKay quiver algebra coincides with the A_n preprojective algebra.

Corollary 4.6. *Let A be the A_n preprojective algebra, and let $\pi : Y \to \mathbb{C}^2/\mu_n$ be the minimal resolution of the A_n surface singularity. The irreducible component E_i of $\pi^{-1}(0)$, associated to the vertex $i \in Q_0$ by the McKay correspondence, shrinks to the vertex simple A-module S_i.*

4.3. D_n **and** E_6 **surface singularities.** Consider the linear action of the binary dihedral group of order $4n$,

$$\mathrm{BD}_{4n} := \left\langle g, j \mid g^{2n} = e, g^n = j^2, gjg = j \right\rangle,$$

on $\mathbb{C}[x,y]$ by the representation

$$\rho(g) = \begin{bmatrix} e^{\pi i/n} & 0 \\ 0 & e^{-\pi i/n} \end{bmatrix}, \qquad \rho(j) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

that is, $g \cdot (x,y) = \left(e^{\pi i/n}x, e^{-\pi i/n}y\right)$ and $j \cdot (x,y) = (y,-x)$. Similarly, consider the linear action on $\mathbb{C}[x,y]$ of the binary tetrahedral group

$$\mathrm{BT} := \left\{ \pm 1, \pm i, \pm j, \pm k, \frac{1}{2}\left(1 \pm i \pm j \pm k\right) \right\} \subset \mathbb{H},$$

where all possible sign combinations occur, by the representation

$$\rho(i) = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \ \rho(j) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \ \rho(k) = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}.$$

The ring of invariants $R = \mathbb{C}[x,y]^{\rho(\mathrm{BD}_{4n})}$ and $R = \mathbb{C}[x,y]^{\rho(\mathrm{BT})}$ are the respective coordinate rings for the D_{n+2} and E_6 Kleinian singularities $\operatorname{Max} R := \mathbb{C}^2/\rho(\mathrm{BD}_{4n})$ and $\mathbb{C}^2/\rho(\mathrm{BT})$.

Denote by Q the McKay quiver of $(\mathrm{BD_{4n}}, \rho)$ (resp. $(\mathrm{E_6}, \rho)$), shown in figure 10 (resp. figure 11), and let A be the preprojective algebra $A = \mathbb{C}Q/\left\langle \sum_i [a_i, \bar{a}_i] \right\rangle$. A is module-finite over its center Z, and $Z \cong R$ (this follows since A is Morita equivalent to the corresponding skew group ring $\mathbb{C}[x,y] * \mathrm{BD}_{4n}$ or $\mathbb{C}[x,y] * \mathrm{BT}$ [RV, proof of Proposition 2.13] which has center R, and Morita equivalent rings have isomorphic centers). Moreover, the smooth locus of $\operatorname{Max} R$ parameterizes the large A-modules,[6] and this fact is extended in Theorem 4.10, where we give strong evidence that Conjecture 2.5 holds for the D_{n+2} and E_6 surface singularities and their respective noncommutative coordinate rings A.

The following lemma is known, but we give a proof for completeness.

Lemma 4.7. *Let $A = \mathbb{C}Q/\left\langle \sum_i [a_i, \bar{a}_i] \right\rangle$ be the preprojective algebra of a quiver Q' whose underlying graph is extended Dynkin. Let d_i be the dimension of the irreducible representation of G corresponding to vertex i. Then the dimension vector of any large A-module is $d = (d_i)_{i \in Q_0}$.*

[6]This follows, for example, since the moduli space of θ-stable modules with $\theta = 0$ and dimension vector d coincides with the smooth locus of $\operatorname{Max} R$, and the only nonzero stable modules with $\theta = 0$ are simple.

Proof. By [C2, p. 18, (1) and (3)] when Q' is extended Dynkin the real roots of the corresponding (positive semi-definite) Tits form $q : \mathbb{Z}^{|Q_0|} \to \mathbb{Z}$ are the coordinate vectors ϵ_i, while the imaginary roots are the nonzero integer multiples of d. Thus for $m \in \mathbb{Z}$, $p(md) := 1 + q(md) = 1$, and so for $m \geq 2$ we have $p(md) = 1 < m = mp(d)$. Apply [C, Theorem 1.2]. □

Lemma 4.8. *Let $A = \mathbb{C}Q/I$ be a quiver algebra and let V be an A-module with pulled-apart supporting subquiver $\widetilde{Q}$ (with respect to some basis). Suppose that (i) $k \in \widetilde{Q}_0$ is a sink in $\widetilde{Q}$, (ii) there is a path in $\widetilde{Q}$ from each vertex $i \in \widetilde{Q}_0$ to k, and (iii) if $\mathrm{h}(a) = \mathrm{h}(b)$ for distinct $a, b \in \widetilde{Q}_1$ then a and b correspond to distinct arrows in Q_1. Then the socle of V is isomorphic to the vertex simple S_k.*

Proof. Let $j \in Q_0$ and consider a nonzero vector $v \in e_jV$ that is not in the 1-dimensional vector space at k. If $a \in Q_1e_j$ satisfies $0 \neq av =: w$ then consider w in place of v; otherwise if $av = 0$ then by (ii) and (iii) there exists a $b \in Q_1e_j$ such that $0 \neq bv =: w$. Since Q is finite, by (ii) we may iterate this process a finite number of times until $w \neq 0$ is in the 1-dimensional vector space at k. The isomorphism $\operatorname{Soc} V \cong S_k$ then follows by (i). (Note that if the dimension vector of V is not $(1, \ldots, 1)$ then (i) and (ii) alone are not sufficient to imply $\operatorname{Soc} V \cong S_k$.) □

