A New Extension of Hořava-Lifshitz Gravity and Curing Pathologies of the Scalar Graviton

Taeyoon Moon[a1], Phillial Oh[b2], and Mu-In Park[c3]

[a] *Center for Quantum Space-time, Sogang University, Seoul, 121-742, Korea*

[b] *Department of Physics and Institute of Basic Science, Sungkyunkwan University, Suwon 440-746 Korea*

[c] *The Institute of Basic Sciences, Kunsan National University, Kunsan, 573-701, Korea*

Abstract

We consider an extension of the Hořava-Lifshitz gravity with extra conformal symmetry by introducing a scalar field with higher order curvature terms. Relaxing the exact local Weyl symmetry, we construct an action with three free parameters which breaks local anisotropic Weyl symmetry but still preserves residual global Weyl symmetry. At low energies, it reduces to a Lorentz-violating scalar-tensor gravity. With a constant scalar field background and particular choices of the parameters, it reduces to the Hořava-Lifshitz (HL) gravity, but any perturbation from these particular configurations produces some non-trivial extensions of HL gravity. The perturbation analysis of the new extended HL gravity in the Minkowski background shows that the pathological behaviors of scalar graviton, i.e., ghost or instability problem, and strong coupling problem do not emerge up to cubic order as well as quadratic order.

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[1]e-mail address: dpproject@skku.edu
[2]e-mail address: ploh@skku.edu
[3]e-mail address: muinpark@gmail.com

1 Introduction

The renormalizability of Einstein gravity has been a long standing problem in quantum gravity. The power counting renormalizability of curvature-squared gravity was obtained by adding the most general, covariant, higher-derivative action containing only dimensionless couplings to the Einstein-Hilbert action with a cosmological constant term [1]. Rigorous renormalizability was established by using BRS invariance [2]. Also, conformal gravity with Weyl tensor was first considered by Bach [3] in 1921. Subsequently, it has been shown that this theory is renormalizable and asymptotic free [4, 5]. However, these curvature-squared gravities have the pathologies of ghost and unitarity problem. For example, conformal gravity have two ghost modes consisting of a massless vector and a massless runaway tensor. The four extra ghost degrees of freedom are due to the higher-derivative nature [6].

Recently, Hořava [7] proposed a renormalizable theory of quantum gravity which is known as Hořava-Lifshitz (HL) gravity. The basic idea is to abandon the Lorentz invariance and equal-footing treatments of space and time in UV. This theory is an Einstein gravity with a Lorentz violating parameter λ which reduces to the usual Einstein gravity with $\lambda = 1$ at low (IR) energies. It is power-counting renormalizable without ghosts for the usual transverse traceless graviton mode and contains 6th-order spatial derivatives. It is based on the detailed balance and the projectability which restrict the lapse function to be a function of time only, $N = N(t)$.

The original HL gravity with the projectability is known to have a few serious problems. These are the existence of the extra degree of freedom of graviton (*scalar graviton*) and the strong coupling problem at the Einstein gravity limit, i.e., $\lambda \to 1$, in IR [8, 9]. As was shown in [10], the scalar graviton can be a ghost or leads to instability and moreover the couplings of cubic order terms blow up at the Einstein gravity limit (strong coupling) in the Minkowski background that makes the metric perturbation break down for the scalar graviton. The strong coupling might not be a problem but only means the necessity of non-perturbative analysis for the Einstein gravity limit. Actually, it was argued in [11, 12] that this strong coupling problem can be eliminated through a non-perturbative effect, like the Vainshtein mechanism in massive gravity [13].

There are two main alternative models of extended HL gravity. The first one was suggested by Sotiriou, Visser, and Weinfurtner (SVW) [14] and the other is the so called healthy extension of HL gravity by Blas, Pujolas, and Sibiryakov (BPS) [15]. SVW were motivated by the fact that HL gravity had a non-zero cosmological constant of the wrong sign to be incompatible with observation. To overcome the problem, firstly they have constructed the gravity model by abandoning the detailed balance condition and restoring parity invariance but with projectability. As pointed out in [14, 16], SVW approach has still the pathology of the scalar graviton. (For related issues in de Sitter background, see [17, 18].)

On the other hand the motivation of BPS was to improve the IR behavior without detailed balance condition and projectability. To do so they first introduced a new 3-vector $a_i = \partial_i N/N$ and its higher derivative terms into the Lagrangian. Here, N became a dynamical scalar field and at low energies, it reduced to a Lorentz-violating scalar-tensor gravity theory. This new model endowed the scalar graviton with a regular action. Consequently the pathology of scalar graviton, such as ghost or instability problem can be cured in BPS extension but it is known that this extension also could have strong coupling problem at the Einstein limit ($\lambda \to 1$) in IR when one considers cubic order action [19, 20]. (See also [21] for low-dimensional analogous.) However, it is also possible to avoid the strong coupling if higher-derivative terms in the action become important below the strong coupling energy scale [22, 23], by assigning hierarchy between the Planck scale and a new low energy scale.

Recently, another extension of HL gravity was performed with a conformally invariant manner and the local anisotropic Weyl gravity was constructed [24]. It extends the original anisotropic Weyl invariance of HL gravity at UV to that of all energy scales using an extra scalar field which compensates the local scale transformation. The action is invariant under the local anisotropic transformations of the space and time metric components with an arbitrary value of the critical exponent z. It turns out that this theory coincides with the low-energy limit of the non-projectable HL gravity and it permits the extra scalar graviton mode which inherits the pathologies of the HL gravity.

