Ghosts of Critical Gravity

Massimo Porrati and Matthew M. Roberts

Center for Cosmology and Particle Physics
Department of Physics
New York University
4 Washington Place
New York, NY 10003, USA

Abstract

Recently proposed "critical" higher-derivative gravities in AdS_D $D > 3$ are expected to carry logarithmic representation of the Anti de Sitter isometry group. In this note, we quantize linear fluctuations of these critical gravities, which are known to be either identical with linear fluctuations of Einstein's gravity or else satisfy logarithmic boundary conditions at spacial infinity. We identify the scalar product uniquely defined by the symplectic structure implied by the classical action, and show that it does not posses null vectors. Instead, we show that the scalar product between any two Einstein modes vanishes, while the scalar product of an Einstein mode with a logarithmic mode is generically nonzero. This is the basic property of logarithmic representation that makes them neither unitary nor unitarizable.

1 Introduction

It has been known for many years that power-counting renormalizable theories of gravity can be obtained by adding to the Einstein-Hilbert action appropriate terms, quadratic in the Ricci and Weyl tensors. In the absence of a cosmological constant, these theories admit Minkowski space as background, but they are also perturbatively non-unitary [1]. Since the whole point of having a power-counting renormalizable theory of gravity is to make perturbative calculations possible, these theories have been justly abandoned long since. Recently, quadratic-curvature actions *with* cosmological constant were re-examined in four [2] and D [3] dimensions. In either case, it was found that there exist a choice of parameters for which these theories possess one AdS background on which neither massive fields, nor massless scalars or vectors propagate. Moreover, on the AdS background, the standard graviton, i.e. the massless tensor mode of Einstein-Hilbert gravity, also propagates and has vanishing energy (the energy of course depends on the action, not just on the form of the mode) [2, 3].

Besides those that satisfies the homogenous Einstein equations on AdS_D, other tensor modes propagate in the “critical” theory [4, 5, 6]. Their asymptotic behavior at space-like infinity differs from standard Einstein-Hilbert modes by terms logarithmic in the AdS radial coordinate. A complete set of propagating modes for critical D-dimensional gravity was presented in [6].

In this note, we show that there exists an unambiguous manner to define the energy and the norm of all modes of log gravity. With that definition, the scalar product of two modes that solve the homogenous Einstein equation vanishes.

Next we come to out main result: the scalar product of a homogenous mode with some of the logarithmic modes is nonzero. In other words, homogenous modes *are not* null vectors and cannot be factored out to yield a (positive-norm) Hilbert space, except if we restrict the physical space to homogenous modes only, and then factor them out. This procedure leave a profoundly uninteresting theory made only the vacuum state. This picture should be compared to the case of $D = 3$, where CFTs possess two copies of the Virasoro algebra. There, in Topologically Massive Gravity (TMG) at the critical point [7], restriction to homogenous modes –which can be promoted to a bona fide non-perturbative constraint on the Hilbert space– selects the vacuum of one such algebra, but allows for nontrivial states of the other [7, 8].

This result shows that the “critical” (a.k.a. log) theory is neither unitary nor does it contain a unitary subspace other than the vacuum [1]. Though the lack of unitarity proven here is bad news for log gravity to give a viable quantum theory of gravity in AdS_D, it

[1]The ghost pole in one-particle exchange amplitudes between physical sources may cancel [9]; this is not enough to rescue unitarity as we shall discuss in the last section.

is consistent with (and indeed required by) it being dual to a logarithmic conformal field theory in $D-1$ dimensions. Such a theory could be of interest in statistical mechanics; its basic properties are summarized in the next section.

Kinematics of Logarithmic CFTs

The isometry group of AdS_D is $SO(2, D-1)$, which is also the conformal group in $D-1$ space-time dimensions. So, the Hilbert space [2] of a consistent quantum gravity in AdS_D decomposes into a direct sum of representations of $SO(2, D-1)$.

