Equality of bulk wave functions and edge correlations in topological superconductors: A spacetime derivation.

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For certain systems, the N-particle ground-state wavefunctions of the bulk happen to be exactly equal to the N-point space-time correlation functions at the edge, in the infrared limit. We show why this had to be so for a class of topological superconductors, beginning with the p+ip state in D=2+1. Varying the chemical potential as a function of Euclidean *time* between weak and strong pairing states is shown to extract the wavefunction. Then a Euclidean rotation that exchanges time and space and approximate Lorentz invariance lead to the edge connection. We illustrate straightforward extension to other dimensions (eg. ³He- B phase in D=3+1) and to correlated states like fractionalized topological superconductors.

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The boundaries or edges of condensed matter systems received scant attention until recent developments showed them to be fertile areas of research both in the Fractional Quantum Hall Effect (FQHE)[1,2]. and in topological insulators and superconductors[3–9].

In two spatial dimensions, the edge dynamics is described by conformal field theory[2] which was also used to produce wave functions in the bulk[11,12]. Moore and Read[11] showed that one may view the FQHE wavefunctions and the quasi-hole excitations as conformal blocks in which both electrons and the quasiparticle coordinates are treated on the same footing and their charges and braiding properties are severely constrained. For an exhaustive review of many related topics see Nayak *et al*[13].

What are the minimal ingredients necessary to establish equality of edge correlations and bulk wavefunctions? Are analytic functions or d=2 conformal invariance required? We show that our edge-bulk equality follows for a class of topological superconductors in various dimensions invoking *only* approximate Lorentz symmetry. The connections obtained here using an effective low energy hamiltonian differ from CS theory[10] in which the hamiltonian vanishes and only non-dynamical particles enter via Wilson loops, as reviewed in Ref.13.

We shall first write down an operator expression for $Z(J)$, the generating function of N-body wavefunctions of the bulk. This is shown to be accomplished by introducing a time dependent chemical potential that changes abruptly at some Euclidean time. We then drop some high derivative terms which do not matter in the infrared, and express $Z(J)$ as a Grassmann integral over a Lorentz invariant action. Rotating by 90 degrees to exchange time and a spatial direction we obtain the *same* topological superconductor but with a spatial edge induced by the jump in chemical potential. We find that the same $Z(J)$ has now morphed into the generating function for the correlation functions of the edge excitations. Three examples are given: the $p + ip$ superconductor in $D = 2 + 1$, ³ He B phase in $D = 3 + 1$ and a p-wave superconductor (the Ising model) in $D = 1 + 1$.

Extracting Wavefunctions: Recall that given a second-quantized N-body state $|\Phi\rangle$ with wavefunction $\phi(x_1, x_2, ..x_N)$ we extract ϕ using

$$\phi(x_1, x_2, ..x_N) = \langle \emptyset | \Psi(x_1)...\Psi(x_N)|\Phi\rangle. \tag{1}$$

where $\langle \emptyset |$ is the Fock vacuum and Ψ is the canonical electron destruction operator. For problems with variable number of particles, let us define the generating function

$$Z(J) = \langle \emptyset | e^{\int dx J(x)\Psi(x)}|\Phi\rangle \tag{2}$$

which yields N-body wavefunctions upon differentiating N- times with respect to the Grassmann source $J(x)$.

We want to express $Z(J)$ as a path integral when $|\Phi\rangle$ is the ground state of a Hamiltonian H without conserved particle number. Since Euclidean time evolution for long times projects to the ground state, we can obtain $|\Phi\rangle$ as

$$|\Phi\rangle = U(0^-, -\infty)|i\rangle \tag{3}$$

where $|i\rangle$ is a generic initial state and $U(0^-, -\infty)$ is the imaginary time propagator from $-\infty$ to 0^-. Then we insert the operator $\exp\left[\int J(x)\Psi(x)dx\right]$ at time 0. Finally, we obtain the Fock vacuum by evolving a generic state $\langle f|$ from time $+\infty$ to 0^+ using a hamiltonian H' with a huge negative μ that empties out fermions so that we may write $\langle \emptyset | = \langle f|U(\infty, 0^+)$. Thus

$$Z(J) = \langle f|U(\infty, 0^+)e^{\int J(x)\Psi(x)dx}U(0^-, -\infty)|i\rangle \tag{4}$$

which has a path integral representation.