In this paper, we relax the exact local conformal invariance and consider anisotropic $z = 3$ Weyl gravity including the higher order derivative terms. It includes three parameters which represents the breaking of the local Weyl invariance, but still preserves the global conformal invariance. When all of these parameters become 1, the theory has local invariance. With fixing the scalar field to a constant value and particular choices of the parameters, the theory reduces to the HL gravity, but any perturbation from these particular configurations produces some non-trivial extensions of HL gravity. We study the behaviors of the scalar graviton in the perturbation analysis of this new extended HL gravity and show that, in the Minkowski background, the pathological behaviors of scalar graviton, i.e., ghost or instability problem, and strong coupling problem do not emerge up to cubic order as well as quadratic order.

2 Anisotropic Weyl-invariant action with higher derivatives: New extended HL gravity

In order to construct anisotropic Weyl-invariant action with higher derivative terms, let us first consider $z = 3$ anisotropic Weyl-invariant gravity [24]

$$S_{aW} = \int dt d^3x N\sqrt{g}\,\left\{\frac{2}{\kappa^2}\left(B_{ij}B^{ij} - \lambda B^2\right) + \varphi^8\left(R - 8\frac{\nabla_i\nabla^i\varphi}{\varphi}\right) - V_\nu(\varphi)\right\}, \qquad (2.1)$$

where κ^2, λ are dimensionless constant parameters, R is the 3-curvature, and B_{ij} is given by

$$B_{ij} = K_{ij} - \frac{2}{N\varphi}g_{ij}(\dot{\varphi} - \nabla_i\varphi N^i), \qquad (2.2)$$

with the extrinsic curvature $K_{ij} = -(\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i)/2N$ (the dot ($\dot{}$) denotes the derivative with respect to t). One can easily check that for

$$V_\nu(\varphi) = \nu\varphi^{12} \tag{2.3}$$

(with a constant coefficient ν) the above action (2.1) is invariant under anisotropic Weyl transformation

$$N \to e^{3\omega}N, \quad N_i \to e^{2\omega}N_i, \quad g_{ij} \to e^{2\omega}g_{ij}, \quad \varphi \to e^{-\frac{\omega}{2}}\varphi. \tag{2.4}$$

In these transformations, ω is a function of space and time, $\omega = \omega(t, \mathbf{x})$. Note that assuming ω as a function of time only, i.e., $\omega = \omega(t)$, is unnatural from the point of view of the above local transformations. This implies that the lapse function N must be a function of space and time also, and this favors the non-projectable case in our construction. Later, we will consider breaking of the above local anisotropic Weyl invariance but keep only the global invariance so that the projectable case is still possible. However, even in this case we will consider only the non-projectable case in order to study whether the scalar graviton problem in the BPS extension [19, 20] can be cured in our new construction.

We also note that, for $\lambda = 1$ and $\varphi = \varphi_0 =$ const., the action (2.1) is reduced to (Lorentz-invariant) Einstein-Hilbert action with the following conditions [25]:

$$\frac{2}{\kappa^2} = \frac{c^2}{16\pi G_N}, \quad \varphi_0^8 = \frac{c^4}{16\pi G_N}, \quad V_\nu(\varphi_0) = \nu\varphi_0^{12} = \frac{2\Lambda c^2}{16\pi G_N}, \quad \nu = \frac{\sqrt{16\pi G_N}2\Lambda}{c^4}, \tag{2.5}$$

where G_N is the gravitational constant, c is the speed of light, and Λ is the cosmological constant. However, we should point out that, for $\lambda \neq 1$ or $\varphi^8 \neq c^4/16\pi G_N$, the action is not invariant under the full diffeomorpism (*Diff*) but invariant under the foliation preserving *Diff*:

$$\begin{aligned}
\delta x^i &= -\zeta^i(t, \mathbf{x}), \ \delta t = -f(t), \\
\delta g_{ij} &= \partial_i \zeta^k g_{jk} + \partial_j \zeta^k g_{ik} + \zeta^k \partial_k g_{ij} + f\dot{g}_{ij}, \\
\delta N_i &= \partial_i \zeta^j N_j + \zeta^j \partial_j N_i + \dot{\zeta}^j g_{ij} + f\dot{N}_i + \dot{f}N_i, \\
\delta N &= \zeta^j \partial_j N + f\dot{N} + \dot{f}N, \\
\delta\varphi &= \zeta^k \partial_k \varphi + f\dot{\varphi}.
\end{aligned} \tag{2.6}$$

Here it is important to note that this *Diff* exists for arbitrary spacetime-dependent N, N_i, g_{ij}, φ. This implies that the equations of motion by varying N, N_i, g_{ij}, φ are all the "local" equations as in the usual Lorentz invariant Einstein or scalar-tensor gravity. This is compatible with the local Weyl invariance (2.4). So, there are two sources of the IR Lorentz violation: One comes from the parameter $\lambda \neq 1$ and another from any fluctuation of φ from the background $\varphi_0^8 = c^4/16\pi G_N$.