If we do not demand that the representation be unitary, then the Hilbert space, H, can contain a reducible but *indecomposable* representation. Let us consider in detail the case of AdS_4. Its isometry group, $SO(2,3)$, admits a Cartan decomposition into four positive roots E^{α_a}, $a = 1, 2, 3, 4$, four negative roots $E^{-\alpha_a}$, and two Cartan generators H_1, H_2. [3]. In the Cartan basis the $SO(2,3)$ algebra is:

$$[H_i, H_j] = 0, \qquad [H_i, E^{\alpha_a}] = \alpha_a^i E^{\alpha_a}, \qquad [E^{-\alpha_a}, E^{\alpha_a}] = \frac{2}{|\alpha_a|^2}\alpha_a \cdot H, \tag{1}$$

with $|\alpha_a|^2 = \sum_{i=1}^2 (\alpha_a^i)^2$.

$SO(2,3)$ representations with energy H_1 bounded below posses a ground state ψ, annihilated by all $E^{\alpha_a}\psi = 0$. Lowest weight vectors can also define logarithmic representations if they are not eigenstates of H_1, but obey instead [4]

$$H_1\psi = E_0\psi + \phi, \qquad H_1\phi = E_0\phi. \tag{2}$$

When the generators H_i, E^α, are self-adjoint with respect to a scalar product, $\langle\ ,\ \rangle$, not necessarily positive definite, then the vector ϕ has zero norm:

$$0 = \langle\psi, (H_1 - E_0)^2\psi\rangle = \langle (H_1 - E_0)\psi, (H_1 - E_0)\psi\rangle = \langle\phi, \phi\rangle. \tag{3}$$

The standard procedure to obtain a non-degenerate scalar product is to identify vectors modulo null vectors. So, if the vector ϕ is null, i.e. if $\langle\chi, \phi\rangle = 0$ for all χ in the representation V, eqs. (2) actually define a standard lowest-weight representation on the quotient space. So, to obtain a truly new representation, the scalar product of ϕ with some vector $\chi \in V$ (such that $\langle\chi, \chi\rangle \neq 0$) *must* be non-vanishing.

[2]By Hilbert space we mean here a vector space H, endowed with a non-degenerate but not necessarily positive bilinear form $\langle u, v\rangle$ such that $u \in H$, $v \in H \mapsto \langle u, v\rangle \in \mathbb{C}$.

[3]The basis used in ref. [6] is: $\alpha_1 = (-1, 1)$, $\alpha_2 = (0, 1)$, $\alpha_3 = (-1, -1)$, $\alpha_4 = (-1, 0)$.

[4]One could consider in principle also poly-logarithmic representations defined by $(H_1 - E_0)\psi^k = \psi^{k+1}$, $k = 0, ..., n$.

Of course, the latter property is incompatible with unitariy, i.e. with the scalar product being positive definite. For if $\langle \phi, \chi \rangle = A \neq 0$, the norm of $z\phi + \chi$ is $\langle \chi, \chi \rangle + 2\mathrm{Re}\, Az$, which can have either sign when z ranges over the complex plane.

Two dimensional logarithmic conformal field theories too are characterized by having zero norm non-null vectors [10] (for a review see [11]). Topologically Massive Gravity (TMG) at the critical point, which is conjectured to be dual to a logarithmic CFT, indeed contains such vectors [12, 13, 14].

Four dimensional critical gravity is in many ways the higher dimensional analog of TMG at the critical point; so, the next question to ask is: does critical gravity too contain logarithmic representations of $SO(2,3)$?

The Example of $D = 4$ Critical Gravity

Ref. [6] gives a complete set of modes for critical gravity. With some obvious changes of notations and simplifications, the $4D$ action of [6] is

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left[R - 2\Lambda - \frac{1}{2} f^{\mu\nu} G_{\mu\nu} + \frac{m^2}{8} (f^{\mu\nu} f_{\mu\nu} - f^2) \right], \tag{4}$$

with $G_{\mu\nu}$ the Einstein tensor and $f_{\mu\nu}$ an auxiliary symmetric tensor field. Elimination of $f_{\mu\nu}$ through its algebraic equations of motion gives an action quadratic in curvatures. Critical gravity is obtained when the cosmological constant is

$$\Lambda = -3m^2. \tag{5}$$

In $4D$, Λ is the usual cosmological constant and action (4) admits an Anti de Sitter background $\bar{g}_{\mu\nu}$ with $R^{\mu\nu}_{\rho\sigma} = (\Lambda/3)(\delta^\mu_\rho \delta^\nu_\sigma - \mu \leftrightarrow \nu)$.

