

# Realization of chiral symmetry in the ERG

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## Abstract

We discuss within the framework of the ERG how chiral symmetry is realized in a linear  $\sigma$  model. A generalized Ginsparg-Wilson relation is obtained from the Ward-Takahashi identities for the Wilson action assumed to be bilinear in the Dirac fields. We construct a family of its non-perturbative solutions. The family generates the most general solutions to the Ward-Takahashi identities. Some special solutions are discussed. For each solution in this family, chiral symmetry is realized in such a way that a change in the Wilson action under non-linear symmetry transformation is canceled with a change in the functional measure. We discuss that the family of solutions reduces via a field redefinition to a family of the Wilson actions with some composite object of the scalar fields which has a simple transformation property. For this family, chiral symmetry is linearly realized with a continuum analog of the operator extension of  $\gamma_5$  used on the lattice. We also show that there exist some appropriate Dirac fields which obey the standard chiral transformations with  $\gamma_5$  in contrast to the lattice case. Their Yukawa interactions with scalars, however, becomes non-linear.

## §1. Introduction

The discovery of chiral symmetry on the lattice<sup>1)–4)</sup> is considered as a new realization of symmetry which is incompatible with a given regularization used in a perturbative and/or non-perturbative approach to field theory. The crucial issue in this realization lies in an algebraic non-linear constraint on the Dirac operators, the Ginsparg-Wilson (GW) relation.<sup>1)</sup> It makes possible to define symmetry transformations<sup>4)</sup> which are not simply specified by the standard  $\gamma_5$  matrix, but its non-trivial extension that depends on the Dirac operator as well as the lattice spacing, avoiding the no-go theorem.<sup>5)</sup> It has been also shown that this formulation of chiral symmetry correctly gives the index theorem related to the chiral anomaly<sup>3).4)</sup>

The GW relation is essentially derived for free-field theories. Gauge field can be easily included as an external field in the relation. It is non-trivial to discuss a generalized GW relation in the presence of interactions of fermions with other fields which undergo chiral transformations.

In this paper we discuss this issue in the context of the exact renormalization group (ERG)<sup>6)–11)</sup> approach<sup>\*)</sup> to continuum theory. The original GW relation was derived for a macroscopic action of the Dirac fields on a coarse lattice obtained via the block-spin transformations from a microscopic action on a fine lattice. In parallel with this, the Wilson action in the ERG defined at IR scale  $\Lambda$  is obtained by blocking of an action given at a UV scale  $\Lambda_0$ . We consider a symmetry assumed to be realized in the standard form at  $\Lambda_0$ . It suffers from a deformation when our reference scale  $\Lambda$  is going down from  $\Lambda_0$  as we take a symmetry-breaking blocking. Even though the symmetry cannot remain in the standard form, it is realized in a non-trivial way. In general, the deformation of symmetry and its properties are described by the Ward-Takahashi (WT) identities for the Wilson action, and by their extended version, the Quantum Master Equation (QME) in the Batalin-Vilkovisky<sup>20)</sup> antifield formalism<sup>\*\*)</sup>.

If we confine ourselves to those for the Wilson action that is bilinear in the Dirac fields, the WT identities for the fermionic sector lead to a relation which is quadratic in the Dirac operator. For free-field theories, we have discussed that the WT identities lead to the standard GW relation for chiral symmetry and that for SU(2) global symmetry.<sup>27)</sup>

In the presence of interactions between the Dirac fields and some other fields, the quadratic relation

<sup>\*)</sup> Recent reviews on the ERG are refs. 12) – 19).

<sup>\*\*)</sup> See, e.g., refs. 21) 22) for the generic idea of formulation of exact (gauge) symmetry in ERG, and refs. 23) – 26) for some concrete discussions for QED and Yang-Mills theory.

in the Dirac operator may be called as a generalized GW relation which contains the fields interacting with the Dirac fields. This generally happens, not restricted to chiral symmetry. Consideration of such a GW relation in chiral symmetry, therefore, may give some important insights into realization of other symmetries including that of gauge symmetry.

We wish to discuss in this paper chiral symmetry in a linear  $\sigma$  model. We take symmetry-breaking blocking with a mass term for the Dirac fields to discuss non-trivial realization. Derivation of the WT identities for a generic linear symmetry has been discussed in ref. 24), and also in ref. 28). In the functional integral method we use, the Wilson action for the IR fields is formally defined by an integration of the blocked UV action to construct the WT identities for the IR action. Once we obtain the identities, we try to discuss the most general solutions to them in a purely algebraic way. The UV action is only used as a boundary condition for the solutions, and is not used to be integrated out. Therefore, we are not confined ourselves to those solutions obtained from explicit integration of the UV fields. Our main task is to solve the WT identities for the Wilson action for the  $\sigma$  model which is assumed to be bilinear in the Dirac fields. Although this is a truncation in the fermionic sector, the WT identities we obtain are exact if we neglect a loop contribution arising from four-fermi interactions. The resulting identity for the Dirac fields leads to a generalized GW relation for the Dirac operator which has a non-linear dependence on the scalar fields. We obtain the most general non-perturbative solutions to the GW relation. They include as a special case the continuum counter part of the solutions obtained on the lattice in ref. 29), where the scalar fields are introduced as auxiliary fields in the Wilson action. There is also some other choice which corresponds to the solution obtained by explicit integration over the UV Dirac fields. It corresponds to a perfect action if no blocking for the scalar fields is performed.

It should be stressed that the chiral transformations for the Dirac fields depend on the Dirac operator, and become non-linear. They induce a Jacobian factor that must be canceled with a counter action of the scalar fields. Thus, the Wilson action constructed from a solution to the GW relation contains the counter action which is not chiral invariant. Chiral symmetry is not realized as invariance of the Wilson action, and inclusion of the contribution from change in the functional measure is crucial in this realization.

On the lattice, an operator extension of  $\gamma_5$  matrix denoted as  $\hat{\gamma}_5$  is extensively used. It depends on the free Dirac operator and the lattice spacing. In the ERG approach to continuum theory considered in this paper, there exists a field redefinition which reduces our family of the solutions to a family of the Wilson actions for which the chiral transformation on the Dirac fields is expressed in terms of  $\hat{\gamma}_5$ . Each Wilson action consists of some composite object of the scalar fields which has a simple transformation properties. Using new fermionic variables, chiral transformation is linearly realized, and symmetry is simply realized as invariance of the Wilson action. We also show that appropriate field redefinition gives the Dirac fields which obey the standard chiral transformation with  $\gamma_5$ . However, their Yukawa interactions with the scalar fields inevitably become non-linear. Such a field redefinition does not make sense on the lattice because of the singularity at the momentum region where the species doublers appear.

We have two ways in realization of chiral symmetry, one with non-linear symmetry transformations, and another with linear transformations. Even for the latter, the Wilson actions which are solutions to the WT identities are non-linear in scalar fields.

This paper is organized as follows. We briefly summarize a functional integral method for derivation of the WT identities for a global symmetry in the next section, and obtain a generalized GW relation for a  $\sigma$  model in section 3. We give a family of non-perturbative solutions, and discuss some concrete solutions in section 4. We then discuss that the family of the Wilson actions constructed from the solutions given in section 4 can be reduced to a family of the Wilson actions for which chiral symmetry is linearly realized. The last section is devoted to summary and outlook.

## §2. Derivation of the WT identities for a global symmetry

Let us consider a generic renormalizable theory with action  $S_0[\phi]$  in 4-dimensional Euclidean space. It is a functional of some fields that are collectively denoted by  $\phi^A$ . The index  $A$  represents

the Lorentz indices of vector fields, the spinor indices of the fermions, and/or indices distinguishing different types of generic fields. The Grassmann parity for  $\phi^A$  is expressed as  $\epsilon(\phi^A) = \epsilon_A$ , so that  $\epsilon_A = 0$  if the field  $\phi^A$  is Grassmann even (bosonic) and  $\epsilon_A = 1$  if it is Grassmann odd (fermionic).