Lemma 4.9. *Let $A = \mathbb{C}Q/I$ be the preprojective algebra of an extended Dynkin quiver, and let $d = (d_i)_{i \in Q_0}$ be the dimension vector of a large A-module. Suppose $V \in \operatorname{Rep}_d A$ and $d_k = 1$. If there is a cycle $c \in e_{\mathrm{t}(c)}A$ such that $c^n \notin \operatorname{ann}_A V$ for all $n \geq 1$ then $\operatorname{Soc} V$ cannot be isomorphic to the vertex simple S_k.*

Proof. If $c^nV \neq 0$ for all $n \geq 1$ then there is an eigenvector $v \in e_{\mathrm{t}(c)}V \subset V$ such that $c^m v = \gamma v$ for some $m \geq 1$ and $\gamma \in \mathbb{C}^*$. But then for sufficiently large r, e_k is a subpath of a term of $(c^m)^r$ modulo I (using the preprojective relations I), and the lemma follows since $c^{mr}v = \gamma^r v \neq 0$. □

Denote by $\mathcal{P}$ the path-like set $Q_{\geq 0} \cup \{0\}$. In the following theorem, let P_1 denote the $\mathcal{P}$-annihilator of an A-module with pulled-apart supporting subquiver given in figure 12 for the D_{n+2} case and in figure 13 for the E_6 case. We will assume that the chain

$$0 \subsetneq P_1 \tag{24}$$

is maximal in the sense of Definition 2.4, which is expected by Lemma 4.9.

Theorem 4.10. *Let $A = \mathbb{C}Q/I$ be the D_{n+2} (resp. E_6) preprojective algebra, let*

$$\pi : Y \to \mathbb{C}^2/\rho(\mathrm{BD}_{4n}) \qquad \big(\textit{resp. } \; \pi : Y \to \mathbb{C}^2/\rho(\mathrm{BT})\big)$$

be the minimal resolution of the Gorenstein D_{n+2} (resp. E_6) surface singularity, and fix a vertex $k \in \{0, 1, n+1, n+2\}$ (resp. $k \in \{0, 5, 6\}$). If the chain (24) is maximal (which is expected), then the exceptional locus $\pi^{-1}(0)$ parameterizes the almost large A-modules with socles isomorphic to the vertex simple S_k. Furthermore, the

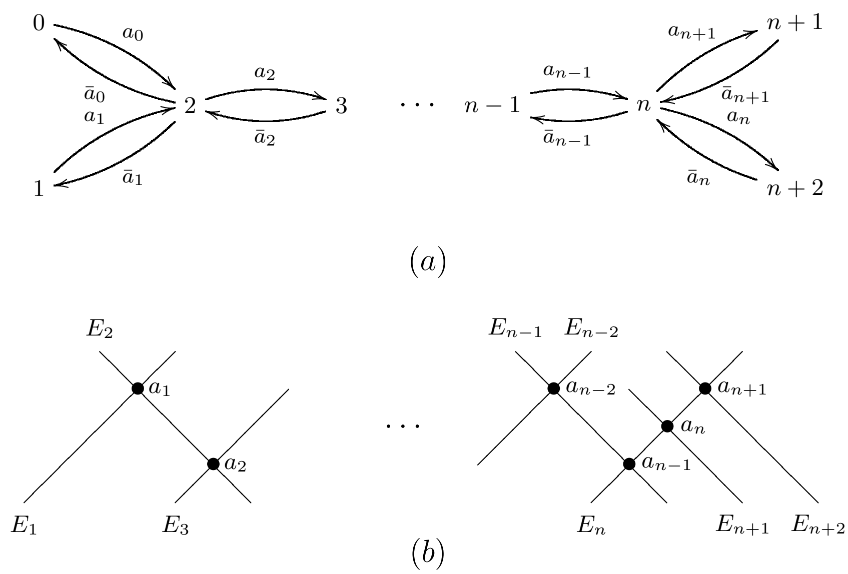


FIGURE 10. (a) The D_{n+2} McKay quiver Q. (b) The exceptional locus of the minimal resolution of the D_{n+2} singularity (each edge is a $\mathbb{P}^1$). By the McKay correspondence, there is an arrow $i \to j$ in Q iff the intersection $E_i \cap E_j$ is nonempty.

irreducible component E_i of $\pi^{-1}(0)$, associated to the vertex $i \in Q_0$ by the McKay correspondence, shrinks to the vertex simple A-module S_i.

Proof. Denote by $\widetilde{Y}$ the space that parameterizes the isoclasses of almost large modules whose socles are isomorphic to S_k.

Claim I: $Y \subseteq \widetilde{Y}$.
(i) $\mathbb{P}^1$*-families.* Each $\widetilde{Q}^i$ in figure 12 (resp. figure 13) is the support of a $\mathbb{P}^1$-family, minus the two points $(1:0)$ and $(0:1)$: apply the method "Trivialize J_0" in section 3.1 to determine the monomorphism

$$\sigma^i : A \to \mathrm{Mat}_{2n}(\mathbb{C}[s_i, t_i]),$$
$$\big(\text{resp.} \quad \sigma^i : A \to \mathrm{Mat}_{12}(\mathbb{C}[s_i, t_i])\ \big)$$

which is given by the labeling of $\widetilde{Q}^i$ in figure 12 (resp. figure 13). Here the unlabeled arrows are represented by ± 1, the sign being chosen so that the preprojective relations hold. By lemma 3.1, given any representation ρ supported on $\widetilde{Q}^i$ there is some $z \in \mathbb{C}^2$ such that ρ is isomorphic to $\epsilon_z \cdot \sigma$. In the D_{n+2} case: it is straightforward to check that the parameters (s,t) in the example given in figures 5.i and 5.iii coincide schematically with the respective parameters (s_i, t_i) for $2 \leq i \leq n$ and $i = 1, n+1, n+2$. In the E_6 case: one may check that the parameters (s,t) in the