Expanding $g_{\mu\nu}$ and $f_{\mu\nu}$ around the AdS background as

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \qquad f_{\mu\nu} = -2(\bar{g}_{\mu\nu} + h_{\mu\nu}) + \frac{2}{3m^2} k_{\mu\nu}, \tag{6}$$

Action (4) reduces to a constant term plus the quadratic action [6]

$$6m^2 S_2 = \int d^4x \sqrt{-\bar{g}} \left[2h^{\mu\nu} \mathcal{G}_{\mu\nu}(k) - \frac{1}{3} (k^{\mu\nu} k_{\mu\nu} - k^2) \right]. \tag{7}$$

The linearized Einstein operator $\mathcal{G}_{\mu\nu}$ reduces to $-(1/2)(\Box + 2m^2)$ on transverse-traceless modes. The equations of motion following from action (7) are

$$\mathcal{G}_{\mu\nu}(h) = \frac{1}{3}(k_{\mu\nu} - \bar{g}_{\mu\nu} k), \qquad \mathcal{G}_{\mu\nu}(k) = 0. \tag{8}$$

Thanks to the Bianchi identity, $k_{\mu\nu}$ is transverse and traceless. In the gauge $D^\mu h_{\mu\nu} - D_\nu h = 0$ eqs. (8) become

$$-\frac{1}{2}(\Box + 2m^2)h_{\mu\nu} = \frac{1}{3}k_{\mu\nu}, \qquad -\frac{1}{2}(\Box + 2m^2)k_{\mu\nu} = 0. \tag{9}$$

In global coordinates (co-latitude θ, longitude ϕ, radius ρ, and time t) the AdS_4 metric is

$$m^2 ds^2 = -\cosh^2(\rho)dt^2 + d\rho^2 + \sinh^2(\rho)[d\theta^2 + \sin^2(\theta)d\phi^2]. \tag{10}$$

The Cartan generators of $SO(2,3)$ are $H_1 = i\partial_t$, $H_2 = -i\partial_\phi$ [6]. The explicit expressions for the E^α generators are also given in [6]. Imposing $E^{\alpha_a}\psi = 0$ one finds two particularly interesting classes of solutions to (9).

One, $\psi^E_{\mu\nu}$, solves the homogeneous linearized Einstein equation $\mathcal{G}_{\mu\nu}(\psi^E) = 0$. In global coordinates it has the form $\psi^E_{\mu\nu} = e^{-iE_0 t + 2i\phi}F(\rho,\theta)_{\mu\nu}$. For large ρ one finds

$$F(\rho,\theta)_{\rho\rho} \sim e^{-(E_0+2)\rho}, \qquad F(\rho,\theta)_{\rho *} \sim e^{-E_0\rho}, \qquad F(\rho,\theta)_{**} \sim e^{-(E_0-2)\rho}. \tag{11}$$

Here $*$ denotes coordinates other than ρ and we did not spell out the θ dependence in F. Normalizability of ψ^E for $\rho \to \infty$ gives $E_0 = 3$ [6].

The other one is

$$f(t,\rho)\psi^E_{\mu\nu}, \qquad f(t,\rho) = it + \log\sinh\rho. \tag{12}$$

It obeys eq. (8) with $k_{\mu\nu} = -(9m^2/2)\psi^E_{\mu\nu}$ [5] .

After this brief review of the result of [6] we come to the definition of the scalar product and energy for linearized critical gravity.

The Inner Product of Quadratic Theories

Consider a general quadratic action

$$S = \int dt \frac{1}{2}(-\dot{q}^T L\dot{q} + +\dot{q}^T Qq + q^T Kq), \tag{13}$$

where L and K are symmetric matrices while Q is antisymmetric and commuting with L: $[L,Q] = 0$. They are defined in terms of a matrix Ω obeying

$$L\Omega - \Omega^T L = 2iQ, \qquad \Omega^T L\Omega = K. \tag{14}$$

Reality of K follows from $[L,Q] = 0$; together with the first of the equations above, it implies that $L\Omega = S + iQ$, with S a real symmetric matrix. If Ω satisfies eqs. (14)

[5] Our inhomogeneous mode equals that in [6] plus a homogenous mode.

so does $-\Omega^*$. When some of the eigenvalues of the matrix Ω coincide, Ω may be non-diagonalizable but it can always be put it in Jordan form. We choose Ω to have "positive frequency" by demanding that its eigenvalues are positive definite.

