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Instanton counting with a surface operator and the chain-saw quiver

Hiroaki Kanno[†] and Yuji Tachikawa[‡] *

[†] Graduate School of Mathematics and KMI[§],
Nagoya University, Nagoya, 464-8602, Japan

[‡] School of Natural Sciences, Institute for Advanced Study,
Princeton, New Jersey 08504, USA

abstract

We describe the moduli space of $SU(N)$ instantons in the presence of a general surface operator of type $N = n_1 + \cdots + n_M$ in terms of the representations of the so-called chain-saw quiver, which allows us to write down the instanton partition function as a summation over the fixed point contributions labeled by Young diagrams. We find that the instanton partition function depends on the ordering of n_I which fixes a choice of the parabolic structure. This is in accord with the fact that the Verma module of the W -algebra also depends on the ordering of n_I . By explicit calculations, we check that the partition function agrees with the norm of a coherent state in the corresponding Verma module.

*on leave from IPMU, the University of Tokyo

[§]Kobayashi-Maskawa Institute for the Origin of Particles and the Universe

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1 Introduction

The instanton partition function of four-dimensional $\mathcal{N} = 2$ gauge theory, as defined in [1, 2], is long known to have a close relationship with two-dimensional conformal field theory [3, 4]. Recently it was realized that the instanton partition function for the gauge group $G = \text{SU}(N)$ is governed by the W-algebra symmetry W_N [5, 6, 7]. In particular, the partition function of the pure $\text{SU}(N)$ gauge theory is given by the norm of a coherent state in the Verma module [8, 9, 10]. The relation between instantons and W-algebras was further explored in e.g. [11, 12, 13, 14, 15, 16, 17].

The appearance of the W-symmetry, more natural for a two-dimensional theory, is better viewed from a higher-dimensional perspective. Consider the six-dimensional $\mathcal{N} = (2, 0)$ theory of type A_{N-1} on $\mathbb{R}_t \times S^1 \times \mathbb{R}^4$, with \mathbb{R}_t as the time direction. First compactify along S^1 to obtain five-dimensional super $\text{SU}(N)$ Yang-Mills on $\mathbb{R}_t \times \mathbb{R}^4$. The BPS sector of its dynamics can be reduced to the supersymmetric quantum mechanics on the instanton moduli space. After Nekrasov's deformation, the Hilbert space of the BPS

sector \mathcal{H}_{BPS} is the equivariant cohomology of the moduli space of instantons. The partition function is the norm of a state in \mathcal{H}_{BPS} . By exchanging the order of compactification it should also be possible to regard \mathcal{H}_{BPS} as the Hilbert space of S^1 compactification of the two-dimensional theory obtained by the compactification of the six-dimensional theory on \mathbb{R}^4 with Nekrasov's deformation. Then the W_N -algebra referred to above should be the symmetry of this two-dimensional theory.

The correspondence can be further generalized by adding surface operators on the four-dimensional side.¹ A class of surface operators in $SU(N)$ gauge theory is characterized by a partition $N = n_1 + n_2 + \dots + n_M$, which we denote as $[n_I]$. For the partition $[1, 1, \dots, 1]$, it was found that the instanton partition function in the presence of the surface operator is governed by the affine symmetry $\widehat{SU}(N)$ [18, 19, 20, 21]. For the surface operator of a more general type, it was conjectured [22] that the two-dimensional symmetry is the W-algebra $W(\widehat{SU}(N), [n_I])$ obtained by the quantum Drinfeld-Sokolov reduction [23, 24, 25] for the embedding $\rho_{[n_I]} : SU(2) \rightarrow SU(N)$ corresponding to the partition $[n_I]$, and that the instanton partition function of the pure gauge theory in the presence of the surface operator is equal to the norm of a coherent state in a Verma module. A few checks of this conjecture were performed in [26, 27, 28].

The aim of this paper is to provide further checks of this proposal. We first describe the moduli space of instantons in the presence of a general surface operator in terms of the representations of a quiver, which mathematicians call the *chain-saw quiver* [29, 30]. The crucial point is the equivalence between the instanton moduli space with a surface operator and the instanton moduli space on an orbifold $\mathbb{C} \times (\mathbb{C}/\mathbb{Z}_M)$. Then the quiver describing the instanton moduli space is obtained by performing the \mathbb{Z}_M orbifold operation to the standard ADHM quiver. With the quiver description at hand, it is straightforward to enumerate and calculate the contribution from the fixed points on the moduli space under the action of the spacetime rotation and the gauge rotation, confirming the formula proposed in [27]. We will see that the quiver and the fixed point formula depend on the ordering of n_I . Therefore, in this paper, we do *not* assume $n_I \geq n_{I+1}$. When the ordering is ignored, we denote the data by $[n_I]$, called a *partition* of N . When the ordering is meaningful, we denote the data by (n_I) , called a *composition* of N , following the terminology in the combinatorics.

¹Another generalization is to consider SO or E -type 6d theory, see e.g. [17].

We then compare the structure of the fixed points with the Verma module of the W -algebra. In order to define the Verma module, the W -algebra generators need to be split into creation and annihilation operators. Although there is essentially a unique choice of such splitting for $W(\widehat{\text{SU}}(N), [1, 1, \dots, 1]) = \widehat{\text{SU}}(N)$ and for $W(\widehat{\text{SU}}(N), [N]) = W_N$, we will find that for general $W(\widehat{\text{SU}}(N), [n_I])$ the splitting depends on the ordering of n_i , or equivalently on the composition (n_I) . The coherent state condition needs to be imposed accordingly. Once this dependence on the composition (n_I) of both the quiver and the Verma module is understood, we will find that the instanton partition function nicely agrees with the norm of the coherent state.

The rest of the paper is organized as follows. In Sec. 2, we start by recalling the data defining the surface operator. The equivalence of the instanton moduli space with a surface operator and the moduli space of orbifolded instantons is then explained. This equivalence is then used to construct a quiver describing the instanton moduli space with a surface operator from the standard ADHM quiver. Finally we study the structure of the fixed points using the resulting quiver, and write down the instanton partition function as a summation over fixed point contributions. In Sec. 3, we first describe the structure of general W -algebras, and then study the structure of their Verma modules. After comparing the graded dimension of the Verma module with the generating functions of the number of the fixed points on the instanton moduli space, we determine the conditions we put on the coherent state. We report on the explicit checks performed using a computer. We conclude with a few remarks in Sec. 4. In the Appendix the structure of the W -algebra we use is determined explicitly.

2 Instanton moduli space with a surface operator

2.1 Defining data of a surface operator

The surface operator as a half-BPS object in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory was introduced by Gukov and Witten [31]. We can employ almost the same definition of the surface operator also in $\mathcal{N} = 2$ theories. In this section we review the basic properties of the surface operator following [31]².

²See also [32, 33, 34, 35, 36, 37] for the surface operators in $\mathcal{N} = 2$ theories.

The surface operator is defined by prescribing a singular behavior of the gauge field. Let us consider $SU(N)$ gauge field A_μ on $\mathbb{R}^4 \simeq \mathbb{C}^2$ with complex coordinates (z, w) and put a surface operator at $w = 0$, filling the z -plane. Let (r, θ) be the polar coordinates of the transverse w -plane, $w = re^{i\theta}$. In the presence of the surface operator the gauge field diverges as

$$A_\mu dx^\mu \sim \text{diag}(\alpha_1, \alpha_2, \dots, \alpha_N) id\theta, \quad (2.1)$$

near the surface operator. By a gauge transformation one can assume $2\pi > \alpha_i \geq \alpha_{i+1} \geq 0$. The data $\vec{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_N)$ give an element of the Lie algebra of the Cartan subgroup $U(1)^{N-1}$ of the gauge group $G = SU(N)$. The commutant of $\vec{\alpha}$ defines a subgroup $\mathbb{L} \subset G$ which is called the Levi subgroup. This means that the gauge group G is broken to \mathbb{L} on the surface.

Suppose $\vec{\alpha}$ has the structure

$$\vec{\alpha} = \underbrace{(\alpha_{(1)}, \dots, \alpha_{(1)})}_{n_1 \text{ times}}, \underbrace{(\alpha_{(2)}, \dots, \alpha_{(2)})}_{n_2 \text{ times}}, \dots, \underbrace{(\alpha_{(M)}, \dots, \alpha_{(M)})}_{n_M \text{ times}}, \quad (2.2)$$

where $\alpha_{(I)} > \alpha_{(I+1)}$. Here (n_I) is a composition of N . Then the Levi subgroup is

$$\mathbb{L} = S[U(n_1) \times U(n_2) \times \dots \times U(n_M)]. \quad (2.3)$$

Note that the Levi subgroup of $SU(N)$ only depends on the partition $[n_I]$, or equivalently, it does not depend on the ordering of (n_I) .