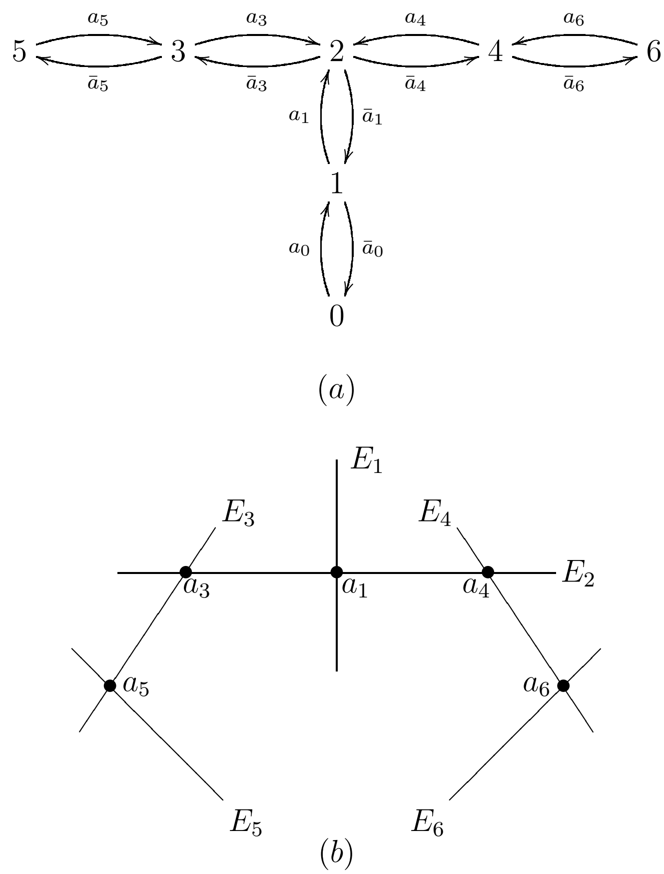


FIGURE 11. (a) The E_6 McKay quiver Q. (b) The exceptional locus of the minimal resolution of the E_6 singularity (each edge is a $\mathbb{P}^1$).

example given in figure 5.iii coincides schematically with the parameters (s_i, t_i) for $i = 1, \ldots, 6$; specifically, the two dimensional vector space in figure 5.iii sits inside the vector space at vertex $2, 3, 2, 3, 4 \in Q_0$ respectively. $\square$

(ii) $\ell_{\mathcal{P}} = 1$ *almost large modules.* By lemma 4.7, the almost large modules have dimension vector $d = (1, 1, 2, \ldots, 2, 1, 1)$ (resp. $(1, 2, 3, 2, 2, 1, 1)$). By lemma 4.8, any module supported on a pulled-apart subquiver given in figure 12 (resp. figure 13) has socle S_k. Here we assume the chain (24) is maximal by Lemma 4.9. $\square$

(iii) $\ell_{\mathcal{P}} = 2$ *almost large modules.* Each intersection point $E_i \cap E_j$ in the minimal resolution corresponds to a (unique) almost large module isoclass V that belongs to two $\mathbb{P}^1$-families. Although these two families have different pulled-apart supporting subquivers, namely $\widetilde{Q}^i$ and $\widetilde{Q}^j$, V is parameterized by the vanishing of a coordinate in each $\mathbb{P}^1$-family, and so the support of V is properly contained in both $\widetilde{Q}^i$ and $\widetilde{Q}^j$, as shown below.

In the D_{n+2} case:

$$\begin{array}{rlcll} a_1 = E_1 \cap E_2: & \widetilde{Q}_1^1 \setminus \{\xrightarrow{t_1}\} & = & \widetilde{Q}_1^2 \setminus \{\xleftarrow{s_2}, \xrightarrow{s_2}\}, & t_1 = s_2 = 0 \\ & \vdots & & & \vdots \\ a_j = E_j \cap E_{j+1}: & \widetilde{Q}_1^j \setminus \{\xrightarrow{t_j}\} & = & \widetilde{Q}_1^{j+1} \setminus \{\xleftarrow{s_{j+1}}, \xrightarrow{s_{j+1}}\}, & t_j = s_{j+1} = 0 \\ & \vdots & & & \vdots \\ a_{n-1} = E_{n-1} \cap E_n: & \widetilde{Q}_1^{n-1} \setminus \{\xrightarrow{t_{n-1}}\} & = & \widetilde{Q}_1^n \setminus \{\xleftarrow{s_n}\}, & t_{n-1} = s_n = 0 \\ a_n = E_n \cap E_{n+1}: & \widetilde{Q}_1^n \setminus \{\xrightarrow{t_n}\} & = & \widetilde{Q}_1^{n+1} \setminus \{\xleftarrow{s_{n+1}}\}, & t_n = s_{n+1} = 0 \\ a_{n+1} = E_n \cap E_{n+2}: & \widetilde{Q}_1^n \setminus \{\xrightarrow{-s_n - t_n}\} & = & \widetilde{Q}_1^{n+2} \setminus \{\xleftarrow{s_{n+2}}\}, & s_n + t_n = s_{n+2} = 0 \end{array}$$

In the E_6 case:

$$\begin{array}{rlcll} a_1 = E_1 \cap E_2: & \widetilde{Q}_1^1 \setminus \{\xrightarrow{s_1}\} & = & \widetilde{Q}_1^2 \setminus \{\xrightarrow{s_2 + t_2}\}, & s_1 = s_2 + t_2 = 0 \\ a_3 = E_3 \cap E_2: & \widetilde{Q}_1^3 \setminus \{\xrightarrow{s_3}\} & = & \widetilde{Q}_1^2 \setminus \{\xrightarrow{s_2}\}, & s_3 = s_2 = 0 \\ a_4 = E_4 \cap E_2: & \widetilde{Q}_1^4 \setminus \{\xrightarrow{t_4}\} & = & \widetilde{Q}_1^2 \setminus \{\xrightarrow{t_2}\}, & t_4 = t_2 = 0 \\ a_5 = E_3 \cap E_5: & \widetilde{Q}_1^3 \setminus \{\xleftarrow{t_3}, \xrightarrow{t_3}\} & = & \widetilde{Q}_1^5 \setminus \{\xrightarrow{s_5}\}, & t_3 = s_5 = 0 \\ a_6 = E_4 \cap E_6: & \widetilde{Q}_1^4 \setminus \{\xleftarrow{s_4}, \xrightarrow{s_4}\} & = & \widetilde{Q}_1^6 \setminus \{\xrightarrow{t_6}\}, & s_4 = t_6 = 0. \end{array}$$

Claim II: $Y \supseteq \widetilde{Y}$.