Next we split the vector q into $q = 2^{-1/2}(A + A^*)$ with $i\dot{A} = \Omega A$ $(i\dot{A}^* = -\Omega^* A^*)$. Eqs. (14) guarantee that A solves the equations of motion $L\ddot{A} - 2Q\dot{A} + KA$. The canonical momentum p conjugate to q is $p = L\dot{q} - Qq$. The canonical momenta conjugate to A, A^* are

$$P = L\dot{A}^* - QA^* = iL\Omega^* A^* - QA^*, \qquad P^* = L\dot{A} - QA = -iL\Omega A - QA. \tag{15}$$

By using eqs. (14,15) it is straightforward to find that the conserved energy is

$$H = \frac{1}{2}(\dot{q}^T L\dot{q} + q^T K q) = \frac{1}{2}(P^T \dot{A} + P^{*T} \dot{A}^*) \tag{16}$$

In canonical quantization, A^* is replaced by the Hermitian conjugate $A^\dagger$ and the non-zero canonical commutation relations are $[A^I, P_J] = i\delta^I_J$, $[A^{\dagger I}, P^\dagger_J] = i\delta^I_J$. The standard Fock vacuum obeys $A|0\rangle = 0$, so a classical positive-frequency solution of the equations of motion, $i\dot{\Phi}(t) = \Omega\Phi(t)$, defines a one-particle state $|\Phi\rangle = -i\Phi^I P_I|0\rangle$, which obeys $A(t)|\Phi\rangle = \Phi(t)|0\rangle$. The scalar product of two states $|\Phi\rangle$, $|\Psi\rangle$ is then

$$\langle\Psi|\Phi\rangle = i(\Psi^{*T} L\dot{\Phi} - \Psi^{*T} Q\Phi). \tag{17}$$

An example directly related to action (7) is

$$L = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \Omega = \begin{pmatrix} \omega & 0 \\ 1/2\omega & \omega \end{pmatrix}, \qquad A(t) = \begin{pmatrix} 2\omega i\alpha e^{-i\omega t} \\ (\alpha t + \beta)e^{-i\omega t} \end{pmatrix} \tag{18}$$

Notice that, though $A(t)$ contains terms linear in t, the scalar product is time-independent. Explicitly, on two solutions defined by constants α, β, α', β', the scalar product formula (17) reduces to $\langle\Psi|\Phi\rangle = 2\omega\alpha'^*\alpha + 2\omega^2 i(\beta'^*\alpha - \alpha'^*\beta)$. States with $\alpha = \alpha' = 0$ have vanishing norm, but these states are not null: the norm of state $\alpha = 0, \beta \neq 0$ with an $\alpha' \neq 0$ state does not vanish.

Action (7) has the form (13). To see that, we can use the fact that $k_{\mu\nu}$ is transverse-traceless and integrate by part in time [6]

$$6m^2 S_2 = \int d^4x\sqrt{-\bar{g}}\left[\bar{g}^{00} D_0 h^{\mu\nu} D_0 k_{\mu\nu} - h^{\mu\nu}\left(\sum_{i=1}^{3} D_i D^i + 2m^2\right) k_{\mu\nu} - \frac{1}{3}(k^{\mu\nu}k_{\mu\nu} - k^2)\right], \tag{19}$$

[6] Initial time and final time configurations are held fixed when varying the action, so we can always add a total *time* derivative to the action without changing the equations of motion. Adding a total divergence of space coordinate, instead, changes the boundary conditions at the AdS boundary, so in general it changes the equations of motion by modifying the boundary behavior of the fields.

Now, a straightforward application of formula (17) gives the scalar product of any two positive-frequency modes of log gravity as

$$\langle\psi|\phi\rangle = \frac{3i}{4}\int d^3x\sqrt{-\bar{g}}\bar{g}^{00}[(\Box+2m^2)\psi^*_{\mu\nu}D_0\phi^{\mu\nu} + \psi^*_{\mu\nu}D_0(\Box+2m^2)\phi^{\mu\nu}] \tag{20}$$

Evidently, the scalar product between two solutions of the homogeneous Einstein equations, $\phi=\phi^E$, $\psi=\psi^E$ vanishes. Also, the potentially dangerous term $\propto t$ in the scalar product of two "log" modes $\phi^s = f(t,\rho)\phi^E$, $\psi = f(t,\rho)\psi^E$ (see definition in eq. (12)) vanishes.