Example 1: p+ip : The mean-field hamiltonian is[15,16]:

$$H = \sum_k (c_k^\dagger, c_{-k}) \begin{pmatrix} \alpha k^2 - \mu & \Delta \cdot (k_1 - ik_2) \\ \Delta^* \cdot (k_1 + ik_2) & -(\alpha k^2 - \mu) \end{pmatrix} \begin{pmatrix} c_k \\ c_{-k}^\dagger \end{pmatrix} \tag{5}$$

here 1, 2 are spatial indices and x_3 will be time. We employ the minimum k dependence in the pairing function, and set the coefficient $\Delta = 1$ for convenience so the gap function is: $\Delta(k_1, k_2) = k_1 - ik_2$

The αk^2 term is needed to ensure the nontrivial topology of the weak-coupling phase[15] and to populate it with electrons for $\mu > 0$. We shall remember this association but drop the 'k^2' term in the computations since it does not affect infrared correlations.
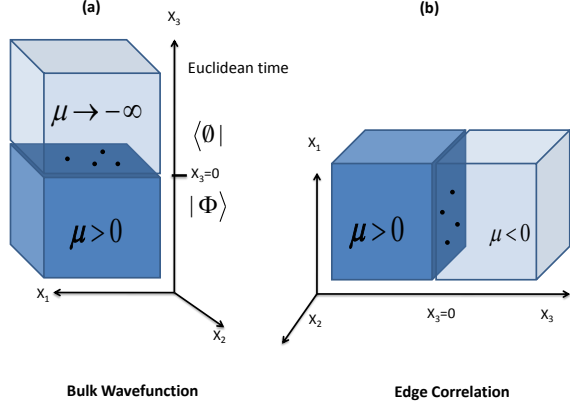
FIG. 1: (a) Wavefunction: The original superconductor with $\mu = \mu_- > 0$ lies in the $x_1 - x_2$ plane and evolves in Euclidean time x_3 from $-\infty$ to 0^-, projecting out the ground state $|\Phi\rangle$. At $x_3 = 0^+$ the chemical potential drops abruptly to a large negative value μ^-, leading to the Fock vacuum. (b) Correlation functions: A Lorentz rotation makes x_1 the new time and x_3 a the spatial coordinate along which the system has an edge at $x_3 = 0$. The world-sheet of the edge lies in the $x_1 - x_2$ plane at $x_3 = 0$.

Now the mean field Hamiltonian in real space:

$$H = \int d^2x \left[\Psi^\dagger(-\mu)\Psi + \frac{1}{2}(\Psi^\dagger(-i\partial_1 - \partial_2)\Psi^\dagger + h.c) \right].$$
(6)

leads to corresponding Grassmann action for $U(0, -\infty)$:

$$S = \int_{-\infty}^{\infty} d^2x \int_{-\infty}^{0} dx_3 \left[\bar\psi \mathcal{D}\psi + \bar\psi i \partial \bar\psi + \psi i \bar\partial \psi \right]$$
(7)

$$\mathcal{D} = (-\partial_3 + \mu) \qquad \partial = \frac{\partial}{\partial z} \qquad \bar\partial = \frac{\partial}{\partial \bar z}$$
(8)

For the $0^+ < x_3 < \infty$, we choose $\mu = \mu_+$, a very large negative number, associated with the Fock vacuum and obtain, for all x_3, the action including the source J:

$$S(J) = \int_{-\infty}^{\infty} d^3x \left[\bar\psi \mathcal{D}\psi + \bar\psi i \partial \bar\psi + \psi i \bar\partial \psi + J\psi\delta(x_3) \right]$$
(9)

where $\mathcal{D}$ now contains a time-dependent $\mu(x_3)$ that jumps at $x_3 = 0$ from $\mu_- > 0$ to $\mu_+ \to -\infty$.