In order to regularize the theory, we introduce an IR momentum cutoff  $\Lambda$  and through a positive function that behaves as

$$K\left(\frac{p^2}{\Lambda^2}\right) \approx \begin{cases} 1 & (p^2 < \Lambda^2), \\ 0 & (p^2 > \Lambda^2). \end{cases} \quad (2.1)$$

We also introduce a UV cutoff  $\Lambda_0 > \Lambda$  and  $K_0(p) \equiv K(p^2/\Lambda_0^2)$ .

Introducing sources  $J_A$  for the fields  $\phi^A$ , the generating functional is written as

$$\mathcal{Z}_\phi[J] = \int \mathcal{D}\phi \exp\left(-\mathcal{S}[\phi; \Lambda_0] + K_0^{-1} J \cdot \phi\right), \quad (2.2)$$

where the action  $\mathcal{S}$  defined at the scale  $\Lambda_0$  is written as the sum of the kinetic and interaction terms

$$\mathcal{S}[\phi; \Lambda_0] = \frac{1}{2} \phi \cdot K_0^{-1} D_0 \cdot \phi + \mathcal{S}_I[\phi; \Lambda_0]. \quad (2.3)$$

Here we use the matrix notation in momentum space:

$$\begin{aligned} J \cdot \phi &= \int \frac{d^4 p}{(2\pi)^4} J_A(-p) \phi^A(p), \\ \phi \cdot D \cdot \phi &= \int \frac{d^4 p}{(2\pi)^4} \phi^A(-p) D_{AB}(p) \phi^B(p). \end{aligned} \quad (2.4)$$

In the ERG approach, one introduces IR fields  $\Phi^A$  whose action, the Wilson action  $S[\Phi]$ , is given as a functional integral of the original UV fields  $\phi^A$ . This can be done to rewrite the partition function (2.2) up to field and source independent constant as

$$\begin{aligned} \mathcal{Z}_\phi[J] &= \int \mathcal{D}\phi \mathcal{D}\Phi \exp -\frac{1}{2} \left( \Phi - f \phi - J \cdot (K\alpha)^{-1} \right) \cdot \alpha \cdot \left( \Phi - f \phi - (-)^{\epsilon_J} (K\alpha)^{-1} \cdot J \right) \\ &\quad \times \exp\left(-\mathcal{S}[\phi; \Lambda_0] + K_0^{-1} J \cdot \phi\right), \end{aligned} \quad (2.5)$$

where  $f = K/K_0$ , and matrix  $\alpha = \alpha_{AB}$  is the kernel of the gaussian integral over the IR fields. We obtain the following expressions for the Wilson action:

$$\begin{aligned} \mathcal{Z}_\phi[J] &= N_J \mathcal{Z}_\Phi[J], \\ \mathcal{Z}_\Phi[J] &= \int \mathcal{D}\Phi \exp\left(-S[\Phi; \Lambda] + J \cdot K^{-1} \Phi\right), \\ \exp(-S[\Phi; \Lambda]) &= \int \mathcal{D}\phi \exp - \left\{ \frac{1}{2} (\Phi - f \phi) \cdot \alpha \cdot (\Phi - f \phi) + \mathcal{S}[\phi; \Lambda_0] \right\}, \end{aligned} \quad (2.6)$$

where the normalization factor  $N_J$  is given by

$$\ln N_J = -\frac{(-)^{\epsilon_A}}{2} J_A K^{-2} (\alpha^{-1})^{AB} J_B. \quad (2.7)$$

We discuss symmetry properties of the Wilson action. Consider a change of variables defined at the UV scale  $\Lambda_0$ :

$$\phi^A \rightarrow \phi'^A = \phi^A + \delta\phi^A, \quad \delta\phi^A = \mathcal{R}^A[\phi]. \quad (2.8)$$

The generating functional (2.2) is invariant under the change of the integration variable. It gives the relation

$$\int \mathcal{D}\phi \left( K_0^{-1} J \cdot \delta\phi - \Sigma[\phi; \Lambda_0] \right) \exp\left(-\mathcal{S}[\phi; \Lambda_0] + K_0^{-1} J \cdot \phi\right) = 0, \quad (2.9)$$

where  $\Sigma[\phi; \Lambda_0]$  is the WT operator given as

$$\Sigma[\phi; \Lambda_0] \equiv \frac{\partial^r \mathcal{S}}{\partial \phi^A} \delta \phi^A - (-)^{\epsilon_A} \frac{\partial}{\partial \phi^A} \delta \phi^A. \quad (2.10)$$

$\Sigma[\phi, \Lambda_0]$  is the sum of the change of the action  $\mathcal{S}[\phi; \Lambda_0]$

$$\delta \mathcal{S} = \frac{\partial^r \mathcal{S}}{\partial \phi^A} \delta \phi^A, \quad (2.11)$$

and that of the functional measure  $\mathcal{D}\phi$

$$\delta \ln \mathcal{D}\phi = (-)^{\epsilon_A} \frac{\partial^r}{\partial \phi^A} \delta \phi^A. \quad (2.12)$$

The relation (2.9) may be rewritten as

$$\begin{aligned} \langle \Sigma[\phi; \Lambda_0] \rangle_{\phi, K_0^{-1}J} &= K_0^{-1}J \cdot \langle \delta \phi \rangle_{\phi, K_0^{-1}J} \\ &= K_0^{-1}J \cdot \langle \mathcal{R}[\phi; \Lambda_0] \rangle_{\phi, K_0^{-1}J} \\ &= K_0^{-1}J \cdot \mathcal{R}[K_0 \partial^l J; \Lambda_0] \mathcal{Z}_\phi[J] \\ &= N_J \left\{ N_J^{-1} \left( K_0^{-1}J \cdot \mathcal{R}[K_0 \partial^l J; \Lambda_0] N_J \right) \right. \\ &\quad \left. + K_0^{-1}J \cdot \mathcal{R}[K_0 \partial^l J; \Lambda_0] \right\} \mathcal{Z}_\phi[J] \\ &= \langle \Sigma[\Phi; \Lambda] \rangle_{\Phi, K^{-1}J}. \end{aligned} \quad (2.13)$$

In this paper we assume that the theory given by (2.2) and (2.3) admits a linear global symmetry described by

$$\begin{aligned} \Sigma[\phi; \Lambda_0] &= \delta \mathcal{S} = \frac{\partial^r \mathcal{S}}{\partial \phi^A} \delta \phi^A = 0, \\ \delta \phi^A &= \mathcal{R}^A{}_B[\Lambda_0] \phi^B, \end{aligned} \quad (2.14)$$

where  $\mathcal{R}^A{}_B$  do not depend on the fields. Renormalization affects and deforms this symmetry. To see this, we compute

$$\begin{aligned} K_0 \frac{\partial^l}{\partial J_A} \mathcal{Z}_\phi[J] &= K_0 \frac{\partial^l}{\partial J_A} N_J \mathcal{Z}_\phi[J] \\ &= N_J \left[ (-)^{\epsilon_A+1} K_0 K^{-2} (\alpha^{-1})^{AB} J_B + K_0 \frac{\partial^l}{\partial J_A} \right] \mathcal{Z}_\phi[J] \\ &= N_J \left\langle f^{-1} \left\{ \Phi^A - (\alpha^{-1})^{AB} \frac{\partial^l S}{\partial \Phi^B} \right\} \right\rangle_{\Phi, K^{-1}J}, \end{aligned} \quad (2.15)$$

which can be used to obtain the WT identity for the IR fields

$$\begin{aligned} \Sigma[\Phi; \Lambda] &= \frac{\partial^r S}{\partial \Phi^A} \delta \Phi^A - (-)^{\epsilon_A} \frac{\partial}{\partial \Phi^A} \delta \Phi^A, \\ \delta \Phi^A &= \mathcal{R}^A{}_B[\Lambda_0] \left\{ \Phi^A - (\alpha^{-1})^{AB} \frac{\partial^l S}{\partial \Phi^B} \right\}. \end{aligned} \quad (2.16)$$

We consider in the next section non-perturbative realization of chiral symmetry for a linear sigma model in the ERG approach.

