Proving AGT conjecture as HS duality: extension to five dimensions

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ABSTRACT

We extend the proof from [25],which interprets the AGT relation as the Hubbard-Stratonovich duality relation to the case of $5d$ gauge theories. This involves an additional q-deformation. Not surprisingly, the extension turns out to be trivial: it is enough to substitute all relevant numbers by q-numbers in all the formulas, Dotsenko-Fateev integrals by the Jackson sums and the Jack polynomials by the MacDonald ones. The problem with extra poles in individual Nekrasov functions continues to exist, therefore, such a proof works only for $\beta = 1$, i.e. for $q = t$ in MacDonald's notation. For $\beta \neq 1$ the conformal blocks are related in this way to a non-Nekrasov decomposition of the LMNS partition function into a double sum over Young diagrams.

1 Introduction

The AGT relation [1]-[25] is a particular version of the AdS/CFT correspondence and, more generally, of a gauge/string duality, which is very interesting, because it is a very concrete and explicit *quantitative* relation between the $2d$ conformal blocks [26] and the instanton partition functions [27]. At the same time, it is highly non-trivial, both conceptually and technically, and a clear proof is still unavailable. A proof is known in some simple particular cases [4, 5], while in general it is reduced to various technically involved recursion schemes in [15, 11], [24] and [23]. Recently, in [25] we used one of the approaches, based on the Dotsenko-Fateev-style representation of conformal blocks [4, 9, 13, 12, 16, 14, 17] and the character calculus [28] from matrix model theory, to cook up a proof based on the standard duality argument. Namely, one can find a quantity, which involves a double sum, and two different summation orders provide the two sides of the AGT relation. In this particular case this is a sum over characters, also averaged over time-variables: if the sum is taken first, one obtains Dotsenko-Fateev integrals of [14] in the form of [16]; if the average is taken first, one obtains sum of the Nekrasov functions [29]. Unfortunately, it works so simple only for $\beta = 1$, otherwise, particular Nekrasov functions have extra poles, which somehow disappear from the sum and are not seen at the conformal block side of the AGT relation: what this really means and how these fictitious poles should be interpreted and handled within the AGT context, remains a mystery.

Instead for $\beta \neq 1$ the Hubbard-Stratonovich duality provides another, non-Nekrasov decomposition of the LMNS partition function [27] into a double sum over Young diagrams, which may have its own significance (one natural way to proceed in this direction is to extend the results of [25] from the spherical 4-point to the arbitrary conformal block). In this letter we consider a natural q-deformation of [25], which corresponds to the straightforward generalization of Seiberg-Witten theory [30, 31], of Nekrasov calculus and of the AGT relation from $4d$ to $5d$ theories. Such an extension has already been addressed in the literature: in [32, 33] and [10, 18, 19]. It is well-known to be straightforward and should not bring any surprises. At the same time, it involves some technicalities in character calculus, because it involves the MacDonald polynomials in the role of characters and the Jackson sums in the role of open-contour integrals. As usual, q-deformation is the level, where all technical features look most natural and all formulas become most transparent. Also it is a natural step towards further generalization: to somewhat more general Kerov polynomials and to $6d$ theories, the very interesting in the AGT context. The last, but not least: the $5d$ deformation seems to play a role in "3d" extensions of the AGT relation [34, 35], which are supposed to involve $3d$ Chern-Simons theory [36] and knot invariants [37, 38].

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$$V_{\Delta_2}(\Lambda) \qquad\qquad V_{\Delta_3}(1)$$

$$\Delta$$

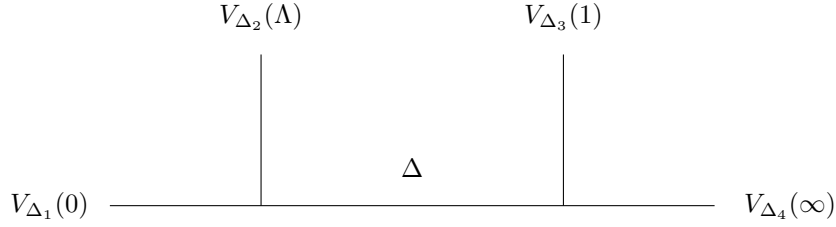$$V_{\Delta_1}(0) \qquad\qquad\qquad V_{\Delta_4}(\infty)$$

Figure 1: Feynman diagram for the 4-point conformal block.

As expected, since all the formulas of [25] for the $N_f = 2N_c = 4$ are nicely factorizable, they are directly generalized to $q \neq 1$, by substitution of all the factors by their q-number counterparts:

$$n \to [n]_q = \frac{1 - q^n}{1 - q} \tag{1}$$

We do not consider here the "pure gauge limit" part of the story: it is again straightforward, but the proper q-version of the Brezin-Gross-Witten unitary β-ensemble [21] deserves separate consideration.

2 Four dimensions

We start with outlining the main aspects of the proof of the standard AGT conjecture in four dimensional case for $\beta = 1$. In $SU(2)$ case the AGT conjecture claims that the instanton part of the four-dimensional $\mathcal{N} = 2$ superconformal field theory coincides with the 4-point conformal block in $2d$ CFT[1]:

$$Z_{Nek}^{4d}(\epsilon_i, \mu_i, a | \Lambda) = B(\Delta_i, \Delta, c | \Lambda) \tag{2}$$

under certain identification of the parameters $\{\epsilon_i, \mu_i, a\}$ and $\{\Delta_i, \Delta, c\}$. The Nekrasov partition function has the form of double expansion over two sets of Young diagrams:

$$Z_{Nek}^{4d}(\Lambda) = \sum_{A,B} N_{A,B}(\epsilon_i, \mu_i, a)\, \Lambda^{|A|+|B|} \tag{3}$$

where the coefficients $N_{A,B}$ are the Nekrasov functions corresponding to the Young diagrams A and B.

It is well-known that the Λ-expansion of the conformal block based on the operator product expansion (OPE) has the form of the sum over two Young diagrams. This OPE procedure is extensively reviewed in the CFT literature [26, 3, 6, 7]; in the particular 4-point case shown in the Fig.1, it gives:

$$B(\Delta_1, \Delta_{\Delta_2}, \Delta_3, \Delta_{\Delta_4}, \Delta, c \,|\, \Lambda) = \sum_{A,B} \Lambda^{\frac{1}{2}(|A|+|B|)}\, \gamma_{\Delta_1\Delta_2\Delta;A} Q_\Delta^{-1}(A,B) \gamma_{\Delta\Delta_3\Delta_4;B} \tag{4}$$

where $\gamma_{\Delta_1\Delta_2\Delta_3;A}$ are the structure coefficients of the OPE algebra, and Q is the Shapovalov form of the Virasoro algebra:

$$Q_\Delta(A,B) = \, < \Delta | L_A L_{-B} | \Delta > \tag{5}$$

$\gamma_{\Delta_1\Delta_2\Delta_3;A}$ are known explicitly, while $Q_\Delta(A,B)$ can be calculated level by level (see, e.g., [3]) and one can directly construct the Λ-expansion. However, this expansion *does not coincide (!)* with the double expansion of the Nekrasov partition function (3). Indeed, the Shapovalov form $Q_\Delta(A,B)$ is not zero only for descendants of the same level, which means that only the Young diagrams with $|A| = |B|$ contribute to the sum (4), but there is no such a restriction in (3).

The appropriate double expansion of the 4-point conformal block comes from the free field representation of correlator. As was shown in [16, 17, 25], utilizing the Dotsenko-Fateev integral representation [39], the conformal block can be represented as a double average over the two independent Selberg ensembles:

$$B(\Delta_i, \Delta, c | \Lambda) = \left\langle \left\langle \prod_{i=1}^{N_+}(1 - \Lambda x_i)^{v_-} \prod_{j=1}^{N_-}(1 - \Lambda y_j)^{v_+} \prod_{i=1}^{N_+}\prod_{j=1}^{N_-}(1 - \Lambda x_i y_j)^{2\beta} \right\rangle_+ \right\rangle_- \tag{6}$$

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[1]Here $B(\Delta_i, \Delta, c | \Lambda)$ is the 4-point conformal block with fields located at 0, Λ, 1 and ∞. We use Λ to denote the double ratio of four coordinates instead of the more conventional q or x, because these letters are used for other purposes in the present text. Physically, $\Lambda = e^{2\pi i\tau}$, where τ is the bare coupling constant, it turns into dimensional Λ_{QCD} after dimensional transmutation when some of the masses $m_1, \ldots, m_4$ tend to infinity.