Consider the moduli space $\mathcal{M}_d^\theta(A)$ of stable A-modules with generic stability parameter $\theta = \left(-1 + \sum_{i \in Q_0} d_i, -1, \ldots, -1\right) \in \mathbb{Z}^{|Q_0|}$, where the first component is θ_k. This choice of θ is equivalent to restricting to modules in $\operatorname{Rep}_d(A)$ whose socles are isomorphic to S_k, and so any almost large module with socle S_k will be θ-stable. But $\mathcal{M}_d^\theta(A)$ is precisely Y by [K, Corollary 3.12], proving our claim. This is also implies that the path-like set $\mathcal{P} = Q_{\geq 0} \cup \{0\}$ is sufficient for determining all almost large modules, since the almost large modules with socle S_k obtained from $Q_{\geq 0} \cup \{0\}$ exhaust the set of all modules in $\operatorname{Rep}_d A$ with socle S_k.

□

4.4. **A non-isolated quotient singularity.** Consider the linear action of the finite abelian group $G = \mu_r^{\oplus 2} = \langle g_1, g_2 \rangle$ on $\mathbb{C}[x, y, z]$ by the representation

$$\rho(G) = \left\langle \rho(g_1) = \operatorname{diag}\left(\omega, \omega^{-1}, 1\right), \rho(g_2) = \operatorname{diag}\left(1, \omega^{-1}, \omega\right) \right\rangle \subset \mathrm{SU}_3(\mathbb{C}),$$

where ω is a primitive rth root of unity. The ring of invariants $R := \mathbb{C}[x, y, z]^{\rho(G)}$ is the coordinate ring for the non-isolated quotient singularity $\mathbb{C}^3/\rho(G) := \operatorname{Max} R$, which is a 3 dimensional version of the A_n singularity (see [R, Example 2.2]). Here we take $r = 4$.

We will find that the resolution of $\mathbb{C}^3/\rho(G)$ determined by the basic triangulation of its toric diagram, given in figure 14, parameterizes the large modules and almost