### §3. The WT identities for chiral symmetry

The model we deal with in this paper consists of a Dirac field,  $\psi$ ,  $\bar{\psi}$ , and complex scalar fields,  $\phi$ ,  $\phi^\dagger$ , collectively denoted as  $\phi^A = \{\psi, \bar{\psi}, \phi, \phi^\dagger\}$ . The UV action is given by

$$\begin{aligned}\mathcal{S}[\phi] &= \mathcal{S}[\bar{\psi}, \psi, \theta] + \mathcal{S}[\phi, \phi^\dagger], \\ \mathcal{S}[\bar{\psi}, \psi, \theta] &= \int_{p,q} \bar{\psi}(-p) \left[ K_0(p)^{-1} \not{p} \delta(p-q) + \theta(p-q) \right] \psi(q), \\ \mathcal{S}[\phi, \phi^\dagger] &= \int_p K_0(p)^{-1} \phi^\dagger(-p) p^2 \phi(p) + \mathcal{S}_I[\phi, \phi^\dagger],\end{aligned}\tag{3.1}$$

where

$$\begin{aligned}\theta(p) &= P_+ \phi(p) + P_- \phi^\dagger(p), \\ P_\pm &= \frac{1 \pm \gamma_5}{2}.\end{aligned}\tag{3.2}$$

The action (3.1) is invariant under chiral transformation

$$\begin{aligned}\delta\psi(p) &= i\varepsilon\gamma_5\psi(p), & \delta\bar{\psi}(-p) &= \bar{\psi}(-p)i\varepsilon\gamma_5, \\ \delta\phi(p) &= -2i\varepsilon\phi(p), & \delta\phi^\dagger(-p) &= 2i\varepsilon\phi^\dagger(-p),\end{aligned}\tag{3.3}$$

with a Grassmann even parameter  $\varepsilon$ . Since we wish to consider a non-trivial realization of chiral symmetry, we will take a chiral non-invariant blocking kernel for the Dirac field

$$\alpha_D(p) = \frac{M K_0(p)}{K(p)(K_0(p) - K(p))},\tag{3.4}$$

where  $M$  is a constant with mass dimension 1. For the scalar field, , we take a blocking kernel

$$\alpha_S(p) = \frac{K_0(p)p^2}{K(p)(K_0(p) - K(p))}.\tag{3.5}$$

For the chiral symmetry, the WT identity (2.16) for the IR fields  $\Phi^A = \{\Psi, \bar{\Psi}, \varphi, \varphi^\dagger\}$  becomes

$$\begin{aligned}\Sigma[\Phi] &= \int_p \left[ \frac{\partial^r S}{\partial \Psi(p)} i\varepsilon\gamma_5 \left\{ \Psi(p) - \alpha_D^{-1}(p) \frac{\partial^l S}{\partial \bar{\Psi}(-p)} \right\} + \left\{ \bar{\Psi}(-p) - \alpha_D^{-1}(p) \frac{\partial^r S}{\partial \Psi(p)} \right\} i\varepsilon\gamma_5 \frac{\partial^l S}{\partial \bar{\Psi}(-p)} \right. \\ &\quad - \frac{\partial^r}{\partial \Psi(p)} 2i\varepsilon\gamma_5 \alpha_D^{-1}(p) \frac{\partial^l S}{\partial \bar{\Psi}(-p)} + \frac{\partial S}{\partial \varphi(p)} (-2i) \left\{ \varphi(p) - \alpha_S^{-1}(p) \frac{\partial S}{\partial \varphi^\dagger(-p)} \right\} \\ &\quad \left. + \left\{ \varphi^\dagger(-p) - \alpha_S^{-1}(p) \frac{\partial S}{\partial \varphi(p)} \right\} (2i) \frac{\partial S}{\partial \varphi^\dagger(-p)} \right] \\ &= \int_p \left[ \frac{\partial^r S}{\partial \Psi(p)} i\varepsilon\gamma_5 \left\{ \Psi(p) - 2\alpha_D^{-1}(p) \frac{\partial^l S}{\partial \bar{\Psi}(-p)} \right\} + \bar{\Psi}(-p) i\varepsilon\gamma_5 \frac{\partial^l S}{\partial \bar{\Psi}(-p)} \right. \\ &\quad \left. - \frac{\partial^r}{\partial \Psi(p)} 2i\varepsilon\gamma_5 \alpha_D^{-1}(p) \frac{\partial^l S}{\partial \bar{\Psi}(-p)} + \frac{\partial S}{\partial \varphi(p)} (-2i)\varphi(p) + \varphi^\dagger(-p) (2i) \frac{\partial S}{\partial \varphi^\dagger(-p)} \right] \\ &= 0.\end{aligned}\tag{3.6}$$

Note that the deformation of chiral symmetry only appears in the fermionic sector. Furthermore, We define the chiral transformation in such a way that non-trivial deformation only appears in  $\delta\Psi$ :

$$\begin{aligned}\delta\Psi(p) &= i\varepsilon\gamma_5 \left\{ \Psi(p) - 2\alpha_D^{-1}(p) \frac{\partial^l S}{\partial \bar{\Psi}(-p)} \right\}, & \delta\bar{\Psi}(-p) &= \bar{\Psi}(-p)i\varepsilon\gamma_5, \\ \delta\varphi(p) &= -2i\varepsilon\varphi(p), & \delta\varphi^\dagger(-p) &= 2i\varepsilon\varphi^\dagger(-p).\end{aligned}\tag{3.7}$$

In the following section, we try to construct solutions to (3.6) regarding it as purely algebraic equation. Therefore, we are not confined ourselves to the specific solution of Wilson action obtained explicit integration of the blocked UV action. For simplicity, we shall use  $\alpha = \alpha_D$  below.

### §4. Solutions to the WT identities

#### 4.1. GW relation in free-field theory

Let us first consider a free-field Wilson action  $S_0$  for the IR fields  $\Phi^A = \{\Psi, \bar{\Psi}, \varphi, \varphi^\dagger\}$

$$\begin{aligned} S_0[\Phi] &= S_0[\bar{\Psi}, \Psi] + S_0[\varphi, \varphi^\dagger], \\ S_0[\bar{\Psi}, \Psi] &= \int_p \bar{\Psi}(-p) \mathcal{D}_0(p) \Psi(p), \\ S_0[\varphi, \varphi^\dagger] &= \int_p K^{-1}(p) \varphi^\dagger(-p) p^2 \varphi(p). \end{aligned} \quad (4.1)$$

Using (3.7), we obtain chiral transformation for free-field theory,

$$\begin{aligned} \delta\Psi(p) &= i\varepsilon\gamma_5 (1 - 2\alpha^{-1}\mathcal{D}_0)(p)\Psi(p), & \delta\bar{\Psi}(-p) &= \bar{\Psi}(-p)i\varepsilon\gamma_5, \\ \delta\varphi(p) &= -2i\varepsilon\varphi(p), & \delta\varphi^\dagger(-p) &= 2i\varepsilon\varphi^\dagger(-p). \end{aligned} \quad (4.2)$$

The WT identity for the free-field action

$$\Sigma[\Phi] = \frac{\partial^r S_0}{\partial\Phi^A} \delta\Phi^A = 0 \quad (4.3)$$

leads to the GW relation

$$\{\gamma_5, \mathcal{D}_0\} = 2\mathcal{D}_0\gamma_5\alpha^{-1}\mathcal{D}_0. \quad (4.4)$$

For our purpose, we don't need to specify the Dirac operator  $\mathcal{D}_0$  for free Dirac fields, and we only assume that it satisfies the GW relation (4.4).