Crucially, the homogeneous modes ψ^E are *not* null vectors, since their scalar product with the log mode constructed from ψ^E, $\phi^s = f(t,\rho)\psi^E$ is nonzero:

$$\langle\psi^E|\phi^s\rangle = -\frac{9im^2}{4}\int d^3x\sqrt{-\bar{g}}\bar{g}^{00}\psi^{E\,*}_{\mu\nu}D_0\psi^{E\,\mu\nu} \neq 0. \tag{21}$$

Non vanishing of eq. (21) can be proven by a simple direct calculation or by noticing that (21) is proportional to the norm of the transverse-traceless mode $\psi^E_{\mu\nu}$ in standard Einstein gravity [7].

The Einstein modes do have zero scalar product with special logarithmic modes: the spin-1 "Proca" modes [6], which have the form $\psi^s_{\mu\nu} = f\psi^E$, $\psi^E = D_{(\mu}A_{\nu)}$. Transversality and tracelessness of $\psi^E_{\mu\nu}$ hold when A_μ obeys the massive spin-1 Proca equation

$$D^\nu F_{\nu\mu} = 6m^2 A_\mu. \tag{22}$$

Vanishing of $\langle\psi^E|fD_{(\mu}A_{\nu)}\rangle$ follows immediately from the fact that this quantity is proportional to the Einstein gravity scalar product $\int d^3x\sqrt{-\bar{g}}\bar{g}^{00}\psi^{E\,*}_{\mu\nu}D_0D^\mu A^\nu$, which vanishes because in Einstein gravity transverse modes are orthogonal to pure gauge modes.

In $3D$, the Proca modes are the only inhomogeneous solutions of eq. (8), hence the Einstein modes are true null vectors that can be modded out to yield a positive-metric Hilbert space. Actually, the normalizable Proca modes decay so rapidly at infinity that they too are null in the norm induced by the NMG action. So, the modding out by these null vectors yields a trivial theory in 3D too [8]. Of course, one can also define the norm of the Proca fields using the Proca action; this is what makes new massive gravity [15] at the critical point potentially nontrivial and consistent, at least at linear order. In $D>3$, the Proca modes [6] and the truly transverse-traceless spin-2 logarithmic modes mix under the action of $SO(D-1)\subset SO(2,D-1)$; therefore, one cannot consistently keep the Proca modes, which are transverse to the Einstein modes, without also keeping the spin-2 modes, which are not.

[7]See e.g. ref. [13] for the analogous calculation in 3D chiral gravity.

[8]This fact became clear in conversations with O. Hohm.

Linearized Energy

A reasonable way to define the energy of a solution which is asymptotically AdS is to construct the linearized stress tensor of a metric perturbation, and integrate the tt component over a spacial slice, which by the Gauss' law constraint reduces to a boundary term. Including the nonlinear terms schematically in the r.h.s of the equations of motion, we find the equations

$$\mathcal{G}(k_{\mu\nu}) = \Theta^{(NL)}_{\mu\nu},\ \mathcal{G}(h_{\mu\nu}) - \frac{1}{3}(k_{\mu\nu} - \bar{g}_{\mu\nu}k) = \Xi^{(NL)}_{\mu\nu}. \tag{23}$$