The Levi subgroup \mathbb{L} is a discrete label of the surface operator. This surface operator has $(M - 1)$ continuous parameters $\alpha_{(M)}$ as we already saw. We can turn on magnetic fluxes \mathbf{m}^I , ($I = 1, \dots, M$), on the surface operator since the gauge group \mathbb{L} on the surface operator has $(M - 1)$ $U(1)$ factors. Correspondingly, there are parameters $\eta_{(I)}$, giving rise to a phase factor $\exp(im^I \eta_I)$ in the path integral. Under supersymmetry, it is natural to combine $\vec{\alpha}$ and $\vec{\eta}$ to complex parameters $\vec{t} := \vec{\eta} + i\vec{\alpha}$. Half-BPS surface operators of $\mathcal{N} = 4$ theory has additional parameters $\vec{\beta}$ and $\vec{\gamma}$.

Let us introduce a few examples of surface operators. When $\vec{\alpha}$ is the most generic, $\mathbb{L} = U(1)^{N-1}$, and the surface operator is called full. If $\vec{\alpha} \propto (\alpha, \alpha, \dots, (1 - N)\alpha)$, $\mathbb{L} = SU(N - 1) \times U(1)$, and the surface operator is called simple. When $N = 2$ the simple surface operator is the same as the full surface operator; When $N = 3$ a nontrivial surface operator is either simple or full. If $\vec{\alpha} = 0$, $\mathbb{L} = SU(N)$, and this corresponds to the absence of the surface operator.

Let \mathfrak{p} be a subalgebra of $\mathfrak{g}_{\mathbb{C}}$ spanned by elements x satisfying

$$[\alpha, x] = i\lambda x, \quad \lambda \geq 0. \quad (2.4)$$

\mathfrak{p} is called a parabolic subalgebra, and the corresponding subgroup $\mathcal{P} \subset G_{\mathbb{C}}$ is called a parabolic subgroup. As examples let us consider the case $N = 4$ and $\mathbb{L} = S[\mathrm{U}(2) \times \mathrm{U}(1)^2]$, corresponding to the partition $[2, 1, 1]$. When $\vec{\alpha} = (\alpha_{(1)}, \alpha_{(1)}, \alpha_{(2)}, \alpha_{(3)})$ and the composition is $(2, 1, 1)$, \mathcal{P} is the subgroup generated by the elements of the form

$$\left(\begin{array}{cc|cc} * & * & * & * \\ * & * & * & * \\ \hline 0 & 0 & * & * \\ \hline 0 & 0 & 0 & * \end{array} \right), \quad (2.5)$$

and when $\vec{\alpha} = (\alpha_{(1)}, \alpha_{(2)}, \alpha_{(2)}, \alpha_{(3)})$ and the composition is $(1, 2, 1)$, \mathcal{P} is the subgroup generated by the elements of the form

$$\left(\begin{array}{c|ccc} * & * & * & * \\ \hline 0 & * & * & * \\ 0 & * & * & * \\ \hline 0 & 0 & 0 & * \end{array} \right). \quad (2.6)$$

These two parabolic subgroups are *not* conjugate to each other inside $\mathrm{SL}(N)$, and the choice depends on the composition (n_I) of N . Note that when the Levi subgroup is $\mathbb{L} = \mathrm{U}(1)^{N-1}$, there is a unique parabolic subgroup associated, consisting of all the semi-upper-triangular matrices. This parabolic subgroup is called the Borel subgroup and denoted by \mathcal{B} .

The surface operator can also be described as a coupling of the four-dimensional gauge theory with a two-dimensional theory on the surface [31]: One can take a non-linear sigma model whose target space is the partial flag manifold $G_{\mathbb{C}}/\mathcal{P}$. As Riemannian manifolds, $G_{\mathbb{C}}/\mathcal{P}$ is always the same with G/\mathbb{L} , but they can be different as complex manifolds, and therefore can give rise to different two-dimensional $\mathcal{N} = (2, 2)$ supersymmetric sigma models, see e.g. [38] for a recent discussion. From this point of view, the parameters \vec{t} are the complexified Kähler parameters. In contrast, a half-BPS surface operator of $\mathcal{N} = 4$ theory is given by the coupling to the two-dimensional supersymmetric sigma model on $T^*(G_{\mathbb{C}}/\mathcal{P})$. In this case the hyperKähler structures for different \mathcal{P} with the same \mathbb{L} can be continuously interpolated by changing $(\vec{\alpha}, \vec{\beta}, \vec{\gamma})$ [31].

2.2 Instantons with a surface operator as orbifolded instantons

We consider an instanton configuration $F = -\star F$ with the singularity (2.1) together with a choice of the composition (n_1, n_2, \dots, n_M) of N , see (2.2). The topological data are the instanton number d and the magnetic fluxes \mathbf{m}^I of the $U(1)$ factors of the Levi subgroup. As always, $\sum_I \mathbf{m}^I = 0$ because the overall $U(1)$ is decoupled. To define the instanton number, we define a smooth gauge field \bar{A}_μ on \mathbb{R}^4 by

$$A_\mu dx^\mu = \bar{A}_\mu dx^\mu + f(r) \text{diag}(\alpha_1, \dots, \alpha_N) id\theta \quad (2.7)$$

where $f(r)$ is a smooth function that interpolates between 1 at $w = 0$ and 0 at $w = \infty$. We define the instanton number d of the configuration as the instanton number of \bar{A}_μ . Then the integral of $\text{tr} F \wedge F$ is given by

$$\frac{1}{8\pi^2} \int_{\mathbb{C}^2 \setminus \{w=0\}} \text{tr} F \wedge F = d + \frac{1}{2} \sum_I \alpha_{(I)} \mathbf{m}^I. \quad (2.8)$$

The topological data are more conveniently parameterized by \vec{d} defined by

$$d_M = d, \quad d_{I+1} = d_I + \mathbf{m}^I \quad (2.9)$$

where the index I is taken modulo M . The moduli space of such configurations is denoted by $\mathcal{M}_{\mathbb{L}, \vec{\alpha}, \vec{d}}$. When $d = 0$, the moduli space describes excitations purely on the surface, given by the supersymmetric sigma model whose target space is $G_{\mathbb{C}}/\mathcal{P}$. This system was analyzed e.g. in [22, 39, 40, 41, 42].

In mathematics literature, it is known [43, 44] that the smooth part of this moduli space, as far as its complex structure is concerned, agrees with the smooth part of the moduli space of $U(N)$ instantons on $\mathbb{C} \times (\mathbb{C}/\mathbb{Z}_M)$ where \mathbb{Z}_M acts on the spacetime as

$$(z, w) \rightarrow (z, \omega w) \quad (2.10)$$

and on the N -dimensional representation of $U(N)$ as the multiplication by

$$\underbrace{(\omega, \dots, \omega)}_{n_1 \text{ times}}, \underbrace{(\omega^2, \dots, \omega^2)}_{n_2 \text{ times}}, \dots, \underbrace{(\omega^M, \dots, \omega^M)}_{n_M \text{ times}} \quad (2.11)$$

where $\omega = e^{2\pi i/M}$. As we will see shortly this equivalence allows us to describe the moduli space of instantons with a surface operator in terms of a quiver, obtained from a standard orbifold operation applied to the ADHM construction of the instantons on \mathbb{C}^2 .

Before proceeding, we would like to sketch the derivation of the equivalence in two ways, one is mathematical and the other is string theoretical. Mathematical derivation goes as follows. The singularities of the moduli space of differential-geometric objects can be resolved by translating them into algebro-geometric language. In this case, the corresponding objects in algebraic geometry is a rank- N torsion-free sheaves on $\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1$ with coordinates (z, w) , with framing given at $\{z = \infty\} \cup \{w = \infty\}$ and with parabolic structure of type \mathcal{P} given at $\{w = 0\}$, whose precise definition can be found in [29, 30]. Another standard mathematical trick, also described in [45, 29, 30], to study this moduli space is to use the one-to-one mapping between a parabolic sheaf on $\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1$ of type \mathcal{P} and a \mathbb{Z}_M -equivariant sheaf on $\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1$, with precisely the conditions (2.10), (2.11). This can be mapped back to a differential-geometric object as an anti-self-dual configuration on $\mathbb{C} \times (\mathbb{C}/\mathbb{Z}_M)$, completing the derivation.