Here the average goes over two ensembles (labeled by symbols $+$ and $-$) of variables $x_1, ... x_{N_+}$ and $y_1, ..., y_{N_-}$ ("eigenvalues of matrix models"):

$$\left\langle f(x_1, \ldots, x_{N_+}) \right\rangle_+ = \frac{1}{Z_+} \int\limits_0^1 dx_1 \ldots \int\limits_0^1 dx_{N_+} \prod_{i<j} (x_i - x_j)^{2\beta} \prod_i x_i^{u_+} (x_i - 1)^{v_+} \; f(x_1, \ldots, x_{N_+})$$

$$\left\langle f(y_1, \ldots, y_{N_-}) \right\rangle_- = \frac{1}{Z_-} \int\limits_0^1 dy_1 \ldots \int\limits_0^1 dy_{N_-} \prod_{i<j} (y_i - y_j)^{2\beta} \prod_i y_i^{u_-} (y_i - 1)^{v_-} \; f(y_1, \ldots, y_{N_-})$$

with the normalization constants

$$Z_\pm = \int\limits_0^1 dz_1 \ldots \int\limits_0^1 dz_{N_\pm} \prod_{i<j} (z_i - z_j)^{2\beta} \prod_i z_i^{u_\pm} (z_i - 1)^{v_\pm}$$

This matrix model representation of the conformal block is very convenient for analysis of its Λ-expansion, moreover, utilizing the standard matrix model technique of character expansion for each set of variables one can rewrite (6) as a double expansion over two sets of Young diagrams. Indeed, let us denote by I the function which is averaged in (6), then one has:

$$I = \prod_{i=1}^{N_+} (1 - qx_i)^{v_-} \prod_{j=1}^{N_-} (1 - qy_j)^{v_+} \prod_{i=1}^{N_+} \prod_{j=1}^{N_-} (1 - qx_iy_j)^{2\beta} =$$

$$= \exp\left(v_- \sum_{i=1}^{N_+} \ln(1 - \Lambda x_i) + v_+ \sum_{j=1}^{N_-} \ln(1 - \Lambda y_j) + 2\beta \sum_{i=1}^{N_+} \sum_{j=1}^{N_-} \ln(1 - \Lambda x_i y_j) \right)$$

$$= \exp\left(-\sum_{k=1}^\infty \frac{\Lambda^k}{k} p_k v_- - \sum_{k=1}^\infty \frac{\Lambda^k}{k} \widetilde{p}_k v_+ - 2\beta \sum_{k=1}^\infty \frac{\Lambda^k}{k} p_k \widetilde{p}_k \right) \tag{7}$$

where in the last step we expanded the logarithms into the powers of Λ and denoted

$$p_k = \sum_{i=1}^{N_+} x_i^k, \quad \widetilde{p}_k = \sum_{j=1}^{N_-} y_j^k, \quad \text{such that} \quad \sum_{i=1}^{N_+} \ln(1 - \Lambda x_i) = -\sum_{i=1}^{N_+} \sum_{k=1}^\infty \frac{\Lambda^k x_i^k}{k} = -\sum_{k=1}^\infty \frac{\Lambda^k}{k} p_k \tag{8}$$

We rewrite (7) in the form [16, 17]

$$I = \exp\left(\beta \sum_{k=1}^\infty \frac{\Lambda^k}{k} p_k \left(-\widetilde{p}_k - \frac{v_-}{\beta} \right) \right) \exp\left(\beta \sum_{k=1}^\infty \frac{\Lambda^k}{k} \widetilde{p}_k \left(-p_k - \frac{v_+}{\beta} \right) \right) \tag{9}$$

The final step that one needs in order to expand (7) into the sum of characters is the Cauchy completeness formula for the Jack polynomials:

$$\exp\left(\beta \sum_{k=1}^\infty \frac{p_k \widetilde{p}_k}{k} \right) = \sum_R j_R(p_k) j_R(\widetilde{p}_k) \tag{10}$$

where j_R is the normalized Jack polynomial (with deformation parameter β) corresponding to the representation R, and the sum runs over all representations of $GL(\infty)$ (over all the Young diagrams R). Utilizing this formula for (9) one finally finds

$$I = \sum_{A,B} \Lambda^{|A|+|B|} j_B(p_k) j_B\left(-\widetilde{p}_k - \frac{v_-}{\beta} \right) j_A(\widetilde{p}_k) j_A\left(-p_k - \frac{v_+}{\beta} \right) \tag{11}$$

Note that, due to presence of the term $2\beta p_k \widetilde{p}_k$ in (7), the expansion goes over a set of two Young diagrams A and B. We find that the Λ-expansion of the conformal block takes the form similar to the expansion of the Nekrasov partition function:

$$\boxed{ B(\Lambda) = \sum_k B_k \Lambda^k = \sum_{A,B} \Lambda^{|A|+|B|} \left\langle j_A(-p_k - v_+) j_B(p_k) \right\rangle_+ \left\langle j_A(\widetilde{p}_k) j_B(-\widetilde{p}_k - v_-) \right\rangle_- } \tag{12}$$

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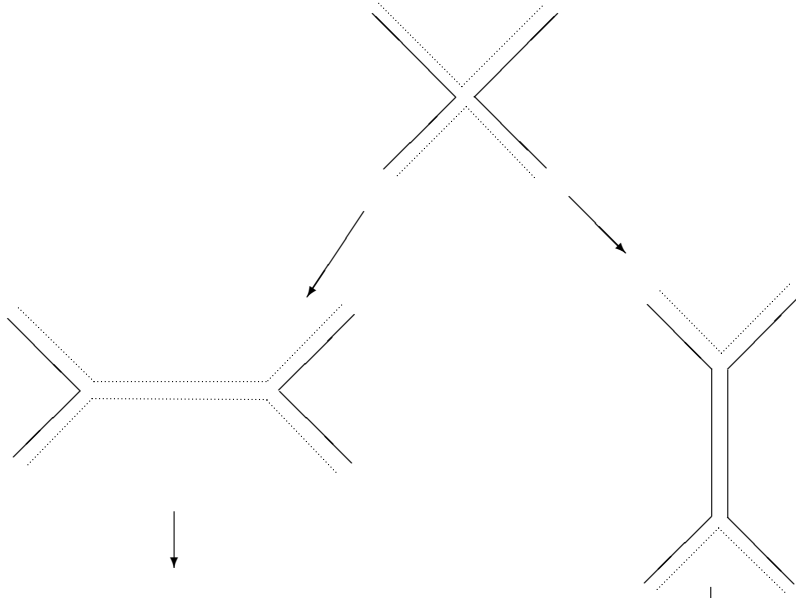
$$\sum_{A,B} N_{AB} \ = \ \overbrace{\sum_{AB} \int_x j_A(x)j_B(x) \int_y j_A(y)j_B(y)} \ = \ \overbrace{\int_{x,y} \sum_A j_A(x)j_A(y) \sum_B j_B(x)j_B(y)} \ = B(\Lambda)$$

Figure 2: Picture of the Nekrasov functions/conformal block duality expressed by the Hubbard-Stratonovich type formula (15). The symbol $\int_z$ here denotes integration with the Selberg measure over variables z_i, and the symbol $\sum_A$ denotes summation over all Young diagrams A.