When we focus on the following Weyl invariant object:

$$\begin{aligned}
\bar{R}_{ij} &\equiv R_{ij} + 6\frac{\nabla_i\varphi\nabla_j\varphi}{\varphi^2} - 2\frac{\nabla_i\nabla_j\varphi}{\varphi} - 2g_{ij}\frac{\nabla_k\varphi\nabla^k\varphi}{\varphi^2} - 2g_{ij}\frac{\nabla_k\nabla^k\varphi}{\varphi} \\
&\equiv R_{ij} + f_{ij}(\nabla\varphi), \quad \bar{R} \equiv g^{ij}\bar{R}_{ij}
\end{aligned} \tag{2.7}$$

we can further extend S_{aW} (2.1) to the power-counting renormalizable and local Weyl invariant action including the higher-derivative terms as

$$S_{haW} = \int dtd^3xN\sqrt{g}\left\{\frac{2}{\kappa^2}\left(B_{ij}B^{ij} - \lambda B^2\right) - V_\nu(\varphi) + \varphi^8\bar{R} + \beta_1\varphi^4\bar{R}^2 + \beta_2\varphi^4(\bar{R}_{ij})^2 \right.$$
$$\left. +\beta_3\bar{R}^3 + \beta_4\bar{R}(\bar{R}_{ij})^2 + \beta_5\bar{R}_{ij}\bar{R}^{jk}\bar{R}^i_k + \beta_6\bar{\nabla}_i\bar{R}_{jk}\bar{\nabla}^i\bar{R}^{jk} + \beta_7(\bar{\nabla}_i\bar{R})^2\right\}, \qquad (2.8)$$

where $\beta_{1\sim7}$ are arbitrary constant parameters and $\bar{\nabla}_i\bar{R}_{jk} = \nabla_i\bar{R}_{jk} - \Psi_{ij}{}^l\bar{R}_{lk} - \Psi_{ik}{}^l\bar{R}_{jl}$ with $\Psi_{ij}{}^l = -2\varphi^{-1}\left(\nabla^l\varphi g_{ij} - \nabla_i\varphi\delta^l_j - \nabla_j\varphi\delta^l_i\right)$. Note that here one can always choose a gauge φ = const as in (2.5) such that it reduces to the SVW action [14]. Then, the physical contents of the original SVW action, like as the scalar graviton problem, would be the same in this extended gravity also. So, there would be no fundamental advantage of this extension to resolve the scalar graviton problem of the original SVW approach.

For this reason, we consider a new extended action with three deformation parameters (ξ_1, ξ_2, ξ_3) which break, for $\xi_{1\sim3} \neq 1$, the local anisotropic Weyl invariance but still preserve residual global Weyl invariance as follows:

$$S = \int dtd^3xN\sqrt{g}\left\{\frac{2}{\kappa^2}\left(\tilde{B}_{ij}\tilde{B}^{ij} - \lambda\tilde{B}^2\right) - V_\nu(\varphi) + \varphi^8\tilde{R} + \beta_1\varphi^4\tilde{R}^2 + \beta_2\varphi^4(\tilde{R}_{ij})^2 \right.$$
$$\left. +\beta_3\tilde{R}^3 + \beta_4\tilde{R}(\tilde{R}_{ij})^2 + \beta_5\tilde{R}_{ij}\tilde{R}^{jk}\tilde{R}^i_k + \beta_6\bar{\nabla}_i\tilde{R}_{jk}\bar{\nabla}^i\tilde{R}^{jk} + \beta_7(\bar{\nabla}_i\tilde{R})^2\right\}, \qquad (2.9)$$

where

$$\tilde{B}_{ij} \equiv K_{ij} - \frac{2}{N\varphi}g_{ij}(\xi_1\dot{\varphi} - \xi_2\nabla_i\varphi N^i), \qquad (2.10)$$
$$\tilde{R}_{ij} \equiv R_{ij} + \xi_3 f_{ij}(\nabla\varphi), \quad \tilde{R} = g^{ij}\tilde{R}_{ij}. \qquad (2.11)$$

Here, the parameter ξ_1 is associated with the breaking of local Weyl invariance along the time slice of the extrinsic curvature scalar. Whereas ξ_2 and ξ_3 are associated with the non-invariances along the spatial directions of the extrinsic curvature and 3-dimensional curvature scalar, respectively. Note that the local Weyl invariance is not completely broken, but there are some residual local symmetries left, depending on the parameters; for example, when $\xi_1 = 1$, $\xi_{2,3} \neq 1$, the transformation function ω can be an arbitrary function of time; for $\xi_1 \neq 1$, $\xi_{2,3} = 1$, ω can be an arbitrary function of space.

It is important to note that, due to lack of local Weyl invariance, one can not choose the "gauge" $\varphi = \varphi_0$ = const always to reduce the theory to the HL gravity. This means that there is the additional, physical, scalar degree of freedom φ. But, in the absence of the scalar fluctuation mode around the background φ_0, this theory should be reduced to the Einstein-Hilbert action in IR such that the conditions in (2.5) is to be satisfied again[1]. More generally, we can also check that in the case of the parameters given by

$$\varphi_0 = \left[\frac{\kappa^2\mu^2(\Lambda_W - \omega)}{8(1-3\lambda)}\right]^{1/8}, \quad \nu = \left[\frac{72(1-3\lambda)\Lambda_W^4}{\kappa^2\mu^2(\Lambda_W-\omega)^3}\right]^{1/2},$$
$$\beta_1 = \frac{|\kappa\mu|(1-4\lambda)}{16}\left[\frac{2}{(\Lambda_W-\omega)(1-3\lambda)}\right]^{1/2}, \quad \beta_2 = -\frac{|\kappa\mu|}{2}\left[\frac{1-3\lambda}{2(\Lambda_W-\omega)}\right]^{1/2}$$
$$\beta_3 = -\frac{\kappa^2}{4W^4}, \quad \beta_4 = \frac{5\kappa^2}{4W^4}, \quad \beta_5 = -\frac{3\kappa^2}{2W^4}, \quad \beta_6 = -\frac{\kappa^2}{2W^4}, \quad \beta_7 = \frac{3\kappa^2}{16W^4}, \tag{2.12}$$