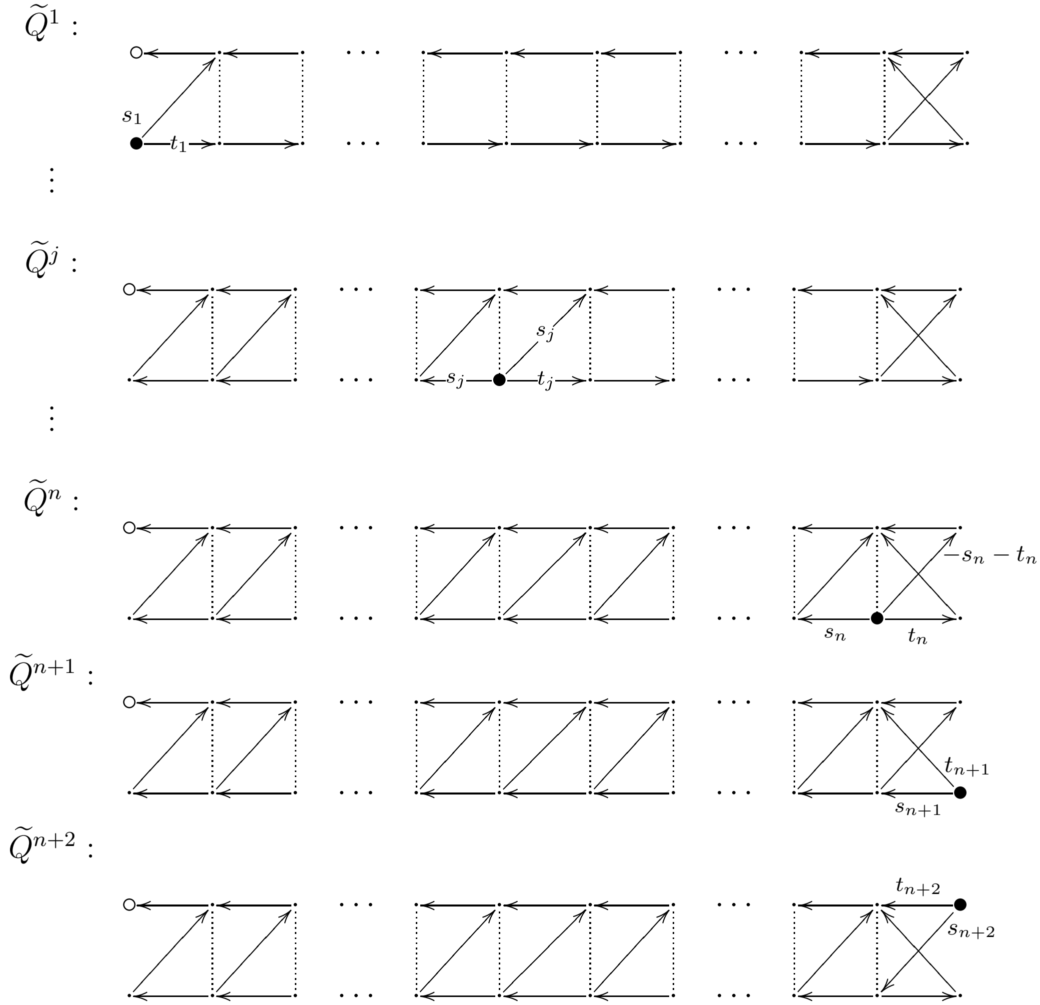


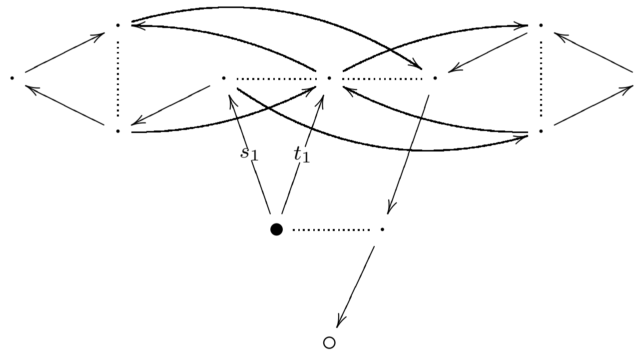
FIGURE 12. The $n+2$ pulled-apart supporting subquivers of the almost large modules over the D_{n+2} preprojective algebra, up to isomorphism. Vertices connected by a dotted edge correspond to the same vertex in Q_0. Vertex k is denoted $\circ$, and each $\mathbb{P}^1$ shrinks to the vertex simple at the vertex denoted $\bullet$.

large modules with isomorphic 1-dimensional socles over the McKay quiver algebra of (G, ρ).[7] The McKay quiver Q of (G, ρ) is determined by noting that there are r^2 irreducible representations ρ_{ij} of G, all of which are 1-dimensional,

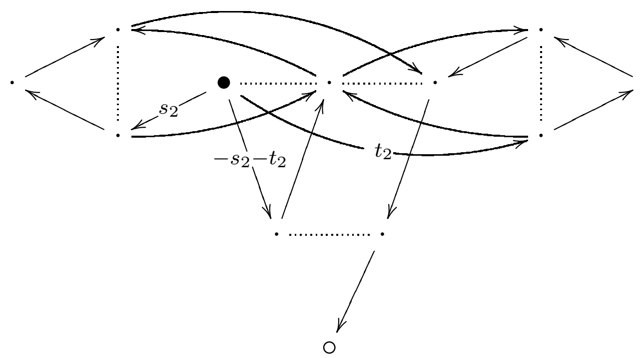
$$\rho_{ij}(g_1) = \omega^i, \quad \rho_{ij}(g_2) = \omega^j.$$

[7]Note that the 3 regular hexagons in the triangulation correspond to 3 del Pezzo surfaces of degree 6, that is, 3 $\mathbb{P}^2$'s blown up at 3 points not on a line.

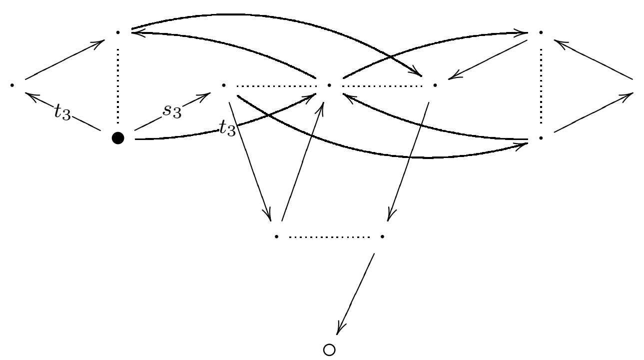
$\widetilde{Q}^1$:



$\widetilde{Q}^2$:



$\widetilde{Q}^3$:

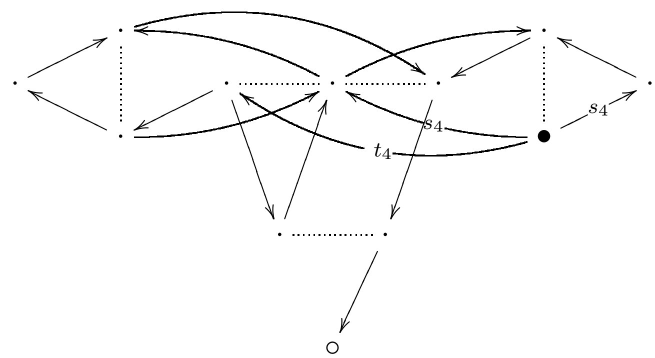


Q may be drawn on a two-torus as shown in figure 15. Denote by $a_i, b_i, c_i \in Q_1 e_i$ the respective arrows that head up, right, and downward to the left, and set $a := \sum_{i \in Q_0} a_i$, $b := \sum_{i \in Q_0} b_i$, and $c := \sum_{i \in Q_0} c_i$. The McKay quiver algebra of (G, ρ) is then

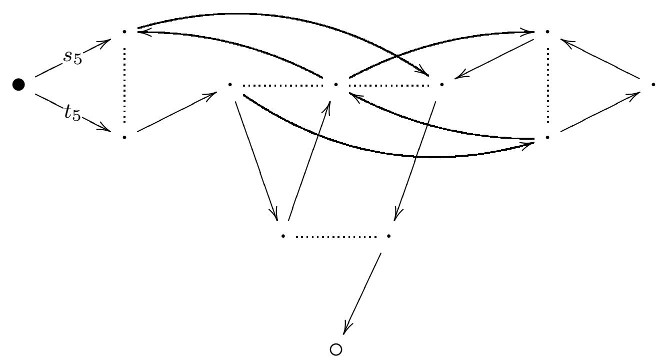
$$A = \mathbb{C}Q / \langle ab - ba, bc - cb, ca - ac \rangle .$$

By [B, Theorem 3.7, with $(x, y, z) = (x_1, y_1, x_2 y_2)$], the large A-modules have dimension vector $(1, \ldots, 1)$, and an impression $(\tau, \mathbb{C}[x, y, z])$ of A is given by the labeling

$\widetilde{Q}^4$:



$\widetilde{Q}^5$:



$\widetilde{Q}^6$:

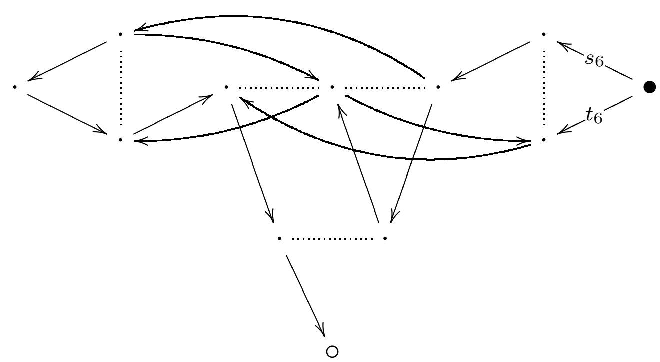


FIGURE 13. The 6 pulled-apart supporting subquivers of the almost large modules over the E_6 preprojective algebra, up to isomorphism. Vertices connected by a dotted edge correspond to the same vertex in Q_0. Vertex k is denoted $\circ$, and each $\mathbb{P}^1$ shrinks to the vertex simple at the vertex denoted $\bullet$.

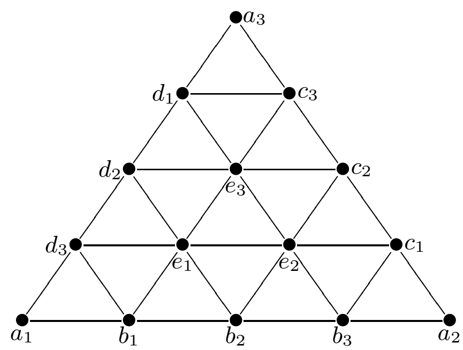


FIGURE 14. The basic triangulation of the toric diagram for the resolution of $\mathbb{C}^3/\rho(G)$ that parameterizes the large modules and almost large modules with isomorphic 1-dimensional socles.

of arrows

$$\bar{\tau}(a_i) = x, \quad \bar{\tau}(b_i) = y, \quad \bar{\tau}(c_i) = z.$$

The following proposition extends the fact that the large A-modules are parameterized by the smooth locus of $\mathbb{C}^3/\rho(G)$.

Proposition 4.11. *Let $A = \mathbb{C}Q/I$ be the McKay quiver algebra for $\left(\mu_4^{\oplus 2}, \rho\right)$, and let $\pi : Y \to \mathbb{C}^3/\rho(\mu_4^{\oplus 2})$ be the resolution determined by the basic triangulation of the toric diagram in figure 14. Then the exceptional locus E parameterizes the almost large A-modules with socle isomorphic to any fixed vertex simple.*

Proof. Recall that the large A-modules have dimension vector $(1, \ldots, 1)$. Denote by $\mathcal{P}$ the path-like set $Q_{\geq 0} \cup \{0\}$. Since $\dim Z = 3$, we must consider $\ell_{\mathcal{P}} = 1, 2, 3$ almost large modules. Fix a vertex $0 \in Q_0$, denoted $\circ$ in figures 16 - 18; here each subquiver is drawn on a two-torus.

$\ell_{\mathcal{P}} = 1$ *almost large modules.* The supporting subquivers for the $\ell_{\mathcal{P}} = 1$ large modules are displayed in figure 16, while the supporting subquivers for $\ell_{\mathcal{P}} = 1$ almost large modules with socle S_0 are displayed in figures 17 and 18, where by a "$\mathbb{P}^n$-family" we really mean a family parameterized by $\mathbb{P}^n$ minus the $n+1$ points of where one of the coordinates is zero. These subquivers are determined as follows: Let $V \in \operatorname{Rep}_{(1,\ldots,1)} A$ be an $\ell_{\mathcal{P}} = 1$ almost large module. Then there is some arrow a that annihilates V since the dimension vector of V is $(1, \ldots, 1)$. For each $i \in Q_0$, denote by $\gamma_i \in e_i A e_i$ the unique cycle (modulo I) at vertex i of length 3. Since a annihilates V, the cycle γ_j containing a as a subpath also annihilates V, and since $\sum_{i \in Q_0} \gamma_i$ is in the center of A by [B, Theorem 2.7], each cycle γ_i must annihilate V. But again since the dimension vector of V is $(1, \ldots, 1)$, at least one arrow in each cycle γ_i must annihilate V. Thus the supporting subquivers for the $\ell_{\mathcal{P}} = 1$ modules have at least one arrow removed from each cycle of length 3. If a cycle of length 3 has two arrows removed, then we will find below that such a subquiver supports an $\ell_{\mathcal{P}} = 2$ almost large module.

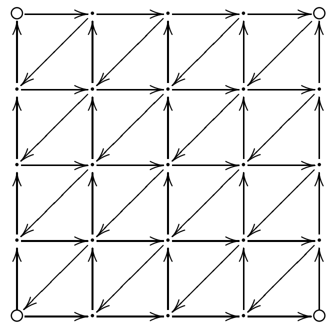


FIGURE 15. The McKay quiver for (G, ρ), drawn on a two-torus. The vertices denoted $\circ$ are identified.

$\ell_{\mathcal{P}} = 2$ *almost large modules.* The supporting subquivers for all $\ell_{\mathcal{P}} = 2$ almost large modules with socle S_0 are also displayed in figures 16 - 18, and they are obtained as follows. Suppose two vertices in the toric diagram (figure 14), say g and h, are connected by an edge. Then the irreducible components of the exceptional locus corresponding to g and h have nonempty intersection, and an open subset of this intersection parameterizes the $\ell_{\mathcal{P}} = 2$ almost large module isoclasses with supporting subquivers having vertex set Q_0 and arrow set

$$g \cap h : Q_1^g \setminus \{ \text{ arrows labeled by } i\} = Q_1^h \setminus \{ \text{ arrows labeled by } j\},$$

where Q^g, Q^h, and the labels i and j are displayed in figures 16 - 18. The following table verifies this explicitly.