Comparing both sides of (3) and (12), the AGT conjecture states that

$$\sum_{A,B} N_{A,B} = \sum_{A,B} \left\langle j_A(-p_k - \frac{v_+}{\beta})j_B(p_k) \right\rangle_+ \left\langle j_A(\widetilde{p}_k)j_B(-\widetilde{p}_k - \frac{v_-}{\beta}) \right\rangle_- \tag{13}$$

But really exciting is that the identity becomes termwise in the case of $\beta = 1$ (corresponding to the case of $\epsilon_1 + \epsilon_2 = 0$ on the side of the Nekrasov function) [25]:

$$N_{A,B}|_{\epsilon_1+\epsilon_2=0} = \left\langle j_A(-p_k - \frac{v_+}{\beta})j_B(p_k) \right\rangle_+ \left\langle j_A(\widetilde{p}_k)j_B(-\widetilde{p}_k - \frac{v_-}{\beta}) \right\rangle_- \Bigg|_{\beta=1} \tag{14}$$

In this way, the AGT relation is interpreted as a standard duality of the Hubbard-Stratonovich type, see Fig.2:

$$\sum_{a,b} \left(\sum_i X_i^a X_i^b\right) \left(\sum_j X_j^a X_j^b\right) = \sum_{a,b,i,j} X_i^a X_i^b X_j^a X_j^b = \sum_{i,j} \left(\sum_a X_i^a X_j^a\right) \left(\sum_b X_i^b X_j^b\right) \tag{15}$$

In our case the role of X_i^a is played by the symmetric polynomials $j_A(p_k)$, summation over a, b corresponds to the summation over the Young diagrams and summation over i and j is the averaging over two independent ensembles. Unfortunately, relation (14) is broken at $\beta \neq 1$ (relation (13), of course, remains true in this case as well). In this case the individual Nekrasov function has more poles then the whole sum (13). These extra poles puzzle [25] remains unresolved and the interpretation of the original AGT conjecture as a Hubbard-Stratonovich duality is still missed in the case of $\beta \neq 1$. Instead, (12) provides an alternative (modified) AGT conjecture which is, perhaps, even more interesting and useful than the original one. The items of the bi-Selberg decomposition (12) have no extra poles, but the numerators do *not* factorize into linear factors, as in the Nekrasov decomposition. The example of the first level $|A| + |B| = 1$ is already fully representative:

$$B_1 = \frac{(a+m_1)(a+m_2)(a+m_3)(a+m_4)}{2a(2a+\epsilon)} + \frac{(a-m_1)(a-m_2)(a-m_3)(a-m_4)}{2a(2a-\epsilon)} =$$

$$= \frac{\big((a+m_1)(a+m_2) - \epsilon(m_1+m_2)\big)(a+m_3)(a+m_4)}{(4a^2-\epsilon^2)} + \frac{(a-m_1)(a-m_2)\big((a-m_3)(a-m_4) - \epsilon(m_3+m_4)\big)}{(4a^2-\epsilon^2)} \tag{16}$$

where the first line is the Nekrasov decomposition, while the second line is the bi-Selberg one in (12). Clearly, the two decompositions are different, but coincide for $\epsilon = \epsilon_1 + \epsilon_2 = 0$, i.e. for $\beta = 1$. In fact, in addition to

(12), there is also an alternative decomposition:

$$B(\Lambda) = \sum_{A,B} \Lambda^{|A|+|B|} \left\langle j_A(p_k + \frac{v_+}{\beta}) j_B(p_k) \right\rangle_+ \left\langle j_A(-\widetilde{p}_k) j_B(-\widetilde{p}_k - \frac{v_-}{\beta}) \right\rangle_-$$

However, at level 1 it is indistinguishable from (12) and we do not add the extra line to (16). Note that no one of the three correlators: $\left\langle j_A(p_k + v/\beta) j_B(p_k) \right\rangle$, $\left\langle j_A(-p_k - v/\beta) j_B(p_k) \right\rangle$, $\left\langle j_A(-p_k - v/\beta) j_B(-p_k) \right\rangle$ is factorizable at $\beta \neq 1$. The only factorizable correlator is $\left\langle j_A(p_k + w) j_B(p_k) \right\rangle$, however, $w \neq v/\beta$ for $\beta \neq 1$ (see (103) below).

Leaving this problem, the generalization of (14) to the five-dimensional case is straightforward. As was noted in [32, 33] every $4d$ Seiberg-Witten theory can be generalized to the $5d$ case by an appropriate q-deformation, with the deformation parameter $q = e^{-\hbar R}$, with R being radius of the compact fifth dimension, so that in the case of $R = 0$ or $q = 1$ one returns to the standard four-dimensional theory. In particular, the deformation of the four-dimensional Nekrasov function to five dimensions is very simple: all the factors of the four-dimensional Nekrasov function are substituted by their q-number counterparts

$$n \to [n]_q = \frac{1 - q^n}{1 - q} \tag{17}$$

The aim of this paper is to describe the appropriate q-deformation of relation (14). Some progress in this direction has been already made in [18] where the q-deformed conformal block is fixed by the q-Virasoro algebra. The free field representation for the q-deformed vertex operators can be found in [10].

Here we do not consider all the preliminary steps, and start directly from q-deformation of the double average (6). Such a q-deformation can be straightforwardly written using the usual properties of q-deformation. All one needs, is to change the factors and integrals in (6) by their q-counterparts, the rules are as follows

- all power-like factors in (6) are substituted with the products:

$$(1 - x)^a \to \prod_{k=0}^{a-1} (1 - q^k x) \tag{18}$$

- the Van-der-Monde determinant (the Jack measure) is replaced by the MacDonald measure:

$$\prod_{1 \leq i < j \leq N} (x_i - x_j)^{2\beta} \to \Delta^{MC}(x) \to \prod_{i \neq j} \prod_{k=0}^{\beta-1} (x_i - q^k x_j) \tag{19}$$

- The integrals in the Selberg average are replaced by the q-Jackson integrals (see (91) in the Appendix for the definition):

$$\int_0^1 dz \to \int_0^1 d_q z \tag{20}$$

In complete analogy with the four-dimensional case, these simple rules lead to the Jackson integral representation of the five-dimensional conformal block and, further, the Nekrasov functions. Similar to the four-dimensional case, formula (14) works only at $\beta = 1$, and the problem of extra poles of the Nekrasov functions remains unresolved.

As a by product of this research, we found a nice, completely factorized formula for the average of two MacDonald polynomials (101). Similar to the Nekrasov functions, this average is completely factorized into linear multiples, but gives the Nekrasov function only at $\beta = 1$.

3 AGT in five dimensions

3.1 Nekrasov Functions

The instanton part of the five-dimensional $SU(N)$ partition function with $N_f = 2N$ fundamentals has form of the sum over N Young diagrams $Y_i, (i = 1...N)$:

$$Z_{Nek}^{5d}(\Lambda) = \sum_{Y_1,...,Y_N} N_{Y_1,...,Y_N} \tilde{\Lambda}^{|Y_1|+...+|Y_N|} \tag{21}$$

and the coefficients of expansion are [10]

$$N_{Y_1,\ldots,Y_N} = \left(v^{-N}\prod_{j=1}^{N}(Q_j^+)^{\frac{1}{2}}(Q_j^-)^{-\frac{1}{2}}\right)^{|Y_1|+\ldots+|Y_N|}\prod_{i,j=1}^{N}\frac{\mathcal{N}_{Y_i,[]}(vQ_i/Q_j^+)\mathcal{N}_{[],Y_i}(vQ_j^-/Q_i)}{\mathcal{N}_{Y_i,Y_j}(Q_i/Q_j)} \tag{22}$$

with

$$\mathcal{N}_{A,B}(Q) = \prod_{(i,j)\in A}\left(1 - Qq^{\mathrm{Leg}_A(i,j)}t^{\mathrm{Arm}_B(i,j)+1}\right)\prod_{(i,j)\in B}\left(1 - Qq^{-\mathrm{Leg}_B(i,j)-1}t^{-\mathrm{Arm}_A(i,j)}\right) \tag{23}$$

where $v = (q/t)^{1/2}$ and $[]$ denotes the empty Young diagram. The first multiplier in (22) can be put unit by rescaling the expansion parameter Λ, we keep it in order to make the Nekrasov functions (22) symmetric in masses.

The parameters t and q are related with the Ω-background parameters as $q = e^{R\epsilon_2}$ and $t = e^{-R\epsilon_1}$, where R is the radius of the compact fifth dimension. The remaining parameters in (22) are related with the v.e.v. of scalar fields a_i and the masses of fundamentals $m_i = \mu_i\sqrt{\epsilon_1\epsilon_2}$ as follows:
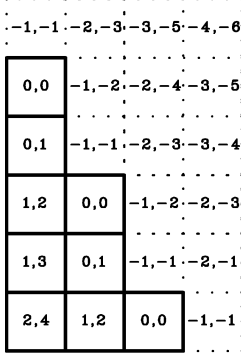
$$Q_i = q^{a_i}, \quad Q_i^+ = q^{-\mu_i}, \quad Q_i^- = q^{-\mu_{N+i}} \tag{24}$$

Note that in [25] we used different normalization for the v.e.v.'s a_i and the masses μ_i:

$$a_i \to \epsilon_2 a_i, \quad \mu_i \to \epsilon_2 \mu_i$$

For the arbitrary Young diagram Y, the symbols $\mathrm{Arm}_Y(i,j)$ and $\mathrm{Leg}_Y(i,j)$ denote the arm-length and leg-length of the box (i,j) in the Young diagram Y respectively. Algebraically, these lengths are given by the expressions

-1,-1	-2,-3	-3,-5	-4,-6	
0,0	-1,-2	-2,-4	-3,-5	
0,1	-1,-1	-2,-3	-3,-4	
1,2	0,0	-1,-2	-2,-3	
1,3	0,1	-1,-1	-2,-1	
2,4	1,2	0,0	-1,-1	

Fig.3

$$\mathrm{Arm}_Y(i,j) = Y'_j - i, \quad \mathrm{Leg}_Y(i,j) = Y_i - j \tag{25}$$

where Y' stands for the transposed Young diagram. Note that functions $\mathrm{Arm}_Y(i,j)$ and $\mathrm{Leg}_Y(i,j)$ can take negative values for (i,j) outside the Young diagram Y. In Fig.3 we give an example of the Young diagram $Y = [5,3,1]$ with the corresponding lengthes $(\mathrm{Leg}_Y(i,j), \mathrm{Arm}_Y(i,j))$ both within the diagram Y and outside it.