The generating function of the BCS wavefunctions is

$$Z(J) = \frac{\int [d\bar\psi d\psi] e^{S(J)}}{\int [d\bar\psi d\psi] e^{S(0)}}$$
(10)

The story is depicted in the left half of Figure 1: the fermions travel unsuspectingly along in Euclidean time x_3 and slam like bugs onto the windshield at $x_3 = 0^-$ when $\delta(x_3)J\psi$ kills them.

Since ψ and $\bar\psi$ in Eq. 9 are independent Grassmann variables, we integrate out $\bar\psi$ to obtain the effective action for just ψ to which alone J couples:

$$S_{eff}(\psi, J) = \int d^3x \left(\psi i \bar\partial \psi + J\psi + \psi \frac{1}{4i\partial} \mathcal{D}^T \mathcal{D}\psi \right)$$

$$\equiv S_0(J) + S_{ind}.$$
(11)

For the infrared limit we keep just the Jackiw-Rebbi zero mode[17] of the hermitian operator

$$\mathcal{D}^T \mathcal{D}(x_3) = (\partial_3 + \mu(x_3))(-\partial_3 + \mu(x_3)),$$
(12)

that obeys $\mathcal{D}f_0 = 0$

$$f_0(x_3) = f_0(0)e^{\int_0^{x_3} \mu(x')dx'}$$
(13)

in the mode expansion of the Grassmann field:

$$\psi(x_1, x_2, x_3) = f_0(x_3)\psi(x_1, x_2).$$
(14)

This kills S_{ind}, and upon integrating f_0^2 over x_3,

$$S_{eff}(J) = \int dx_1 dx_2 \, \psi(i\bar\partial + Jf_0(0))\psi$$
(15)

While this is indeed the action of a chiral majorana fermion living in the $1 - 2$ plane we are not done: we need to show that this fermion and this action also arise at the edge of the *same* $p + ip$ system. But so far we have no edge! It will be introduced shortly, but first a summary of results on the wavefunction.

Pfaffian Wavefunction: Integrating over ψ in Eq. 15, and suppressing the constant $f_0^2(0)$) we find

$$Z(J) = \exp\left[\left[\int d^2\mathbf{r} J(\mathbf{r}) \left[\frac{1}{4i\bar\partial} \right]_{\mathbf{rr'}} J(\mathbf{r'}) \right] \right]$$
(16)

The two-particle wavefunction $\phi(\mathbf{r}_1 - \mathbf{r}_2)$ can be written in terms of many related quantities:

$$\phi = \frac{\partial^2 Z(J)}{\partial J_1 \partial J_2} = \left[\frac{1}{2i\bar\partial} \right]_{\mathbf{r}_1\mathbf{r}_2} = \Delta_{\mathbf{r}_1\mathbf{r}_2}^{*-1} = \frac{1}{z_1 - z_2}$$
(17)

and the N-particle wavefunction is $\text{Pf}(\frac{1}{z_i - z_j})$. In the Supplementary Material we relate $Z(J)$ and the conventional BCS wavefunction:

$$|BCS\rangle = \exp\left(\frac{1}{2} \int \Psi^\dagger(x)g(x - y)\Psi^\dagger(y)dxdy \right) |\emptyset\rangle$$
(18)

and see that $\phi = -g(\mathbf{r}_1 - \mathbf{r}_2)$.

