

The generating function of amplitudes with N twisted and M untwisted states

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ABSTRACT: We show that the generating function of all amplitudes with N twisted and M untwisted states, i.e. the Reggeon vertex for magnetized branes on \mathbb{R}^2 can be computed once the correlator of N non excited twisted states and the corresponding Green function are known and we give an explicit expression as a functional of the these objects.

KEYWORDS: D-branes, Conformal Field Theory.

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1. Introduction and conclusions

In the late 80s a lot of work was done in computing the generating functions of all amplitudes for the bosonic string and superstring. Many methods were (further) developed such as the sewing method ([1]), the group theoretic method ([2]) and conserved charges method ([3]). Following the main idea of ([4],[5]) in this paper we would like to compute the generating function for N generic excited twisted states and M generic untwisted states on \mathbb{R}^2 for the open string in presence of magnetic fields in the upper half plane using the path integral approach. Much work has been already done in computing non excited twisted states correlation functions, especially on T^2 (see for example [6], [7], [8] and [9]) but not so much on the computation of correlators involving excited twisted fields ([15], [16] see for earlier work) which remain quite mysterious.

In this paper we want to show that there is a quite simple way of labeling excited twisted states which is deeply connected with the operator-state map and that few ingredients are actually needed for computing all correlators involving excited twisted state and arbitrary untwisted ones on \mathbb{R}^2 . To obtain any correlator is only necessary the knowledge of the full (i.e. classical and quantum) N non excited twist correlator on the disk¹

$$C(x_1, \dots, x_N) = \langle \sigma_{\epsilon_1}(x_1, \bar{x}_1) \dots \sigma_{\epsilon_N}(x_N, \bar{x}_N) \rangle_{disk, full} \quad x_t \in \mathbb{R} \quad (1.1)$$

¹The twist fields in this and the following correlators are actually $\sigma_{\epsilon, \kappa=0}(x, \bar{x})$, see the in the main text.

and the boundary Green function in presence of such operators

$$G_{bou}^{ij}(x; y; \{x_t\}_{t=1\dots N}) = G_{bou}^{ji}(y; x; \{x_t\}_{t=1\dots N}) = G^{ij}(x, \bar{x}; y, \bar{y}; \{x_t\}_{t=1\dots N}) \quad (1.2)$$

which can be derived from

$$G^{ij}(z, \bar{z}; w, \bar{w}; \{x_t\}_{t=1\dots N}) = \frac{\langle X^i(z, \bar{z}) X^j(w, \bar{w}) \sigma_{\epsilon_1}(x_1, \bar{x}_1) \dots \sigma_{\epsilon_N}(x_N, \bar{x}_N) \rangle_{disk}}{\langle \sigma_{\epsilon_1}(x_1, \bar{x}_1) \dots \sigma_{\epsilon_N}(x_N, \bar{x}_N) \rangle_{disk}}. \quad (1.3)$$

by setting $z = x, w = y \in \mathbb{R}$.

The main result of the paper is the generating function for the above mentioned amplitudes given in eq.s (1.29) and (1.30). These two expressions have exactly the same content but the latter is written in a more usual way, i.e. using auxiliary expansion variables while the former has an expression like those used in the previous literature ([1]). Let us now explain the building blocks of this last version of the main formula (1.29).

- To any (excited) twisted operator inserted at x_t ($t = 1 \dots N$) in the amplitude we associate an auxiliary Hilbert space \mathcal{H}_t . On \mathcal{H}_t act the quantum fields $X_{(t)}^i(z, \bar{z})^2$ ($i = 1, 2$ or $i = z, \bar{z}$)

$$\begin{aligned} Z_{(t)}(z, \bar{z}) &= X_{(t)}^z(z, \bar{z}) = \frac{X_{(t)}^1 + iX_{(t)}^2}{\sqrt{2}} = \frac{1}{2} (Z_{L(t)}(z) + Z_{R(t)}(\bar{z})), \\ \bar{Z}_{(t)}(z, \bar{z}) &= X_{(t)}^{\bar{z}}(z, \bar{z}) = \frac{X_{(t)}^1 - iX_{(t)}^2}{\sqrt{2}} = \frac{1}{2} (\bar{Z}_{(t)L}(z) + \bar{Z}_{(t)R}(\bar{z})) \end{aligned} \quad (1.4)$$

which have expansions

$$\begin{aligned} Z_{(t)L}(z) &= z_{(t)0} + i\sqrt{2\alpha'} e^{-i\gamma_t} \sum_{n=0}^{\infty} \left[\frac{\bar{\alpha}_{(t)n+1-\epsilon_t}}{n+1-\epsilon_t} z^{-(n+1-\epsilon_t)} - \frac{\alpha_{(t)n+\epsilon_t}^\dagger}{n+\epsilon_t} z^{+(n+\epsilon_t)} \right] \\ Z_{(t)R}(\bar{z}) &= z_{(t)0} + i\sqrt{2\alpha'} e^{+i\gamma_t} \sum_{n=0}^{\infty} \left[\frac{\bar{\alpha}_{(t)n+1-\epsilon_t}}{n+1-\epsilon_t} \bar{z}^{-(n+1-\epsilon_t)} - \frac{\alpha_{(t)n+\epsilon_t}^\dagger}{n+\epsilon_t} \bar{z}^{+(n+\epsilon_t)} \right] \end{aligned} \quad (1.5)$$

and

$$\begin{aligned} \bar{Z}_{(t)L}(z) &= \bar{z}_{(t)0} + i\sqrt{2\alpha'} e^{+i\gamma_t} \sum_{n=0}^{\infty} \left[-\frac{\bar{\alpha}_{(t)n+1-\epsilon_t}^\dagger}{n+1-\epsilon_t} z^{+(n+1-\epsilon_t)} + \frac{\alpha_{(t)n+\epsilon_t}}{n+\epsilon_t} z^{-(n+\epsilon_t)} \right] \\ \bar{Z}_{(t)R}(\bar{z}) &= \bar{z}_{(t)0} + i\sqrt{2\alpha'} e^{-i\gamma_t} \sum_{n=0}^{\infty} \left[-\frac{\bar{\alpha}_{(t)n+1-\epsilon_t}^\dagger}{n+1-\epsilon_t} \bar{z}^{+(n+1-\epsilon_t)} + \frac{\alpha_{(t)n+\epsilon_t}}{n+\epsilon_t} \bar{z}^{-(n+\epsilon_t)} \right] \end{aligned} \quad (1.6)$$

The previous fields satisfy the boundary conditions³

$$e^{i\gamma_t} \partial Z_{(t)L}|_x = e^{-i\gamma_t} \bar{\partial} Z_{(t)R}|_x \quad x \in \mathbb{R}^+ \quad (1.7)$$

$$e^{+i\gamma_t-1} \partial Z_{(t)L}|_y = e^{-i\gamma_t-1} \bar{\partial} Z_{(t)R}|_y \quad y = |y| e^{i\pi} \in \mathbb{R}^- \quad (1.8)$$

²In the following quantum fields have attached the label of the Hilbert space they act on, e.g. $X_{(t)}^i(z, \bar{z})$ while classical fields in path integral have no label, i.e. $X^i(z, \bar{z})$.

³These can also be written as $e^{i\gamma_t} \partial Z_{(t)L}(x) = \frac{1}{\cos \gamma_t} \partial Z_{(t)}(x, x)$ when $x > 0$ and $e^{i\gamma_t-1} \partial Z_{(t)L}(y) = \frac{1}{\cos \gamma_{t-1}} \partial Z_{(t)}(y, \bar{y})$ when $y < 0$. These expressions are those used to connect the open string operators when naturally expressed as function of $X(x, \bar{x})$ to their expressions as functional of $X_L(x)$.

where we have defined the phases $(-\frac{\pi}{2} < \gamma_t < \frac{\pi}{2})$

$$\begin{aligned} e^{i\gamma_t} &= \frac{1 + iB_t}{\sqrt{1 + B_t^2}} \rightarrow B_t = \tan \gamma_t = 2\pi\alpha' q_{(0)} F_{12(0)} \\ e^{i\gamma_{t-1}} &= \frac{1 + iB_{t-1}}{\sqrt{1 + B_{t-1}^2}} \rightarrow B_{t-1} = \tan \gamma_{t-1} = 2\pi\alpha' q_{(\pi)} F_{12(\pi)} \end{aligned} \quad (1.9)$$

where $B_{t-1} = 2\pi\alpha' q_{(\pi)} F_{12(\pi)}$ and $B_t = 2\pi\alpha' q_{(0)} F_{12(0)}$ are the adimensional magnetic fields which are on the $x < 0$ ($\sigma = \pi$) and $x > 0$ ($\sigma = 0$) boundaries. In the field expansion the shift ϵ_t is given by

$$\epsilon_t = \begin{cases} \frac{1}{\pi}(\gamma_t - \gamma_{t-1}) & \gamma_t > \gamma_{t-1} \\ 1 + \frac{1}{\pi}(\gamma_t - \gamma_{t-1}) & \gamma_t < \gamma_{t-1} \end{cases} \quad 0 \leq \epsilon_t < 1 \quad (1.10)$$