the "background" action for the scalar field $S(\varphi_0)$ can be reduced to the IR-modified HL action (without parity violation),

$$S_{HL} = \int dtd^3x\,\sqrt{g}N\left\{\frac{2}{\kappa^2}(K_{ij}K^{ij} - \lambda K^2) + \frac{\kappa^2\mu^2(\Lambda_W-\omega)}{8(1-3\lambda)}R - \frac{3\kappa^2\mu^2\Lambda_W^2}{8(1-3\lambda)}\right.$$
$$\left. -\frac{\kappa^2}{2W^4}C_{ij}C^{ij} + \frac{\kappa^2\mu^2(1-4\lambda)}{32(1-3\lambda)}R^2 - \frac{\kappa^2\mu^2}{8}R_{ij}R^{ij}\right\}, \tag{2.13}$$

where ω is an arbitrary constant parameter which breaks the detailed balance softly in IR [7, 28, 29]. But, we stress that there is no symmetry which can gauge away the scalar fluctuation mode generally, i.e., for arbitrary parameters ξ_1, ξ_2, ξ_3 the pathological scalar graviton problem may be different as one can see in the following sections. And this scalar mode carries the Lorentz violation effect even in IR, in addition to the IR Lorentz violation parameter λ in HL gravity; the action (2.9) can be considered as a (power-counting) renormalizable, Lorentz-violating scalar-tensor theory. Note also that in the action (2.9) we have 3 new parameters, $\xi_{1\sim 3}$ compared to HL gravity, and the coefficient of the Cotton term $-\kappa^2 C_{ij}C^{ij}/2W^4$ in (2.13) is determined from specific values of $\beta_{3\sim 7}$.

3 Scalar graviton mode in the quadratic action

In order to check explicitly the scalar graviton problem in the Minkowski background, we first consider the terms that contribute to the quadratic action for the linear perturbation as

$$S = \int dtd^3xN\sqrt{g}\left\{\frac{2}{\kappa^2}\left(\tilde{B}_{ij}\tilde{B}^{ij} - \lambda\tilde{B}^2\right) - V_\nu(\varphi) + \varphi^8\tilde{R} + \beta_1\varphi^4\tilde{R}^2 + \beta_2\varphi^4(\tilde{R}_{ij})^2\right.$$
$$\left. +\beta_6\bar{\nabla}_i\tilde{R}_{jk}\bar{\nabla}^i\tilde{R}^{jk} + \beta_7(\bar{\nabla}_i\tilde{R})^2\right\}. \tag{3.1}$$

[1]There exist some subtleties in the identification for $\lambda \neq 1$ [25]. But here we consider the λ-deformed Einstein-Hilbert action $S_{\lambda \mathrm{EH}} = (c^4/16\pi G_N)\int dtd^3x\sqrt{g}N[c^{-2}(K_{ij}K^{ij} - \lambda K^2) + R^{(3)} - 2c^{-2}\Lambda]$ following [7, 25–27].

Then we consider the following scalar perturbations of the metric and the matter field for the Minkowski background with $\Lambda = 0$ $[\varphi_0 = (c^4/16\pi G_N)^{1/8},\ V_\nu(\varphi_0) = 2\Lambda c^2/16\pi G_N = 0]$, up to the linear order ($\triangle \equiv \partial_i \partial^i$)

$$N = 1 + \phi, \quad N_i = \partial_i B, \quad g_{ij} = (1 - 2\psi)\delta_{ij} + 2\frac{\partial_i \partial_j}{\triangle}E, \quad \varphi = \varphi_0 + \tilde{\varphi}. \tag{3.2}$$

Substitution of the above perturbations into the action (3.1) leads to the following quadratic action (by adopting the convention $\kappa^2 = 2$)