The Edge: To relate $Z(J)$ in Eqn. 9 to a problem with the edge we rewrite $S(J)$ in Lorentz invariant form:

$$S(J) = \int d^3x \left[\bar\Psi \left(\bar\partial\!\!\!/ - \mu \right) \Psi + J^T\Psi \right] \quad \text{where}$$
(19)

$$\Psi = \begin{pmatrix} \psi \\ \bar\psi \end{pmatrix} \qquad \bar\Psi = \Psi^T\varepsilon; \quad \varepsilon = i\sigma_2 \quad \partial\!\!\!/ = \gamma_\mu \partial_\mu$$
(20)

$$\gamma_1 = \sigma_2 \quad \gamma_2 = -\sigma_1 \quad \gamma_3 = \sigma_3$$
(21)

$$J^T = J\delta(x_3)(1\,0).$$
(22)

Look at the left half of Figure 1. We see our current description of the superconductor: translationally invariant in the $x_1 - x_2$ plane, regarded as the space in which the $p_1 + ip_2$ superconductor lives, and with a jump in μ at "time" $x_3 = 0$. In this description, the functional integral is saturated by one mode $f_0(x_3)$, glued to the interface, exactly like the electron gas at a heterojunction.

Extracting $H(x_1, x_2)$ from the Lorentz invariant action is like taking the row-to row transfer matrix. To derive the hamiltonian that governs the column-to -column dynamics, we rotate the three dimensional spacetime by $-\frac{\pi}{2}$ around the x_2 axis to obtain the view shown in the right half of Figure 1. The points carry the same labels as before but the spinor undergoes a rotation:

$$\Psi = \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix} = e^{i\frac{\pi}{4}i\gamma_3\gamma_1}\begin{pmatrix} \psi' \\ \bar{\psi}' \end{pmatrix} = e^{i\frac{\pi}{4}\sigma_1}\Psi' \quad (23)$$

Upon performing this transformation we end up with

$$S(\Psi', J) = \int d^3x \left[\bar{\Psi}' \left[\sigma_3\partial_1 - \sigma_1\partial_2 - \sigma_2\partial_3 - \mu \right] \Psi' \right.$$
$$\left. + J\delta(x_3)(\frac{\psi' + i\bar{\psi}'}{\sqrt{2}}) \right] \quad (24)$$

which describes exactly the same $p + ip$ superconductor but in the $2-3$ plane (with $1 \to 3, 3 \to -1$) with an edge at $x_3 = 0$. An $\alpha(k_2^2 + k_3^2)$ term may now be added without affecting infrared edge correlations. This is required to complete our identification of regions with (and without) fermions with μ positive (negative).

To see that the field $\frac{\psi' + i\bar{\psi}'}{\sqrt{2}}$ that J couples to is precisely the Majorana field that arises at the edge, consider solving the equation for the zero mode which follows from Eq. 24 on dropping all x_1, x_2 dependence:

$$(\sigma_2\partial_3 + \mu(x_3))\chi_0' = 0 \Rightarrow \chi_0'(x_3) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix} f_0(x_3). \quad (25)$$

the normalizable spinor solution indeed corresponds to the operator $\frac{1}{\sqrt{2}}(\psi' + i\psi'^\dagger)$.

We are done, for we have shown that $Z(J)$ is at once the generators of electronic wavefunction in the bulk and of correlation functions of the Majorana field at the edge.

For completeness, the edge Majorana field action follows from saturating the x_3 dependence of Ψ' as follows:

$$\Psi'(x_1, x_2, x_3) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix} f_0(x_3)\psi'(x_1, x_2) \quad (26)$$

Plugging this into the action $S(\Psi', J)$ one finds, upon integrating the normalized function $f_0'(x_3)$ over x_3

$$S(\Psi', J) \to \int dx_1 dx_2 \left[\psi'i\bar{\partial}\psi' + Jf_0(0)\psi' \right] \quad (27)$$

exactly as in Eqn.15, for the wavefunction.