The previous operators act on the \mathcal{H}_t twisted ground state defined by

$$\bar{\alpha}_{(t)n+1-\epsilon_t} |T_t\rangle = \alpha_{(t)m+\epsilon_t} |T_t\rangle = x_{(t)0}^2 |T_t\rangle = 0 \quad (1.11)$$

and have the non vanishing commutation relations ⁴

$$\begin{aligned} [\bar{\alpha}_{(t)n+1-\epsilon_t}, \bar{\alpha}_{(t)m+1-\epsilon_t}^\dagger] &= (n+1-\epsilon_t)\delta_{n,m} \quad n, m \geq 0 \\ [\alpha_{(t)n+\epsilon_t}, \alpha_{(t)m+\epsilon_t}^\dagger] &= (n+\epsilon_t)\delta_{n,m} \quad n, m \geq 0 \\ [z_{(t)0}, \bar{z}_{(t)0}] &= \frac{2\pi\alpha'}{B_t - B_{t-1}} \end{aligned} \quad (1.12)$$

Notice that the choice of the definition of the zero modes vacuum is somewhat arbitrary since they do not change the energy, our choice is dictated by our gauge choice for the background magnetic field $A = B x^1 dx^2$ which implies the translational invariance $X^2 \rightarrow X^2 + \epsilon$ and by the observation that is almost the proper choice in toroidal compactifications. The existence of the zero modes imply that the vacuum is degenerate since all the states $|T_t, \kappa_t\rangle = e^{i\kappa_t x_{(t)0}^1} |T_t\rangle$ have exactly the same energy of the vacuum and therefore there exists a one parameter family of twist fields ([12]) $\sigma_{\epsilon_t, \kappa_t}(x, \bar{x})$.

⁴Since the annihilator and creator operators have flat indexes this holds independently of our choice of the taking the metric diagonal; in particular from definition of the complex fields we have $(dX^1)^2 + (dX^2)^2 = 2dZd\bar{Z}$, i.e. $G_{z\bar{z}} = 1$.

Given the previous vacuum definition we have the following twisted Green functions

$$\begin{aligned}
G_{T(t)}^{zz}(z, \bar{z}; w, \bar{w}) &= [Z^{(+)}(z, \bar{z}), Z^{(-)}(w, \bar{w})]|_{an.cont} = \frac{\pi\alpha'}{B_t - B_{t-1}} \\
G_{T(t)}^{\bar{z}\bar{z}}(z, \bar{z}; w, \bar{w}) &= [\bar{Z}^{(+)}(z, \bar{z}), \bar{Z}^{(-)}(w, \bar{w})]|_{an.cont} = -\frac{\pi\alpha'}{B_t - B_{t-1}} \\
G_{T(t)}^{z\bar{z}}(z, \bar{z}; w, \bar{w}) &= [Z^{(+)}(z, \bar{z}), \bar{Z}^{(-)}(w, \bar{w})]|_{an.cont} \\
&= \frac{\pi\alpha'}{B_t - B_{t-1}} \\
&\quad - \frac{\alpha'}{2} \left[g_{\epsilon_t} \left(\frac{w}{z} \right) + g_{\epsilon_t} \left(\frac{\bar{w}}{\bar{z}} \right) + e^{-2i\gamma_t} g_{\epsilon_t} \left(\frac{\bar{w}}{z} \right) + e^{2i\gamma_t} g_{\epsilon_t} \left(\frac{w}{\bar{z}} \right) \right] \\
G_{T(t)}^{\bar{z}z}(z, \bar{z}; w, \bar{w}) &= [\bar{Z}^{(+)}(z, \bar{z}), Z^{(-)}(w, \bar{w})]|_{an.cont} \\
&= -\frac{\pi\alpha'}{B_t - B_{t-1}} \\
&\quad - \frac{\alpha'}{2} \left[g_{1-\epsilon_t} \left(\frac{w}{z} \right) + g_{1-\epsilon_t} \left(\frac{\bar{w}}{\bar{z}} \right) + e^{2i\gamma_t} g_{1-\epsilon_t} \left(\frac{\bar{w}}{z} \right) + e^{-2i\gamma_t} g_{1-\epsilon_t} \left(\frac{w}{\bar{z}} \right) \right]
\end{aligned} \tag{1.13}$$

which can be obtained by analytically continuing their operatorial expression from $|z| > |w|$ to the whole upper plane in such a way to preserve the symmetry $G^{ij}(z, \bar{z}; w, \bar{w}) = G^{ij}(w, \bar{w}; z, \bar{z})$. In the previous expressions we have defined $g_\nu(z)$ as the analytic continuation of

$$g_{\nu,s}(z) = - \sum_{n-\nu>0} \frac{1}{n-\nu} z^{n-\nu} \quad |z| < 1, \quad -\pi + 2\pi s < \phi = \arg(z) \leq \pi + 2\pi s. \tag{1.14}$$

in the properly chosen sheet s . Notice that the symmetry of the Green function $G^{ij}(z, \bar{z}; w, \bar{w}) = G^{ji}(w, \bar{w}; z, \bar{z})$ is not obvious in the zero modes sector, i.e. for the constant terms but it holds due to the g transformation property ([12])

$$g_{\nu,s}(z) = C_{\nu,s}(\phi) + g_{1-\nu,-s} \left(\frac{1}{z} \right), \quad C_{\nu,s}(\phi) = \begin{cases} \frac{\pi e^{-i\pi\nu}}{\sin\pi\nu} e^{-i2\pi\nu s} & 2\pi s < \phi < \pi + 2\pi s \\ \frac{\pi e^{+i\pi\nu}}{\sin\pi\nu} e^{-i2\pi\nu s} & -\pi + 2\pi s < \phi < 2\pi s \end{cases} \tag{1.15}$$

This fact implies that we cannot really completely separate the zero modes and non zero modes also for the twisted sector as it already happens for the untwisted one.

For $x > 0 > y$ and $|y/x| < 1$ the previous Green functions become on the boundary⁵

$$\begin{aligned}
G_{T(t) \text{ bou}}^{zz}(x; y) &= \frac{\pi\alpha'}{B_t - B_{t-1}} \\
G_{T(t) \text{ bou}}^{\bar{z}\bar{z}}(x; y) &= -\frac{\pi\alpha'}{B_t - B_{t-1}} \\
G_{T(t) \text{ bou}}^{z\bar{z}}(x; y) &= \frac{\pi\alpha'}{B_t - B_{t-1}} - 2\alpha' \cos \gamma_t \cos \gamma_{t-1} e^{i\gamma_t - i\gamma_{t-1}} g_{\epsilon_t} \left(\frac{y}{x} \right) \\
G_{T(t) \text{ bou}}^{\bar{z}z}(x; y) &= -\frac{\pi\alpha'}{B_t - B_{t-1}} - 2\alpha' \cos \gamma_t \cos \gamma_{t-1} e^{-i\gamma_t + i\gamma_{t-1}} g_{1-\epsilon_t} \left(\frac{y}{x} \right) \quad (1.16)
\end{aligned}$$

The other cases can be obtained with the substitution rule $x > 0 \cos \gamma_t e^{i\gamma_t} \leftrightarrow x < 0 \cos \gamma_{t-1} e^{-i\gamma_{t-1}}$ and the same for y in the $G^{z\bar{z}}$ propagator. For the $G^{\bar{z}z}$ propagator one takes the complex conjugate of the previous substitution rule.

- In a similar way to any untwisted operator we insert in the amplitude we associate an auxiliary Hilbert space \mathcal{H}_{a,t_a} . This Hilbert space as well as the position where the untwisted vertex is inserted x_{a,t_a} are better labeled by both a counting label $a = 1 \dots M$ and a further label $t_a \in \{1, \dots, N\}$ which specify which is the magnetic field felt by the untwisted state (dipole string). This could seem irrelevant but it is important in defining the regularized Green functions (1.27) and in computing the non commutative phases. In the following we will use a lighter notation as $x_{a,t_a} \rightarrow x_a$ when there is not possibility of confusion.