Note that thanks to the Bianchi identity $\Theta^{(NL)}_{\mu\nu}$ is automatically covariantly conserved, while $\Xi^{(NL)}_{\mu\nu}$ is not. We therefore can use $\Theta = \mathcal{G}(k)$ to construct conserved charges following the method of [16] and replacing the metric perturbation $h_{\mu\nu}$ with $k_{\mu\nu} = 3\mathcal{G}_{\mu\nu}(h) - \bar{g}_{\mu\nu}\mathcal{G}^{\sigma}_{\sigma}(h)$, which gives that the conserved charge for a killing vector ξ is

$$E(\bar{\xi}) = \frac{1}{8\pi G}\oint dS_i\sqrt{-\bar{g}}\left[D_\beta K^{0i\nu\beta} - K^{0j\nu i}D_j\right]\bar{\xi}_\nu, \tag{24}$$

where

$$K^{\mu\alpha\nu\beta} = \frac{1}{2}\left[\bar{g}^{\mu\beta}H^{\nu\alpha} + \bar{g}^{\nu\alpha}H^{\mu\beta} - \bar{g}^{\mu\nu}H^{\alpha\beta} - \bar{g}^{\alpha\beta}H^{\mu\nu}\right],\ H^{\mu\nu} = k^{\mu\nu} - \frac{1}{2}\bar{g}^{\mu\nu}k. \tag{25}$$

It is clear from this formulation that any solution that is purely an Einstein mode will have $k_{\mu\nu} = 0$ and therefore will have all conserved charges vanish, in agreement with other calculations of the energy in critical gravity. To find a nonzero energy we must turn on a nonzero $k_{\mu\nu}$ mode. At large radius the only static, spherically symmetric solution to (8) is simple in terms of $k_{\mu\nu} = \tilde{k}_{\mu\nu} + D_{(\mu}\xi_{\nu)}$, the Coulomb tail of a mass in AdS ($\tilde{k}_{\mu\nu}$), along with a vector mode[9] ($D_{(\mu}\xi_{\nu)}$). The Coulomb taill of a mass in AdS_4 behaves as $\tilde{k}_{tt} = -\bar{g}_{tt}\tilde{k}_{\rho\rho} = \tilde{M}/\sinh\rho$. We must include a vector mode to ensure the consistency of the equation for h, because $D^\mu\mathcal{G}_{\mu\nu} = 0$ and we must therefore require that $D^\mu(k_{\mu\nu} - \bar{g}_{\mu\mu}k) = 0$ so that the equation of motion is covariantly conserved. This is ensured with $\xi^r = \tilde{M}/6\sinh^2\rho\cosh\rho$, which gives

$$k_{tt} = -k_{\rho\rho}\sinh^2\rho\cosh^2\rho = 2k_{\theta\theta} = 2k_{\phi\phi}/\sin^2\theta = M/\sinh\rho, \tag{26}$$

where $M = 2\tilde{M}/3$. It is easy to check that the mass as defined in (24) is simply $M/2G$. We also note that this nonzero $k_{\mu\nu}$ sources a logarithmic falloff in h,

$$h_{tt} = \frac{M\log\cosh\rho}{3\sinh\rho} + \frac{M\cosh^2\rho}{3}\left(\pi - 4\arctan[\tanh(\rho/2)] - \frac{1+\tanh^2\rho}{\sinh\rho}\right),$$

[9]This is of course not an honest linear diffeomorphism , as by eq. (8) and the diff invariance of $\mathcal{G}$ we see that k does not transform.

$$h_{\rho\rho} = \frac{M \log\cosh\rho}{3\cosh^2\rho\sinh\rho}. \tag{27}$$

We can also turn on a homogeneous mode in h, that is one satisfying $\mathcal{G}(h) = 0$, but this will clearly not contribute to the energy.

Miscellaneous Remarks

The one-particle Hilbert space obtained by quantizing critical gravity on its AdS background splits into a sum of Einstein modes and log modes: $H = H^s \oplus H^E$. The norm N on such space vanishes on H^E but it is off diagonal. Schematically, for $|\psi\rangle = |\psi^s\rangle \oplus |\psi^E\rangle$, $|\phi\rangle = |\phi^s\rangle \oplus |\phi^E\rangle$, one has:

$$\langle\psi|N|\phi\rangle = \left(\langle\psi^s|, \langle\psi^E|\right) \begin{pmatrix} 1 & \alpha \\ \alpha & 0 \end{pmatrix} \begin{pmatrix} |\phi^s\rangle \\ |\phi^E\rangle \end{pmatrix}, \qquad \alpha \neq 0. \tag{28}$$

A computation of the energy using formula (16) shows that, on a mode of frequency ω, its *unnormalized* expectation value is proportional to the norm N: $\langle\psi|H|\phi\rangle = \omega\langle\psi|N|\phi\rangle$. The Fock space contain only states with positive frequency, so that whenever the norm is nonzero, the normalized energy, $\langle\psi|H|\phi\rangle/\langle\psi|N|\phi\rangle$, is positive and equal to the frequency ω, as expected. When the norm vanishes, by consistency one must define the energy so that it also vanishes. Restricting the multi-particle Hilbert space to the Einstein modes is thus equivalent to selecting the zero-energy sub-sector of critical gravity, in perfect analogy with 3D chiral gravity [8]. To obtain a properly defined Hilbert space, the restriction to Einstein modes must be followed by modding out by null states.