In the case of $N = 2$, the partition function takes the form

$$Z_{Nek}^{5d}(\Lambda) = \sum_{A,B} N_{A,B}\tilde{\Lambda}^{|A|+|B|} \tag{26}$$

and the coefficients can be rewritten in the form used in [25]:

$$N_{A,B} = \frac{\prod_{k=1}^{2} f_A^+(\mu_k + a)f_B^+(\mu_k - a)\prod_{k=3}^{4} f_A^-(\mu_k + a)f_B^-(\mu_k - a)}{g_{A,A}(0)g_{A,B}(-2a)g_{B,A}(2a)g_{B,B}(0)} q^{\frac{-\mu_1-\mu_2+\mu_3+\mu_4-2(1-\beta)}{2}(|A|+|B|)} \tag{27}$$

such that all the functions are some products of q-numbers:

$$f_A^\pm(x) = \prod_{(i,j)\in A}[\pm x \mp i\beta \pm j \mp \tfrac{1}{2}(1-\beta)]_q, \tag{28}$$

and

$$g_{A,B}(x) = \prod_{(i,j)\in A}[x + \beta\mathrm{Arm}_A(i,j) + \mathrm{Leg}_B(i,j) + \beta]_q[-x - \beta\mathrm{Arm}_A(i,j) - \mathrm{Leg}_B(i,j) - 1]_q \tag{29}$$

where we used the following definition of β:

$$t = q^\beta, \quad \beta = -\frac{\epsilon_1}{\epsilon_2} \tag{30}$$

As we shall see in the case of $N = 2$ $\tilde{\Lambda}$ is actually slightly different from the Λ-parameter of the conformal block, that is,

$$\boxed{\tilde{\Lambda} = \Lambda q^\gamma} \tag{31}$$

with

$$\gamma = \sum_{k=1}^{4} \frac{\mu_k}{2} + 1 \tag{32}$$

3.2 Dotsenko-Fateev integral

The Dotsenko-Fateev integral representation for the $5d$ conformal block is an appropriate q-deformation of the four-dimensional double average (6). Similar to four dimensions, this representation can be constructed by utilizing the free field representation of the conformal block, the corresponding q-deformed vertex operators being described in [10]. In fact, the q-deformations of all factors in (6) are well-known, and, hence, the proper q-version of (6) can be obtained directly by the usual rules (18)-(20). In this way, one easily finds[2]:

$$B^{5d}(\Lambda) = \left\langle \left\langle \prod_{i=1}^{N_+} \prod_{k=0}^{v_- -1} (1 - q^k \Lambda x_i) \prod_{j=1}^{N_-} \prod_{k=0}^{v_+ -1} (1 - q^k \Lambda y_j) \prod_{i=1}^{N_+} \prod_{j=1}^{N_-} \prod_{k=0}^{\beta -1} (1 - q^k \Lambda x_i y_j)^2 \right\rangle_+ \right\rangle_- \tag{33}$$

The averages are taken over two independent sets (labeled by symbols $+$ and $-$) of variables $x_1, ..., x_{N_+}$ and $y_1, ..., y_{N_-}$ ("eigenvalues in the matrix model terms") as follows:

$$\left\langle f \right\rangle_+ = \frac{1}{S_+} \int_0^1 d_q x_1 \ldots \int_0^1 d_q x_{N_+} \prod_{i \neq j} \prod_{k=0}^{\beta -1} (x_i - q^k x_j) \prod_i x_i^{u_+} \prod_{k=0}^{v_+ -1} (1 - q^k x_i) \, f(x_1, \ldots, x_{N_+}) \tag{34}$$

$$\left\langle f \right\rangle_- = \frac{1}{S_-} \int_0^1 d_q y_1 \ldots \int_0^1 d_q y_{N_-} \prod_{i \neq j} \prod_{k=0}^{\beta -1} (y_i - q^k y_j) \prod_i y_i^{u_-} \prod_{k=0}^{v_- -1} (1 - q^k y_i) \, f(y_1, \ldots, y_{N_-}) \tag{35}$$

with the normalization constants

$$S_\pm = \int_0^1 d_q z_1 \ldots \int_0^1 d_q z_N \prod_{i \neq j} \prod_{k=0}^{\beta -1} (z_i - q^k z_j) \prod_i z_i^{u_\pm} \prod_{k=0}^{v_+ -1} (1 - q^k z_i) \tag{36}$$

which guarantee $\left\langle 1 \right\rangle_+ = \left\langle 1 \right\rangle_- = 1$. We show in section 3.4 that the q-deformed β-ensemble (33), indeed, correctly reproduces the $5d$ Nekrasov partition function.

3.3 The AGT conjecture

As we show in the next subsection, in the case of $N = 2$ there is a simple identity between the five-dimensional conformal block (33) and the five-dimensional partition function (26):

$$B^{5d}(\Lambda) = Z^{5d}_{Nek}(\Lambda) \tag{37}$$

with the following identification of parameters:

$$N_+ = \frac{\epsilon_2}{\epsilon_1}(a - \mu_2) = \frac{\mu_2 - a}{\beta}, \; N_- = -\frac{\epsilon_2}{\epsilon_1}(a + \mu_4) = \frac{a + \mu_4}{\beta}$$

$$u_+ = \mu_1 - \mu_2 - 1 - \frac{\epsilon_1}{\epsilon_2} = \mu_1 - \mu_2 - 1 + \beta, \; u_- = \mu_3 - \mu_4 - 1 - \frac{\epsilon_1}{\epsilon_2} = \mu_3 - \mu_4 - 1 + \beta \tag{38}$$

$$v_+ = -\mu_1 - \mu_2, \; v_- = -\mu_3 - \mu_4$$

Note that this AGT-identification does not depend on q.

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[2] Hereafter, for the sake of simplicity we write all the formulas for integer values of parameters v_+, v_- and β. Note, however, that the extension to non-integer quantities is very straightforward, see (94) and (95) in the Appendix.

3.4 Bi-Selberg expansion of the conformal block

The proof of the AGT conjecture for $\beta = 1$ is much similar to the $4d$ case outlined in the Introduction, where the proof was based on the expansion of the Dotsenko-Fateev integrand into the Jack polynomials. Obviously, in the $5d$ case the expansion should be into the MacDonald polynomials, which are the appropriate q-deformation of the Jack functions. Denote by I the integrand of (33), then:

$$I = \prod_{i=1}^{N_+}\prod_{k=0}^{v_- - 1}(1 - q^k \Lambda x_i)\prod_{j=1}^{N_-}\prod_{k=0}^{v_+ - 1}(1 - q^k \Lambda y_j)\prod_{i=1}^{N_+}\prod_{j=1}^{N_-}\prod_{k=0}^{\beta-1}(1 - q^k \Lambda x_i y_j)^2 =$$

$$= \exp\left(\sum_{i=1}^{N_+}\sum_{k=0}^{v_- - 1}\ln(1 - q^k\Lambda x_i) + \sum_{j=1}^{N_-}\sum_{k=0}^{v_+ - 1}\ln(1 - q^k\Lambda y_j) + 2\sum_{i=1}^{N_+}\sum_{j=1}^{N_-}\sum_{k=0}^{\beta-1}\ln(1 - q^k\Lambda x_i y_j)\right) =$$

$$= \exp\left(-\sum_{i=1}^{N_+}\sum_{k=0}^{v_- - 1}\sum_{m=1}^{\infty}\frac{q^{km}\Lambda^m x_i^m}{m} - \sum_{j=1}^{N_-}\sum_{k=0}^{v_+ - 1}\sum_{m=1}^{\infty}\frac{q^{km}\Lambda^m y_j^m}{m} - 2\sum_{i=1}^{N_+}\sum_{j=1}^{N_-}\sum_{k=0}^{\beta-1}\sum_{m=1}^{\infty}\frac{q^{km}\Lambda^m x_i^m y_j^m}{m}\right) =$$

$$= \exp\left(-\sum_{m=1}^{\infty}\frac{\Lambda^m}{m}\Big(p_m\,[v_-]_{q^m} + p_m\,[v_+]_{q^m} + 2\,[\beta]_{q^m}p_m\,\widetilde{p}_m\Big)\right) \tag{39}$$

where in the last step we used the notations

$$p_m = \sum_{i=1}^{N_+}x_i^m, \quad \widetilde{p}_m = \sum_{j=1}^{N_-}y_j^m, \quad [v_\pm]_{q^m} = \frac{1 - q^{mv_\pm}}{1 - q^m} = 1 + q^m + q^{2m}... + q^{(v_\pm - 1)m} \tag{40}$$