g	h	i	j	g	h	i	j	g	h	i	j	g	h	i	j	g	h	i	j
a_1	d_3	2	2	b_2	e_2	4	6	c_1	e_2	4	2	c_3	d_1	4	4	d_2	e_1	3	2
a_1	b_1	1	1	b_2	b_3	2	1	c_1	c_2	3	1	c_3	a_3	3	1	d_2	d_3	2	1
b_1	d_3	3	3	b_3	e_2	3	3	c_2	e_2	2	5	a_3	d_1	2	1	d_3	e_1	4	4
b_1	e_1	4	3	b_3	c_1	4	2	c_2	e_3	4	5	d_1	e_3	3	3	e_1	e_2	5	1
b_1	b_2	2	1	b_3	a_2	2	1	c_2	c_3	3	1	d_1	d_2	2	1	e_2	e_3	4	4
b_2	e_1	3	1	a_2	c_1	2	1	c_3	e_3	2	2	d_2	e_3	4	6	e_3	e_1	1	6

One may check that all other $\ell_{\mathcal{P}} = 2$ almost large modules do not have socle S_0.

$\ell_{\mathcal{P}} = 3$ *almost large modules.* There are 8 $\ell_{\mathcal{P}} = 3$ almost large module isoclasses, and these correspond to the faces of the basic triangles in the toric diagram. The supporting subquiver for such a module is obtained by intersecting the three subquivers corresponding to the vertices of the corresponding basic triangle, and all other $\ell_{\mathcal{P}} = 3$ almost large modules do not have socle S_0. We leave the verification to the reader.

It is clear that the almost large modules with socle S_0 obtained from the path-like set $\mathcal{P} = Q_{\geq 0} \cup \{0\}$ exhaust the set of all modules in $\text{Rep}_{(1,\cdots,1)} A$ with socle S_0, and so no other path-like set need be considered. $\square$

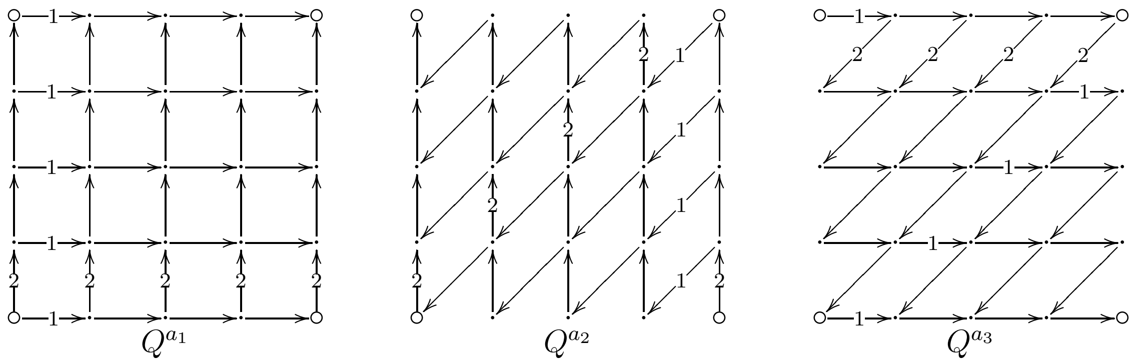


FIGURE 16. Q^{a_1} (resp. Q^{a_2}, Q^{a_3}) supports the $\mathbb{C}^* \times \mathbb{C}^*$-family of $\ell_{\mathcal{P}} = 1$ large modules parameterized by the vanishing of the single coordinate $z = 0$ (resp. $x = 0$, $y = 0$) in the smooth locus of $\operatorname{Max} Z$, corresponding to the vertex a_1 (resp. a_2, a_3) in the toric diagram (figure 14). For each $1 \le i \le 2$, the subquiver obtained by removing all arrows from Q^{a_j} labeled i supports the $\mathbb{C}^*$-family of $\ell_{\mathcal{P}} = 2$ almost large modules corresponding to an edge emanating from a_j in the toric diagram.

Remark 4.12. This example shows that the irreducible components of the exceptional locus need not shrink to the annihilator of a vertex simple module: Each $\mathbb{P}^1$-family supported on a subquiver in figure 17 shrinks to the annihilator of a simple module supported on a subquiver with vertex set given by the bold vertices in the figure. Such a point in $\operatorname{Max} A$, which we view as a point-like sphere, sits over a point of $\operatorname{Max} R$ with one non-vanishing coordinate (x, y, or z). Furthermore, each $\mathbb{P}^2$-family supported on a subquiver in figure 18 collapses to two points in $\operatorname{Max} A$, namely the annihilators of the two vertex simples at the bold vertices in the figure. Both of these points sit over the origin of $\operatorname{Max} R$.

Remark 4.13. This example and the conifold quiver algebra from section 4.1 are examples of square superpotential algebras (see [B, Definition 1.1]). The supporting subquivers for the $\ell_{\mathcal{P}} = 1$ (resp. $\ell_{\mathcal{P}} = 2$; $\ell_{\mathcal{P}} = 3$) large and almost large modules coincide with the subquivers obtained by removing all the arrows from Q that occur in a so called perfect matching (resp. the intersection of two perfect matchings; the intersection of three perfect matchings). In this sense perfect matchings may be viewed as a special case of almost large modules over a particular class of quiver algebras whose centers are toric Gorenstein singularities, and whose relations are derived from a potential. This observation will be addressed in a forthcoming paper, [B2].