#### 4.2. The generalized GW relation for interacting theory

We next include interaction terms of the Dirac fields with the scalars expressed by

$$\vartheta(p) \equiv P_+\varphi(p) + P_-\varphi^\dagger(p), \quad (4.5)$$

where  $P_\pm = (1 \pm \gamma_5)/2$ . Our basic assumption is that IR action is bilinear in the fermionic fields:

$$\begin{aligned} S[\Phi] &= S_1[\bar{\Psi}, \Psi, \vartheta] + S_2[\varphi, \varphi^\dagger], \\ S_1[\bar{\Psi}, \Psi, \vartheta] &= \int_{p,q} \bar{\Psi}(-p) \mathcal{D}(p, q) \Psi(q), \\ S_2[\varphi, \varphi^\dagger] &= S_0[\varphi, \varphi^\dagger] + S_I[\varphi, \varphi^\dagger] + S_{\text{counter}}[\vartheta], \end{aligned} \quad (4.6)$$

where  $S_0[\varphi, \varphi^\dagger]$  is the free-field action (4.1) for scalars.  $S_0 + S_I$  is chiral invariant, while  $S_{\text{counter}}[\vartheta]$  is not. The latter is determined below. The Dirac operator  $\mathcal{D}$  is assumed to take the form

$$\mathcal{D}(p, q) = \mathcal{D}_0(p)\delta(p - q) + \eta(p)\mathcal{V}(p, q)\eta(q), \quad (4.7)$$

where  $\mathcal{D}_0$  is the Dirac operator which satisfies the GW relation (4.4) for free-fields, and

$$\eta(p) = 1 - \alpha^{-1}(p)\mathcal{D}_0(p). \quad (4.8)$$

$\mathcal{V}$  is some functional of  $\vartheta$ . We try to find most general solutions to (3.6), assuming the IR action is bilinear in the Dirac fields.

For the action (4.6), we divide the WT identity (3.6) into two parts:

$$\begin{aligned} \Sigma[\Phi] &= \Sigma_1[\Phi] + \Sigma_2[\Phi], \\ \Sigma_1[\Phi] &= \int_p \left[ \frac{\partial^r S_1}{\partial\Psi(p)} i\varepsilon\gamma_5 \left\{ \Psi(p) - 2\alpha^{-1}(p) \frac{\partial^l S_1}{\partial\bar{\Psi}(-p)} \right\} + \bar{\Psi}(-p) i\varepsilon\gamma_5 \frac{\partial^l S_1}{\partial\bar{\Psi}(-p)} \right] \end{aligned}$$

$$\begin{aligned}
& + \frac{\partial S_1}{\partial \varphi(p)}(-2i)\varphi(p) + \varphi^\dagger(-p)(2i)\frac{\partial S_1}{\partial \varphi^\dagger(-p)} \Big], \\
\Sigma_2[\Phi] = & \int_p \left[ -\frac{\partial^r}{\partial \bar{\Psi}(p)} 2i\varepsilon\gamma_5\alpha^{-1}(p)\frac{\partial^l S_1}{\partial \bar{\Psi}(-p)} + \frac{\partial S_2}{\partial \varphi(p)}(-2i)\varphi(p) \right. \\
& \left. + \varphi^\dagger(-p)(2i)\frac{\partial S_2}{\partial \varphi^\dagger(-p)} \right]. \tag{4.9}
\end{aligned}$$

Since  $\Sigma_1$  is bilinear in the Dirac fields while  $\Sigma_2$  only contains scalar fields, each should separately vanish

$$\begin{aligned}
\Sigma_1[\Phi] &= 0, \\
\Sigma_2[\Phi] &= 0. \tag{4.10}
\end{aligned}$$

Let us first consider the WT identity  $\Sigma_1[\Phi] = 0$ . It leads to a generalized GW relation

$$\{\gamma_5, \mathcal{D}(p, q)\} = 2 \int_k \mathcal{D}(p, k)\gamma_5\alpha^{-1}(k)\mathcal{D}(k, q) - (i\varepsilon)^{-1}\delta\mathcal{D}(p, q). \tag{4.11}$$

This generates an equation for chiral transformation of  $\mathcal{V}$

$$\begin{aligned}
\delta\mathcal{V}(p, q) &= -i\varepsilon \left[ \{\gamma_5, \mathcal{V}(p, q)\} - 2 \int_k \mathcal{V}(p, k)\eta(k)\gamma_5\alpha^{-1}(k)\eta(k)\mathcal{V}(k, q) \right] \\
&= -i\varepsilon \left[ \{\gamma_5, \mathcal{V}(p, q)\} - \int_k \mathcal{V}(p, k)\alpha^{-1}(k)\{\gamma_5, \eta(k)\}\mathcal{V}(k, q) \right], \tag{4.12}
\end{aligned}$$

where we have used the relation

$$2 \eta(p)\gamma_5\eta(p) = \{\gamma_5, \eta(p)\}, \tag{4.13}$$

which follows from the GW relation (4.4) for free-fields.

In order to solve (4.12), we introduce a composite object  $\Theta(p)$  which is assumed to obey the same chiral transformation as  $\vartheta$ :

$$\delta\Theta(p, q) = -i\varepsilon\{\gamma_5, \Theta(p, q)\} \tag{4.14}$$

Using this object, we express  $\mathcal{V}$  as

$$\begin{aligned}
\mathcal{V}(p, q) &= \left[ \Theta(p, q) - \int_k \Theta(p, k)B(k)\Theta(k, q) \right. \\
&\quad \left. + \int_{k,l} \Theta(p, k)B(k)\Theta(k, l)B(l)\Theta(l, q) + \dots \right] \\
&\equiv \left( \Theta \frac{1}{1 + B\Theta} \right)(p, q) = \left( \frac{1}{1 + \Theta B} \Theta \right)(p, q). \tag{4.15}
\end{aligned}$$

The form of  $\mathcal{V}$  is suggested in ref. 29) <sup>\*)</sup>. We discuss in the next section how this form emerges from the point of view of linear realization of chiral symmetry.

Using (4.14), and taking the average of two expressions for  $\delta\mathcal{V}$  obtained from (4.15), we find

$$\begin{aligned}
\delta\mathcal{V}(p, q) &= -i\varepsilon \left[ \gamma_5\mathcal{V}(p, q) + \mathcal{V}(p, q)\gamma_5 \right. \\
&\quad \left. - \int_k \mathcal{V}(p, k)\{\gamma_5, B(k)\}\mathcal{V}(k, q) \right]. \tag{4.16}
\end{aligned}$$

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<sup>\*)</sup> We may define  $\mathcal{V}(p, q) = A(p)\left(\Theta \frac{1}{1+B\Theta}\right)(p, q)C(q)$  by introducing  $A$  and  $C$  which commute with  $\gamma_5$ . However,  $A, C$  can be absorbed by redefinition of  $\Theta$  and  $B$  as  $\Theta(p, q) \rightarrow \Theta'(p, q) = A(p)\Theta(p, q)C(q)$  and  $B(p) \rightarrow B'(p) = C^{-1}(p)B(p)A^{-1}(p)$ .

Comparing (4.12) with (4.16), we have the conditions

$$\{\gamma_5, B\} = 2\alpha^{-1}\eta\gamma_5\eta. \quad (4.17)$$

Note that the composite object  $\Theta$  is subjected to chiral transformation (4.14), otherwise arbitrary. It can be expanded in terms of the original field  $\vartheta$  as

$$\Theta(p, q) = \beta_1(p)\vartheta(p-q)\xi_1(q) + \int_{k,l} \beta_2(p)\vartheta(p-k)\kappa(k,l)_2\vartheta(l-q)\xi_2(q) + \dots, \quad (4.18)$$

where the coefficients functions  $\beta$ 's,  $\xi$ 's and the matrices  $\kappa$ 's are commuting or anticommuting functions of momentum variables. They are subjected to the boundary condition  $\Theta \rightarrow \vartheta$  as  $\Lambda \rightarrow \Lambda_0$ , otherwise arbitrary. Thus, we conclude that (4.15) is the most general set of solutions to (4.12) for  $B$  satisfying (4.17). The Dirac operator with this  $\mathcal{V}$  is given by

$$\mathcal{D}(p, q) = \mathcal{D}_0(p)\delta(p-q) + \eta(p)\left(\Theta\frac{1}{1+B\Theta}\right)(p, q)\eta(q). \quad (4.19)$$

We may rewrite it as

$$\mathcal{D}(p, q) = \mathcal{D}_0(p)\delta(p-q) + \gamma_5\eta^{-1}(p)\gamma_5\mathcal{V}'(p, q)\eta(q), \quad (4.20)$$

where

$$\mathcal{V}'(p, q) = \gamma_5\eta(p)\gamma_5\eta(p)\left(\Theta\frac{1}{1+B\Theta}\right)(p, q). \quad (4.21)$$