Unlike 3D chiral gravity, here the end result of this procedure leaves only one state in the theory, the Fock vacuum.

The results described so far were obtained by studying linearized critical gravity. Yet, the general structure we found can be promoted to a full non-linear analysis. In particular, the vanishing of energy in solutions which asymptotically become Einstein modes has been shown to hold for black holes too in [2, 3]. A general proof that all asymptotically-Einstein solutions have zero energy should be possible using the general definition of energy in quadratic-curvature gravities given in [17]. Conversely, in the previous section we just proved that non-vanishing energy requires the asymptotic behavior of log modes.

It was conjectured in [18] and proven in [19] that TMG in $3D$ is perturbatively stable around certain warped AdS_3 vacua, even for non-critical values of the Chern-Simons coupling constant. Stability is achieved by restricting the asymptotic boundary conditions in such a way that only boundary gravitons of one chirality propagate. This is in perfect analogy with the behavior of TMG at the critical point on non-warped AdS_3. In this paper, we argued that in $D > 3$, restriction to the Einstein asymptotics can give us only a trivial Hilbert space made only of the vacuum. So, if an analog to the results of [18, 19] could be found for $D > 3$ higher-derivative gravity, it would probably require asymptotics on fields that leave no physical state beyond the vacuum.

It may happen that negative-norm ghost poles cancel against some positive norm poles in one-particle exchange diagrams between physical sources [9]. This is not enough to ensure unitarity, since in the full non-linear theory defined by eq. (4) ghosts can obviously also be pair-produced. Moreover, even at the level of one-particle diagrams, one must still mod out by null states, arriving again at an empty theory.

Finally, we must point out explicitly that the Hilbert space of Einstein modes is the only physically meaningful subspace without negative norm states (unfortunately, it also has no positive norm states). Obviously, by diagonalizing the metric N in eq. (28) one can obtain a positive-norm subspace. Unfortunately, that subspace is not closed under $SO(2,3)$ transformations, since logarithmic representations are indecomposable. It is amusing to check explicitly this property by verifying that a positive-norm subspace is not even closed under time evolution. In fact, its vectors must be linear combinations of log modes and gravity modes. Consider in particular a mode of frequency ω: $|\psi\rangle = |\psi^s\rangle + |\phi^E\rangle$, $\psi^s = f(t,\rho)\psi^E$. Under a time translation $t \to t+\tau$, it transforms into $\psi(t) \to \psi(t+\tau) = \exp(-i\omega\tau)(\psi + i\tau\psi^E)$. Equation (21) says that $\langle\psi^s|\psi^E\rangle = c\langle\psi^E|\psi^E\rangle_E$, where $\langle|\rangle_E$ is the scalar product in Einstein gravity and c is a positive constant; so the scalar product $\langle\psi(t+\tau)|N|\psi(t+\tau)\rangle$ is independent of τ. Now consider the linear combination $\psi(t) + C\psi(t+\tau)$. Its norm is

$$\begin{aligned}\langle\psi(t) + C\psi(t+\tau)|N|\psi(t) + C\psi(t+\tau)\rangle &= (CC^*+1)\langle\psi|\psi\rangle + 2\langle\psi|\psi\rangle\mathrm{Re}\,(Ce^{-i\omega\tau}) + \\ &\quad -2c\langle\psi^E|\psi^E\rangle_E\mathrm{Im}\,(\tau Ce^{-i\omega\tau}).\end{aligned} \tag{29}$$

The right hand side in this equation becomes negative for $C \neq 0$ and τ large.