String theoretical derivation goes as follows. When we realize $\mathcal{N} = 4$ super Yang-Mills theory by compactifying the six-dimensional $\mathcal{N} = (2, 0)$ theory on T^2 , a surface operator of type \mathbb{L} in four dimensions can be obtained by placing a codimension-two defect of type \mathbb{L} in six dimensions along T^2 . When the original theory is of type A_{N-1} and can be realized as the low-energy theory of coincident N M5-branes, it was shown in [46] (also see Sec. 3 of [47]) that the codimension-two defect of type \mathbb{L} corresponding to the partition $N = n_1 + \dots + n_M$ can be realized by considering a Type IIA configuration with N D4-branes ending on M D6-branes such that n_I D4-branes end on the I -th D6-brane. Lifting the configuration to M-theory, we have N M5-branes hitting the singularity of a multi Taub-NUT space. As far as the complex structure is concerned, the situation is now described by the eleven-dimensional spacetime $\mathbb{R}^2 \times \mathbb{C} \times (\mathbb{C}^2/\mathbb{Z}_M) \times \mathbb{R}^3$ with coordinates $(x^0, x^1, z, w, w', x^7, x^8, x^9)$ where the orbifold is given by

$$(w, w') \rightarrow (\omega w, \omega^{-1} w'). \quad (2.12)$$

N M5-branes are on $w' = x^7 = x^8 = x^9 = 0$, and the codimension-two defect is the orbifold singularity $w = 0$. Compactifying this system along x^0 and x^1 , we end up with $\mathcal{N} = 4$ Yang-Mills theory on the orbifold given in (2.10).

2.3 Orbifolded instantons and the chain-saw quiver

Let us recall how the moduli space of $SU(N)$ instantons on \mathbb{R}^4 with instanton number d is given by the ADHM construction. Let us introduce two vector spaces V and W with $\dim_{\mathbb{C}} V = d$ and $\dim_{\mathbb{C}} W = N$. In the language of D-brane configuration the system we consider is d D0-branes bound to N D4-branes. As an effective theory on D4-branes we have $U(N)$ gauge theory and d D0-branes describe the gas of d point-like instantons. In this picture the ADHM construction is given by a dual description where we consider an effective $0 + 1$ -dimensional theory on D0 brane [48, 49, 50, 51]. We have $A, B \in \text{Hom}(V, V)$ from “0-0” string. From “0-4” and “4-0” string we have $P \in \text{Hom}(W, V)$ and $Q \in \text{Hom}(V, W)$. The ADHM equation arises as the BPS condition for this D-brane system and given by

$$\mathcal{E}_{\mathbb{C}} := [A, B] + PQ = 0, \quad (2.13)$$

$$\mathcal{E}_{\mathbb{R}} := [A, A^\dagger] + [B, B^\dagger] + PP^\dagger - Q^\dagger Q = 0. \quad (2.14)$$

Then the moduli space is given by

$$\mathcal{M}_{\text{ADHM}} := \{(A, B, P, Q) \mid \mathcal{E}_{\mathbb{C}} = 0, \mathcal{E}_{\mathbb{R}} = 0\} / U(d), \quad (2.15)$$

where $g \in U(d)$ acts on the ADHM data as follows;

$$(A, B, P, Q) \rightarrow (gAg^{-1}, gBg^{-1}, gP, Qg^{-1}). \quad (2.16)$$

It is known that the resolution of singularity of the ADHM moduli space is given by the following affine algebro-geometric quotient [52, 53];

$$\widetilde{\mathcal{M}}_{\text{ADHM}} := \{(A, B, P, Q) \mid \mathcal{E}_{\mathbb{C}} = 0, \text{ stability}\} // \text{GL}(d, \mathbb{C}). \quad (2.17)$$

This is a smooth space of complex dimension $2Nd$.

To obtain the quiver describing the moduli space of instantons on $\mathbb{C} \times (\mathbb{C}/\mathbb{Z}_M)$, we perform the standard orbifolding procedure as in [54, 55, 56]. The vector spaces V and W are decomposed according to the representation under \mathbb{Z}_M action,

$$W = \bigoplus_{I=1}^M W_I, \quad V = \bigoplus_{I=1}^M V_I. \quad (2.18)$$

We also have

$$\dim W_I = n_I, \quad \dim V_I = d_I. \quad (2.19)$$

In the language of branes, W_I and V_I are the Chan-Paton spaces of D4- and D0-branes on which the orbifold action acts as ω^I . In particular, d_I characterizes the number of fractional instantons.

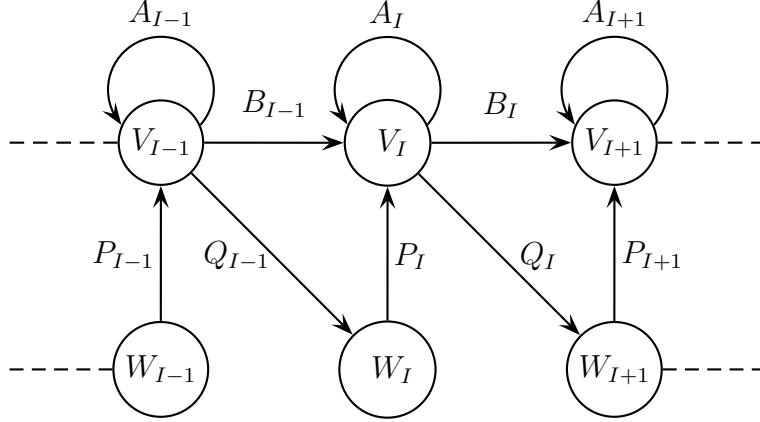


Figure 1: A part of \mathbb{Z}_M chain-saw quiver

The isometry $(z, w) \rightarrow (e^{\epsilon_1} z, e^{\epsilon_2} w)$ of \mathbb{C}^2 acts on the ADHM data as $(A, B, P, Q) \rightarrow (e^{\epsilon_1} \cdot A, e^{\epsilon_2} \cdot B, P, e^{\epsilon_1 + \epsilon_2} \cdot Q)$. Therefore, under the \mathbb{Z}_M orbifold action $(z, w) \rightarrow (z, \omega w)$, the components which survive are $A_I \in \text{Hom}(V_I, V_I)$, $B_I \in \text{Hom}(V_I, V_{I+1})$, $P_I \in \text{Hom}(W_I, V_I)$ and $Q_I \in \text{Hom}(V_I, W_{I+1})$, and the equation of motion is

$$A_{I+1}B_I - B_I A_I + P_{I+1}Q_I = 0. \quad (2.20)$$

Here in the following, the index I is taken modulo M , therefore $V_{M+1} = V_1$, etc. The resulting quiver is the chain-saw quiver [30] shown in Fig. 1. The resulting moduli space is a smooth complex manifold of dimension

$$\sum_{I=1}^M (n_I + n_{I+1})d_I. \quad (2.21)$$

2.4 Instanton counting with a surface operator

For $\mathcal{N} = 2$ pure $SU(N)$ gauge theory, it was shown in [2] that the instanton contribution to the partition function can be expressed by the geometrical data specifying how the Cartan torus $U(1)^2 \times U(1)^N$ of the symmetry $SO(4) \times U(N)$ acts on the instanton moduli

space. We first recall how it was done for the case without the surface operator. Then the extension to the case with general surface operator is straightforward.

2.4.1 Before the orbifolding

The action $(z, w) \rightarrow (e^{\epsilon_1} z, e^{\epsilon_2} w)$ on \mathbb{C}^2 induces an action on the ADHM data given by

$$(A, B, P, Q) \rightarrow (e^{\epsilon_1} \cdot A, e^{\epsilon_2} \cdot B, P, e^{\epsilon_1 + \epsilon_2} \cdot Q), \quad (2.22)$$

and $g \in U(N)$ acts as

$$(A, B, P, Q) \rightarrow (A, B, P g^{-1}, g Q). \quad (2.23)$$

Fixed points under these actions are isolated and labeled by N -tuples of Young diagrams $\vec{\lambda} = (\lambda^1, \dots, \lambda^N)$; for a Young diagram λ we denote the height of the j -th column by λ_j .