The chiral transformation of  $\mathcal{V}'$  is given by

$$\delta\mathcal{V}'(p, q) = -i\epsilon\left(\{\gamma_5, \mathcal{V}'(p, q)\} - 2\int_k \mathcal{V}'(p, k)\gamma_5\alpha^{-1}(k)\mathcal{V}'(k, q)\right). \quad (4.22)$$

Let us discuss the WT identity  $\Sigma_2[\Phi] = 0$ . The WT operator  $\Sigma_2[\Phi]$  contains the Jacobian factor  $J$  associated with the chiral transformation for the Dirac fields  $\delta\Psi$ . It is given by

$$\begin{aligned} J &= -\int_p \frac{\partial^r}{\partial\Psi(p)} 2i\epsilon\gamma_5\alpha^{-1}(p) \frac{\partial^l S_1}{\partial\bar{\Psi}(-p)} = -2i\epsilon\text{Tr}\left[\gamma_5\alpha^{-1}\mathcal{D}\right] \\ &= -2i\epsilon\text{Tr}\left[\gamma_5\alpha^{-1}\mathcal{V}'\right]. \end{aligned} \quad (4.23)$$

where  $\text{Tr}$  should be taken over the spinor and the momentum.

In order to cancel this contribution, the scalar sector action  $S_2$  should contain a counter term

$$S_{\text{counter}}[\mathcal{V}'] = \text{Tr} \log \left[1 - \alpha^{-1}\mathcal{V}'\right]. \quad (4.24)$$

It reduces to

$$\begin{aligned} S_{\text{counter}}[\mathcal{V}'] &= \text{Tr} \log \left[1 - \alpha^{-1}\mathcal{V}'\right] \\ &= \text{Tr} \log \left(1 - \frac{\gamma_5}{2}\{\gamma_5, B\}\Theta\frac{1}{1+B\Theta}\right) \\ &= \text{Tr} \log \left(1 - B\Theta\frac{1}{1+B\Theta} + \frac{\gamma_5}{2}[\gamma_5, B]\Theta\frac{1}{1+B\Theta}\right) \\ &= \text{Tr} \log \left[\left(1 + \frac{\gamma_5}{2}[\gamma_5, B]\Theta\right)\frac{1}{1+B\Theta}\right]. \end{aligned} \quad (4.25)$$

We define

$$S_{\text{counter}}[\Theta] = -\text{Tr} \log(1 + B\Theta). \quad (4.26)$$



Note that the difference  $S_{\text{counter}}[\mathcal{V}'] - S_{\text{counter}}[\Theta]$  is chiral invariant:

$$\delta\left(S_{\text{counter}}[\mathcal{V}'] - S_{\text{counter}}[\Theta]\right) = \delta\text{Tr} \log\left(1 + \frac{\gamma_5}{2}[\gamma_5, B]\Theta\right) = 0. \quad (4.27)$$

Since the counter action is determined up to a chiral invariant term, we may take  $S_{\text{counter}}[\Theta]$  as a counter action:

$$\Sigma_2[\Phi] = J + \delta S_{\text{counter}}[\mathcal{V}'] = J + \delta S_{\text{counter}}[\Theta] = 0. \quad (4.28)$$

In this way we have obtained generic IR action which solves the WT identity  $\Sigma[\Phi] = 0$ .

We shall discuss a set of solutions obtained by taking appropriate  $\Theta$ , and special functions for  $B$ .

1)  $\Theta(p, q) = \eta^{-1}(p)\gamma_5\eta^{-1}(p)\gamma_5\vartheta(p - q)$ ,  $B = \alpha^{-1}\gamma_5\eta\gamma_5\eta$ .

It gives the Dirac operator

$$\mathcal{D}(p, q) = \mathcal{D}_0(p)\delta(p - q) + \gamma_5\eta^{-1}(p)\gamma_5\left(\vartheta\frac{1}{1 + \alpha^{-1}\vartheta}\right)(p, q)\eta(q). \quad (4.29)$$

2)  $\Theta(p, q) = \vartheta(p - q)\gamma_5\eta^{-1}(q)\gamma_5\eta^{-1}(q)$ ,  $B = \alpha^{-1}\eta\gamma_5\eta\gamma_5$

It gives the Dirac operator where the factors  $\eta$  and  $\gamma_5\eta^{-1}\gamma_5$  appeared in (4.29) is exchanged:

$$\mathcal{D}(p, q) = \mathcal{D}_0(p)\delta(p - q) + \eta(p)\left(\vartheta\frac{1}{1 + \alpha^{-1}\vartheta}\right)(p, q)\gamma_5\eta^{-1}(q)\gamma_5. \quad (4.30)$$

The set of the solutions 1) and 2) is the continuum analog of the one given in ref. 29) on the lattice. We stress that the factor  $\eta$  is equal to 1 at  $\Lambda = \Lambda_0$  and it is regular for  $\Lambda < \Lambda_0$  in continuum theory considered here. On the lattice, the corresponding factor vanishes at the momentum region where species doublers appear. For both of 1) and 2), the counter action (4.24) reduces to

$$S_{\text{counter}}[\vartheta] = -\text{Tr} \log(1 + \alpha^{-1}\vartheta). \quad (4.31)$$

3)  $\Theta(p, q) = f^{-1}(p)\vartheta(p - q)f^{-1}(q)$ ,  $B = \alpha^{-1}\eta$ ,

This solution which gives the Dirac operator

$$\mathcal{D}(p, q) = \mathcal{D}_0(p)\delta(p - q) + f^{-1}(p)\eta(p)\left(\vartheta\frac{1}{1 + f^{-2}\alpha^{-1}\eta\vartheta}\right)(p, q)f^{-1}(q)\eta(q), \quad (4.32)$$

is related to the Wilson action derived from explicit integration over the UV Dirac fields. Actually, for the UV Dirac action

$$\begin{aligned} \mathcal{S}[\bar{\psi}, \psi, \theta] &= \int_{p, q} \bar{\psi}(-p)D(p, q)\psi(q), \\ D(p, q) &= K_0^{-1}(p)\not{p}\delta(p - q) + \theta(p - q), \end{aligned} \quad (4.33)$$

we may integrate the blocked action:

$$\begin{aligned} e^{-S[\bar{\Psi}, \Psi, \theta]} &= \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\left[-\int_{p, q} \left(\bar{\Psi}(-p) - f(p)\bar{\psi}(-p)\right)\alpha(p)\delta(p - q)\left(\Psi(q) - f(q)\psi(q)\right) \right. \\ &\quad \left. - \int_{p, q} \bar{\psi}(-p)D(p, q)\psi(q)\right]. \end{aligned} \quad (4.34)$$

The resulting Wilson action is given by

$$\begin{aligned} S[\bar{\Psi}, \Psi, \theta] &= \int_{p, q} \bar{\Psi}(-p)\mathcal{D}(p, q)\Psi(q) - \log \text{Det}(D + f^2\alpha), \\ \mathcal{D}(p, q) &= \alpha(p)\delta(p - q) - f(p)\alpha(p)(D + f^2\alpha)^{-1}(p, q)f(q)\alpha(q). \end{aligned} \quad (4.35)$$

The log Det term is generated as the Lee-Yang term of the functional integration over the UV Dirac fields.