Thus, one obtains

$$I = \exp\left(\sum_{m=1}^{\infty}\frac{[\beta]_{q^m}\Lambda^m}{m}\widetilde{p}_m\Big(-p_m - \frac{[v_+]_{q^m}}{[\beta]_{q^m}}\Big)\right)\exp\left(\sum_{m=1}^{\infty}\frac{[\beta]_{q^m}\Lambda^m}{m}p_m\Big(-\widetilde{p}_m - \frac{[v_-]_{q^m}}{[\beta]_{q^m}}\Big)\right) \tag{41}$$

Now to proceed to the expansion into a sum over the Young diagrams, we use the Cauchy completeness formula for the MacDonald polynomials:

$$\exp\Big(\sum_{m=1}^{\infty}\frac{[\beta]_{q^m}}{m}p_m\widetilde{p}_m\Big) = \sum_{R}\frac{C_R}{C'_R}M_R(p_m)M_R(\widetilde{p}_m) \tag{42}$$

Here $M(p_m)$ are the normalized MacDonald polynomials, the hook lengths C_R and C'_R are defined by (77) and the summation goes over all Young diagrams R. Using this, one finally obtains

$$I = \sum_{A,B}\Lambda^{|A|+|B|}\frac{C_A C_B}{C'_A C'_B}M_A(\widetilde{p}_m)M_A\Big(-p_m - \frac{[v_+]_{q^m}}{[\beta]_{q^m}}\Big)M_B(p_m)M_B\Big(-\widetilde{p}_m - \frac{[v_-]_{q^m}}{[\beta]_{q^m}}\Big) \tag{43}$$

Therefore, the $5d$ Dotsenko-Fateev integral takes the form:

$$\boxed{B^{5D}(\Lambda) = \sum_{A,B}\Lambda^{|A|+|B|}\frac{C_A C_B}{C'_A C'_B}\left\langle\,M_A\Big(-p_m - \frac{[v_+]_{q^m}}{[\beta]_{q^m}}\Big)M_B(p_m)\,\right\rangle_+\left\langle\,M_B\Big(-\widetilde{p}_m - \frac{[v_-]_{q^m}}{[\beta]_{q^m}}\Big)M_A(\widetilde{p}_m)\,\right\rangle_-}$$

$$\tag{44}$$

This quantity has no form of (101) and, therefore, does not factorize. On the other hand, it avoids the problem of extra poles emerging in the Nekrasov decomposition, see [25].

3.5 The case of $\beta = 1$

The situation is completely different if $\beta = 1$, when every double average in (44) factorizes and literally reproduces the corresponding Nekrasov function which have no extra pole at $\beta = 1$. In this case, the MacDonald

polynomials are reduced to the usual Schur functions, however, the Selberg averages are still given by the Jackson integrals $M_A(p_k)|_{\beta=1} = \chi_A(p_k)$. In order to calculate

$$B^{5D}(\Lambda)\big|_{\beta=1} = \sum_{A,B} \Lambda^{|A|+|B|} \left\langle \chi_A\left(-p_m - [v_+]_{q^m}\right)\chi_B(p_m) \right\rangle_+ \left\langle \chi_B\left(-\widetilde{p}_m - [v_-]_{q^m}\right)\chi_A(\widetilde{p}_m) \right\rangle_- \quad (45)$$

one uses formula (101) of the Appendix which is reduced in this case to the form

$$\left\langle \chi_A(p_k + [v]_k)\,\chi_B(p_k) \right\rangle = \left(\prod_{(i,j)\in A} q^{i-1} \prod_{(k,s)\in B} q^{k-1+v} \right) \frac{[v+N,A]_q\,[u+v+N,A]_q\,[u+N,B]_q\,[N,B]_q}{G_{AA}^+(0)G_{A'B}^+(2N+u+v)G_{BA'}^-(2N+u+v)G_{BB}^+(0)} \quad (46)$$

where now

$$[x,A]_q = \prod_{(i,j)\in A} [x-i+j]_q, \quad \text{and} \quad G_{AB}^\pm(x) = \prod_{(i,j)\in A} [x \pm \mathrm{Arm}_A(i,j) \pm \mathrm{Leg}_B(i,j) \pm 1]_q$$

Consider the double average appearing in (45):

$$\widetilde{N}_{A,B} = \left\langle \chi_A\left(-p_m - [v_+]_{q^m}\right)\chi_B(p_m) \right\rangle_+ \left\langle \chi_B\left(-\widetilde{p}_m - [v_-]_{q^m}\right)\chi_A(\widetilde{p}_m) \right\rangle_- =$$

$$= (-1)^{|A|+|B|} \left\langle \chi_{A'}\left(p_m + [v_+]_{q^m}\right)\chi_B(p_m) \right\rangle_+ \left\langle \chi_{B'}\left(\widetilde{p}_m + [v_-]_{q^m}\right)\chi_A(\widetilde{p}_m) \right\rangle_- \quad (47)$$

where we used the formula for the characters of negative argument:

$$\chi_A(-p) = (-1)^{|A|}\chi_{A'}(p) \quad (48)$$

The usage of (38) at the point $\epsilon_1 + \epsilon_2$ gives

$$v_+ + N_+ = -\mu_1 - a, \quad v_- + N_- = a - \mu_3 \quad (49)$$

$$u_+ + v_+ + N_+ = -\mu_2 - a, \quad u_- + v_- + N_- = a - \mu_4 \quad (50)$$

$$u_+ + N_+ = \mu_1 - a, \quad u_- + N_- = a + \mu_3 \quad (51)$$

$$2N_+ + u_+ + v_+ = -2a, \quad 2N_- + u_- + v_- = 2a \quad (52)$$

$$N_+ = \mu_2 - a, \quad N_- = \mu_4 + a \quad (53)$$

Thus, (47) takes the form

$$\widetilde{N}_{A,B} = q^{(-\mu_1-\mu_2-2)|B|+(-\mu_3-\mu_4-2)|A|} \prod_{(i,j)\in A} q^{i+j} \prod_{(i,j)\in B} q^{i+j} \times$$

$$\times \frac{\prod_{k=1}^{2}[-\mu_k - a, A']_q[\mu_k - a, B]_q \prod_{k=3}^{4}[\mu_k + a, A]_q[-\mu_k + a, B']_q}{G_{A'A'}^+(0)G_{AB}^+(-2a)G_{BA}^-(-2a)G_{BB}^+(0)G_{B'B'}^+(0)G_{BA}^+(2a)G_{AB}^-(2a)G_{AA}^+(0)} \quad (54)$$

Note that at $\beta = 1$ all the factors here can be expressed through the functions (28) and (29):

$$[x,A] = f_A^+(x), \quad [-x,A'] = f_A^-(x), \quad g_{A,B}(x) = G_{A,B}^+(x)G_{A,B}^-(-x) \quad (55)$$

and one can reduce the expression to the Nekrasov functions (27). Finally, with the use of the following simple identities:

$$\prod_{(i,j)\in A} q^{\mathrm{Leg}_{i,j}(A)} = \prod_{(i,j)\in A} q^{j-1} \quad (56)$$

$$\prod_{(i,j)\in A} q^{\mathrm{Arm}_{i,j}(A)} = \prod_{(i,j)\in A} q^{i-1} \quad (57)$$

$$\prod_{(i,j)\in A} q^{B_i} = \prod_{(i,j)\in B} q^{A_i} \quad (58)$$

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one finds

$$\boxed{\widetilde{N}_{A,B} = N_{A,B}} \tag{59}$$

where $N_{A,B}$ is the Nekrasov function defined by (27) and restricted to $\epsilon = \epsilon_1 + \epsilon_2 = 0$. Therefore, finally we arrive at

$$B^{5D}(\Lambda)\big|_{\beta=1} = \sum_{A,B} N_{A,B}\big|_{\epsilon_1+\epsilon_2=0} \Lambda^{|A|+|B|} = Z_{Nek}^{5D}(\Lambda)\big|_{\epsilon_1+\epsilon_2=0} \tag{60}$$