Example 2: $^3He - B$ *in D=3+1:* In a simplified model of superfluid $^3He - B$, Cooper pairs have spin 1, whose projection lies perpendicular to the momenta $\pm\mathbf{k}$[19,20]. The winding of this axis around the Fermi surface in the weak pairing phase leads to its topological properties[20,21]. The mean-field Hamiltonian for this time-reversal invariant class DIII system[19,20] is:

$$H = \sum_{\mathbf{p}\sigma\sigma'} \Psi_{\mathbf{p}\sigma}^\dagger(\frac{\mathbf{k}^2}{2m} - \mu)\Psi_{\mathbf{k}\sigma} \quad (28)$$
$$+ \{\Delta_{\mathbf{k}\sigma\sigma'}\psi_{\mathbf{k}\sigma}\psi_{-\mathbf{k}\sigma'} + \text{h.c.}\}$$
$$\Delta_{\mathbf{k}\sigma\sigma'} = [\varepsilon\mathbf{k}\cdot\boldsymbol{\sigma}]_{\sigma\sigma'}$$

The $d = 3$ problem is just the $d = 2$ problem on steroids: Δ goes from being a complex number to a quaternion, and the spinless fermion is replaced by a two-component spinor. Hence the weak-pairing wavefunction is $g_{\sigma_i\sigma_j}(\mathbf{r}_{ij}) \sim \frac{[\mathbf{r}_{ij}\cdot\boldsymbol{\sigma}\varepsilon]_{\sigma_i\sigma_j}}{r_{ij}^3}$ and the many-body wavefunction is the corresponding $\text{Pf}(g)$ as noted in Ref. 8 .

The Lorentz invariant action for the wavefunction is

$$S = \int d^4x \frac{1}{2}\bar{\Psi}[\slashed{\partial} - \mu]\Psi \quad \text{where} \quad (29)$$

$$\gamma_0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad \boldsymbol{\gamma} = \begin{pmatrix} 0 & i\boldsymbol{\sigma}\varepsilon \\ i\varepsilon\boldsymbol{\sigma} & 0 \end{pmatrix} \quad (30)$$

$$\bar{\Psi} = \Psi^T\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \quad (31)$$

Now the 0 and 1 directions are exchanged by $R = \exp\left[\frac{i\pi}{2}\frac{i\gamma_0\gamma_1}{2}\right]$, so that J now couples to $\frac{\psi' + i\sigma_3\psi'^\dagger}{\sqrt{2}}$ which is readily verified, as before, to be the gapless edge mode of the rotated theory. The action for the edge theory obtained by saturating with the zero mode is

$$S_{edge} = \int d^3x \frac{1}{2}\bar{\psi}\slashed{\partial}\psi \quad \slashed{\partial} = \sigma_j\partial_j \quad \bar{\psi} = \psi^T(-\sigma_2) \quad (32)$$

Example 3: We could equally well go *down* a dimension, to a spinless p-wave superconductor in $d = 1 + 1$[14] where $\Delta = k_x$, which is also related to the quantum Ising model, via the Jordan-Wigner mapping. The edge theory is $0 + 1$ dimensional, corresponding to a Majorana zero mode, with Lagrangian $\mathcal{L} = \frac{1}{2}\psi\partial_x\psi$.

Fractionalized Topological Superconductors: We construct a fractionalized superconducting phase in D=2+1 that bears the same relation to the $p + ip$ superconductor as the Laughlin $m = 3$ quantum Hall state bears to the integer Hall effect. Consider splitting the electron operator at each site into three fermions ('partons') $c_r = if_{1r}f_{2r}f_{3r}$ and $c_r^\dagger = if_{1r}^\dagger f_{2r}^\dagger f_{3r}^\dagger$[18,23] with the following $p + ip$ mean field action for the partons:

$$S(J) = \int d^3x \left[\mathcal{L}_0 + iJf_1f_2f_3\right] \quad (33)$$

$$\mathcal{L}_0 = \frac{1}{2}\sum_{a=1}^3 \left(\bar{f}_a \ f_a\right)\begin{pmatrix} -\partial_3 + \mu & i\partial_1 + \partial_2 \\ i\partial_1 - \partial_2 & -\partial_3 - \mu \end{pmatrix}\begin{pmatrix} f_a \\ \bar{f}_a \end{pmatrix}$$

When the gauge theory is in a deconfined phase, the partons accurately describe the low energy dynamics. The SO(3) symmetry of the action, a remnant of the SU(3) gauge redundancy implied by $c_r = if_{1r}f_{2r}f_{3r}$[22], is the gauge symmetry here. When the gapped bulk is integrated out, it generates an SO(3)$_1$ (or equivalently SU(2)$_2$)[24] Chern-Simons term which renders the gauge field massive thereby liberating the partons with the action in Eq. 33.