$$\begin{aligned} S^{(2)} = \int dt d^3x \Bigg\{ & -6\dot{\psi}^2 + 16\xi_1 \dot{\psi} \frac{\dot{\tilde{\varphi}}}{\varphi_0} - 24\xi_1^2 \frac{\dot{\tilde{\varphi}}^2}{\varphi_0^2} + 4\psi \triangle \dot{B} - 4\psi \ddot{E} - 8\xi_1 \triangle \dot{B} \frac{\tilde{\varphi}}{\varphi_0} - 8\xi_1 \ddot{\psi} \frac{\tilde{\varphi}}{\varphi_0} \\ & + 8\xi_1 \ddot{E} \frac{\tilde{\varphi}}{\varphi_0} + (1-\lambda)\left(3\dot{\psi} - 6\xi_1 \frac{\dot{\tilde{\varphi}}}{\varphi_0} + \triangle B - \dot{E}\right)^2 \\ & - 2\varphi_0^8 \left(\psi \triangle \psi - 2\phi \triangle \psi - 16\frac{\tilde{\varphi}}{\varphi_0} \triangle \psi\right) - 8\xi_3 \varphi_0^8 \left(\phi + 7\frac{\tilde{\varphi}}{\varphi_0}\right) \frac{\triangle \tilde{\varphi}}{\varphi_0} \\ & + \beta_1 \varphi_0^4 \left(4\triangle \psi - 8\xi_3 \frac{\triangle \tilde{\varphi}}{\varphi_0}\right)^2 + \beta_2 \varphi_0^4 \left(\partial_i \partial_j \psi + \delta_{ij} \triangle \psi - 2\xi_3 \frac{\partial_i \partial_j \tilde{\varphi}}{\varphi_0} - 2\xi_3 \delta_{ij} \frac{\triangle \tilde{\varphi}}{\varphi_0}\right)^2 \\ & - 6\beta_6 \left(\psi - 2\xi_3 \frac{\tilde{\varphi}}{\varphi_0}\right) \triangle^3 \left(\psi - 2\xi_3 \frac{\tilde{\varphi}}{\varphi_0}\right) - 16\beta_7 \left(\psi - 2\xi_3 \frac{\tilde{\varphi}}{\varphi_0}\right) \triangle^3 \left(\psi - 2\xi_3 \frac{\tilde{\varphi}}{\varphi_0}\right) \Bigg\}. \end{aligned} \tag{3.3}$$

Varying the quadratic action with respect to ϕ and B, we obtain the (local) Hamiltonian and momentum constraints (assuming regular boundary conditions)[2]

$$\psi - 2\xi_3 \frac{\tilde{\varphi}}{\varphi_0} = 0, \tag{3.4}$$

$$\frac{1 - 3\lambda}{1 - \lambda}\left(\dot{\psi} - 2\xi_1 \frac{\dot{\tilde{\varphi}}}{\varphi_0}\right) + \triangle B - \dot{E} = 0. \tag{3.5}$$

Note that the above action does not have the contributions for the higher derivative terms due to the Hamiltonian constraints (3.4)[3], in contrast to the HL case, and nor the ξ_2 dependence. Note also that, for non-vanishing ξ_3, the momentum constraint (3.5) further reduces to

$$\frac{1 - 3\lambda}{1 - \lambda}\left(1 - \frac{\xi_1}{\xi_3}\right)\dot{\psi} + \triangle B - \dot{E} = 0. \tag{3.6}$$

[2]This corresponds to varying the action with respect to $\triangle\phi$ and $\triangle B$, instead of ϕ and B, with the appropriate integration by parts.

[3]For the usual tensor graviton modes, however, we have the same higher-derivative contributions as in HL gravity such that the (power-counting) renormalizability is not lost with our new extension. And even for the scalar graviton mode, it is generally expected that the renormalizability can be maintained again due to non-linear corrections: The cancelation of higher-derivative terms is peculiar to the linear perturbation but not generally true in higher-order perturbations.

By substituting $\tilde{\varphi}$ and $\triangle B - \dot{E}$ with ψ, from the constraints (3.4) and (3.6) with the appropriate integrations by parts, the above quadratic action becomes

$$S^{(2)} = 2\int dtd^3x \left\{ -\frac{1}{c_\psi^2}\dot{\psi}^2 + \frac{1-\xi_3}{\xi_3}c^2\psi\triangle\psi \right\}, \tag{3.7}$$

where

$$c_\psi^2 = \frac{1-\lambda}{3\lambda-1}\left(\frac{\xi_1}{\xi_3}-1\right)^{-2}. \tag{3.8}$$

We note that, when $\xi_3 \to \infty$, we obtain the same result as in the HL gravity

$$S_{HL}^{(2)} = 2\int dtd^3x \left\{ -\frac{1}{c_{HL}^2}\dot{\psi}^2 - c^2\psi\triangle\psi \right\}, \tag{3.9}$$

with $c_{HL}^2 = (1-\lambda)/(3\lambda-1)$ if we ignore the higher-derivatives terms which were kept there.[4] In this case, it is known that scalar graviton ψ has several pathological behaviors: ψ would be either unstable when $c_{HL}^2 < 0$ or be a ghost when $c_{HL}^2 > 0$; moreover, there are strongly coupled interactions for $c_{HL} \to 0$ $(\lambda \to 1)$, i.e., the perturbation around the Minkowski background can not be defined in the Einstein gravity limit $(\lambda \to 1)$ [10].

But in our case, we can cure the instability/ghost problem with [5]

$$c_\psi^2 < 0 \;\; (1 < \lambda), \;\;\; 0 < \xi_3 < 1. \tag{3.10}$$

This situation may be compared with the BPS extension [15,19], where the non-projectable lapse function $N(t, \mathbf{x})$ becomes a dynamical scalar field by adding the potential term $V = \eta\nabla_i N\nabla^i N/N^2$+(higher-derivative terms). In this case the resulting scalar graviton action in the quadratic order becomes

$$S_{BPS}^{(2)} = 2\int dtd^3x \left\{ -\frac{1}{c_{HL}^2}\dot{\psi}^2 - \frac{\eta-2}{\eta}c^2\psi\triangle\psi \right\}, \tag{3.11}$$

and the above problems of the instability/ghost can be also cured for $c_{HL}^2 < 0,\; 0 < \eta < 2$. But regarding the strong coupling problem which persists in the BPS extension still [19–23], we can cure this problem also in our construction, as will be shown in the next section.