The ADHM data associated to an N -tuple $\vec{\lambda}$ is explicitly given as follows. Let w_1, \dots, w_N a basis of W , such that $\text{diag}(e^{a_1}, \dots, e^{a_N}) \in U(N)$ acts by

$$w^I \rightarrow e^{a_I} w^I. \quad (2.24)$$

Let us associate a basis $v_{(i,j)}^I$ of V for each box $(i, j) \in \lambda^I$, i.e. for a box at the i -th row, the j -th column of the Young diagram λ^I . In particular, $\dim V$ is the total number of boxes in $\vec{\lambda}$. For notational simplicity, we define $v_{(i,j)}^I = 0$ if $(i, j) \notin \lambda^I$. The toric action on this basis is given by

$$v_{(i,j)}^I \rightarrow e^{(1-i)\epsilon_1 + (1-j)\epsilon_2} v_{(i,j)}^I. \quad (2.25)$$

We then have

$$A v_{(i,j)}^I = v_{(i+1,j)}^I, \quad B v_{(i,j)}^I = v_{(i,j+1)}^I, \quad P w^I = v_{(1,1)}^I, \quad Q = 0. \quad (2.26)$$

The tangent space of the moduli space at the fixed point $\vec{\lambda}$ is given by the cohomology of the following complex [52, 57, 3]:

$$\begin{array}{ccccc} & & \text{Hom}(V, V) \otimes (T_1 \oplus T_2) & & \\ & & \oplus & & \\ \text{Hom}(V, V) & \xrightarrow{\sigma} & \text{Hom}(W, V) & \xrightarrow{\tau} & \text{Hom}(V, V) \otimes T_1 \otimes T_2, \quad (2.27) \\ & & \oplus & & \\ & & \text{Hom}(V, W) \otimes T_1 \otimes T_2 & & \end{array}$$

where T_1, T_2 are one dimensional modules on which $U(1)^2$ act as the multiplication by e^{ϵ_1} and e^{ϵ_2} , respectively. Hence the character of the tangent space at a fixed point $\vec{\lambda}$ under $U(1)^2 \times U(1)^N$ is given by

$$\chi_{\vec{\lambda}} = -V^*V(1 - e^{\epsilon_1})(1 - e^{\epsilon_2}) + W^*V + V^*W e^{\epsilon_1 + \epsilon_2}. \quad (2.28)$$

Here, we abused the notation and identified the vector spaces and their characters, i.e.

$$V = \sum_I \sum_{(i,j) \in \lambda^I} e^{a_I + (1-i)\epsilon_1 + (1-j)\epsilon_2}, \quad W = \sum_I e^{a_I} \quad (2.29)$$

and the asterisk $*$ in the superscript reverses the sign of the exponents.

In the expression (2.28), all the terms with negative coefficient cancel out and there are $2Nd$ remaining terms,

$$\chi_{\vec{\lambda}} = \sum_{i=1}^{2Nd} \exp(w_i(\vec{\lambda})). \quad (2.30)$$

Each weight w_i is an integral linear combination of the equivariant parameters $\epsilon_{1,2}$ and a_I . Then Nekrasov's partition function of the pure $SU(N)$ theory is given by

$$Z(\epsilon_{1,2}, a_I, q) = \sum_{\vec{\lambda}} q^{|\vec{\lambda}|} z(\vec{\lambda}). \quad (2.31)$$

where

$$z(\vec{\lambda}) = \prod_{i=1}^{2Nd} \frac{1}{w_i(\vec{\lambda})}. \quad (2.32)$$

In the physical terms, $q = \Lambda^{2N}$ is the dynamical scale, a_I gives the vev of the adjoint scalar in the vector multiplet, and $\epsilon_{1,2}$ is the deformation parameter in Nekrasov's Ω -background.

2.4.2 After the orbifolding

The fixed points after the orbifolding can be classified in a similar manner, and still labeled by N -tuples of Young diagrams, $\vec{\lambda} = (\lambda^{s,I})$ where $I = 1, \dots, M$, $s = 1, \dots, n_I$.

We first redefine the equivariant parameters $\epsilon_{1,2}$ so that it acts on (z, w) via

$$(z, w) \rightarrow (e^{\epsilon_1} z, e^{\frac{1}{M}\epsilon_2} w) \quad (2.33)$$

because the physical space corresponds to the wedge in the w -plane of the angle $2\pi/M$. We also rename the equivariant parameters a_1, \dots, a_N by $a_{s,I}$ where $I = 1, \dots, M$ and $s = 1, \dots, n_I$.

We then introduce a basis $w_{s,I}$ of W_I such that $U(1)^2 \times U(1)^N$ acts by

$$w^{s,I} \rightarrow e^{-\frac{I}{M}\epsilon_2 + a_{s,I}} w^{s,I}. \quad (2.34)$$

We also introduce a basis $v_{(i,j)}^{s,I+j}$ of V_{I+j} for each $(i,j) \in \lambda^{s,I}$ such that $U(1)^2 \times U(1)^N$ acts by

$$v_{(i,j)}^{s,I+j} \rightarrow e^{-\frac{I}{M}\epsilon_2 + a_{s,I}} e^{(1-i)\epsilon_1 + \frac{1}{M}(1-j)\epsilon_2} v_{(i,j)}^{s,I+j} \quad (2.35)$$

Then the fixed configuration is given by

$$A_I v_{(i,j)}^{s,I} = v_{(i+1,j)}^{s,I}, \quad B_I v_{(i,j)}^{s,I} = v_{(i,j+1)}^{s,I+1}, \quad P_I w^{s,I} = v_{(1,1)}^{s,I}, \quad Q_I = 0. \quad (2.36)$$

Note that the dimension of V_I is then given by

$$d_I(\vec{\lambda}) = \sum_{j=1}^{\infty} \sum_{s=1}^{n_{I+1-j}} \lambda_j^{s,I+1-j}, \quad (2.37)$$

i.e. the boxes in the j -th column of the Young diagram $\lambda^{s,I}$ contributes to the dimension of V_{I+j-1} . To illustrate this point, an example of a fixed-point configuration for the \mathbb{Z}_2 orbifold is shown in Fig. 2.

The character of the tangent space of the moduli space at $\vec{\lambda}$ is given by taking the \mathbb{Z}_M invariant part of the complex (2.27). The result is

$$\chi_{\vec{\lambda}} = \sum_{I=1}^M \left[-(1 - e^{\epsilon_1}) V_I^* V_I + (1 - e^{\epsilon_1}) e^{\frac{1}{M}\epsilon_2} V_{I-1}^* V_I + W_I^* V_I + e^{\epsilon_1 + \frac{1}{M}\epsilon_2} V_{I-1}^* W_I \right]. \quad (2.38)$$

As before, we abused the notation and identified the vector space and its character, i.e.

$$V_I = \sum_{J=1}^M \sum_{s=1}^{n_{I-J+1}} e^{\lfloor \frac{I-J}{M} \rfloor \epsilon_2 - \frac{I}{M}\epsilon_2 + a_{s,I-J+1}} \sum_{(i,j) \in \lambda^{s,I-J+1}} e^{(1-i)\epsilon_1 - j\epsilon_2}, \quad (2.39)$$

$$W_I = \sum_{s=1}^{n_I} e^{-\frac{I}{M}\epsilon_2 + a_{s,I}}, \quad (2.40)$$

where $\lfloor \cdot \rfloor$ denotes the floor function. Since $1 \leq I, J \leq M$, it gives either 0 for $I \geq J$ or -1 for $I < J$. Expansion of (2.38) reproduces the conjecture made by Wyllard in [27].

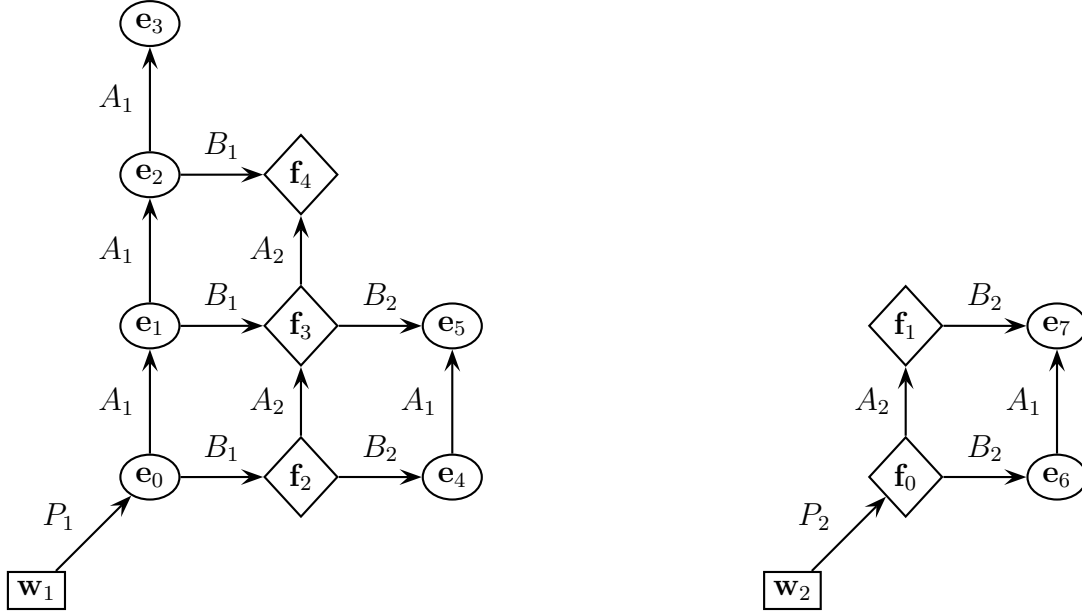


Figure 2: A representation of the chain saw quiver for $(n_I) = (1, 1)$, $\lambda^1 = (4, 3, 2)$, $\lambda^2 = (2, 2)$ with $d_1 = 8$ and $d_2 = 5$. e_i forms the basis of V_1 , f_i forms the basis of V_2 , and $w_{1,2}$ is the basis of $W_{1,2}$.