On the auxiliary Hilbert space \mathcal{H}_{a,t_a} act the quantum fields

$$Z_{(a,t_a)}(z, \bar{z}) = \frac{1}{\sqrt{2}} \left(X_{(a,t_a)}^1(z, \bar{z}) + iX_{(a,t_a)}^2(z, \bar{z}) \right) = \frac{1}{2} (Z_{(a,t_a)L}(z) + Z_{(a,t_a)R}(\bar{z})) \quad (1.17)$$

which have expansions

$$\begin{aligned}
Z_{(a,t_a)L} &= e^{-i\gamma_{t_a}} \left(z_{(a,t_a)0} - 2\alpha' \bar{p}_{(a,t_a)} i \ln(z) + i\sqrt{2\alpha'} \sum_{n=1}^{\infty} + \frac{\bar{\alpha}_{(a,t_a)n}}{n} z^{-n} - \frac{\alpha_{(a,t_a)n}^\dagger}{n} z^n \right) \\
Z_{(a,t_a)R} &= e^{+i\gamma_{t_a}} \left(z_{(a,t_a)0} - 2\alpha' \bar{p}_{(a,t_a)} i \ln(\bar{z}) + i\sqrt{2\alpha'} \sum_{n=1}^{\infty} + \frac{\bar{\alpha}_{(a,t_a)n}}{n} \bar{z}^{-n} - \frac{\alpha_{(a,t_a)n}^\dagger}{n} \bar{z}^n \right) \quad (1.18)
\end{aligned}$$

⁵When $|y/x| > 1$ we must be more careful since we want to evaluate the g on a cut; for example when $0 < x, y$ the expression which is valid for all ranges is $G_{T(t) \text{ bou}}^{z\bar{z}}(x; y) = \frac{\pi\alpha'}{B_t - B_{t-1}} - \alpha' \cos \gamma_t [e^{-i\gamma_t} g_{\epsilon_t} (\frac{y}{x} e^{-i0}) + e^{i\gamma_t} g_{\epsilon_t} (\frac{y}{x} e^{+i0})]$. In any case we can always use the symmetry property for the Green functions to reduce the computation in the range where we can apply the given expressions.

and

$$\begin{aligned}\bar{Z}_{(a,t_a)L} &= e^{+i\gamma_{t_a}} \left(\bar{z}_{(a,t_a)0} - 2\alpha' p_{(a,t_a)} i \ln(z) + i\sqrt{2\alpha'} \sum_{n \neq 0} -\frac{\bar{\alpha}_{(a,t_a)n}^\dagger}{n} z^n + \frac{\alpha_{(a,t_a)n}}{n} z^{-n} \right) \\ \bar{Z}_{(a,t_a)R} &= e^{-i\gamma_{t_a}} \left(\bar{z}_{(a,t_a)0} - 2\alpha' p_{(a,t_a)} i \ln(\bar{z}) + i\sqrt{2\alpha'} \sum_{n \neq 0} -\frac{\bar{\alpha}_{(a,t_a)n}^\dagger}{n} \bar{z}^n + \frac{\alpha_{(a,t_a)n}}{n} \bar{z}^{-n} \right)\end{aligned}\tag{1.19}$$

The previous quantum fields satisfy the boundary conditions

$$\begin{aligned}e^{+i\gamma_{t_a}} \partial Z_{(a,t_a)}|_x &= e^{-i\gamma_{t_a}} \bar{\partial} \bar{Z}_{(a,t_a)}|_x \quad x \in \mathbb{R}^+ \\ e^{+i\gamma_{t_a}} \partial Z_{(a,t_a)}|_y &= e^{-i\gamma_{t_a}} \bar{\partial} \bar{Z}_{(a,t_a)}|_y \quad y = |y|e^{i\pi} \in \mathbb{R}^-\end{aligned}$$

where we have defined the angle γ_{t_a} , in a similar way for the twisted scalar (dicharged string), as

$$e^{i\gamma_{t_a}} = \frac{1 + iB_{t_a}}{\sqrt{1 + B_{t_a}^2}} \Rightarrow B_{t_a} = \tan \gamma_{t_a}, \quad -\frac{\pi}{2} < \gamma_{t_a} < \frac{\pi}{2}\tag{1.20}$$

The creation and destruction operators act on the dipole ground state defined by

$$\bar{\alpha}_{(a,t_a)n} |0_{(a,t_a)}\rangle = \alpha_{(a,t_a)n} |0_{(a,t_a)}\rangle = \bar{p}_{(a,t_a)} |0_{(a,t_a)}\rangle = p_{(a,t_a)} |0_{(a,t_a)}\rangle = 0\tag{1.21}$$

and have non trivial commutation relations

$$\begin{aligned}[z_{(a,t_a)0}, \bar{z}_{(a,t_a)0}] &= 2\pi\alpha' B_{t_a} \\ [z_{(a,t_a)0}, p_{(a,t_a)}] &= i \\ [\alpha_{(a,t_a)n}, \alpha_{(a,t_a)m}^\dagger] &= n\delta_{m,n} \\ [\bar{\alpha}_{(a,t_a)n}, \bar{\alpha}_{(a,t_a)m}^\dagger] &= n\delta_{m,n}\end{aligned}\tag{1.22}$$

The normal ordering is the usual one but it worth noticing that in the zero modes sector is defined as

$$: e^{i(\bar{k}Z_{(a)zm} + k\bar{Z}_{(a)zm})(x, \bar{x})} := \begin{cases} e^{i \cos \gamma_t (\bar{k}z_{(a)0} + k\bar{z}_{(a)0})} e^{2\alpha' \ln(|x|) \cos \gamma_t (\bar{k}\bar{p}_{(a)0} + kp_{(a)0})} & x > 0 \\ e^{i \cos \gamma_t (\bar{k}\hat{z}_{(a)0} + k\hat{\bar{z}}_{(a)0})} e^{2\alpha' \ln(|x|) \cos \gamma_t (\bar{k}\bar{p}_{(a)0} + kp_{(a)0})} & x < 0 \end{cases}\tag{1.23}$$

with $\hat{z}_{(a)0} = z_{(a)0} - i2\pi\alpha' \tan \gamma_t \bar{p}$ which have the property that their commutation relations are the opposite of the $z_{(a)0}$ ones. Finally the untwisted Green functions in a magnetic background B_{t_a} are given by ($0 < \arg(z - \bar{w}) < \pi$)

$$\begin{aligned}G_{U(t_a)}^{zz}(z, \bar{z}, w, \bar{w}) &= G_{U(t_a)}^{\bar{z}\bar{z}}(z, \bar{z}, w, \bar{w}) = 0 \\ G_{U(t_a)}^{\bar{z}\bar{z}}(z, \bar{z}, w, \bar{w}) &= [Z^{(+)}(z, \bar{z}), \bar{Z}^{(-)}(w, \bar{w})]|_{an.cont} \\ &= +\frac{1}{2}\pi\alpha' \sin(2\gamma_{t_a}) - \alpha' [\ln|z - w| + \cos(2\gamma_{t_a}) \ln|z - \bar{w}| + \sin(2\gamma_{t_a}) \arg(z - \bar{w})] \\ G_{U(t_a)}^{zz}(z, \bar{z}, w, \bar{w}) &= [\bar{Z}^{(+)}(z, \bar{z}), Z^{(-)}(w, \bar{w})]|_{an.cont} \\ &= -\frac{1}{2}\pi\alpha' \sin(2\gamma_{t_a}) - \alpha' [\ln|z - w| + \cos(2\gamma_{t_a}) \ln|z - \bar{w}| - \sin(2\gamma_{t_a}) \arg(z - \bar{w})]\end{aligned}\tag{1.24}$$

The constant terms can be obtained by rewriting

$$z_{(a)0} = z_{(a)00} + i\pi\alpha' \tan \gamma_t \bar{p} \quad (1.25)$$

so that $[z_{(a)00}, \bar{z}_{(a)00}] = 0$ and considering the additional term proportional to \bar{p} coming from this rewriting as belonging to $Z^{(+)}(z, \bar{z})$. Notice however once again that the constant terms are needed to ensure the symmetry $G^{ij}(z, \bar{z}; w, \bar{w}) = G^{ji}(w, \bar{w}; z, \bar{z})$.

The previous Green functions become on the boundary $z = x, w = y \in \mathbb{R}$ ⁶

$$\begin{aligned} G_{U(t_a), bou}^{z\bar{z}}(x; y) &= \frac{1}{2}\pi\alpha' \sin(2\gamma_{t_a}) - 2\alpha' \left[\cos^2(\gamma_{t_a}) \ln|x-y| + \frac{1}{2} \sin(2\gamma_{t_a}) \arg(x-\bar{y}) \right] \\ G_{U(t_a), bou}^{\bar{z}z}(x; y) &= -\frac{1}{2}\pi\alpha' \sin(2\gamma_{t_a}) - 2\alpha' \left[\cos^2(\gamma_{t_a}) \ln|x-\bar{y}| - \frac{1}{2} \sin(2\gamma_{t_a}) \arg(x-\bar{y}) \right] \end{aligned} \quad (1.26)$$

From these expressions we can read the open string metric $\mathcal{G}_{(t)}^{z\bar{z}} = \cos^2(\gamma_t)$ and the non commutativity parameter $\Theta_{(t)}^{z\bar{z}} = \frac{1}{2} \sin(2\gamma_t)$, we can also read the \mathbb{R}^2 vielbein $\mathcal{V}_{(t)}^{\bar{z}z} = \mathcal{V}_{(t)}^{zz} = \frac{1}{\cos(\gamma_t)}$ where $\underline{z}, \bar{\underline{z}}$ are the flat indexes which are also implicit in the creation and destruction operators.

- We define the boundary Green function regularized by the untwisted Green function for a background B_{t_a} as

$$G_{bou, reg U(t_a)}^{ij}(x; y; \{x_v\}) = G_{bou}^{ij}(x; y; \{x_v\}) - G_{U(t_a), bou}^{ij}(x; y) \quad x, y \in \mathbb{R} \quad (1.27)$$

where $G_{U(t_a), bou}^{ij}(x, y)$ are defined in eq.s (1.26). The choice of the background B_t in the regularization would seem arbitrary but it is not since these regularized Green functions (and their derivatives) enter only where an untwisted dipole state is emitted and this is on a well defined interval of the boundary.

We also define the analogous twisted boundary Green function regularized by the twisted Green function at the twist insertion point t as⁷

$$G_{bou, reg T(t)}^{ij}(x, y; \{x_v\}) = G_{bou}^{ij}(x, y; \{x_v\}) - G_{T(t), bou}^{ij}(x, y; \{x_0 = x_t, x_\infty = \infty\}) \quad (1.28)$$

where $G_{t, bou}^{ij}$ are given in eq.s (1.16) with the substitution $\frac{y}{x} \rightarrow \frac{y-x_t}{x-x_t}$.