We finally remark that for the special case of $\xi_3 = \xi_1$, the $\dot{\psi}^2$ term is disappearing even though the spatial derivatives term remains. This means that there is no dynamical scalar graviton at the quadratic order.

[4] This would be clear in the action (3.1), where the matter perturbation $\tilde{\varphi}$ is decoupled from the gravity part in the $\xi_3 \to \infty$ limit; if we consider first the $\xi_3 \to \infty$ limit before implementing the Hamiltonian constraint (3.4), we can recover the higher-derivative terms also.

[5] The ghost can be also avoided with $\lambda < 1/3$ but we do not consider this possibility here since the Einstein gravity with $\lambda = 1$ can not be obtained.

4 Strong coupling in the cubic action

We now turn to the cubic-order perturbation of the action to check whether the strong coupling problem can be resolved in our framework. Since the issue is about the non-linear perturbation at low energies (IR), it is enough to consider the action at IR limit:

$$S^{(IR)} = \int dtd^3xN\sqrt{g}\ \left\{\tilde{B}_{ij}\tilde{B}^{ij} - \lambda\tilde{B}^2 + \varphi^8\tilde{R} - V_\nu(\varphi)\right\}. \tag{4.1}$$

In order to study the cubic order interaction terms we consider the non-linear scalar perturbations around the Minkowski background ($V_\nu(\varphi_0) = 0$), without loss of generality[6] as follows [10, 18, 19]

$$N = e^\phi, \quad N_i = \partial_i B, \quad g_{ij} = e^{-2\psi}\delta_{ij}, \quad \varphi = \varphi_0 + \tilde{\varphi}. \tag{4.2}$$

Here we choose the $E = 0$ gauge in the most general scalar perturbation (3.2) to simplify the computations. After some manipulations one can find the cubic-order action given by

$$\begin{aligned}
S^{(3)} = &\\
&\int dtd^3x\Bigg\{ -16\varphi_0^7\tilde{\varphi}(\partial\psi)^2 + 2\varphi_0^8\psi(\partial\psi)^2 - 2\phi\varphi_0^8(\partial\psi)^2 + 112\varphi_0^6\tilde{\varphi}^2\partial^2\psi - 32\varphi_0^7\psi\tilde{\varphi}\partial^2\psi\\
&+32\varphi_0^7\phi\tilde{\varphi}\partial^2\psi + 2\varphi_0^8\phi^2\partial^2\psi + 2\varphi_0^8\psi^2\partial^2\psi - 4\varphi_0^8\phi\psi\partial^2\psi - 8\xi_3\Big(\frac{\varphi_0^7}{2}\phi^2\triangle\tilde{\varphi} + \frac{\varphi_0^7}{2}\psi^2\triangle\tilde{\varphi}\\
&-\phi\varphi_0^7\psi\triangle\tilde{\varphi} - 7\varphi_0^6\tilde{\varphi}\psi\triangle\tilde{\varphi} + 7\varphi_0^6\phi\tilde{\varphi}\triangle\tilde{\varphi} + 21\varphi_0^5\tilde{\varphi}^2\triangle\tilde{\varphi} + \varphi^7\psi\partial_i\tilde{\varphi}\partial_i\psi - \varphi_0^7\phi\partial_i\psi\partial_i\tilde{\varphi}\\
&-7\varphi_0^6\tilde{\varphi}\partial_i\tilde{\varphi}\partial_i\psi\Big) - 9(1-3\lambda)\psi\dot{\psi}^2 - 2(1-3\lambda)\psi\dot{\psi}\triangle B - 2(1-3\lambda)\dot{\psi}\partial_k\psi\partial_k B\\
&-2(1-\lambda)\triangle B\partial_k\psi\partial_k B + \psi\partial_i\partial_j B\partial_i\partial_j B + 4\partial_i\partial_j B\partial_i B\partial_j\psi - \lambda\psi(\triangle B)^2 - 3(1-3\lambda)\phi\dot{\psi}^2\\
&-2(1-3\lambda)\phi\dot{\psi}\triangle B - \phi\partial_i\partial_j B\partial_i\partial_j B + \lambda\phi(\triangle B)^2 - 4(1-3\lambda)\varphi_0^{-1}\Big(-6\xi_1\psi\dot{\psi}\dot{\tilde{\varphi}}\\
&-3\xi_2\dot{\psi}\partial_i\tilde{\varphi}\partial_i B - \xi_1\partial_i B\partial_i\psi\dot{\tilde{\varphi}} - \xi_2\triangle B\partial_i\tilde{\varphi}\partial_i B - 3\xi_1\phi\dot{\psi}\dot{\tilde{\varphi}} - \xi_1\phi\triangle B\dot{\tilde{\varphi}} - 3\xi_1\psi\dot{\psi}\dot{\tilde{\varphi}}\\
&-\xi_1\psi\triangle B\dot{\tilde{\varphi}} - 3\xi_1\varphi_0^{-1}\tilde{\varphi}\dot{\psi}\dot{\tilde{\varphi}} - \xi_1\varphi_0^{-1}\triangle B\dot{\tilde{\varphi}}\tilde{\varphi}\Big) + 12(1-3\lambda)\varphi_0^{-2}\left(-2\xi_1\xi_2\dot{\tilde{\varphi}}\partial_i\tilde{\varphi}\partial_i B\right.\\
&\left.-\xi_1^2\phi\dot{\tilde{\varphi}}^2 - 3\xi_1^2\psi\dot{\tilde{\varphi}}^2 - 2\xi_1^2\varphi_0^{-1}\tilde{\varphi}\dot{\tilde{\varphi}}^2\right)\Bigg\}
\end{aligned} \tag{4.3}$$