After expansion, $\chi_{\vec{\lambda}}$ has the form

$$\chi_{\vec{\lambda}} = \sum_{i=1}^{\dim(\vec{\lambda})} e^{w_i(\vec{\lambda})} \quad (2.41)$$

where w_i are integral linear combinations of $a_{s,I}$ and $\epsilon_{1,2}$ and

$$\dim(\vec{\lambda}) \equiv \sum (n_I + n_{I+1})d_I(\vec{\lambda}). \quad (2.42)$$

Then Nekrasov's partition function of the pure $SU(N)$ gauge theory with a general surface operator is given by

$$Z(\epsilon_{1,2}, a_{s,I}, q_I) = \sum_{\vec{\lambda}} z(\vec{\lambda}) \prod_I q_I^{d_I(\vec{\lambda})}. \quad (2.43)$$

where

$$z(\vec{\lambda}) = \prod_{i=1}^{\dim(\vec{\lambda})} \frac{1}{w_i(\vec{\lambda})}. \quad (2.44)$$

It is natural to assign mass dimension 1 to the equivariant parameters $\epsilon_{1,2}$ and $a_{s,I}$. To make Z dimensionless, we should then assign mass dimension $n_I + n_{I+1}$ to q_I . Then

the mass dimension of

$$\Lambda^{2N} \equiv q_1 q_2 \cdots q_M \quad (2.45)$$

is $2N$, and Λ can be naturally identified with the dynamical scale of the gauge theory.

3 Comparison with the W-algebra

3.1 Structure of the W-algebra

The W-algebra we denote as $W(\widehat{\text{SU}}(N), [n_I])$ is obtained by the quantum Drinfeld-Sokolov reduction applied to the affine Lie algebra $\widehat{\text{SU}}(N)$ with an $\text{SU}(2)$ embedding

$$\rho : \text{SU}(2) \rightarrow \text{SU}(N), \quad (3.1)$$

where the fundamental N -dimensional representation of $\text{SU}(N)$ is decomposed as

$$N = \underline{n_1} \oplus \underline{n_2} \oplus \cdots \oplus \underline{n_M} \quad (3.2)$$

under $\rho(\text{SU}(2))$. Here \underline{m} stands for an m -dimensional irreducible representation of $\text{SU}(2)$.

The concrete procedure of the reduction is concisely explained in [25]. For simplicity of exposition, we consider $W(\widehat{\text{U}}(N), [n_I])$ instead of $W(\widehat{\text{SU}}(N), [n_I])$. The difference is a decoupled free boson. The algebra is generated by fields

$$U_{J,(s)}^I(z), \quad 1 \leq I, J \leq M, \quad s = \frac{|n_I - n_J|}{2} + 1, \dots, \frac{|n_I + n_J|}{2} \quad (3.3)$$

where s denotes the scaling dimension. The Miura transformation realizes this algebra as a subalgebra of the current algebra

$$\widehat{\text{U}}(m_1) \times \widehat{\text{U}}(m_2) \times \cdots \times \widehat{\text{U}}(m_{M'}) \quad (3.4)$$

where $[m_I]$ is the dual partition of $[n_I]$. The level of $\widehat{\text{U}}(m_I)$ is $k + N - m_I$, where k is the level of the original $\widehat{\text{U}}(N)$ current algebra. It is also to be noted that $\widehat{\text{U}}(1)$ parts of $\widehat{\text{U}}(m_I)$ and $\widehat{\text{U}}(m_J)$ have non-zero off-diagonal components in the level matrix.

Let $\mathcal{I}_n = \{I \mid n_I = n\}$. Then $\mathcal{J}_J^I(z) \equiv U_{J,(1)}^I(z)$ for $I, J \in \mathcal{I}_n$ generate $\widehat{\text{U}}(|\mathcal{I}_n|)$ current subalgebra of $W(\widehat{\text{U}}(N), [n_I])$, and fields with $s > 1$ transform in the bifundamental representation under these current subalgebra.

We consider the algebra in the R-sector, where every field has integer moding. Then the universal enveloping algebra is generated by operators $U_{J,(s),n}^I, n \in \mathbb{Z}$. The Cartan subalgebra is generated by N generators $U_{I,(s)} \equiv U_{I,(s),0}^I$ (no summation on I), and in particular $U_{I,(1)}$ satisfies

$$[U_{I,(1)}, U_{J',(s),n}^J] = (\delta_{IJ} - \delta_{IJ'}) U_{J',(s),n}^J. \quad (3.5)$$

We declare $U_{J,(s),n}^I$ with $I > J, n \geq 0$ and those with $I \leq J, n > 0$ to be the annihilation operators. The Verma module is then generated by the highest weight vector $|u_{I,(s)}\rangle$ where $u_{I,(s)}$ is the eigenvalue of $U_{I,(s)}$.

Note that the structure of the Verma module depends on the composition (n_I) , not just the partition $[n_I]$. To see this, let us assign a factor of y_I^q for a state of charge q under $U_{I,(1)}$, and a factor of z^n for a state at grade n , and call them the multiplicative charge of states or operators. Then the multiplicative charge of $U_{J,(s),-n}^I$ is $z^n y_J / y_I$. We define

$$x_1 = y_1 / y_2, \dots, x_{k-1} = y_{M-1} / y_M, x_M = z y_M / y_1. \quad (3.6)$$

Then the charge of every creation operator is a monomial with non-negative powers of x_I , and the graded dimension of the Verma module is given by

$$\text{tr } z^{\Delta L_0} \prod_I y_I^{\Delta U_{I,(1)}} = \prod_{i=1}^M \prod_{J=1}^{\infty} \left(1 - \prod_{s=I}^{I+J-1} x_s \right)^{-\min(n_I, n_{I+J})}. \quad (3.7)$$

Here, ΔL_0 and $\Delta U_{I,(1)}$ measure the difference of the respective conserved charges between the state concerned and the highest weight vector. Thus the graded dimension clearly depends on the composition (n_i) . Note that this is for the Verma module of $W(\widehat{\text{U}}(N), [n_I])$; the one for $W(\widehat{\text{SU}}(N), [n_I])$ is obtained by dropping a free boson, i.e. dropping a factor of $\prod_n (1 - (x_1 \cdots x_M)^n)^{-1}$.

3.2 Instantons and states in the Verma module

Let us now compare the structure of the W-algebra with the instanton calculation. We consider first the five-dimensional maximally supersymmetric Yang-Mills with gauge group $\text{SU}(N)$ on $\mathbb{R}_t \times \mathbb{C}_z \times \mathbb{C}_w$, and introduce a codimension-two defect of type \mathbb{L} associated to a partition $[n_I]$, with monodromy $\vec{\alpha}$ at $w = 0$, as defined in Sec. 2.1. Its BPS sector can be reduced to a supersymmetric quantum mechanics on the instanton

moduli spaces. We perform Nekrasov's deformation, and then the Hilbert space of the BPS states is given by the equivariant cohomology

$$\mathcal{H}_{\text{BPS}} = \bigoplus_{d, \mathfrak{m}} H_{\text{U}(1)^2 \times \text{U}(1)^{N-1}}^*(\mathcal{M}_{\mathbb{L}, \vec{\alpha}, \vec{d}}). \quad (3.8)$$

This should also be the Hilbert space of the two-dimensional theory obtained by compactifying the six-dimensional $\mathcal{N} = (2, 0)$ theory on $\mathbb{C}_z \times \mathbb{C}_w$, further compactified on S^1 . Then it is this two-dimensional theory that has the W-algebra symmetry $W(\widehat{\text{SU}}(N), [n_I])$. Furthermore, \mathcal{H}_{BPS} is expected to become a Verma module \mathcal{V} of the W-algebra, $\mathcal{H}_{\text{BPS}} = \mathcal{V}$.