Acknowledgements

Appendix

MacDonald polynomials

Definition. The MacDonald polynomials is the distinguished basis in the space of symmetric polynomials of $\{x_i\}$. Let us first define the basis

$$p_R = p_{R_1}(x) \dots p_{R_n}(x) = p_1^{m_1}(x) p_2^{m_2}(x) \dots \tag{61}$$

where

$$p_k = \sum_{i=1}^{N} x_i^k \tag{62}$$

with the scalar product

$$\langle p_R | p_{R'} \rangle = \delta_{RR'} \prod_k m_k! k^{m_k} \prod_{i=1}^{n} \frac{1 - q^{R_i}}{1 - t^{R_i}}, \qquad t = q^{\beta} \tag{63}$$

which can be also manifestly realized by

$$\langle f(p_k) | g(p_k) \rangle = f\left(k \frac{1 - q^k}{1 - t^k} \frac{\partial}{\partial p_k} \right) g(p_k) \bigg|_{p_k=0} \tag{64}$$

Introduce the symmetric functions $m_R = \sum_{\sigma} x_1^{R_{\sigma(1)}} x_2^{R_{\sigma(2)}} \dots$ with R_i being the lengths of rows of the Young diagram R and the (partial) ordering of the Young diagrams is defined as $R \geq R'$ iff $|R| = |R'|$ and $\sum_{k=1}^{i} R_k \geq \sum_{k=1}^{i} R'_k$ for all i. Then, the MacDonald polynomials are the polynomials given by the expansion[3]

$$M_R^{q,t}(x_1, ..., x_n) = \sum_{R' < R} c_{RR'} m_{R'} = m_R + \dots \tag{65}$$

with the unit coefficient c_{RR} that satisfy the orthogonality condition

$$\langle M_R^{q,t} | M_{R'}^{q,t} \rangle = 0 \qquad \text{if } R \neq R' \tag{66}$$

Examples. The few first MacDonald polynomials are:

$$M_1 = p_1, \qquad M_2 = \frac{(1-t)(1+q)}{(1-tq)} \frac{p_1^2}{2} + \frac{(1+t)(1-q)}{(1-tq)} \frac{p_2}{2}, \qquad M_{11} = \frac{p_1^2}{2} - \frac{p_2}{2}$$

---

[3]We omit the superscript q, t unless this may lead to a confusion.

$$M_3 = \frac{(1+q)(1-q^3)(1-t)^2}{(1-q)(1-tq)(1-tq^2)}\frac{p_1^3}{6} + \frac{(1-t^2)(1-q^3)}{(1-tq)(1-tq^2)}\frac{p_1 p_2}{2} + \frac{(1-q)(1-q^2)(1-t^3)}{(1-t)(1-tq)(1-tq^2)}\frac{p_3}{3}$$

$$M_{21} = \frac{(1-t)(2qt+q+t+2)}{1-qt^2}\frac{p_1^3}{6} + \frac{(1+t)(t-q)}{1-qt^2}\frac{p_1 p_2}{2} - \frac{(1-q)(1-t^3)}{(1-t)(1-qt^2)}\frac{p_3}{3}, \qquad M_{111} = \frac{p_1^3}{6} - \frac{p_1 p_2}{2} + \frac{p_3}{3}$$

Limiting cases. At the point $t = q$ ($\beta = 1$) the MacDonald polynomials reduces to the Schur polynomials:

$$M(x_i)|_{t=q} = \chi_R(x_i) = \frac{\det_{1\leq i,j\leq N} x_i^{R_j+N-j}}{\Delta(x)} = \det_{ij} S_{R_i-i+j}(p) \tag{67}$$

where $\exp\left(\sum p_k z^k/k\right) = \sum_k S_k(t) z^k$ and the Van-der-Monde determinant $\Delta(x) = \det_{ij} x_i^{N-j} = \prod_{i<j}^N (x_i - x_j)$.

In the intermediate case $q = 1$ the MacDonald polynomials degenerate to the symmetric Jack polynomials which are relevant for the proof of AGT conjecture in $4d$ case :

$$M(x_i)|_{q=1} = J^\beta(x_i) \tag{68}$$

MacDonald polynomials as a set of eigenfunctions. They are also uniquely defined as the common system of eigenfunctions of the commuting set of operators, which are nothing but the Ruijsenaars Hamiltonians [40, 33]:

$$\hat{H}_k = \sum_{i_1<...<i_k} \frac{1}{\Delta(x)} \hat{T}_{i_1}...\hat{T}_{i_k} \Delta(x)\, \hat{Q}_{i_1}...\hat{Q}_{i_k}, \qquad [\hat{H}_k, \hat{H}_m] = 0 \tag{69}$$

where the shift operators are defined as:

$$\hat{T}_k = q^{\beta x_k \partial_{x_k}}, \qquad \hat{Q}_k = q^{(1-\beta)x_k \partial_{x_k}} \tag{70}$$

The spectrum of (69) can be defined from the eigenvalues of spectral operator:

$$\left(\sum_{k=0}^n z^k \hat{H}_k\right) M_R(x_1, ..., x_n) = \prod_{i=1}^\infty (1 + z\, q^{R_i+\beta(n-i)}) M_R(x_1, ..., x_n) \tag{71}$$

Note that at $\beta = 1$, when $\hat{Q}_k = 1$ the spectral operator can be summed exactly:

$$\sum_{k=0}^n z^k \hat{H}_k|_{t=q} = \sum_{k=0}^n z^k \sum_{i_1<...<i_k} \frac{1}{\Delta(x)} \hat{T}_{i_1}...\hat{T}_{i_k} \Delta(x) = \frac{1}{\Delta(x)} \prod_{k=1}^n (1 + z\hat{T}_k)\, \Delta(x) \tag{72}$$

and one obtains

$$\left[\frac{1}{\Delta(x)} \prod_{k=1}^n (1 + z\hat{T}_k)\, \Delta(x)\right] \chi_R(x) = \prod_{i=1}^n (1 + zq^{n-i+R_i}) \chi_R(x) \tag{73}$$

Orthogonality. Besides the scalar product (64), there is another scalar product $<,>^*$ such that the MacDonald polynomials are also orthogonal w.r.t. it, but have other norms. This scalar product is given by the integral with the MacDonald measure:

$$\langle f, g\rangle^* = \oint_{|z_1|=1} \frac{dz_1}{z_1} ... \oint_{|z_N|=1} \frac{dz_N}{z_N} \prod_{m=0}^{\beta-1}\prod_{i\neq j} \left(1 - q^m \frac{z_i}{z_j}\right) f(z_1, ..., z_N) g(z_1^{-1}, ..., z_N^{-1}) \tag{74}$$

and the normalization condition is

$$\langle M_A, M_B\rangle^* = \delta_{A,B} \frac{C'_A}{C_A} \frac{[\beta N, A]}{[\beta N + 1 - \beta, A]} \tag{75}$$

with the q-Pochhammer symbol

$$[x, A]_q = \prod_{(i,j)\in A} [x - i\beta + j + \beta - 1]_q, \tag{76}$$

and

$$C'_A = \prod_{(i,j)\in A} [\beta\mathrm{Arm}_A(i,j) + \mathrm{Leg}_A(i,j) + \beta]_q, \quad C_A = \prod_{(i,j)\in A} [\beta\mathrm{Arm}_A(i,j) + \mathrm{Leg}_A(i,j) + 1]_q \tag{77}$$

Cauchy-Stanley completeness identity. The MacDonald polynomials satisfy the following identity of expansion of the bilinear exponential:

$$\boxed{\exp\Big(\sum_{k=1}^{\infty} \frac{[\beta]_{q^k}}{k} p_k\widetilde{p}_k\Big) = \sum_R \frac{C_R}{C'_R} M_R(p_k)M_R(\widetilde{p}_k)} \tag{78}$$