[6] One can transform these exponential-type perturbations to the other more general power-type perturbations like as $\bar{N} = 1 + \bar{\phi} + c_2\bar{\phi}^2 + \cdots,\ \bar{g}_{ij} = 1 - 2\bar{\psi} + d_2\bar{\psi}^2 + \cdots$ by setting $\phi = \phi^{(1)} + \phi^{(2)} + \cdots,\ \psi = \psi^{(1)} + \psi^{(2)} + \cdots$ and $\bar{\phi} = \bar{\phi}^{(1)} + \bar{\phi}^{(2)} + \cdots,\ \bar{\psi} = \bar{\psi}^{(1)} + \bar{\psi}^{(2)} + \cdots$.

Using the first-order Hamiltonian and momentum constraints (3.4), (3.5) (or (3.6)) obtained in the previous section[7], the above action (4.3) reduces to

$$\begin{aligned} S^{(3)} &= 2\int dtd^3x\left\{\left(-1+\frac{5}{\xi_3}-\frac{7}{\xi_3^2}\right)c^2\psi(\partial_i\psi)^2-\frac{2}{c_\psi^4}\left(\frac{\xi_1}{\xi_3}-1\right)^{-2}\left(\frac{\xi_2}{\xi_3}-1\right)\dot{\psi}\partial_i\psi\partial_i\left(\frac{\dot{\psi}}{\triangle}\right)\right. \\ &\quad -\frac{3}{2c_\psi^4}\left(\frac{\xi_1}{\xi_3}-1\right)^{-2}\psi\left(\frac{\partial_i\partial_j}{\triangle}\dot{\psi}\right)^2+\left[\frac{3}{2c_\psi^4}\left(\frac{\xi_1}{\xi_3}-1\right)^{-2}\right. \\ &\quad \left.\left.-\frac{1}{c_\psi^2}\left(\frac{\xi_1}{\xi_3}-1\right)^{-1}\left(3-3\frac{\xi_1}{\xi_3}-\frac{\xi_1}{\xi_3^2}\right)\right]\psi\dot{\psi}^2\right\}. \end{aligned} \tag{4.4}$$

Note again that when $\xi_3 \to \infty$ the above action can be reduced to the cubic action in the HL gravity as [10, 19]

$$\begin{aligned} S^{(3)}_{HL} = 2\int dtd^3x\Bigg\{ &-c^2\psi(\partial_i\psi)^2+\frac{2}{c_{HL}^4}\dot{\psi}\partial_i\psi\partial_i\left(\frac{\dot{\psi}}{\triangle}\right) \\ &+\frac{3}{2}\left[-\frac{1}{c_{HL}^4}\psi\left(\frac{\partial_i\partial_j}{\triangle}\dot{\psi}\right)^2+\frac{2c_{HL}^2+1}{c_{HL}^4}\psi\dot{\psi}^2\right]\Bigg\}. \end{aligned} \tag{4.5}$$

Now, in order to discuss the strong coupling problem we use the canonically normalized variable $\hat{\psi} = \bar{M}_{Pl}\psi/|c_\psi|$ (by recovering $2/\kappa^2 = cM_{Pl}^2/16\pi\hbar \equiv \bar{M}_{Pl}^2/2$) such that the quadratic action (3.7) becomes

$$S^{(2)} = \int dtd^3x\left\{\dot{\hat{\psi}}^2+\frac{1-\xi_3}{\xi_3}\,c^2|c_\psi|^2\hat{\psi}\triangle\hat{\psi}\right\}. \tag{4.6}$$

Then, the cubic action (4.4) becomes

$$\begin{aligned} &S^{(3)} = \\ &\frac{1}{\bar{M}_{Pl}}\int dtd^3x\left\{\left(-1+\frac{5}{\xi_3}-\frac{7}{\xi_3^2}\right)c^2|c_\psi|^3\hat{\psi}(\partial_i\hat{\psi})^2-\frac{2}{|c_\psi|}\left(\frac{\xi_1}{\xi_3}-1\right)^{-2}\left(\frac{\xi_2}{\xi_3}-1\right)\dot{\hat{\psi}}\partial_i\hat{\psi}\partial_i\left(\frac{\dot{\hat{\psi}}}{\triangle}\right)\right. \\ &\quad -\frac{3}{2|c_\psi|}\left(\frac{\xi_1}{\xi_3}-1\right)^{-2}\hat{\psi}\left(\frac{\partial_i\partial_j}{\triangle}\dot{\hat{\psi}}\right)^2+\left[\frac{3}{2|c_\psi|}\left(\frac{\xi_1}{\xi_3}-1\right)^{-2}\right. \\ &\quad \left.\left.+|c_\psi|\left(\frac{\xi_1}{\xi_3}-1\right)^{-1}\left(3-3\frac{\xi_1}{\xi_3}-\frac{\xi_1}{\xi_3^2}\right)\right]\hat{\psi}\dot{\hat{\psi}}^2\right\}. \end{aligned} \tag{4.7}$$

Note that all the terms but the first term scale as c_ψ^{-1} and so there is strong coupling for $\lambda \to 1$ since $c_\psi \to 0$ naively, from (3.8): All the cubic interaction terms that have the time