We replace the instanton moduli space $\mathcal{M}_{\mathbb{L}, \vec{\alpha}, \vec{d}}$, which is highly singular, with its resolution of singularities, the moduli space of parabolic framed torsion free sheaves $\mathcal{M}_{\mathcal{P}, \vec{d}}$. Here \mathcal{P} is the parabolic subgroup determined by $\vec{\alpha}$, or equivalently by the composition (n_I) . Then we consider the Hilbert space

$$\mathcal{H}'_{\text{BPS}} = \bigoplus_{\vec{d}} H_{\text{U}(1)^2 \times \text{U}(1)^N}^*(\mathcal{M}_{\mathcal{P}, \vec{d}}). \quad (3.9)$$

Since the fixed points of the action of $\text{U}(1)^2 \times \text{U}(1)^N$ on $\mathcal{M}_{\mathcal{P}, \vec{d}}$ are isolated, the equivariant localization theorem tells us that the equivariant cohomology has a basis labeled by the fixed points

$$\mathcal{H}'_{\text{BPS}} = \bigoplus_{\vec{\lambda}} \mathbb{C}|\vec{\lambda}\rangle. \quad (3.10)$$

We expect that $\mathcal{H}'_{\text{BPS}}$ is basically the same as the original Hilbert space \mathcal{H}_{BPS} , except for a factor \mathcal{F} from free bosons introduced by the resolution of singularities. Combining with the expectation $\mathcal{H}_{\text{BPS}} = \mathcal{V}_{\mathcal{P}}$, we arrive at a general conjecture

$$\mathcal{H}'_{\text{BPS}} = \mathcal{V}_{\mathcal{P}} \otimes \mathcal{F}. \quad (3.11)$$

This statement has been checked for a few cases. When $N = 1$, $\mathcal{H}'_{\text{BPS}}$ consists solely of the Fock space of a free boson, and the action of the Heisenberg algebra was constructed geometrically in [58, 59, 60]. For $N > 1$ with no surface operator, or equivalently when $\mathbb{L} = \text{SU}(N)$ and $\mathcal{P} = \text{SL}(N)$, $\mathcal{V}_{\mathcal{P}}$ is the Verma module of the W_N algebra and \mathcal{F} is the Fock space of a free boson [5, 6, 61, 62]. When the surface operator is full, or equivalently when $\mathbb{L} = \text{U}(1)^{N-1}$ and \mathcal{P} is the Borel subgroup \mathcal{B} , $\mathcal{V}_{\mathcal{P}}$ is the Verma module of the affine $\widehat{\text{SU}}(N)$ algebra and again \mathcal{F} is a Fock space [20, 21]. Moreover, in [45, 63], the cohomology

was studied directly before the resolution of singularities when $\mathcal{P} = \mathcal{B}$ and it was shown that they are the Verma module of $\widehat{\text{SL}}(N)$ without an extra boson.

We would like to perform a few more checks of this proposal, for the surface operator of more general type. First, let us consider the generating function of the fixed points,

$$\text{tr}_{\mathcal{H}'} \prod_I x_I^{d_I} = \sum_{\vec{\lambda}} \prod_I x_I^{d_I(\vec{\lambda})}, \quad (3.12)$$

which can be thought of as Nekrasov's partition function of $\mathcal{N} = 4$ super Yang-Mills theory, because taking the trace means compactifying \mathbb{R}_t of the maximally supersymmetric five-dimensional Yang-Mills theory along a circle.

This generating function can be easily calculated by noticing that adding a row of length j to $\lambda^{s,I}$ contributes a factor of $x_I x_{I+1} \cdots x_{I+j-1}$. Since there are n_I choices of s , we find

$$\text{tr}_{\mathcal{H}'} \prod_I x_I^{d_I} = \prod_{I=1}^M \prod_{j=1}^{\infty} \left(1 - \prod_{s=I}^{I+j-1} x_s \right)^{-n_I}. \quad (3.13)$$

Note that this is mostly the same as (3.7) except that we have n_I instead of $\min(n_I, n_{I+J-1})$ in the exponent.

In particular, when all of n_I are equal, we see that the generating function gives the graded dimension of the Verma module of $W(\widehat{\text{U}}(N), \rho)$, or the product of the Verma module of $W(\widehat{\text{SU}}(N), \rho)$ and the Fock space of a free boson. This includes the cases referred to above, $\mathbb{L} = \text{SU}(N)$ and $\mathbb{L} = \text{U}(1)^{N-1}$. It also covers the case when $n_I \leq n_{I+1}$ for all I and the quiver is not taken to be cyclic. This corresponds to the finite analogue dealt with in [22]. However, it is clear that this is not the whole story. In the general case, one finds that in \mathcal{H}' there are $1 + \sum_{1 \leq I < J \leq M} |n_I - n_J|$ bosons, presumably free, in addition to the Verma module $\mathcal{V}_{\mathcal{P}}$.

3.3 Partition function and the Whittaker vector

In order to study a more detailed structure of $\mathcal{H}'_{\text{BPS}}$, let us consider the instanton partition function of the pure $\text{SU}(N)$ gauge theory defined by (2.43). From the point of view of the five-dimensional maximally supersymmetric Yang-Mills, four-dimensional $\mathcal{N} = 2$ pure $\text{SU}(N)$ theory is obtained by making \mathbb{R}_t into a segment with a common half-BPS boundary condition at both ends; recall Witten's construction using D4-branes suspended between NS5-branes [64]. The boundary condition determines a state $|W\rangle$ in \mathcal{H}_{BPS} .

Therefore, we expect the partition function to be given by the inner product, $Z = \langle W|W\rangle$, possibly up to a factor coming from the extra bosons in \mathcal{F} . The condition to be put on $|W\rangle$ can be guessed from the behavior of the Seiberg-Witten curve [8] and is a certain coherent state, as detailed below. The state $|W\rangle$ is usually called the Whittaker vector in the representation theory.

This statement was checked in [8, 10] for the case without the surface operator, and proved for $SU(2)$ in [65, 66, 67, 68]. It was proved for the full surface operator of general $SU(N)$ in [18, 19]. In these papers, it was always the case that for the pure gauge theory, the factor coming from the extra bosons was trivial. Then it is natural to conjecture [26, 27] that for the pure gauge theory

$$Z = \langle W|W\rangle \quad (3.14)$$

where $|W\rangle$ is a coherent state in the Verma module, or the Whittaker vector. Note that for a general surface operator, both sides of (3.14) depend on the choice of the parabolic subgroup, or equivalently on the composition (n_I) .

Let us determine the condition to be put on the coherent state. In general, a coherent state is defined by

$$U_{J,(s),n}^I |W\rangle = u_{J,(s),n}^I |W\rangle \quad (3.15)$$

for annihilation operators $U_{J,(s),n}^I$. The “eigenvalues” $u_{J,(s),n}^I$ need to vanish whenever $U_{J,(s),n}^I$ is given by a commutator of two other annihilation operators. One way to guarantee this condition is to allow nonzero $u_{J,(s),n}^I$ only for the annihilation operators whose multiplicative charge as defined in Sec. 3.1 is x_1, x_2, \dots , or x_n . For each x_I , we have annihilation operators $U_{I,(s),0}^{I+1}$ where we define $U_{M,(s),0}^{M+1} \equiv U_{M,(s),1}^1$ for notational simplicity. Now, the expansion of Z is in terms of q_1, q_2, \dots, q_M where q_I has mass dimension $n_I + n_{I+1}$. As the mass dimension of the four-dimensional theory is translated to the scaling dimension of the two-dimensional fields, which was given by s , the only sensible choice is to impose the conditions

$$U_{I,(\frac{n_I+n_{I+1}}{2}),0}^{I+1} |W\rangle = c_I q_I^{1/2} |W\rangle \quad (3.16)$$

and to let the remaining annihilation operators to annihilate $|W\rangle$. Here c_I are constants depending on the precise normalization of the generators $U_{J,(s),n}^I$, which we expect to be rational numbers when the generators are naturally defined via the quantum Drinfeld-Sokolov reduction. This condition is a natural generalization of the conditions proposed

in [26, 27], and also takes into account the dependence of the Verma module and the instanton partition function on the composition (n_I) .