We rewrite

$$\begin{aligned} (D + f^2\alpha)(p, q) &= (K_0^{-1}(p)\not{p} + f^2(p)\alpha(p)) \delta(p - q) + \theta(p - q) \\ &\equiv \Delta^{-1}(p)\delta(p - q) + \theta(p - q), \end{aligned} \quad (4.36)$$

and take its inverse as

$$(D + f^2\alpha)^{-1}(p, q) = \left[ \delta(p - q) - \Delta(p)\theta(p - q) + \int_k \Delta(p)\theta(p - k)\Delta(k)\theta(k - q) + \dots \right] \Delta(q) \quad (4.37)$$

$$\equiv \Delta(p)\delta(p - q) - \Delta(p) \left( \theta \frac{1}{1 + \Delta\theta} \right) (p, q) \Delta(q), \quad (4.38)$$

We then have

$$\mathcal{D}(p, q) = \mathcal{D}_0(p)\delta(p - q) + f(p)\alpha(p)\Delta(p) \left( \theta \frac{1}{1 + \Delta\theta} \right) (p, q) \Delta(q) f(q)\alpha(q), \quad (4.39)$$

where the Dirac operator for free-fields takes the form

$$\begin{aligned} \mathcal{D}_0(p) &= \alpha(p) - f(p)\alpha(p)\Delta(p)f(p)\alpha(p) \\ &= f^{-1}(p) \left\{ \frac{M\not{p}}{(K_0(p) - K(p))\not{p} + K(p)M} \right\} \end{aligned} \quad (4.40)$$

for  $\alpha$  of (3.4). Using

$$\eta = f^2\alpha\Delta, \quad (4.41)$$

we find that the Dirac operator in (4.32) is obtained from that of (4.39) with a replacement of  $\theta$  by  $\vartheta$ . Furthermore, the log Det (Lee-Yang) term in the action (4.32) is nothing but the counter action  $S_{\text{counter}}[\vartheta]$  when we identify  $\theta = \vartheta$ . The Wilson action with (4.39) and  $S_{\text{counter}}[\vartheta]$  is a perfect action if no blocking is performed for the scalar fields. For bilinear UV action for the Dirac fields, integration over the UV Dirac fields automatically yields a solution to the WT identity  $\Sigma[\Phi] = 0$  if we put  $\theta = \vartheta$  for scalar sector.

## §5. Reduction to linear realization of chiral symmetry

### 5.1. Chiral transformation with $\hat{\gamma}_5$

In the previous section, we have obtained a certain family of the Wilson actions which solves the WT identities. We have given some concrete solutions choosing specific functions for  $B$  and  $\Theta$ . Chiral transformations for each Wilson action are non-linear. In this section, we discuss that the family of the Dirac actions  $\bar{\Psi}\mathcal{D}\Psi$  with the Dirac operators (4.19) reduces, by a field redefinition, to a family of the Wilson actions for which chiral symmetry is linearly realized. Note that our basic assumption that the Wilson action is bilinear in the Dirac fields makes the linearization of symmetry possible.

We begin with the expression for the Dirac operator (4.20), and introduce new Dirac field  $\Psi'$  as

$$\Psi'(p) = \int_q L^{-1}(p, q)\Psi(q). \quad (5.1)$$

We require that its chiral transformation becomes

$$\delta\Psi'(p) = i\varepsilon\hat{\gamma}_5(p)\Psi'(p) \quad (5.2)$$

with

$$\hat{\gamma}_5(p) = \gamma_5 (1 - 2\alpha^{-1}(p)\mathcal{D}_0(p)). \quad (5.3)$$

This  $\hat{\gamma}_5$  is continuum counter part of the one extensively used on the lattice. It satisfies  $(\hat{\gamma}_5)^2 = 1$ , and can be used to define the chiral projection operators:

$$\hat{P}_\pm(p) = \frac{1 \pm \hat{\gamma}_5(p)}{2}. \quad (5.4)$$

Our task is to find the operator  $L$  in (5.1). It follows from (5.1) and (5.2) that it should obey the chiral transformation

$$\delta L(p, q) = i\epsilon \left( \hat{\gamma}_5(p)L(p, q) - L(p, q)\hat{\gamma}_5(q) - 2 \int_k \alpha^{-1}(p)\eta^{-1}(p)\gamma_5\mathcal{V}'(p, k)\eta(k)L(k, q) \right). \quad (5.5)$$

As shown in Appendix A, a solution to this equation is given by

$$L(p, q) = \eta^{-1}(p) \frac{1}{1 - \alpha^{-1}\mathcal{V}'}(p, q)\eta(q). \quad (5.6)$$

We next consider new Dirac operator for  $\Psi'$ :

$$\mathcal{D}'(p, q) = \int_k \mathcal{D}(p, k)L(k, q). \quad (5.7)$$

It is straightforward to compute the r.h.s as shown in Appendix B:

$$\mathcal{D}'(p, q) = \mathcal{D}_0(p)\delta(p - q) + \mathcal{V}''(p, q), \quad (5.8)$$

where

$$\begin{aligned} \mathcal{V}''(p, q) &= \left( \mathcal{V}' \frac{1}{1 - \alpha^{-1}\mathcal{V}'} \right)(p, q)\eta(q) \\ &= \frac{1}{2}\alpha(p)\gamma_5\{\gamma_5, B(p)\} \left( \Theta \frac{1}{1 + \frac{1}{2}\gamma_5[\gamma_5, B]\Theta} \right)(p, q)\eta(q). \end{aligned} \quad (5.9)$$

The chiral transformation of  $\mathcal{V}''$  becomes linear

$$\delta\mathcal{V}''(p, q) = -i\epsilon \left( \gamma_5\mathcal{V}''(p, q) + \mathcal{V}''(p, q)\hat{\gamma}_5(q) \right). \quad (5.10)$$

The field redefinition (5.1) with (5.6) induces a Jacobian factor

$$\begin{aligned} \mathcal{D}\Psi &= \mathcal{D}\Psi' \exp \mathcal{J}, \\ \mathcal{J} &= \text{Tr} \log \left[ 1 - \alpha^{-1}\mathcal{V}' \right], \end{aligned} \quad (5.11)$$

which exactly cancels the counter action (4.24).

In conclusion, the IR theory can be described by

$$\mathcal{D}\Psi' \mathcal{D}\bar{\Psi} \mathcal{D}\varphi \mathcal{D}\varphi^\dagger \exp -S'[\Psi', \bar{\Psi}, \varphi, \varphi^\dagger], \quad (5.12)$$

where the Wilson action is given by

$$S'[\Psi', \bar{\Psi}, \varphi, \varphi^\dagger] = \int_{p, q} \bar{\Psi}(-p)\mathcal{D}'(p, q)\Psi'(q) + \int_p \varphi^\dagger(-p)K^{-1}(p)p^2\varphi(p) + S_I[\varphi, \varphi^\dagger] \quad (5.13)$$

with the Dirac operator given in (5.8). The action  $S'$  is invariant under the chiral transformation

$$\begin{aligned} \delta\Psi'(p) &= i\epsilon\hat{\gamma}_5(p)\Psi'(p), \\ \delta\bar{\Psi}(-p) &= \bar{\Psi}(-p)i\epsilon\gamma_5, \\ \delta\varphi(p) &= -2i\epsilon\varphi(p), \\ \delta\varphi^\dagger(-p) &= 2i\epsilon\varphi^\dagger(-p). \end{aligned} \quad (5.14)$$

Chiral symmetry in this theory is simply expressed by the invariance of the Wilson action  $\delta S' = 0$ .

It should be noticed that the composite object  $\mathcal{V}''$  does not in general commutes with  $\gamma_5$ , and cannot be expressed by using the chiral projection operators. For the special case,  $B = \alpha^{-1}\gamma_5\eta\gamma_5\eta$  and  $\Theta(p, q) = \eta^{-1}(p)\gamma_5\eta^{-1}(p)\gamma_5\vartheta(p - q)$ ,  $\mathcal{V}''$  reduces to

$$\mathcal{V}''(p, q) = \vartheta(p - q)\eta(q) = P_+\varphi(p - q)\hat{P}_+(q) + P_-\varphi^\dagger(p - q)\hat{P}_-(q), \quad (5.15)$$

which is linear in  $\varphi$  and  $\varphi^\dagger$ . This is the continuum analog of the Yukawa couplings constructed on the lattice ref. 29).

### 5.2. Chiral transformation with $\gamma_5$

In the previous subsection, we have discussed a linear realization of chiral symmetry using  $\hat{\gamma}_5$ . Based on the ERG point of view, we discuss here that it is after all possible to realize chiral symmetry as the standard way with the constant  $\gamma_5$  for the fermionic sector. Non-trivial modification only appears in the Yukawa coupling with scalar fields.