⁶When using these Green functions in eq. (3.14) in absence of twist fields we recover the results from the operatorial formalism.

⁷The symmetrization is because we have a symmetric function in $x \leftrightarrow y$, i.e. independent on the way we take the limit $x > y$ or $y > x$.

Given the previous building blocks the main formula is given by ($x_t \neq x_a \quad \forall t, a$)

$$\begin{aligned}
& \langle V_{N+M}(\{x_t\}_{t=1,\dots,N}; \{x_{a,t_a}\}_{a=1,\dots,M}) \rangle = C(x_1, \dots, x_N) \\
& \prod_{a=1}^M \langle 0_{(a)a}, z_{(a)00} = \bar{z}_{(a)00} = 0 \mid \prod_{t=1}^N \langle T_{(t)}, x_{(t)}^1 = 0 \mid \delta(i \sum_a (\alpha_{(a)0} - \bar{\alpha}_{(a)0}) + i \sum_t (z_{(t)0} - \bar{z}_{(t)0})) \rangle \\
& \prod_a \exp \left\{ -\frac{1}{4\alpha'} \alpha_{(a)0}^2 \mathcal{V}_{(t_a)}^2 \bar{z}\bar{z} G_{bou, reg}^{z\bar{z}} U_{(t_a)}(x; y; \{x_v\}) - \frac{1}{4\alpha'} \bar{\alpha}_{(a)0}^2 \mathcal{V}_{(t_a)}^2 \bar{z}\bar{z} G_{bou, reg}^{z\bar{z}} U_{(t_a)}(x; y; \{x_v\}) \right. \\
& \quad \left. - \frac{1}{2\alpha'} \sum_{n,m=0}^{\infty} \alpha_{(a)n} \bar{\alpha}_{(a)m} \mathcal{V}_{(t_a)} \bar{z}\bar{z} \mathcal{V}_{(t_a)} \bar{z}\bar{z} \frac{\partial_x^n}{n!} \frac{\partial_y^m}{m!} G_{bou, reg}^{z\bar{z}} U_{(t_a)}(x; y; \{x_v\}) \right\} \Big|_{x=y=x_a} \\
& \prod_t \exp \left\{ \frac{1}{2} \left(\frac{\tan \gamma_t - \tan \gamma_{t-1}}{2\pi\alpha'} x_{(t)0} \right)_{i=2}^2 G_{bou, reg}^{22} T_{(t)}(x; y; \{x_v\}) \right. \\
& \quad \left. - \frac{1}{2\alpha'} \sum_{n,m=1}^{\infty} \frac{\bar{\alpha}_{(t)n}}{n - \epsilon_t} \frac{\alpha_{(t)m}}{m - 1 + \epsilon_t} \mathcal{V}_{(t)} \bar{z}\bar{z} \mathcal{V}_{(t)} \bar{z}\bar{z} \frac{\partial_x^{n-1}}{(n-1)!} \frac{\partial_y^{m-1}}{(m-1)!} \left[(x - x_t)^{1-\epsilon_t} (y - x_t)^{\epsilon_t} \partial_x \partial_y G_{bou, reg}^{z\bar{z}} T_{(t)}(x; y; \{x_v\}) \right] \right\} \Big|_{x=y=x_t} \\
& \prod_{a < b} \exp \left\{ -\frac{1}{2\alpha'} \alpha_{(a)0} \alpha_{(b)0} \mathcal{V}_{(t_a)} \bar{z}\bar{z} \mathcal{V}_{(t_b)} \bar{z}\bar{z} G_{bou}^{z\bar{z}}(x; y; \{x_v\}) - \frac{1}{2\alpha'} \bar{\alpha}_{(a)0} \bar{\alpha}_{(b)0} \mathcal{V}_{(t_a)} \bar{z}\bar{z} \mathcal{V}_{(t_b)} \bar{z}\bar{z} G_{bou}^{\bar{z}\bar{z}}(x; y; \{x_v\}) \right. \\
& \quad \left. - \frac{1}{2\alpha'} \sum_{n,m=0}^{\infty} \alpha_{(a)n} \bar{\alpha}_{(b)m} \mathcal{V}_{(t_a)} \bar{z}\bar{z} \mathcal{V}_{(t_b)} \bar{z}\bar{z} \frac{\partial_x^n}{n!} \frac{\partial_y^m}{m!} G_{bou}^{z\bar{z}}(x; y; \{x_v\}) \right\} \\
& \quad \left. - \frac{1}{2\alpha'} \sum_{n,m=0}^{\infty} \bar{\alpha}_{(a)n} \alpha_{(b)m} \mathcal{V}_{(t_b)} \bar{z}\bar{z} \mathcal{V}_{(t_a)} \bar{z}\bar{z} \frac{\partial_x^n}{n!} \frac{\partial_y^m}{m!} G_{bou}^{\bar{z}\bar{z}}(x; y; \{x_v\}) \right\} \Big|_{x=x_a, y=x_b} \\
& \prod_{t < u} \exp \left\{ \left(\frac{\tan \gamma_t - \tan \gamma_{t-1}}{2\pi\alpha'} x_{(t)0} \right)_{i=2} \left(\frac{\tan \gamma_u - \tan \gamma_{u-1}}{2\pi\alpha'} x_{(u)0} \right)_{i=2} G_{bou}^{22}(x; y; \{x_v\}) \right. \\
& \quad \left. - \frac{1}{2\alpha'} \sum_{n,m=1}^{\infty} \frac{\bar{\alpha}_{(t)n}}{n - \epsilon_t} \frac{\alpha_{(u)m}}{m - 1 + \epsilon_u} \mathcal{V}_{(t)} \bar{z}\bar{z} \mathcal{V}_{(u)} \bar{z}\bar{z} \frac{\partial_x^{n-1}}{(n-1)!} \frac{\partial_y^{m-1}}{(m-1)!} \left[(x - x_t)^{\epsilon_t} (y - x_u)^{1-\epsilon_u} \partial_x \partial_y G_{bou}^{z\bar{z}}(x; y; \{x_v\}) \right] \right. \\
& \quad \left. - \frac{1}{2\alpha'} \sum_{n,m=1}^{\infty} \frac{\alpha_{(t)n}}{n - 1 + \epsilon_t} \frac{\bar{\alpha}_{(u)m}}{m - \epsilon_u} \mathcal{V}_{(t)} \bar{z}\bar{z} \mathcal{V}_{(u)} \bar{z}\bar{z} \frac{\partial_x^{n-1}}{(n-1)!} \frac{\partial_y^{m-1}}{(m-1)!} \left[(x - x_t)^{1-\epsilon_t} (y - x_u)^{\epsilon_u} \partial_x \partial_y G_{bou}^{z\bar{z}}(x; y; \{x_v\}) \right] \right\} \Big|_{x=x_t, y=x_u}
\end{aligned}$$

$$\begin{aligned}
& \prod_{t,a} \exp \left\{ -\frac{1}{2\alpha'} \left(\frac{\tan \gamma_t - \tan \gamma_{t-1}}{\pi} \frac{x(t)0 \ i=2}{\sqrt{2\alpha'}} \right) \sum_{m=0}^{\infty} \alpha_{(a)m} \mathcal{V}_{(t_a) \ \bar{z}z} \frac{\partial_y^m}{m!} G_{bou}^{2z}(x, y; \{x_v\}) \right. \\
& - \frac{1}{2\alpha'} \left(\frac{\tan \gamma_t - \tan \gamma_{t-1}}{\pi} \frac{x(t)0 \ i=2}{\sqrt{2\alpha'}} \right) \sum_{m=0}^{\infty} \bar{\alpha}_{(a)m} \mathcal{V}_{(t_a) \ \underline{z}\bar{z}} \frac{\partial_y^m}{m!} G_{bou}^{2\bar{z}}(x, y; \{x_v\}) \\
& - \frac{1}{2\alpha'} \sum_{n=1, m=0}^{\infty} \frac{\bar{\alpha}_{(t)n}}{n - \epsilon_t} \alpha_{(a)m} \mathcal{V}_{(t) \ \underline{z}\bar{z}} \mathcal{V}_{(t_a) \ \bar{z}z} \frac{\partial_x^{n-1}}{(n-1)!} \frac{\partial_y^m}{m!} [(x - x_t)^{\epsilon_t} \partial_x G_{bou}^{\bar{z}z}(x, y; \{x_v\})] \\
& \left. - \frac{1}{2\alpha'} \sum_{n=1, m=0}^{\infty} \frac{\alpha_{(t)n}}{n - 1 + \epsilon_t} \bar{\alpha}_{(a)m} \mathcal{V}_{(t) \ \bar{z}z} \mathcal{V}_{(t_a) \ \underline{z}\bar{z}} \frac{\partial_x^{n-1}}{(n-1)!} \frac{\partial_y^m}{m!} [(x - x_t)^{1-\epsilon_t} \partial_x G_{bou}^{\underline{z}\bar{z}}(x, y; \{x_v\})] \right\} \Big|_{x=x_t, y=x_a}
\end{aligned} \tag{1.29}$$