The equality $Z = \langle W|W \rangle$ was checked for a few cases by Wyllard [26, 27] using the explicit commutation relations of the corresponding W-algebras worked out for the partition of the form $[2, 1, \dots, 1]$ in [69, 70]. The equality is also proved for the finite analogue in [22] when (n_I) is ordered so that $n_I \leq n_J$ when $I \leq J$.

In the rest of the paper we will report the check of (3.14) for the partition

$$[2^\mu 1^\nu] = \underbrace{[2, \dots, 2]}_\mu, \underbrace{[1, \dots, 1]}_\nu \quad (3.17)$$

for various ordering. It is performed by first working out the quantum Drinfeld-Sokolov reduction and then implementing the resulting algebra in Mathematica. The detail of the reduction is given in Appendix A and the Mathematica file to calculate the instanton partition function and the Whittaker vector is available on the preprint webpage as an ancillary file so that any reader can check by herself/himself.

We start from $\widehat{U}(N)$ affine current algebra of level k . The resulting W-algebra $W(\widehat{SU}(N), [2^\mu 1^\nu])$ has generators

$$j_b^a(z), \quad J_j^i(z), \quad U_j^a(z), \quad \check{U}_b^i(z), \quad S_b^a(z) \quad (3.18)$$

where $a, b \in \mathcal{I}_2$ and $i, j \in \mathcal{I}_1$. Here $\mathcal{I}_n = \{I \mid n_I = n\}$, where (n_I) is a composition of N in terms of μ twos and ν ones. These fields were respectively denoted by

$$U_{b,(1)}^a(z), \quad U_{j,(1)}^i(z), \quad U_{j,(3/2)}^a(z), \quad U_{b,(3/2)}^i(z), \quad U_{b,(2)}^a(z) \quad (3.19)$$

in Sec. 3.1. These are realized as a subalgebra of $\widehat{U}(\mu+\nu)_{k+\mu} \times \widehat{U}(\mu)_{k+\mu+\nu}$ current algebra, generated by fields $J_J^I(z)$ and $\check{J}_b^a(z)$ where $I, J = 1, \dots, \mu + \nu$ and $a, b \in \mathcal{I}_2$.

Here we write down explicit forms of conditions put on the Whittaker vector, and the mapping of parameters between the W-algebra and the instanton counting for a few explicit examples. The highest weight of the Verma module is given by the eigenvalues of the Cartan of the W-generators, but the same information can be more conveniently encoded by the Cartan of $\widehat{U}(\mu+\nu) \times \widehat{U}(\mu)$, namely $h_I \equiv J_{I,0}^I$ and $\check{h}_a \equiv \check{J}_{a,0}^a$ (no summation on I or a .) Then the conjecture is that

$$Z(\epsilon_{1,2}; a_i; q_I) = \langle W|W \rangle \quad \text{for } |W\rangle \in \mathcal{V}_{\mathcal{P}, h_I, \check{h}_a} \quad (3.20)$$

under a suitable mapping between the parameters $\epsilon_{1,2}$, a_i and h_I, \check{h}_a .

For the composition $(2, 2, 1, 1)$, the nonzero conditions we put on the Whittaker vector are

$$\begin{aligned} S_{1,0}^2|W\rangle &= q_1^{1/2}|W\rangle, & \langle W|S_{2,0}^1 &= \langle W|q_1^{1/2}, \\ \tilde{U}_{2,0}^3|W\rangle &= q_2^{1/2}|W\rangle, & \langle W|U_{3,0}^2 &= \langle W|q_2^{1/2}, \\ j_{3,0}^4|W\rangle &= q_3^{1/2}|W\rangle, & \langle W|j_{4,0}^3 &= -\langle W|q_3^{1/2}, \\ U_{4,1}^0|W\rangle &= q_4^{1/2}|W\rangle, & \langle W|\tilde{U}_{0,-1}^4 &= -\langle W|q_4^{1/2}. \end{aligned} \quad (3.21)$$

Then Z equals $\langle W|W\rangle$ when $k = -6 - \epsilon_2/\epsilon_1$ and

$$(h_1, h_2, h_3, h_4, \check{h}_1, \check{h}_2) = \left(\frac{a_{1,1}}{\epsilon_1}, \frac{a_{1,2}}{\epsilon_1}, \frac{a_{1,3}}{\epsilon_1}, \frac{a_{1,4}}{\epsilon_1}, \frac{a_{2,1} + \epsilon_2}{\epsilon_1}, \frac{a_{2,2} + \epsilon_2}{\epsilon_1} \right) + (1, 2, 3, 4, 5, 6). \quad (3.22)$$

For the composition $(2, 1, 2, 1)$, the nonzero conditions we put on the Whittaker vector are

$$\begin{aligned} \tilde{U}_{1,0}^2|W\rangle &= q_1^{1/2}|W\rangle, & \langle W|U_{2,0}^1 &= \langle W|q_1^{1/2}, \\ U_{2,0}^3|W\rangle &= q_2^{1/2}|W\rangle, & \langle W|\tilde{U}_{3,0}^2 &= -\langle W|q_2^{1/2}, \\ \tilde{U}_{3,0}^4|W\rangle &= q_3^{1/2}|W\rangle, & \langle W|U_{4,0}^3 &= \langle W|q_3^{1/2}, \\ U_{4,1}^0|W\rangle &= q_4^{1/2}|W\rangle, & \langle W|\tilde{U}_{0,-1}^4 &= -\langle W|q_4^{1/2}. \end{aligned} \quad (3.23)$$

Then Z equals $\langle W|W\rangle$ when $k = -6 - \epsilon_2/\epsilon_1$ and

$$(h_1, h_2, h_3, h_4, \check{h}_1, \check{h}_3) = \left(\frac{a_{1,1}}{\epsilon_1}, \frac{a_{1,2}}{\epsilon_1}, \frac{a_{1,3}}{\epsilon_1}, \frac{a_{1,4}}{\epsilon_1}, \frac{a_{2,1} + \epsilon_2}{\epsilon_1}, \frac{a_{2,3} + \epsilon_2}{\epsilon_1} \right) + (1, 2, 3, 4, 5, 6). \quad (3.24)$$

Finally for the composition $(2, 2, 2)$, the nonzero conditions we put on the Whittaker vector are

$$\begin{aligned} S_{1,0}^2|W\rangle &= q_1^{1/2}|W\rangle, & \langle W|S_{2,0}^1 &= \langle W|q_1^{1/2}, \\ S_{2,0}^3|W\rangle &= q_2^{1/2}|W\rangle, & \langle W|S_{3,0}^2 &= \langle W|q_2^{1/2}, \\ S_{3,0}^0|W\rangle &= q_3^{1/2}|W\rangle, & \langle W|S_{0,-1}^3 &= \langle W|q_3^{1/2}. \end{aligned} \quad (3.25)$$

Then Z equals $\langle W|W\rangle$ when $k = -6 - \epsilon_2/\epsilon_1$ and

$$\begin{aligned} (h_1, h_2, h_3, \check{h}_1, \check{h}_2, \check{h}_3) &= \\ & \left(\frac{a_{1,1}}{\epsilon_1}, \frac{a_{1,2}}{\epsilon_1}, \frac{a_{1,3}}{\epsilon_1}, \frac{a_{2,1} + \epsilon_2}{\epsilon_1}, \frac{a_{2,2} + \epsilon_2}{\epsilon_1}, \frac{a_{2,3} + \epsilon_2}{\epsilon_1} \right) + (1, 2, 3, 4, 5, 6). \end{aligned} \quad (3.26)$$

The general rule for the mapping is now clear. The salient points are that we always found the agreement, once the maps $k = -N - \epsilon_2/\epsilon_1$ and $h_i \sim a_i/\epsilon_1$ are used, and that the formula giving the level k agrees with the previous papers [18, 20, 21, 26, 27].

4 Conclusions

In this paper, we provided further checks of a recent conjecture that the instanton partition function of $\mathcal{N} = 2$ pure $SU(N)$ gauge theory in the presence of a surface operator of type $[n_I]$ is governed by the W-algebra $W(\widehat{SU}(N), [n_I])$. We first saw an explicit description of the instanton moduli space in the presence of a surface operator in terms of the representations of the chain-saw quiver and then determined the fixed point contributions. Then we studied the structure of the Verma module of the general W-algebra, and determined the conditions to be put on the Whittaker vector. We saw that the instanton partition function equals the norm of the Whittaker vector once an appropriate mapping between the parameters on the four-dimensional side and on the two-dimensional side was made. Along the way, we found that both the instanton moduli and the Verma module depend on the composition (n_I) , not just on the partition $[n_I]$.