[7] In order to compute the cubic-order interaction one only needs to consider the constraints for the perturbations of N and N_i to the first order [30]. More generally, for the n'th-order interactions, one only needs to consider the $(n-2)$'th order [31].

derivatives of ψ blow up in that limit. However in our construction, due to the presence of another coupling ξ_1, which would be running in principle, this strong coupling problem can be cured by the "fine tuning " in the limit of $\lambda \to 1$. If

$$(\xi_1 - \xi_3) \sim (\lambda - 1)^s \quad (s \leq -1/2), \tag{4.8}$$

then

$$|c_\psi| \sim \frac{\sqrt{\lambda - 1}}{|\xi_1 - \xi_3|} \sim (\lambda - 1)^{1/2-s}, \tag{4.9}$$

and the troublesome interactions which scale as

$$\frac{1}{|c_\psi|}\left(\frac{\xi_1}{\xi_3} - 1\right)^{-2} \sim \frac{1}{(\lambda - 1)^{s+1/2}} \tag{4.10}$$

can be made to be regular. The cubic action becomes finite as

$$S^{(3)} \sim \frac{1}{\bar{M}_{Pl}} \int dtd^3x \left\{ -2\left(\frac{\xi_2}{\xi_3} - 1\right)\dot{\hat{\psi}}\partial_i\hat{\psi}\partial_i\left(\frac{\dot{\hat{\psi}}}{\triangle}\right) - \frac{3}{2}\hat{\psi}\left(\frac{\partial_i\partial_j}{\triangle}\dot{\hat{\psi}}\right)^2 + \frac{3}{2}\hat{\psi}\dot{\hat{\psi}}^2 \right\}. \tag{4.11}$$

for $s = -1/2$ or vanishing for $s < -1/2$. On the other hand, in this case, the quadratic action (4.6) becomes

$$S^{(2)} \sim \int dtd^3x\ \dot{\hat{\psi}}^2 \tag{4.12}$$

such that there is no ghost/instability problem either. Note that the condition (4.8) for the absence of strong coupling for $\lambda \to 1$ is consistent with the condition (3.10) for the absence of ghost/instability for $\lambda > 1$; this is in contrast to BPS gravity case [21]. Moreover, for the special case of $\xi_1 = \xi_3$ in the action (4.4), all the cubic terms that have the time-derivatives of ψ vanish and the above cubic action (4.7) reduces to

$$S^{(3)} = \frac{1}{\bar{M}_{Pl}} \int dtd^3x \left\{ \left(-1 + \frac{5}{\xi_3} - \frac{7}{\xi_3^2}\right) c^2|c_\psi|^3\hat{\psi}(\partial_i\hat{\psi})^2 \right\}. \tag{4.13}$$

such that there is no strong coupling problem either, unless $\xi_3 = 0$. Note that in this case, the quadratic action (3.7) did not have the time-derivative term either.

Finally, we remark that the higher-derivatives terms which have been ignored in the cubic interaction can not change our conclusion. This is because they generate only the spatial derivatives of $\psi, \tilde{\varphi}$, and E, not B. From the constraints (3.4), (3.5) (or (3.6)), only B is related to the time derivative of ψ whose interactions reveal strong coupling as in the BPS case [19].

5 Conclusion and discussion

We have extended the HL gravity with extra conformal invariance by introducing an extra scalar field. In the case of the critical exponent $z = 3$, which breaks the equal-footing treatment of space and time in UV, power counting renormalizability can be achieved without the ghost problem for the transverse traceless graviton modes. Relaxing the exact Weyl symmetry, we considered an action with three new coupling parameters ξ_1, ξ_2, ξ_3 which breaks the *local* anisotropic Weyl symmetry but still preserves residual *global* Weyl invariance. With a constant scalar field and $\xi_3 \to \infty$ limit, it reduces to the HL gravity, but it generally have some more degrees of freedom to cure the pathologies of the scalar graviton. Actually, we have found that, in the perturbation around the Minkowski background, both the instability/ghost problem of scalar graviton at the quadratic order and the strong coupling problem at the cubic order can be cured by the appropriate fine tuning of the (running) couplings as $\lambda \to 1_+,\ 0 < \xi_3 < 1,\ \xi_1 - \xi_3 \sim (\lambda - 1)^s\ \ (s \leq -1/2)$. This implies that the scalar matter field $\tilde{\varphi}$, which drives the scalar graviton ψ as in (3.4), (3.5) (or (3.6)), regularizes the strongly-coupled cubic interactions and makes the scalar graviton healthy in the quadratic propagation as well. This is in contrast to BPS approach where only one new coupling η was relevant in IR and so the strong coupling problem and the instability/ghost problem can not be cured simultaneously, unless a new low energy scale below the Planck scale is introduced.

For the projectable case, we can not use the Hamiltonian constraint (3.4) anymore and the pure scalar graviton terms are the same as in the HL gravity. In other words, one has the same pathologies of scalar graviton as in the (projectable) HL gravity.

Our new action reduces to Lorentz-violating scalar-tensor gravity theory at low energies. It is known that there is very strong constraints for the viable scalar-tensor theories [32]. It is left as an open problem whether this theory can be consistent with other observational and local gravity tests also. It is also challenging to check the closure of algebras from the full Hamiltonian analysis[8].

[8] After the completion of this work, we became aware of the article [33] in which the full Hamiltonian analysis in the *exact* Weyl-invariant, BPS extension of HL gravity was performed.

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