Emptying out the electrons requires removing the f fermions, hence the strong pairing phase of the fs, where their chemical potential is taken to be large and negative, corresponds to the Fock vacuum. The electron correlators involve products of three parton correlators each in

a p+ip state, so the electronic wavefunction is:

$$\Psi(z_1, z_2, \ldots, z_{2N}) = \left\{ \mathrm{Pf} \left[\frac{1}{z_i - z_j} \right] \right\}^3 \qquad (34)$$

Equivalently we can start from the edge where the three Majorana modes are massless by gauge symmetry and have no relevant short range interactions in three spacetime dimensions. Long range gauge interactions do not exist due to the Chern-Simons term. Consequently the bulk effective action must also be described by three noninteracting fermions.

Summary: We have explained why the electronic wavefunctions in the bulk coincided with the massless Majorana correlation functions at the edge in certain problems. We first wrote $Z(J) = \langle \emptyset | e^{J\Psi} | BCS \rangle$ as a path integral in which the chemical potential abruptly jumped at in Euclidean time. Dropping the 'k^2' terms which determined boundary conditions on μ, we obtained a Lorentz invariant action. Upon rotation by $\pi/2$ the same action described a system that had an edge and $Z(J)$ had meanwhile morphed into the generating function for edge correlations. In general, rotating axes will relate bulk wavefunctions to the edge correlations of a different (possibly unnatural) problem. The examples considered here are self-dual in this respect.

Our analysis holds in many dimensions and applies to fractionalized cases as well, as long as varying μ can change the topology. This is possible in the Altland-Zirnbauer classification[8] for models in class D in d=1 and d=2 (like p+ip), in class C in d=2 (like d+id) and class DIII in d=2,3 (He-3 B phase) but not for classes like CI in d=3,[27] which additionally rely on band topology of the weak pairing Fermi surface. We are currently modifying our derivation for Laughlin quantum Hall states, where μ couples to a conserved charge.

The entanglement spectrum of the bulk seems to determine the edge theory[25,26], which we now relate back to the bulk wavefunction. Since the entanglement of a gapped phase appears from near the cut, *the entire bulk wavefunction must be coded holographically in every $d-1$ dimensional sliver probed in the entanglement analysis.*

Previously, the connection between edge states and bulk wavefunctions has played an important role identifying new FQH states[11,28]. Our work suggests a similar approach could be fruitful in identifying interacting topological phases in D=3+1.

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I. SUPPLEMENTARY MATERIAL

Suppose we are given a Majorana hamiltonian

$$H = \frac{1}{2} \sum_{ij} \Psi_i h_{ij} \Psi_j \qquad (35)$$

wherein

$$\{\Psi_j, \Psi_j\} = \delta_{ij}, \qquad (36)$$

where i, j subsume all labels, spatial and internal. If the labels are continuous, the Dirac δ should be used and derivatives ∂ viewed as antisymmetric matrices.

By definition the Grassmann integrals are

$$\int \psi d\psi = 1 \qquad \int 1 \cdot d\,\psi = 0 \qquad (37)$$

The Euclidean path integral corresponding to h is

$$Z = \int [d\psi] e^{\frac{1}{2} \int dt \sum_{ij} \psi_i(t)(-\partial_t \delta_{ij} - h_{ij}) \psi_j(t)} \qquad (38)$$

Since coherent states do not exist for Majorana operators (which square to $\frac{1}{2}$ and not 0), one way to derive this result is to first form Dirac operators from pairs of Majorana operators, use fermion coherent states for the former to obtain a path integral and then undo the transformation back to Majorana fields.