We begin with an IR action

$$S_{IR} = \int_{p,q} \bar{\Psi}(-p) \mathcal{D}(p,q) \Psi(q) + S_{\text{counter}}, \quad (5.16)$$

where the counter action is given by

$$S_{\text{counter}} = \text{Tr} \log(1 - \alpha^{-1} \mathcal{D}). \quad (5.17)$$

Consider the non-linear chiral transformation discussed in 4.2:

$$\begin{aligned} \delta\Psi(p) &= i\epsilon\gamma_5 \int_k (\delta(p-k) - 2\alpha^{-1}(p)\mathcal{D}(p,k))\Psi(k) \\ \delta\bar{\Psi}(-p) &= i\epsilon\bar{\Psi}(-p)\gamma_5. \end{aligned} \quad (5.18)$$

The Dirac operator  $\mathcal{D}$  is subjected to the generalized GW relation (4.11)

$$\delta\mathcal{D}(p,q) = -i\epsilon \left( \{\gamma_5, \mathcal{D}(p,q)\} - 2 \int_k \mathcal{D}(p,k) \gamma_5 \alpha^{-1}(k) \mathcal{D}(k,q) \right). \quad (5.19)$$

We perform a field redefinition

$$\Psi'(p) = \int_k (\delta(p-k) - \alpha^{-1}(p)\mathcal{D}(p,k))\Psi(k). \quad (5.20)$$

In terms of new Dirac fields, chiral transformation becomes the standard form

$$\delta\Psi'(p) = i\epsilon\gamma_5\Psi'(p), \quad (5.21)$$

and the new Dirac operator is given by

$$\mathcal{D}'(p,q) = \int_k \mathcal{D}(p,k) \frac{1}{1 - \alpha^{-1}\mathcal{D}}(k,q). \quad (5.22)$$

Note that the Jacobian factor associated with the non-linear transformation (5.20) exactly cancels the counter action  $S_{\text{counter}}$ . The GW relation (5.19) is simplified as

$$\{\gamma_5, \mathcal{D}'(p,q)\} = -(i\epsilon)^{-1} \delta\mathcal{D}'(p,q). \quad (5.23)$$

This is exactly the standard relation for chiral symmetry. If we take the standard free Dirac operator such as  $D_0(p) \sim \not{p}$  and decompose the total Dirac operator  $\mathcal{D}'$  as

$$\mathcal{D}'(p,q) = D_0(p)\delta(p-q) + \Theta(p,q) \quad (5.24)$$

where  $\Theta(p,q)$  obeys

$$\delta\Theta(p,q) = -i\epsilon\{\gamma_5, \Theta(p,q)\}. \quad (5.25)$$

Therefore,  $\Theta$  can be expanded as (4.18), and identifies with that appeared in the previous section. It is obvious that the Dirac operator (5.24) provides the most general form of the solutions to (5.23).

On the other hand, we may also decompose the Dirac operator in (5.16) as

$$\mathcal{D}(p,q) = \mathcal{D}_0(p)\delta(p-q) + \hat{V}(p,q) \quad (5.26)$$

$$\mathcal{D}_0 = D_0 \frac{1}{1 + \alpha^{-1}D_0}, \quad (5.27)$$

where  $\mathcal{D}_0$  is obtained by integration of the blocked UV action for free-fields if we take  $D_0 = f^{-2}K_0^{-1}\not{p}$ . It satisfies the GW relation for free fields,  $\{\gamma_5, \mathcal{D}_0\} = 2\alpha^{-1}\mathcal{D}_0\gamma_5\mathcal{D}_0$ . Then, using (5.22) for two Dirac operators,  $\mathcal{D}$  and  $\mathcal{D}'$ , we can express  $\hat{V}$  in terms of  $\Theta$  as

$$\hat{V}(p, q) = \eta(p) \left( \Theta \frac{1}{1 + \alpha^{-1}\eta\Theta} \right) (p, q) \eta(q). \quad (5.28)$$

It leads to the Dirac operator

$$\mathcal{D}(p, q) = \mathcal{D}_0(p)\delta(p - q) + \eta(p) \left( \Theta \frac{1}{1 + \alpha^{-1}\eta\Theta} \right) (p, q) \eta(q). \quad (5.29)$$

(5.29) should be compared with (4.19), and are recognized to belong to our most general set of the solutions to the WT identities. The discussion given here motivates to take (4.15). We also notice that the field redefinition (5.20) does make sense in continuum theory, but does not make sense on the lattice. For free-field theory, it becomes  $\Psi'(p) = \eta(p)\Psi(p)$ . On the lattice,  $\eta(p)$  vanishes in the region of momentum where the species doublers appear.

We conclude that the standard chiral symmetry is realized even when symmetry-breaking blocking is performed for the Dirac fields. However, their interactions with the scalar fields inevitably become non-linear.

## §6. Summary and outlook

In this paper we have given the WT identities for chiral symmetry realized in the presence of a symmetry-breaking blocking for the Dirac fields within the ERG framework. The WT identities have been obtained using a functional method where the Wilson action is expressed as functional integral over the blocked UV action. We have solved the WT identities as purely algebraic equations, assuming the Wilson action is bilinear in the Dirac fields. The assumption makes it possible to divide the WT identities into two sets of equations: One corresponds to a generalized GW relation for the fermionic sector. The other is to fix counter actions consisting of the scalar fields which are needed to cancel the contribution arising from the Jacobian factor associated with the non-linear chiral transformation on the Dirac fields. Here, chiral symmetry is realized in such a way that the change of the Wilson action under non-linear symmetry transformations is canceled with the change of the functional measure in the WT identities. Our family of solutions contains some special solutions: One is the continuum analog of the solution discussed in the lattice theory. The other one is a kind of perfect action obtained by integrating the blocked UV action over the Dirac fields and identifying IR scalars with UV ones.

In addition to the non-linear realization of symmetry, we have also discussed linear realization of chiral symmetry. This is achieved by a field redefinition for the fermionic variables. The chiral transformations for new Dirac fields are given with an operator extension of  $\gamma_5$  or even  $\gamma_5$  itself. Accordingly, even if we have started with symmetry-breaking regularization, it is possible for us to end up with the standard representation of chiral symmetry. The whole complexity appears in the Yukawa couplings where the Dirac fields have non-polynomial interactions with scalar fields.

The WT identities considered in this paper can be lifted to the quantum master equation (QME) in the Batalin-Vilkovisky antifield formalism.<sup>20)</sup> The reduction of the WT identities with the Jacobian contributions to those without them corresponds to reduction of the QME to the classical master equation (CME) via a canonical transformation in the space of fields and antifields. It will be shown that the reduction of the QME to the CME happens in chiral symmetry considered here.

Since our method given here is expected to apply to other linear symmetries such as supersymmetry and abelian gauge symmetry, it is interesting to discuss their generalized GW relations.

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## Appendix A

### — Field Redefinition —

We will show that

$$L(p, q) = \eta^{-1}(p) \frac{1}{1 - \alpha^{-1} \mathcal{V}'}(p, q) \eta(q). \quad (\text{A}\cdot 1)$$

is a solution to

$$\delta L(p, q) = i\epsilon \left( \hat{\gamma}_5(p) L(p, q) - L(p, q) \hat{\gamma}_5(q) - 2 \int_k \alpha^{-1}(p) \eta^{-1}(p) \gamma_5 \mathcal{V}'(p, k) \eta(k) L(k, q) \right). \quad (\text{A}\cdot 2)$$

Chiral transformation of  $L(p, q)$  in (A·1) is given by

$$\delta L(p, q) = - \int_{k,l} \eta^{-1}(p) \frac{1}{1 - \alpha^{-1} \mathcal{V}'}(p, l) (-\alpha^{-1}(l) \delta \mathcal{V}'(l, k)) \frac{1}{1 - \alpha^{-1} \mathcal{V}'}(k, q) \eta(q). \quad (\text{A}\cdot 3)$$