where the operator indexes are raised and lowered using the flat metric while Green function indexes are raised and lowered using \mathbb{R}^2 metric. The previous expression can also be written without using the auxiliary operators as a more conventional generating function. In order to do so we introduce the auxiliary parameters $d_{(t)n}$, $\bar{d}_{(t)n}$ and $c_{(a)n}$ and $\bar{c}_{(a)n}$ which roughly correspond to $\alpha_{(t)n+1-\epsilon}$, $\bar{\alpha}_{(t)n-\epsilon}$ and $\alpha_{(a)n}$, $\bar{\alpha}_{(a)n}$ (see eq.s (4.16) and (3.18) for a precise mapping) of the previous expression. Then we can write the generating function as

$$\begin{aligned}
& \mathcal{V}_{N+M}(\{d_{(t)}\}_{t=1, \dots, N}; \{c_{(a)}\}_{a=1, \dots, M}; \{x_t\}_{t=1, \dots, N}; \{x_{a,t_a}\}_{a=1, \dots, M}) = \\
& = \delta \left(Re \left(\sum_t d_{(t)0} + \sum_a c_{(a)0} \right) \right) C(x_1, \dots, x_N) \\
& \prod_a \exp \left\{ \frac{1}{2} c_{(a)0}^2 G_{bou, \text{reg } U(t_a)}^{\bar{z}\bar{z}}(x; y; \{x_v\}) + \frac{1}{2} \bar{c}_{(a)0}^2 G_{bou, \text{reg } U(t_a)}^{zz}(x; y; \{x_v\}) \right. \\
& \quad \left. \sum_{n,m=0}^{\infty} c_{(a)n} \bar{c}_{(a)m} \partial_x^n \partial_y^m G_{bou, \text{reg } U(t_a)}^{\bar{z}\bar{z}}(x; y; \{x_v\}) \right\} \Big|_{x=y=x_a} \\
& \prod_t \exp \left\{ \frac{1}{2} d_{(t)0}^2 G_{bou, \text{reg } T(t)}^{\bar{z}\bar{z}}(x; y; \{x_v\}) + \frac{1}{2} \bar{d}_{(t)0}^2 G_{bou, \text{reg } T(t)}^{zz}(x; y; \{x_v\}) \right. \\
& \quad + d_{(t)0} \bar{d}_{(t)0} G_{bou, \text{reg } T(t)}^{\bar{z}\bar{z}}(x; y; \{x_v\}) \\
& \quad \left. + \sum_{n,m=1}^{\infty} \bar{d}_{(t)n} d_{(t)m} \partial_x^{n-1} \partial_y^{m-1} \left[(x - x_t)^{1-\epsilon_t} (y - x_t)^{\epsilon_t} \partial_x \partial_y G_{bou, \text{reg } T(t)}^{\bar{z}\bar{z}}(x, y; \{x_v\}) \right] \right\} \Big|_{x=y=x_t} \\
& \prod_{a < b} \exp \left\{ \bar{c}_{(a)n} \bar{c}_{(b)m} G_{bou}^{zz}(x; y; \{x_v\}) + c_{(a)n} c_{(b)m} G_{bou}^{\bar{z}\bar{z}}(x; y; \{x_v\}) \right. \\
& \quad + \sum_{n,m=0}^{\infty} \bar{c}_{(a)n} c_{(b)m} \partial_x^n \partial_y^m G_{bou}^{\bar{z}\bar{z}}(x; y; \{x_v\}) \left. \right\} \\
& \quad + \sum_{n,m=0}^{\infty} c_{(a)n} \bar{c}_{(b)m} \partial_x^n \partial_y^m G_{bou}^{zz}(x; y; \{x_v\}) \left. \right\} \Big|_{x=x_a, y=x_b}
\end{aligned}$$

$$\begin{aligned}
& \prod_{t < u} \exp \left\{ d_{(t)0} d_{(u)0} G_{bou}^{\bar{z}\bar{z}}(x_t, x_u; \{x_v\}) + \bar{d}_{(t)0} \bar{d}_{(u)0} G_{bou}^{zz}(x_t, x_u; \{x_v\}) \right. \\
& \quad + d_{(t)0} \bar{d}_{(u)0} G_{bou}^{\bar{z}z}(x_t, x_u; \{x_v\}) + \bar{d}_{(t)0} d_{(u)0} G_{bou}^{z\bar{z}}(x_t, x_u; \{x_v\}) \\
& \quad + \sum_{n,m=1}^{\infty} d_{(t)n} \bar{d}_{(u)m} \partial_x^{n-1} \partial_y^{m-1} [(x-x_t)^{\epsilon_t} (y-x_u)^{1-\epsilon_u} \partial_x \partial_y G_{bou}^{\bar{z}\bar{z}}(x, y; \{x_v\})] \\
& \quad \left. + \sum_{n,m=1}^{\infty} \bar{d}_{(t)n} d_{(u)m} \partial_x^{n-1} \partial_y^{m-1} [(x-x_t)^{1-\epsilon_t} (y-x_u)^{\epsilon_u} \partial_x \partial_y G_{bou}^{z\bar{z}}(x, y; \{x_v\})] \right\} \Big|_{x=x_t, y=x_u} \\
& \prod_{t,a} \exp \left\{ \sum_{m=0}^{\infty} d_{(t)0} \bar{c}_{(a)m} \partial_y^m G_{bou}^{\bar{z}\bar{z}}(x, y; \{x_v\}) + \bar{d}_{(t)0} c_{(a)m} \partial_y^m G_{bou}^{z\bar{z}}(x, y; \{x_v\}) \right. \\
& \quad + \sum_{n=1, m=0}^{\infty} d_{(t)n} \bar{c}_{(a)m} \partial_x^{n-1} \partial_y^m [(x-x_t)^{\epsilon_t} \partial_x G_{bou}^{\bar{z}\bar{z}}(x, y; \{x_v\})] \\
& \quad \left. + \sum_{n=1, m=0}^{\infty} \bar{d}_{(t)n} c_{(a)m} \partial_x^{n-1} \partial_y^m [(x-x_t)^{1-\epsilon_t} \partial_x G_{bou}^{z\bar{z}}(x, y; \{x_v\})] \right\} \Big|_{x=x_t, y=x_a} \quad (1.30)
\end{aligned}$$

Notice that all the previous expressions are meaningful because of the behavior of the Green functions

$$\begin{aligned}
G_{bou}^{zz}(x, y; \{x_v\}) &= const \\
G_{bou}^{\bar{z}\bar{z}}(x, y; \{x_v\}) &= const \\
G_{bou}^{z\bar{z}}(x, y; \{x_v\}) &= G_{bou}^{\bar{z}\bar{z}}(y, x; \{x_v\}) \sim_{x \rightarrow x_t} const + (x-x_t)^{\epsilon_t} [g_0^{\bar{z}\bar{z}}(y; \{x_v\}) + O(x-x_t)] \\
G_{bou}^{\bar{z}z}(x, y; \{x_v\}) &= G_{bou}^{z\bar{z}}(y, x; \{x_v\}) \sim_{y \rightarrow x_u} const + (y-x_u)^{1-\epsilon_u} [g_0^{z\bar{z}}(x; \{x_v\}) + O(y-x_u)] \\
G_{bou}^{\bar{z}z}(x, y; \{x_v\}) &= G_{bou}^{z\bar{z}}(y, x; \{x_v\}) \sim_{x \rightarrow y; x, y \in (x_t, x_{t+1})} const - 2\alpha' \cos^2 \gamma_t \log|x-y| + O(x-y)
\end{aligned} \quad (1.31)$$

where $g_0^{\bar{z}\bar{z}}$ and $g_0^{z\bar{z}}$ are some functions of the given variables and the last line is strictly speaking true when $x \rightarrow y$ but not at the same time when $x \rightarrow x_t$ and $y \rightarrow x_t$. It is anyhow true that $(x-x_t)^{1-\epsilon_t} (y-x_t)^{\epsilon_t} \partial_x \partial_y G_{bou, reg}^{z\bar{z}}(x, y; \{x_v\})$ is well defined for $x = y = x_t$ as discussed in the appendix B.

The rest of the paper is organized in the following way: in the next section we make some examples of the use of the previous formulae and we clarify the operator to state mapping we use in the twisted sector. In section 3 we derive the previous formulae for the case with non excited twisted matter and finally in section 4 we consider excited twisted matter.

2. Examples

We want now apply the main formulae stated in the previous section to some examples while elucidating the nature of excited twisted states.

We start from the simplest example and then move to some more complex ones while in appendix A we check the $N = 2$ not excited states and M tachyons amplitude against the result found in ([12]).