A few different representations of this identity are known in the literature, all of them can be obtained from (78) by simple algebraic manipulations. For example, with $p_k = \sum_i x_i^k$, $\widetilde{p}_k = \sum_j y_j^k$ the l.h.s. of (78) can be rewritten as follows:

$$\exp\Big(\sum_{k=1}^{\infty} \frac{[\beta]_{q^k}}{k} p_k\widetilde{p}_k\Big) = \exp\Big(\sum_{i,j}\sum_{k=1}^{\infty} \frac{1-t^k}{k(1-q^k)} x_i^k y_j^k\Big) = \prod_{i,j} \frac{\exp\Big(-\mathrm{Li}_2(tx_iy_j|q)\Big)}{\exp\Big(-\mathrm{Li}_2(x_iy_j|q)\Big)} \tag{79}$$

where $\mathrm{Li}_2(x|q)$ is the quantum dilogarithm function:

$$\mathrm{Li}_2(x|q) = \sum_{k=1}^{\infty} \frac{x^k}{k(1-q^k)} \tag{80}$$

Using the identity for the quantum dilogarithm, which relates it with the q-exponential

$$\exp\Big(-\mathrm{Li}_2(x|q)\Big) = \prod_{k=0}^{\infty}(1-q^k x) \overset{\mathrm{def}}{=} (x;q)_\infty = \sum_{n=0}^{\infty} \frac{(-1)^n}{[n]_q!(1-q)^n} q^{n(n-1)/2}x^n \overset{\mathrm{def}}{=} \mathrm{E}_q(-x) \tag{81}$$

one obtains the Cauchy completeness identity in the infinite product form or, equivalently, in the q-exponential form:

$$\boxed{\sum_R \frac{C_R}{C'_R} M_R(p_k)M_R(\widetilde{p}_k) = \prod_{i,j} \frac{(tx_iy_j)_\infty}{(x_iy_j)_\infty} = \prod_{i,j} \frac{\mathrm{E}_q(-tx_iy_j)}{\mathrm{E}_q(-x_iy_j)}} \tag{82}$$

Finally, consider (78) at the point $\widetilde{p}_k = -\widetilde{p}_k/[\beta]_{q^k}$:

$$\exp\Big(-\sum_{k=1}^{\infty} \frac{p_k\widetilde{p}_k}{k}\Big) = \sum_R \frac{C_R}{C'_R} M_R(p_k)M_R(-\widetilde{p}_k/[\beta]_{q^k}) \tag{83}$$

Expressing the l.h.s. of this identity through the eigenvalues (**??**), one obtains

$$\exp\Big(-\sum_{k=1}^{\infty} \frac{p_k\widetilde{p}_k}{k}\Big) = \prod_{i,j}\exp\Big(-\sum_{k=1}^{\infty} \frac{x_i^k y_j^k}{k}\Big) = \prod_{i,j}\exp\Big(\ln(1-x_iy_j)\Big) = \prod_{i,j}(1-x_iy_j) \tag{84}$$

The r.h.s. can be transformed by utilizing the identity for the MacDonald polynomial of negative argument

$$M_R^{q,t}\Big(-\frac{p_k}{[\beta]_{q^k}}\Big) = (-1)^{|R|}\frac{C'_R}{C_R} M_{R'}^{t,q}(p_k) \tag{85}$$

where R' stands for the transposed Young diagram (conjugated representation) and we write the deformation parameters q and t explicitly to emphasize that the MacDonald polynomials at the r.h.s. and l.h.s. of this identity are calculated at interchanged t and q. One can easily check that (85) provides an involution transformation by applying it twice which results into unity. In order to proof, one suffices to note that

$$C'_A(\beta) = \beta^{|A|} C_{A'}\Big(\frac{1}{\beta}\Big), \qquad [\beta]_{q^k}[\beta^{-1}]_{t^k} = 1 \tag{86}$$

Applying this involution transformation to (83) and using (84) one gets

$$\sum_R (-1)^{|R|} M_R^{q,t}(p_k) M_{R'}^{t,q}(\tilde{p}_k) = \prod_{i,j}(1 - x_i y_j) \tag{87}$$

Switching again to the eigenvalues and using that $M_R(-y_j) = (-1)^{|R|} M_R(y_j)$ one finally obtains the standard form of the Cauchy completeness identity:

$$\boxed{\sum_R M_R^{q,t}(x_i) M_{R'}^{t,q}(y_j) = \prod_{i,j}(1 + x_i y_j)} \tag{88}$$

q-deformed β-ensembles

We consider the following average for the polynomial $f(x_1,...,x_N)$:

$$\Big\langle f \Big\rangle = \frac{1}{S} \int_0^1 d_q x_1 \ldots \int_0^1 d_q x_N \prod_{i \neq j} \prod_{k=0}^{\beta-1} (x_i - q^k x_j) \prod_i x_i^u \prod_{k=0}^{v-1}(1 - q^k x_i) \, f(x_1,\ldots,x_N) \tag{89}$$

where the normalization

$$S = \int_0^1 d_q x_1 \ldots d_q x_N \prod_{i \neq j} \prod_{k=0}^{\beta-1}(x_i - q^k x_j) \prod_i x_i^u \prod_{k=0}^{v-1}(1 - q^k x_i) \tag{90}$$

provides $\Big\langle 1 \Big\rangle = 1$. Here we use the notion of Jackson integral:

$$\int_0^a f(x) d_q x = (1-q) a \sum_{k=0}^{\infty} q^k f(q^k a), \quad \text{in particular} \quad \int_0^1 f(x) d_q x = (1-q) \sum_{k=0}^{\infty} q^k f(q^k) \tag{91}$$

The Jackson integrals of polynomials are equal to

$$\int_0^1 x^n d_q x = \frac{1}{[n+1]_q}, \quad [n]_q = \frac{1-q^n}{1-q} = 1 + q + \ldots + q^{n-1}$$

The average (89) is the obvious q-deformation of the Selberg β-ensemble considered in our previous paper [25]:

$$\Big\langle f \Big\rangle^{\text{Selb}} = \frac{\int_0^1 dx_1 \ldots \int_0^1 dx_N \prod_{i<j}(x_i - x_j)^{2\beta} \prod_i x_i^u (x_i - 1)^v \, f(x_1,\ldots,x_N)}{\int_0^1 dx_1 \ldots \int_0^1 dx_N \prod_{i<j}(x_i - x_j)^2 \prod_i x_i^u (x_i - 1)^v} \tag{92}$$

For the sake of simplicity, we keep in (89) the parameters β and v integer, extension to non-integer values of the parameters being straightforward. For instance, the MacDonald measure in (89)[4]

$$\Delta^{MC}(x_i) = \prod_{i \neq j} \prod_{m=0}^{\beta-1}(x_i - q^m x_j) \tag{93}$$

can be rewritten in the form:

$$\Delta^{MC}(x_i) = \prod_{i \neq j} \prod_{m=0}^{\infty} \left(\frac{x_i - q^m x_j}{x_i - t q^m x_j} \right) = \prod_{i \neq j} \exp\left(-\sum_{k=1}^{\infty} \frac{1}{k} \frac{1-t^k}{1-q^k} \left(\frac{x_j}{x_i} \right)^k \right), \quad t = q^\beta \tag{94}$$

where β can take non-integer values. Analogously, at non-integer v

$$\prod_{k=0}^{v-1}(1 - q^k x) \longrightarrow \exp\left(-\sum_{m=1}^{\infty} \frac{1}{m} \frac{1-q^{vm}}{1-q^m} x^m \right) \tag{95}$$

---

[4]Note that (69) involves the ordinary Van-der-Monde determinant, not (93).