Rewriting (4·22) for  $\delta \mathcal{V}'(p, q)$  as

$$\begin{aligned} \alpha^{-1}(l) \delta \mathcal{V}'(l, k) &= -2i\epsilon \alpha^{-1}(l) \left( \gamma_5 \mathcal{V}'(l, k) - \int_q \mathcal{V}'(l, q) \gamma_5 \alpha^{-1}(q) \mathcal{V}'(q, k) \right) + i\epsilon \alpha^{-1}(l) [\gamma_5, \mathcal{V}'(l, k)] \\ &= -2i\epsilon \int_q \left( \delta(l - q) - \alpha^{-1}(l) \mathcal{V}'(l, q) \right) \gamma_5 \alpha^{-1}(q) \mathcal{V}'(q, k) + i\epsilon \alpha^{-1}(l) [\gamma_5, \mathcal{V}'(l, k)], \end{aligned} \quad (\text{A}\cdot 4)$$

we find

$$\begin{aligned} \delta L(p, q) &= -2i\epsilon \int_k \eta^{-1}(p) \gamma_5 \alpha^{-1}(p) \mathcal{V}'(p, k) \frac{1}{1 - \alpha^{-1} \mathcal{V}'}(k, q) \eta(q) \\ &\quad + i\epsilon \int_{k,l} \eta^{-1}(p) \frac{1}{1 - \alpha^{-1} \mathcal{V}'}(p, l) [\gamma_5, \alpha^{-1}(l) \mathcal{V}'(l, k)] \frac{1}{1 - \alpha^{-1} \mathcal{V}'}(k, q) \eta(q) \\ &= -2i\epsilon \int_k \alpha^{-1}(p) \eta^{-1}(p) \gamma_5 \mathcal{V}'(p, k) \eta(k) L(k, q) + i\epsilon \eta^{-1}(p) \left[ \gamma_5, \frac{1}{1 - \alpha^{-1} \mathcal{V}'}(p, q) \right] \eta(q) \\ &= i\epsilon \left( \hat{\gamma}_5(p) L(p, q) - L(p, q) \hat{\gamma}_5(q) - 2 \int_k \alpha^{-1}(p) \eta^{-1}(p) \gamma_5 \mathcal{V}'(p, k) \eta(k) L(k, q) \right), \end{aligned} \quad (\text{A}\cdot 5)$$

where we have used  $\eta^{-1}(p) \gamma_5 = \hat{\gamma}_5 \eta^{-1}(p)$ .

## Appendix B

### — The New Dirac Operator —

The new Dirac operator is given by

$$\begin{aligned} \mathcal{D}'(p, q) &= \int_k \mathcal{D}_0(p) \delta(p - k) \eta^{-1}(k) \frac{1}{1 - \alpha^{-1} \mathcal{V}'}(k, q) \eta(q) + \gamma_5 \eta^{-1}(p) \gamma_5 \int_k \mathcal{V}'(p, k) \frac{1}{1 - \alpha^{-1} \mathcal{V}'}(k, q) \eta(q) \\ &= \mathcal{D}_0(p) \delta(p - q) + \left[ \eta^{-1} \alpha^{-1} \mathcal{D}_0 + (\gamma_5 \eta \gamma_5)^{-1} \right] (p) \int_k \mathcal{V}'(p, k) \frac{1}{1 - \alpha^{-1} \mathcal{V}'}(k, q) \eta(q). \end{aligned} \quad (\text{B}\cdot 1)$$

Using the GW relation  $\{\gamma_5, \mathcal{D}_0\} = 2\alpha^{-1} \mathcal{D}_0 \gamma_5 \mathcal{D}_0$ , we find

$$\eta \gamma_5 \eta \gamma_5 = (1 - \alpha^{-1} \mathcal{D}_0) [\gamma_5 (1 - \alpha^{-1} \mathcal{D}_0) \gamma_5] = 1 - \frac{1}{2} \alpha^{-1} (\mathcal{D}_0 + \gamma_5 \mathcal{D}_0 \gamma_5). \quad (\text{B}\cdot 2)$$

It leads to

$$\begin{aligned} \eta^{-1} \alpha^{-1} \mathcal{D}_0 + (\gamma_5 \eta \gamma_5)^{-1} &= \left( \frac{1}{1 - \alpha^{-1} \mathcal{D}_0} \alpha^{-1} \mathcal{D}_0 \right) + \frac{1}{\gamma_5 (1 - \alpha^{-1} \mathcal{D}_0) \gamma_5} \\ &= \left( \frac{1}{1 - \alpha^{-1} \mathcal{D}_0} \right) \left[ \frac{1}{\gamma_5 (1 - \alpha^{-1} \mathcal{D}_0) \gamma_5} \right] \left[ \gamma_5 (1 - \alpha^{-1} \mathcal{D}_0) \gamma_5 \alpha^{-1} \mathcal{D}_0 + 1 - \alpha^{-1} \mathcal{D}_0 \right] = 1. \end{aligned} \quad (\text{B}\cdot 3)$$

Therefore,

$$\mathcal{D}'(p, q) = \mathcal{D}_0(p)\delta(p - q) + \int_k \mathcal{V}'(p, k) \left( \frac{1}{1 - \alpha^{-1}\mathcal{V}'} \right) (k, q) \eta(q). \quad (\text{B}\cdot 4)$$

We next consider the definition of  $\mathcal{V}'$  given in (4.21). It follows that

$$\begin{aligned} \int_k \mathcal{V}'(p, k)(1 + B\Theta)(k, q) &= \frac{1}{2}\alpha(p)\gamma_5\{\gamma_5, B(p)\}\Theta(p, q) \\ \int_k \left(1 - \alpha^{-1}\mathcal{V}'\right)(p, k)(1 + B\Theta)(k, q) &= \left(1 + \frac{1}{2}\gamma_5[\gamma_5, B]\Theta\right)(p, q). \end{aligned} \quad (\text{B}\cdot 5)$$

These relations give

$$\begin{aligned} \int_k \mathcal{V}'(p, k) \left( \frac{1}{1 - \alpha^{-1}\mathcal{V}'} \right) (k, q) &\equiv \left( \mathcal{V}' \frac{1}{1 - \alpha^{-1}\mathcal{V}'} \right) (p, q) = \left( \frac{1}{1 - \mathcal{V}'\alpha^{-1}} \mathcal{V}' \right) (p, q) \\ &= \frac{1}{2} \int_k \alpha(p)\gamma_5\{\gamma_5, B(p)\}\Theta(p, k) \left( \frac{1}{1 + \frac{1}{2}\gamma_5[\gamma_5, B]\Theta} \right) (k, q) \\ &= \frac{1}{2}\alpha(p)\gamma_5\{\gamma_5, B(p)\} \left( \Theta \frac{1}{1 + \frac{1}{2}\gamma_5[\gamma_5, B]\Theta} \right) (p, q). \end{aligned} \quad (\text{B}\cdot 6)$$

Since  $\mathcal{V}'$  transforms as

$$\delta\mathcal{V}' = -i\varepsilon \left( \gamma_5\mathcal{V}' + \mathcal{V}'\gamma_5 - 2\mathcal{V}'\alpha^{-1}\gamma_5\mathcal{V}' \right), \quad (\text{B}\cdot 7)$$

we have

$$\delta \left( \mathcal{V}' \frac{1}{1 - \alpha^{-1}\mathcal{V}'} \right) (p, q) = -i\varepsilon \left[ \gamma_5 \left( \mathcal{V}' \frac{1}{1 - \alpha^{-1}\mathcal{V}'} \right) (p, q) + \left( \mathcal{V}' \frac{1}{1 - \alpha^{-1}\mathcal{V}'} \right) (p, q) \gamma_5 \right]. \quad (\text{B}\cdot 8)$$

Using  $\gamma_5\eta(q) = \eta\hat{\gamma}_5(q)$ , we find

$$\delta\mathcal{V}''(p, q) = \delta \left[ \left( \mathcal{V}' \frac{1}{1 - \alpha^{-1}\mathcal{V}'} \right) (p, q) \eta(q) \right] = -i\varepsilon \left( \gamma_5\mathcal{V}''(p, q) + \mathcal{V}''(p, q)\hat{\gamma}_5(q) \right). \quad (\text{B}\cdot 9)$$

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