2.1 Example 1: N not excited twisted states

From the operator to auxiliary state map

$$\sigma_{\epsilon_t, \kappa_t}(x_t, \bar{x}_t) \leftrightarrow |T_t, \kappa_t\rangle = \lim_{x \rightarrow 0} \sigma_{\epsilon_t, \kappa_t}(x, \bar{x})|0_{SL}\rangle \quad (2.1)$$

we deduce that

$$\begin{aligned} & \langle \sigma_{\epsilon_1, \kappa_1}(x_1, \bar{x}_1) \dots \sigma_{\epsilon_N, \kappa_N}(x_N, \bar{x}_N) \rangle \\ &= \langle V_{N+0}(\{x_t\}_{t=1, \dots, N} | \prod_t |T_t, \kappa_t\rangle \rangle \\ &= \delta\left(\sum_t \kappa_t\right) e^{-\frac{1}{2} \sum_t \kappa_t^2} G_{bou, reg}^{22}(x_t; x_t) e^{-\frac{1}{2} \sum_{u, t} \kappa_t \kappa_u} G_{bou}^{22}(x_t; x_u) C(x_1, \dots, x_N) \end{aligned} \quad (2.2)$$

where the phases proportional to κ probably vanish as they do in the $N = 2$ case but this must be checked with an explicit computation of the Green function which can, in principle, be extracted from ([6]) after T-dualizing. The same computation can be performed using the more conventional generating function as

$$\langle \sigma_{\epsilon_1, \kappa_1}(x_1, \bar{x}_1) \dots \sigma_{\epsilon_N, \kappa_N}(x_N, \bar{x}_N) \rangle = \mathcal{V}_{N+M} \prod_t e^{i\kappa_t \frac{\overleftarrow{\partial}}{\partial d_{(t)}^2}} \Big|_{c=0; d=0}. \quad (2.3)$$

2.2 Example 2: N not excited twisted and 2 untwisted states

Similarly to what done in the previous example from the maps

$$\begin{aligned} \sigma_{\epsilon_t}(x_t, \bar{x}_t) &\leftrightarrow |T_t\rangle = \lim_{x \rightarrow 0} \sigma_{\epsilon_t}(x, \bar{x})|0_{SL}\rangle \\ \partial X^z(y_a, \bar{y}_a) &\leftrightarrow -i\sqrt{2\alpha'} \cos \gamma_{t_a} \alpha_{(a)1}^\dagger |0_{(a)}\rangle = \lim_{y \rightarrow 0^+} \partial Z_{(a)}(y, \bar{y})|0_{SL}\rangle \end{aligned} \quad (2.4)$$

where we have made the choice $x_{t_a} < y_a < x_{t_a} + 1$ (which fixes the magnetic field felt by the untwisted state), we deduce

$$\begin{aligned} & \langle \sigma_{\epsilon_1, \kappa_1}(x_1, \bar{x}_1) \dots \sigma_{\epsilon_N, \kappa_N}(x_N, \bar{x}_N) \partial_{y_1} Z(y_1, \bar{y}_1) \partial_{y_2} \bar{Z}(y_2, \bar{y}_2) \rangle \\ &= \langle V_{N+2}(\{x_t\}_{t=1, \dots, N}, \{x_a\}_{a=1, 2} | \otimes_t |T_t\rangle \otimes (-i\sqrt{2\alpha'} \cos \gamma_{t_1}) \alpha_{(1)1}^\dagger |0_{(1)}\rangle \otimes (-i\sqrt{2\alpha'} \cos \gamma_{t_2}) \bar{\alpha}_{(2)1}^\dagger |0_{(2)}\rangle \rangle \\ &= \delta\left(\sum_t \kappa_t\right) e^{-\frac{1}{2} \sum_t \kappa_t^2} G_{bou, reg}^{22}(x_t; x_t) e^{-\frac{1}{2} \sum_{u, t} \kappa_t \kappa_u} G_{bou}^{22}(x_t; x_u) C(x_1, \dots, x_N) \partial_{y_1} \partial_{y_2} G_{bou}^{z\bar{z}}(y_1; y_2; \{x_t\}) \end{aligned} \quad (2.5)$$

where \bar{y} is a function of y as in eq. (3.2). The previous result is an immediate consequence of the the definition of G given in (1.3) but it can also be interpreted as a ‘‘proof’’ of eq. (1.3) since the Green function entering in the previous formula is the Green function obtained from the path integral.

2.3 Example 3: $N - 1$ not excited twisted, 1 excited twisted and 2 untwisted states

We can now discuss the excited twisted states. The easiest way to denote an excited twisted state is by writing from which untwisted state it can be obtained by OPE, for example $(\partial_x Z \partial_x^2 Z \sigma_{\epsilon, \kappa})(x, \bar{x})$ can be obtained by taking the finite part of the OPE $(\partial_y Z \partial_y^2 Z)(y, \bar{y}) \sigma_{\epsilon, \kappa}(x, \bar{x})$ as $y \rightarrow x$. This limit can be taken in a clearer and easier way when we realize $(\partial_y Z \partial_y^2 Z)(y, \bar{y})$ as a normal ordered operator in a twisted auxiliary Hilbert space where $\sigma_{\epsilon, \kappa}(x, \bar{x}) \leftrightarrow |T, \kappa\rangle$. In particular we can define the finite part of the limit as

$$\lim_{y \rightarrow 0^+} : [y^{1-\epsilon} \partial_y Z(y, y) \partial_y [y^{1-\epsilon} \partial_y Z(y, y)]] : |T, \kappa\rangle = \left(-i\sqrt{2\alpha'} \cos \gamma \right)^2 \alpha_c^\dagger \alpha_{1+c}^\dagger |T, \kappa\rangle \quad (2.6)$$

Similarly we can consider the map

$$(\partial \bar{Z} \sigma_{\epsilon_t, \kappa_t})(x_t, \bar{x}_t) \leftrightarrow -i\sqrt{2\alpha'} \cos \gamma_t \bar{\alpha}_{1-\epsilon}^\dagger |T_t, \kappa_t\rangle \quad (2.7)$$

and compute the correlator

$$\begin{aligned} & \langle (\partial_x \bar{Z} \sigma_{\epsilon_1, \kappa_1})(x_1, \bar{x}_1) \dots \sigma_{\epsilon_N, \kappa_N}(x_N, \bar{x}_N) \partial_{y_1} Z(y_1, \bar{y}_1) \rangle \\ &= \langle V_{N+1}(\{x_t\}_{t=1, \dots, N}, \{x_a\}_{a=1, 2}) (-i\sqrt{2\alpha'} \cos \gamma_1) \bar{\alpha}_{(1)\epsilon_1}^\dagger |T_1, \kappa_1\rangle \otimes_{t>1} |T_t, \kappa_t\rangle \otimes (-i\sqrt{2\alpha'} \cos \gamma_{t_1}) \alpha_{(1)1}^\dagger |0_{(1)}\rangle \rangle \end{aligned} \quad (2.8)$$

to be

$$\begin{aligned} & \delta \left(\sum_t \kappa_t \right) e^{-\frac{1}{2} \sum_t \kappa_t^2 G_{bou, reg}^{22}(x_t; x_t)} e^{-\frac{1}{2} \sum_{u, t} \kappa_t \kappa_u G_{bou}^{22}(x_t; x_u)} C(x_1, \dots, x_N) \\ & \partial_{y_1} \left[(x - x_1)^{\epsilon_1} \partial_x G_{bou}^{\bar{z}z}(x, y_1; \{x_v\}) \right] |_{x=x_1}. \end{aligned} \quad (2.9)$$

The same computation can be performed using the generating function as

$$\langle (\partial_x \bar{Z} \sigma_{\epsilon_1, \kappa_1})(x_1, \bar{x}_1) \dots \sigma_{\epsilon_N, \kappa_N}(x_N, \bar{x}_N) \partial_{y_1} Z(y_1, \bar{y}_1) \rangle = \mathcal{V}_{N+M} \frac{\overleftarrow{\partial}}{\partial d_{(1)1}} e^{i\kappa_1 \frac{\overleftarrow{\partial}}{\partial d_{(1)0}^2}} \prod_{t>1} e^{i\kappa_t \frac{\overleftarrow{\partial}}{\partial d_{(t)0}^2}} \frac{\overleftarrow{\partial}}{\partial c_{(1)1}} \Big|_{c=0; d=0} \quad (2.10)$$

3. Derivation for untwisted matter

The starting point is very similar to ([4],[5]) where it was recognized that the generator for all closed (super)string amplitudes is a quadratic path integrals. The idea in the previous papers is that the appropriate boundary condition for R and/or NS sector can be obtained simply by inserting *linear* sources with the desired boundary conditions. Because of this assumption the quantum fluctuations are the same for all the amplitudes: from the purely NS to the mixed ones. It was later realized that this prescription misses a proper treatment of the quantum fluctuations ([10]) and that when this part is considered the amplitudes factorize correctly ([11]).

Here we consider open strings and we realize the proper twisted boundary conditions by *quadratic* boundary terms which are nothing else but the coupling of the string to the

magnetic field background. Therefore a non excited twist field is realized by a discontinuity in the magnetic field⁸.