1-MacDonald average

The average of the single MacDonald polynomial in the q-deformed β-ensemble, which generalizes the celebrated Kadell formula [41], has the form

$$\left\langle M_A(p) \right\rangle = q^{W_A(v,\beta)} \frac{[N\beta, A]_q [u + N\beta + 1 - \beta, A]_q}{d_q(A)[u + v + 2N\beta + 2 - 2\beta, A]_q} \tag{96}$$

where

$$q^{W_Y(v,\beta)} = \prod_{(i,j)\in Y} q^{v+(i-1)\beta} = q^{|Y|v} \prod_{i=1}^{h(A)} q^{(i-1)\beta Y_i} \tag{97}$$

and

$$d_q(Y) = \prod_{(i,j)\in Y} [\beta + (Y_i - j) + \beta(Y'_j - i)]_q \tag{98}$$

In the case of Jack polynomials this latter quantity could be presented as a particular value of the polynomial:

$$J_A\left(p_k = \delta_{k,1}\right) = \frac{\beta^{|A|}}{\prod\limits_{(i,j)\in Y} \left(\beta + (Y_i - j) + \beta(Y'_j - i)\right)} \tag{99}$$

which led to formula (74) in [25] (there was a misprint in [25]):

$$\left\langle J_A(p_k) \right\rangle^{\mathrm{Selb}} = J_A(\delta_{k,1}) \frac{[N\beta, A][u + N\beta + 1 - \beta, A]}{\beta^{|A|}[u + v + 2N\beta + 2 - 2\beta, A]} \tag{100}$$

However, in the q-deformed case there is no such a simple relation:

$$M_A(\delta_{k,1}) \neq \frac{\beta^{|A|}}{d_q(A)}$$

2-MacDonald average

We have found the following formula for the Selberg average of product of two non-normalized MacDonald polynomials:

$$\left\langle M_A(p_k + w_k) M_B(p_k) \right\rangle = q^{W_{A,B}(v,\beta)} \frac{[v + N\beta + 1 - \beta, A]_q [u + N\beta + 1 - \beta, B]_q}{[N\beta, A]_q [u + v + N\beta + 2 - 2\beta, B]_q} \times$$

$$\times \frac{\prod\limits_{i,j=1}^{N} P_\beta\left(u + v + 2\beta N + 2 - \beta(1 + i + j)\right)}{\left(\prod\limits_{1\leq i<j\leq N} P_\beta(\beta j - \beta i)\right)^2} \frac{\prod\limits_{1\leq i<j\leq N} P_\beta\left(A_i - A_j + \beta(j - i)\right) \prod\limits_{1\leq i<j\leq N} P_\beta\left(B_i - B_j + \beta(j - i)\right)}{\prod\limits_{i,j=1}^{N} P_\beta\left(u + v + 2\beta N + 2 + A_i + B_j - \beta(1 + i + j)\right)}$$

Note that this expression explicitly depends on N, the number of parameters x_i in (89), and we use the rule $A_i = 0$ if i exceeds the number of rows in A. Note that in our normalization $\langle 1 \rangle = 1$ for the empty Young diagrams $M_{[]}(p_k) = 1$:

$$\left\langle M_{[]}(p_k + w_k) M_{[]}(p_k) \right\rangle = \langle 1 \rangle = 1 \tag{101}$$

In formula (101)

$$P_\beta(x) = \frac{\Gamma_q(x + \beta)}{\Gamma_q(x)} = \prod_{k=0}^{\beta-1} [x + k]_q \tag{102}$$

the latter identity being correct in the case of integer β. At last,

$$w_k = -q^{vk}\frac{[\beta - v - 1]_{q^k}}{[\beta]_{q^k}} = -q^{vk}\frac{[(\beta - v - 1)k]_q}{[\beta k]_q} \tag{103}$$

since

$$[n]_{q^k} = \frac{[nk]_q}{[k]_q}$$

Note that the main feature of (101), its complete factorization into q-number factors, happens only at these specific values of w_k.

Example

We now illustrate the use of these formulas in the simplest example of the average $< p_1 + w_1 >$. It can be considered as $< M_1(p) > +w_1$ and evaluated with the help of (96), or as $< M_1(p + w)M_0(p) >$ and evaluated with the help of (101).

In the first case one has:

$$< p_1 + w_1 > \stackrel{(96)}{=} w_1 + q^v\frac{[N\beta]_q}{[\beta]_q}\frac{[u + N\beta + 1 - \beta]_q}{[u + v + 2N\beta + 2 - 2\beta]_q} = \tag{104}$$

$$= \begin{cases} w_1 + q^{-\mu_1 - \mu_2}\frac{[\mu_2 - a]_q[\mu_1 - a]_q}{[\beta]_q[-2a + 1 - \beta]_q} & \text{for} \quad < \ldots >_+ \quad \text{in (38)} \\[3mm] w_1 + q^{-\mu_3 - \mu_4}\frac{[\mu_4 + a]_q[\mu_3 + a]_q}{[\beta]_q[-2a + 1 - \beta]_q} & \text{for} \quad < \ldots >_- \quad \text{in (38)} \end{cases} \tag{105}$$

These expressions are nicely decomposed into a product of two "linear" factors for $w_1 = 0$ and also for

$$w_1 = \begin{cases} -q^{-\mu_1 - \mu_2}\frac{[\beta - 1 + \mu_1 + \mu_2]_q}{[\beta]_q} \\[3mm] -q^{-\mu_3 - \mu_4}\frac{[\beta - 1 + \mu_3 + \mu_4]_q}{[\beta]_q} \end{cases} \tag{106}$$

This distinguished value of w_1 is especially easy to find for $q = 1$: the discriminant of quadratic polynomial $(a - \mu_1)(a - \mu_2) + \beta w_1(-2a + 1 - \beta)$ is the full square:

$$D = (\mu_1 + \mu_2 + 2\beta w_1)^2 - 4\Big(\mu_1\mu_2 + \beta w_1(1 - \beta)\Big) = (\mu_1 - \mu_2)^2 + 4\beta w_1\Big(\beta w_1 + \mu_1 + \mu_2 - (1 - \beta)\Big) =$$

$$= (\mu_1 - \mu_2)^2 \quad \text{for} \quad w_1 = 0 \quad \text{or} \quad w_1 = \frac{-\mu_1 - \mu_2 + 1 - \beta}{\beta} \tag{107}$$

When q is switched on, one has:

$$\frac{1}{[\beta]_q}\Big(q^{-\mu_1 - \mu_2}\frac{[\mu_1 - a]_q[\mu_2 - a]_q}{[-2a + 1 - \beta]_q} - q^{-\mu_1 - \mu_2}[\beta - 1 + \mu_1 + \mu_2]_q\Big) = q^{\beta - 1}\frac{[1 - \beta - \mu_1 - a]_q[1 - \beta - \mu_2 - a]_q}{[\beta]_q[1 - \beta - 2a]_q} \tag{108}$$

Note that the main role of the w-shift is to change the relative sign between a and μ in the numerator, like in (16). However, the value of this shift, which is important for factorization property, is here different from the value of the shift in (44), needed to reproduce the conformal block: the shifts are the same only for $\beta = 1$.

In the second representation of the same average one uses formula (101) with $A = [1]$ and $B = []$. In this case the products of P_β-factors get non-trivial contributions only from $i = 1$:

$$\Big\langle p_1 + w_1 \Big\rangle = q^{\beta - 1}\frac{[v + N\beta + 1 - \beta]_q}{[N\beta]_q}\prod_{j=2}^{N}\frac{P_\beta\big(1 + \beta(j - 1)\big)}{P_\beta\big(\beta(j - 1)\big)}\prod_{j=1}^{N}\frac{P_\beta(u + v + 2\beta N + 2 - \beta(2 + j))}{P_\beta(u + v + 2\beta N + 3 - \beta(2 + j))} \tag{109}$$

Using the property of $P_\beta(x)$:

$$\frac{P_\beta(x + 1)}{P_\beta(x)} = \frac{[x + \beta]_q}{[x]_q}$$

which is obvious from its definition (102), we find

$$\Big\langle p_1 + w_1 \Big\rangle = q^{\beta - 1}\frac{[v + N\beta + 1 - \beta]_q}{[N\beta]_q}\prod_{j=2}^{N}\frac{[\beta j]_q}{[\beta(j - 1)]_q}\prod_{j=1}^{N}\frac{[u + v + 2\beta N + 2 - \beta(j - 2)]_q}{[u + v + 2\beta N + 2 - \beta(j - 1)]_q} =$$

$$= q^{\beta-1} \frac{[N\beta]_q}{[\beta]_q} \frac{[u+v+\beta N+2-2\beta]_q}{[u+v+2\beta N+2-2\beta]_q} \frac{[v+N\beta+1-\beta]_q}{[N\beta]_q} = q^{\beta-1} \frac{[u+v+\beta N+2-2\beta]_q[v+N\beta+1-\beta]_q}{[\beta]_q[u+v+2\beta N+2-2\beta]_q} =$$

$$= q^{\beta-1} \frac{[1-\beta-\mu_2-a]_q[1-\beta-\mu_1-a]_q}{[\beta]_q[1-\beta-2a]_q} \tag{110}$$

where at the last stage we substituted parameters (38) for the $< \ldots >_+$ average. The result is exactly the same as (108).

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