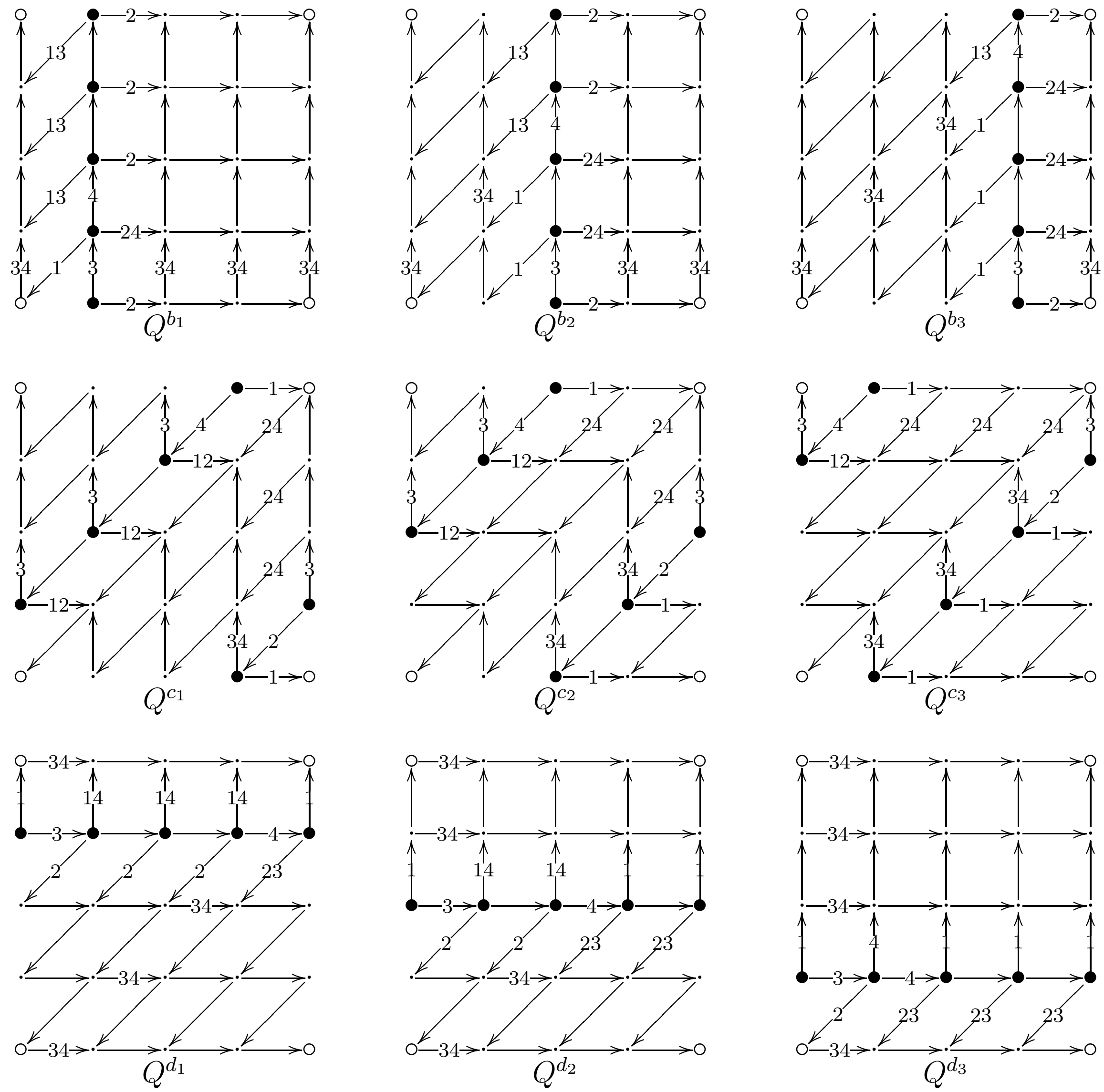


FIGURE 17. For $g \in \{b_j, c_j, d_j\}$, Q^g supports the $\mathbb{C}^*$-family of $\mathbb{P}^1$-families of $\ell_{\mathcal{P}} = 1$ almost large modules parameterized by the irreducible component of the exceptional locus corresponding to the vertex g on the perimeter of the toric diagram (figure 14). For each $1 \leq i \leq 4$, the subquiver obtained by removing all arrows from Q^g labeled i supports the $\mathbb{C}^*$- or $\mathbb{P}^1$-family of $\ell_{\mathcal{P}} = 2$ almost large modules corresponding to an edge emanating from g in the toric diagram ($\mathbb{C}^*$ iff the edge is along the perimeter).

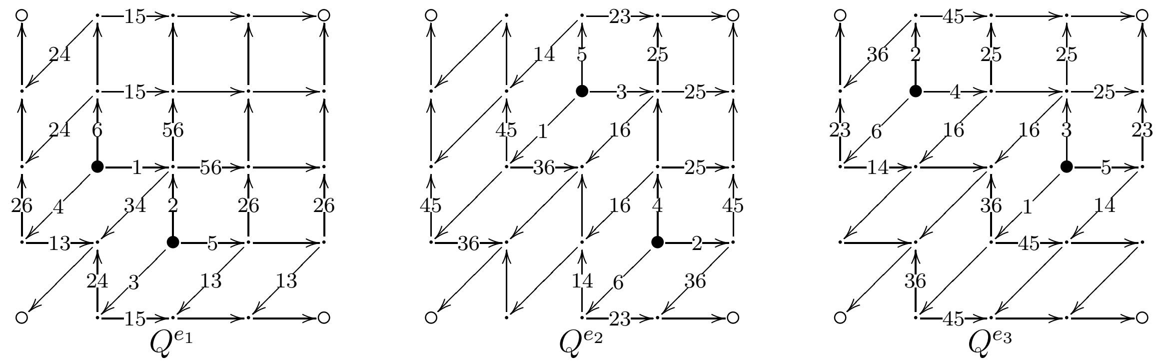


FIGURE 18. Q^{e_j} supports the $\mathbb{P}^2$-family of $\ell_{\mathcal{P}} = 1$ almost large modules parameterized by the irreducible component of the exceptional locus corresponding to the vertex e_j in the toric diagram (figure 14). For each $1 \leq i \leq 6$, the subquiver obtained by removing all arrows from Q^{e_j} labeled i supports the $\mathbb{P}^1$-family of $\ell_{\mathcal{P}} = 2$ almost large modules corresponding to an edge emanating from e_j in the toric diagram.

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