The N non excited twist field amplitudes with Euclidean worldsheet metric is then computed by the quadratic path integral^{9 10}

$$C(x_1, \dots, x_N) \delta(0) = \mathcal{N} \int \mathcal{D}X e^{-\frac{1}{2\pi\alpha'} [\int_H d^2z \frac{1}{2} G_{ij} \partial_z X^i \partial_{\bar{z}} X^j - i \int_{\partial H} dx B(x) X^1(x, \bar{x}) \partial_x X^2(x, \bar{x})]} \quad (3.1)$$

where $dx \partial_x X^2(x, \bar{x})$ must be interpreted as the pullback on the boundary of dX^2 in such a way that \bar{x} depends on x as

$$\bar{x} = \begin{cases} x & x = |x| > 0 \\ x e^{-i2\pi} & x = |x| e^{i\pi} < 0 \end{cases}, \quad (3.2)$$

H is the superior half plane and the adimensional magnetic field $B(x)$ is given by

$$B(x) = 2\pi\alpha' q F_{12}(x) = \sum_{t=1}^N B_t \theta(x - x_t) \theta(x_{t+1} - x) \quad (3.3)$$

so that the dicharged string at $x = x_t$ feels a magnetic field B_{t-1} on the left, i.e. $\sigma = \pi$ and B_t on the right, i.e. $\sigma = 0$. Here we have set $B_{-1} = B_{N+1}$ and $x_{-1} = -\infty$, $x_{N+1} = +\infty$.

In the previous equation (3.1) we have chosen the gauge

$$A_1 = 0, \quad A_2 = Bx^1 \quad (3.4)$$

in order to make clear that we have only one zero mode which is associated with the shift $X^2 \rightarrow X^2 + \epsilon$ and therefore there is only one conserved momentum as it is the case in the Landau levels problem on \mathbb{R}^2 .

Now we can add the untwisted states vertexes. This can be done by considering the generating function of all untwisted vertexes at $x = x_a \in \mathbb{R}$ as¹¹

$$\begin{aligned} \mathcal{S}(c_{(a)}) &= \exp\left\{ \sum_{n=0}^{\infty} c_{(a)n_i} \partial_x^n |_{x=x_a} X^i(x, \bar{x}) \right\} = \exp\left\{ \int_{\partial H} dx J_i(x; x_a) X^i(x, \bar{x}) \right\} \\ J_i(x; x_a) &= \sum_{n=0}^{\infty} c_{(a)n_i} \partial_x^n \delta(x - x_a) \end{aligned} \quad (3.5)$$

where c are arbitrary complex numbers (or functions as we use in the next section). To understand how the previous generating vertex work let us take the example from ([12]). Consider the vertex which describes the fluctuations of the gauge vector around the dipole

⁸And in particular the transition from an eigenstate $|\alpha\rangle$ in magnetic field B_t to an eigenstate $|\beta\rangle$ in magnetic B_{t+1} at worldsheet time τ_t can be computed as in usual quantum mechanics as $\langle\beta|\alpha\rangle = \int \mathcal{D}X \langle X(\sigma), \tau_t^- | \alpha \rangle (\langle X(\sigma), \tau_t^+ | \beta \rangle)^*$.

⁹We define $d^2z = 2dz d\bar{z}$ and $z = e^{\tau E + i\sigma} \in H$.

¹⁰This is the path integral corresponding to all twist fields with zero ‘‘momentum’’ and therefore it is proportional to a $\delta(0)$ which arises from X^2 zero mode. The general case is treated in the next section.

¹¹This formulation involving the full field on the boundary is only right for NN boundary condition. For DD boundary condition one should use a slightly different one ([13]).

string background we can derive it from a generating functional for the dipole string as ¹²
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$$V(x_a; \epsilon, k) = \epsilon_i \partial_x X^i(x, \bar{x}) e^{ik_j X^j(x, \bar{x})} \Big|_{x=x_a} = S(c_{(a)}, x_a) \epsilon_i \frac{\overleftarrow{\partial}}{\partial c_{(a)1i}} e^{ik_j \frac{\overleftarrow{\partial}}{\partial c_{(a)0j}}} \Big|_{c_{(a)}=0} \quad (3.6)$$

As a matter of facts the previous vertex gives an indefinite result when inserted in the path integral even when $B = 0$. We must therefore regularize it and consider

$$\begin{aligned} \mathcal{S}_{reg}(c_{(a)}) &= \mathcal{N}_{(a)}(x_a) \exp \left\{ \sum_{n=0}^{\infty} c_{(a)ni} \partial_x^n \Big|_{x=x_a} \langle X^i(x, \bar{x}) \rangle \right\} \\ &= \mathcal{N}_{(a)}(x_a) \exp \left\{ \int_{\partial H} dx J_{i,reg}(x; x_a) X^i(x, \bar{x}) \right\} \end{aligned} \quad (3.7)$$

where the regularized curred is given by

$$J_{i,reg}(x; x_a) = \sum_{n=0}^{\infty} c_{(a)ni} \partial_{x_a}^n \delta_{reg}(x - x_a), \quad (3.8)$$

the averaged field by

$$\langle X^i(x, x) \rangle = \int_{\partial H} dy \delta_{reg}(x - y) X(y, y) \quad (3.9)$$

and the normalization factor is

$$\begin{aligned} \mathcal{N}_{(a)}(x_a) &= \exp \left\{ -\frac{1}{2} \sum_{n,m=0}^{\infty} c_{(a)ni} c_{(a)mj} \right. \\ &\quad \left. \int_{\partial H} dx \int_{\partial H} dy \partial_x^n \delta_{reg}(x - x_a) \partial_y^m \delta_{reg}(y - x_a) G_{U(t_a), bou}^{ij}(x; y) \right\} \Big|_{x=x_a, y=x_a} \end{aligned} \quad (3.10)$$

with $G_{U(t_a), bou}^{ij}(x; y)$ the boundary Green functions in the dipole case with magnetic field B_{t_a} given in eq.s (1.26). There are two reasons why we have introduced the previous definitions. The first is that it works in reproducing the amplitudes for $N = 2$ as discussed in appendix A. The second is connected to the way the regularization terms is suggested from the operatorial formalism. In operatorial formalism the simplest approach is to consider a point splitting, i.e. $[\exp(c_{(a)i} X_{(a)}^i(x_a, x_a))]_{p.s.} = \exp(c_{(a)i} [X_{(a)}^{i(-)}(x_a e^{-\eta}, x_a e^{-\eta}) + X_{(a)}^{i(+)}(x_a, x_a)])$

¹²Notice that here we are talking about the abstract path integral representation of the vertex and not of the operatorial representation. The operatorial representation of the vertex can be realized in an auxiliary space with both twisted and untwisted boundary conditions. The untwisted auxiliary Hilbert space representation is the usual operatorial representation while the twisted one is the one derived in ([12]). Just because of these different realizations this auxiliary Hilbert space must not be confused with $\mathcal{H}_{(a,t_a)}$ introduced before which is a way of representing the $c_{(a)ni}$, see (3.18).

¹³It is worth noticing how vertexes for dipole strings have the same functional form independently on the magnetic backgrounds B_{t_a} nevertheless they differ because different conditions for physical states, in the previous example we have $k_\mu \eta^{\mu\nu} k_\nu + k_i \mathcal{G}^{ij}(B_{t_a}) k_j = \epsilon_\mu \eta^{\mu\nu} k_\nu + \epsilon_i \mathcal{G}^{ij}(B_{t_a}) k_j = 0$ where $\mu, \nu \neq 1, 2$.

which implies a regularization factor $\mathcal{N}_{(a)}(x_a) = \exp\left(-\frac{1}{2}c_{(a)i}c_{(a)j}G_{bou}^{ij}(x; y)\right)\Big|_{x=x_a, y=x_a} e^{-\eta}$. When we smooth the fields the previous regularization factor becomes

$$\begin{aligned}\mathcal{N}_{(a)}(x_a) &= \exp\left(-\frac{1}{2}c_{(a)i}c_{(a)j} \int_{x>y} dx dy 2 \delta_{reg}(x-x_a)\delta_{reg}(y-x_a)G_{bou}^{ij}(x; y)\right) \\ &= \exp\left(-c_{(a)i}c_{(a)j} \int dx dy \delta_{reg}(x-x_a)\delta_{reg}(y-x_a) \frac{G_{bou}^{ij}(x; y) \theta(x-y) + G_{bou}^{ji}(y; x) \theta(y-x)}{2}\right) \\ &= \exp\left(-\frac{1}{2}c_{(a)i}c_{(a)j} \int dx dy \delta_{reg}(x-x_a)\delta_{reg}(y-x_a)G_{bou}^{ij}(x; y)\right)\end{aligned}\quad (3.11)$$

where the factor 2 in the first line is due to the fact we are using one of the two δ just one half because of the constraint $x > y$ as it can be directly verified by using a step function regularization of the delta. In the last step we have used the property $G_{bou}^{ji}(y; x) = G_{bou}^{ij}(x; y)$.