The Gaussian integral that is repeatedly used is

$$Z(J) = \int e^{\frac{1}{2} \chi A \chi + J \chi} [d\chi] = Pf(A) e^{\frac{1}{2} J A^{-1} J} \qquad (39)$$

where J and χ are $2N$-component Grassmann vectors.

The two-point correlator is

$$\langle \psi_a \psi_b \rangle = \frac{\partial^2 Z(J)}{\partial J_a \partial J_b} \bigg|_{J=0} = A_{ba}^{-1} = -A_{ab}^{-1} \qquad (40)$$

The minus sign in the last term can be avoided if the exponent is written as $e^{-\frac{1}{2} \chi A \chi}$.

Higher correlators are given by Pfaffians.

A. Pfaffian wavefunctions

Let us put these ideas to work in deriving the many-body wave functions from the second quantized BCS state.

Consider the generating function of wavefunctions for any number of particles from which the wavefunctions can be obtained by differentiating with respect to the Grassmann source $J(x)$

$$Z(J) =$$
$$\langle \emptyset | e^{\int dx J(x) \Psi(x)} | BCS \rangle$$
$$= \langle \emptyset | e^{\int dx J(x) \Psi(x)} \cdot I \cdot e^{\frac{1}{2} \int \Psi^\dagger(x) g(x-y) \Psi^\dagger(y) dx dy} | \emptyset \rangle$$
$$\equiv \int [d\bar{\psi} d\psi] \, e^{-\bar{\psi}\psi} \langle \emptyset | e^{J(x)\Psi} | \psi \rangle \langle \bar{\psi} | e^{\frac{1}{2} \Psi^\dagger g \Psi^\dagger} | \emptyset \rangle \qquad (41)$$

where, in the last step we have resorted to a compact notation and inserted the following resolution of the identity in terms of Grassmann coherent states:

$$I = \int |\psi\rangle \langle \bar{\psi} | e^{-\bar{\psi}\psi} [d\bar{\psi} d\psi] \qquad (42)$$

and where it is understood for example that

$$|\psi\rangle = \prod_x |\psi(x)\rangle \qquad [d\bar{\psi} d\psi] = \prod_x [d\bar{\psi}(x) d\psi(x)] \qquad (43)$$

It is important to remember that $\bar{\psi}$ and ψ are independent and dummy variables. Using the defining property of coherent states

$$\Psi|\psi\rangle = \psi|\psi\rangle \qquad \langle \bar{\psi}|\Psi^\dagger = \langle \bar{\psi}|\bar{\psi} \qquad (44)$$

in Eq. 41 we find

$$Z(J) = \int [d\bar{\psi} d\psi] \, e^{-\bar{\psi}\psi} e^{J\psi} e^{\frac{1}{2}\bar{\psi} g \bar{\psi}} \qquad (45)$$

where we have used the fact that

$$\langle \emptyset | \psi \rangle \langle \bar{\psi} | \emptyset \rangle = 1 \qquad (46)$$

since at each site

$$|\psi\rangle = |0\rangle - \psi|1\rangle \qquad \langle \bar{\psi}| = \langle 0| - \langle 1|\bar{\psi} \qquad (47)$$

and $|\emptyset\rangle = |0\rangle \otimes |0\rangle \otimes ....|0\rangle$. Doing the integrals over ψ and $\bar{\psi}$, we find

$$Z(J) = e^{\frac{1}{2} J g J}. \qquad (48)$$

The pair wavefunction is

$$\phi(x_1, x_2) = \frac{\partial^2 Z}{\partial J(x_1) \partial J(x_2)} \bigg|_{J=0} = -g(x_1 - x_2) \qquad (49)$$

Higher correlations follow from Wick's theorem. For example

$$\begin{aligned} \phi(x_1, x_2, x_3, x_4) &= g(x_1 - x_2)g(x_3 - x_4) \\ &- g(x_1 - x_3)g(x_2 - x_4) \\ &+ g(x_1 - x_4)g(x_2 - x_3). \end{aligned} \qquad (50)$$