Our explicit checks were only for surface operators of type $[2^\mu 1^\nu]$. This was due to the technical problem that the quantum Drinfeld-Sokolov reduction for a partition $[n_I]$ involves normal-ordered products of h currents where $h = \max(n_I)$. This makes the calculation very tedious when $h > 2$. Then, one immediate generalization of this work would be to extend the checks to more general surface operators by implementing the quantum Drinfeld-Sokolov reduction itself on the computer. The authors would like to welcome the help of anybody interested. Another direction of study is to incorporate hypermultiplets to the analysis and to consider quiver gauge theories. This should correspond, on the W-algebra side, to the study of actions of primary fields on the Verma modules.

In this paper we viewed the surface operator as coming from the codimension-two defect of the six-dimensional $\mathcal{N} = (2, 0)$ theory. Therefore the presence of the surface operator changed the two-dimensional theory. The surface operators of four-dimensional gauge theory can also be thought of as coming from the codimension-four defects of the six-dimensional theory. From this point of view, the introduction of a surface operator corresponds to an insertion of a degenerate field on the Riemann surface, without changing the two-dimensional theory [71, 13]. The latter viewpoint was more suited to the comparison to the topological string approach in the case of the simple surface operator, as was exemplified in many papers [72, 39, 73, 40, 41]. Therefore it would be interesting to analyze general surface operators from this latter point of view. The relation of these

two points of view should be related to the mapping between the Toda theory and the WZW theory [74, 75, 76, 77, 78], and is also worth further investigation. In [37] the instanton moduli space with a surface operator is described by a coupled system of the ADHM equation with two-dimensional vortex. It was derived from the brane configuration realizing the codimension-four defects and called the enhanced ADHM system. A comparison of the enhanced ADHM quiver of [37] and the chain-saw quiver may shed light on this issue.

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A Details of the Drinfeld-Sokolov reduction

We perform the quantum Drinfeld-Sokolov reduction [23, 24, 25] for the partition

$$N = \underbrace{2 + \cdots + 2}_{\mu} + \underbrace{1 + \cdots + 1}_{\nu}, \quad (\text{A.1})$$

following the procedure described in [25]. For an extensive list of references, see [79, 80]. We consider $U(N)$ instead of $SU(N)$.

First, an index of the adjoint of $U(N)$ is denoted by $\overset{A}{B}$ where $A, B = 1, \dots, N$. This corresponds to a single index a in the paper [25]. The structure constant is then

$$f_{\overset{A}{B} \overset{C}{D} \overset{E}{F}} = -\delta_B^C \delta_E^A \delta_D^F + \delta_D^A \delta_B^F \delta_E^C. \quad (\text{A.2})$$

It is too cumbersome to keep this notation, so we use the convention

$$T_B^A \equiv T_B^{\overset{A}{B}} \equiv T_A^B \quad (\text{A.3})$$

then the left-hand side of the equation above is just f_{BDE}^{ACF} .

We start from the $U(N)$ currents J_B^A ; our unhatted currents are the hatted currents of [25]. The $SU(2)$ embedding for the partition $[2^\mu 1^\nu]$ is given by

$$t_0 = \text{diag}\left(\underbrace{\frac{1}{2}, \dots, \frac{1}{2}}_\mu, \underbrace{0, \dots, 0}_\nu, \underbrace{-\frac{1}{2}, \dots, -\frac{1}{2}}_\mu\right) \quad (\text{A.4})$$

and the grading is done by

$$\delta = \text{diag}\left(\underbrace{1, \dots, 1}_\mu, \underbrace{1, \dots, 1}_\nu, \underbrace{0, \dots, 0}_\mu\right). \quad (\text{A.5})$$

Accordingly, we decompose J_B^A as follows :

$$J_B^A = \left(\begin{array}{c|c|c} \tilde{j}_b^a & U_j^a & V_b^a \\ \hline X_b^i & J_j^i & Y_b^i \\ \hline \mathbf{1} & 0 & \tilde{j}_b^a \end{array} \right) \quad (\text{A.6})$$

where $a, b = 1, \dots, \mu$ and $i, j = 1, \dots, \nu$. Also we introduce $I, J = 1, \dots, \mu + \nu$ and let

$$J_J^I = \left(\begin{array}{c|c} \tilde{j}_b^a & U_j^a \\ \hline X_b^i & J_j^i \end{array} \right), \quad Y_b^I = \left(\begin{array}{c} V_b^a \\ \hline Y_b^i \end{array} \right). \quad (\text{A.7})$$

The c ghosts are

$$c_b^I = \left(\begin{array}{c} c_b^a \\ \hline c_b^i \end{array} \right). \quad (\text{A.8})$$

The Miura transformation realizes the W-algebra as a subalgebra of $\widehat{U}(\mu + \nu)_{k+\mu} \times \widehat{U}(\mu)_{k+\mu+\nu}$ generated by J_J^I and \tilde{j}_b^a . Note that there are also off-diagonal components in the level matrix:

$$J_J^I(z) \tilde{j}_b^a(w) \sim \delta_J^I \delta_b^a \frac{1}{(z-w)^2}. \quad (\text{A.9})$$

The generators of the W-algebras are

$$j_b^a = \tilde{j}_b^a + \check{j}_b^a, \quad J_j^i, \quad U_i^a \quad (\text{A.10})$$

and

$$S_b^a \equiv V_b^{a(1)}, \quad \tilde{U}_b^i \equiv Y_b^{i(1)} \quad (\text{A.11})$$

where $B^{(1)}$ is a solution to the tic-tac-toe

$$D_0(B^{(1)}) = D_1 B. \quad (\text{A.12})$$

At this stage it is already clear that the W-algebra has a current subalgebra $U(\mu)_{2k+N} \times U(\nu)_{k+\mu}$.

To determine S_b^a and \tilde{U}_b^i explicitly, we need the BRST differentials. The differential D_0 is given by

$$D_0(X_b^i) = -c_b^i, \quad D_0(\tilde{j}_b^a) = -c_b^a, \quad D_0(\check{j}_b^a) = +c_b^a \quad (\text{A.13})$$

and the differential D_1 is given by

$$D_1(Y_b^J) = -\check{j}_b^a c_a^J + J_J^I c_b^J + (k+N)\partial c_b^J \quad (\text{A.14})$$

which can be further decomposed to

$$D_1(V_b^a) = -\check{j}_b^c c_c^a + \tilde{j}_b^c c_b^c + U_j^a c_b^j + (k+N)\partial c_b^a, \quad (\text{A.15})$$

$$D_1(Y_b^i) = -\check{j}_b^a c_a^i + X_a^i c_b^a + J_j^i c_b^j + (k+N)\partial c_b^i. \quad (\text{A.16})$$

Therefore we find

$$S_b^a = -U_j^a X_b^j + \check{j}_b^c \tilde{j}_c^a - (k+\mu+\nu)\partial \frac{1}{2}(\tilde{j}_b^a - \check{j}_b^a), \quad (\text{A.17})$$

$$\tilde{U}_b^i = \check{j}_b^a X_a^i - J_j^i X_b^j - (k+\mu+\nu)\partial X_b^i. \quad (\text{A.18})$$

In the expressions above, the normal ordering is implicit in the product of the currents, i.e. $(JJ)_m = \sum_i : J_i J_{m-i} :$ where

$$: J_n J_m := \begin{cases} J_n J_m & \text{if } n \leq -1, \\ J_m J_n & \text{if } n \geq 0. \end{cases} \quad (\text{A.19})$$

Note that this is the OPE normal ordering and not the naive normal ordering.

Summarizing, the W-algebra has affine $U(\mu)_{2k+N} \times U(\nu)_{k+\mu}$ currents j_b^a, J_j^i , the “bi-fundamentals” U_i^a, \tilde{U}_a^i and spin-two fields S_b^a , written in terms of affine $\widehat{U}(\mu + \nu)_{k+\mu} \times \widehat{U}(\mu)_{k+\mu+\nu}$ currents J_J^I and \tilde{j}_b^a via the relations (A.7),(A.9),(A.10),(A.17) and (A.18).

For $m = n = 1, N = 3$, the W-algebra is the Bershadsky-Polyakov algebra $W_3^{(2)}$. Our formula reproduces (3.16) of de Boer-Tjin [25], by assigning their names (3.14) to the components given in (A.6), except for

$$S = W_1^8 + \frac{\widehat{J}^4 \widehat{J}^4}{36} - \frac{\partial \widehat{J}^4}{2}. \quad (\text{A.20})$$

This discrepancy is just a redefinition of the basis in the algebra.

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