Acknowledgments

It is a pleasure to thank Olaf Hohm, Chris Pope and Eric Bergshoeff for useful correspondence and discussions. M.P. is supported in part by NSF grant PHY-0758032, and by ERC Advanced Investigator Grant n.226455 *Supersymmetry, Quantum Gravity and Gauge Fields (Superfields)*; M.M.R. is supported by the Simons Postdoctoral Fellowship Program.

References

[1] K. S. Stelle, "Renormalization of Higher Derivative Quantum Gravity," Phys. Rev. **D16**, 953-969 (1977); "Classical Gravity with Higher Derivatives," Gen. Rel. Grav.

9, 353-371 (1978).

[2] H. Lu, C. N. Pope, "Critical Gravity in Four Dimensions," [arXiv:1101.1971 [hep-th]].

[3] S. Deser, H. Liu, H. Lu, C. N. Pope, T. C. Sisman, B. Tekin, "Critical Points of D-Dimensional Extended Gravities," [arXiv:1101.4009 [hep-th]].

[4] M. Alishahiha, R. Fareghbal, "D-Dimensional Log Gravity," [arXiv:1101.5891 [hep-th]].

[5] I. Gullu, M. Gurses, T. C. Sisman, B. Tekin, "AdS Waves as Exact Solutions to Quadratic Gravity," [arXiv:1102.1921 [hep-th]].

[6] E. A. Bergshoeff, O. Hohm, J. Rosseel, P. K. Townsend, "Modes of Log Gravity," [arXiv:1102.4091 [hep-th]].

[7] W. Li, W. Song, A. Strominger, "Chiral Gravity in Three Dimensions," JHEP **0804**, 082 (2008) [arXiv:0801.4566 [hep-th]].

[8] A. Maloney, W. Song, A. Strominger, "Chiral Gravity, Log Gravity and Extremal CFT," Phys. Rev. **D81**, 064007 (2010) [arXiv:0903.4573 [hep-th]].

[9] T. C. Sisman, I. Gullu, B. Tekin, "All unitary cubic curvature gravities in D dimensions," [arXiv:1103.2307 [hep-th]].

[10] V. Gurarie, "Logarithmic operators in conformal field theory," Nucl. Phys. **B410**, 535-549 (1993) [hep-th/9303160].

[11] M. Flohr, "Bits and pieces in logarithmic conformal field theory," Int. J. Mod. Phys. A **18** (2003) 4497 [arXiv:hep-th/0111228].

[12] K. Skenderis, M. Taylor, B. C. van Rees, "Topologically Massive Gravity and the AdS/CFT Correspondence," JHEP **0909**, 045 (2009) [arXiv:0906.4926 [hep-th]]; "AdS boundary conditions and the Topologically Massive Gravity/CFT correspondence," [arXiv:0909.5617 [hep-th]].

[13] D. Grumiller, I. Sachs, "AdS (3) / LCFT (2) $\rightarrow$ Correlators in Cosmological Topologically Massive Gravity," JHEP **1003**, 012 (2010) [arXiv:0910.5241 [hep-th]].

[14] D. Grumiller, N. Johansson, "Gravity duals for logarithmic conformal field theories," J. Phys. Conf. Ser. **222**, 012047 (2010) [arXiv:1001.0002 [hep-th]].

[15] E. A. Bergshoeff, O. Hohm, P. K. Townsend, "Massive Gravity in Three Dimensions," Phys. Rev. Lett. **102**, 201301 (2009) [arXiv:0901.1766 [hep-th]].

[16] L. F. Abbott and S. Deser, "Stability Of Gravity With A Cosmological Constant," Nucl. Phys. B **195** (1982) 76.

[17] S. Deser, B. Tekin, "Gravitational energy in quadratic curvature gravities," Phys. Rev. Lett. **89**, 101101 (2002) [hep-th/0205318]; "Energy in generic higher curvature gravity theories," Phys. Rev. **D67**, 084009 (2003) [hep-th/0212292].

[18] D. Anninos, W. Li, M. Padi, W. Song, A. Strominger, "Warped AdS(3) Black Holes," JHEP **0903**, 130 (2009) [arXiv:0807.3040 [hep-th]].

[19] D. Anninos, M. Esole, M. Guica, "Stability of warped AdS(3) vacua of topologically massive gravity," JHEP **0910**, 083 (2009) [arXiv:0905.2612 [hep-th]].