In conclusion the path integral we want to compute in order to get the generating function for all the M untwisted correlators in presence of N twists is

$$\begin{aligned}Z(\{x_t\}; \{x_a\}) &= \mathcal{N} \int \mathcal{D}X e^{-\frac{1}{2\pi\alpha'} [\int_H d^2z \frac{1}{2}G_{ij}\partial_z X^i \partial_{\bar{z}} X^j + i \int_{\partial H} dx B(x)X^1(x, \bar{x})\partial_x X^2(x, \bar{x})]} \\ &\quad \times \prod_{a=1}^M \mathcal{N}_{(a)}(x_a) e^{\int_{\partial H} dx J_{i, reg}(x; x_a) X^i(x, \bar{x})}\end{aligned}\quad (3.12)$$

This path integral can then be performed to get

$$\begin{aligned}Z(\{x_t\}; \{x_a\}) &= C(x_1, \dots, x_N) \delta\left(i \sum_a c_{(a)0} i=2\right) \\ &\quad \prod_a \exp\left\{\frac{1}{2} \int_{\partial H} dx \int_{\partial H} dy J_{i, reg}(x; x_a) J_{j, reg}(y; x_a) \right. \\ &\quad \left. [G_{bou}^{ij}(x; y; \{x_v\}) - G_{U(t_a), bou}^{ij}(x; y)]\right\} \\ &\quad \prod_{a, b; a \neq b} \exp\left\{\frac{1}{2} \int_{\partial H} dx \int_{\partial H} dy J_{i, reg}(x; x_a) J_{j, reg}(y; x_b) \right. \\ &\quad \left. G_{bou}^{ij}(x; y; \{x_v\})\right\}\end{aligned}\quad (3.13)$$

where $G_{bou}^{ij}(x; y; \{x_v\})$ is the Green function of the quadratic operator which turns out to be the one defined in (1.2). Notice that the quadratic operator is not even hermitian exactly as it happens for Landau levels in the plain quantum mechanics. The previous

result can be also rewritten after the regularization has been removed as

$$\begin{aligned}
Z(\{x_t\}; \{x_a\}) = & C(x_1, \dots, x_N) \delta\left(i \sum_a c_{(a)0} i=2\right) \\
& \prod_a \exp \left\{ \frac{1}{2} \sum_{n,m=0}^{\infty} c_{(a)ni} c_{(a)mj} \partial_x^n |_{x=x_a} \partial_y^m |_{y=x_a} G_{bou \text{ reg } U(t_a)}^{ij}(x; y; \{x_v\}) \right\} \\
& \prod_{a,b;a \neq b} \exp \left\{ \frac{1}{2} \sum_{n,m=0}^{\infty} c_{(a)ni} c_{(b)mj} \partial_x^n |_{x=x_a} \partial_y^m |_{y=x_b} G_{bou}^{ij}(x; y; \{x_v\}) \right\}
\end{aligned} \tag{3.14}$$

where we have defined the boundary Green function regularized by the untwisted Green function for a background B_{t_a}

$$G_{bou \text{ reg } U(t_a)}^{z\bar{z}}(x, y; \{x_v\}) = \left[G_{bou}^{z\bar{z}}(x; y; \{x_v\}) - G_{U(t_a), bou}^{z\bar{z}}(x; y) \right] \tag{3.15}$$

The previous expression can be simplified a little using the symmetry $G_{bou}^{ij}(x; y) = G_{bou}^{ji}(y; x)$ and by rewriting it in the complex basis as

$$\begin{aligned}
Z(\{x_t\}; \{x_a\}) = & C(x_1, \dots, x_N) \delta \left(\sum_a (c_{(a)0} - \bar{c}_{(a)0}) \right) \\
& \prod_a \exp \left\{ \frac{1}{2} c_{(a)0}^2 G_{bou \text{ reg } U(t_a)}^{z\bar{z}}(x; y; \{x_v\}) + \frac{1}{2} \bar{c}_{(a)0}^2 G_{bou \text{ reg } U(t_a)}^{z\bar{z}}(x; y; \{x_v\}) \right. \\
& \quad \left. \sum_{n,m=0}^{\infty} \bar{c}_{(a)n} c_{(a)m} \partial_x^n |_{x=x_a} \partial_y^m |_{y=x_a} G_{bou \text{ reg } U(t_a)}^{z\bar{z}}(x; y; \{x_v\}) \right\} \\
& \prod_{a < b} \exp \left\{ \bar{c}_{(a)n} \bar{c}_{(b)m} G_{bou}^{z\bar{z}}(x_a; x_b; \{x_v\}) + c_{(a)n} c_{(b)m} G_{bou}^{z\bar{z}}(x_a; x_b; \{x_v\}) \right. \\
& \quad + \sum_{n,m=0}^{\infty} \bar{c}_{(a)n} c_{(b)m} \partial_x^n |_{x=x_a} \partial_y^m |_{y=x_b} G_{bou}^{z\bar{z}}(x; y; \{x_v\}) \left. \right\} \\
& \quad + \sum_{n,m=0}^{\infty} c_{(a)n} \bar{c}_{(b)m} \partial_x^n |_{x=x_a} \partial_y^m |_{y=x_b} G_{bou}^{z\bar{z}}(x; y; \{x_v\}) \left. \right\}
\end{aligned} \tag{3.16}$$

with $\bar{c}_{(a)n} = c_{(a)n\bar{z}}$ and $c_{(a)n} = c_{(a)n\bar{z}}$.

We can now give a different formulation of the previous result if we realize the algebra

$$[c_{(a)ni}, \overleftarrow{\frac{\partial}{\partial c_{(b)mj}}}] = \delta_i^j \delta_{n,m} \delta_{a,b} \tag{3.17}$$

on the untwisted scalar (dipole string) auxiliary Hilbert spaces \mathcal{H}_{a,t_a} with backgrounds B_{t_a}

introduced before as

$$\begin{aligned}
& 1 \rightarrow \langle z_{(a)00} = \bar{z}_{(a)00} = 0 | \langle 0_{(a)} | \\
\bar{c}_{(a)n} & \rightarrow \frac{i}{\sqrt{2\alpha'}} \frac{\alpha_{(a)n}}{\cos \gamma_{t_a} n!}, \quad c_{(a)n} \rightarrow \frac{i}{\sqrt{2\alpha'}} \frac{\bar{\alpha}_{(a)n}}{\cos \gamma_{t_a} n!} \quad n \geq 0 \\
\frac{\overleftarrow{\partial}}{\partial \bar{c}_{(a)m}} & \rightarrow -i\sqrt{2\alpha'} (m-1)! \cos \gamma_{t_a} \alpha_{(a)m}^\dagger, \quad \frac{\overleftarrow{\partial}}{\partial c_{(a)m}} \rightarrow -i\sqrt{2\alpha'} (m-1)! \cos \gamma_{t_a} \bar{\alpha}_{(a)m}^\dagger \quad m > 0
\end{aligned} \tag{3.18}$$

where the ‘‘strange’’ choice of the normalization is due to the last expressions which arise from the desire of identifying

$$-i\sqrt{2\alpha'} (m-1)! \cos \gamma_{t_a} \alpha_{(a)m}^\dagger \sim \partial^m X_{(a)}^{(-)z}(x, \bar{x})|_{x=0}. \tag{3.19}$$

Using these auxiliary Hilbert spaces we can now rewrite the previous expression for the M untwisted correlators (3.16) as

$$\begin{aligned}
Z(\{x_t\}; \{x_a\}) &= C(x_1, \dots, x_N) \delta \left(i \sum_a (\alpha_{(a)0} - \bar{\alpha}_{(a)0}) \right) \prod_{a=1}^M \langle z_{(a)00} = \bar{z}_{(a)00} = 0 | \langle 0_{(a)} | \\
& \prod_a \exp \left\{ -\frac{1}{4\alpha'} \alpha_{(a)0}^2 \mathcal{V}_{(t_a) \underline{z}\bar{z}}^2 G_{bou, reg}^{\bar{z}\bar{z}} U_{(t_a)}(x; y; \{x_v\}) \right. \\
& \quad - \frac{1}{4\alpha'} \bar{\alpha}_{(a)0}^2 \mathcal{V}_{(t_a) \bar{z}z}^2 G_{bou, reg}^{zz} U_{(t_a)}(x; y; \{x_v\}) \\
& \quad \left. - \frac{1}{2\alpha'} \sum_{n,m=0}^{\infty} \alpha_{(a)n} \bar{\alpha}_{(a)m} \mathcal{V}_{(t_a) \bar{z}z} \mathcal{V}_{(t_a) \underline{z}\bar{z}} \frac{\partial_x^n}{n!} \frac{\partial_y^m}{m!} G_{bou, reg}^{z\bar{z}} U_{(t_a)}(x; y; \{x_v\}) \right\} \Big|_{x=y=x_a} \\
& \prod_{a < b} \exp \left\{ -\frac{1}{2\alpha'} \sum_{n,m=0}^{\infty} \alpha_{(a)n} \bar{\alpha}_{(b)m} \mathcal{V}_{(t_a) \bar{z}z} \mathcal{V}_{(t_b) \underline{z}\bar{z}} \frac{\partial_x^n}{n!} \frac{\partial_y^m}{m!} G_{bou}^{z\bar{z}}(x; y; \{x_v\}) \right. \\
& \quad \left. - \frac{1}{2\alpha'} \sum_{n,m=0}^{\infty} \bar{\alpha}_{(a)n} \alpha_{(b)m} \mathcal{V}_{(t_b) \bar{z}z} \mathcal{V}_{(t_a) \underline{z}\bar{z}} \frac{\partial_x^n}{n!} \frac{\partial_y^m}{m!} G_{bou}^{\bar{z}z}(x; y; \{x_v\}) \right\} \Big|_{x=x_a, y=x_b}
\end{aligned} \tag{3.20}$$

where \mathcal{V} are the \mathbb{R}^2 vielbein which connect the α flat index with the Green function G curved index.

4. Derivation for twisted matter

The strategy we are going to follow is to consider the amplitude derived in previous section with $N+M$ untwisted states at the positions $\{x_a\}_{a=1\dots M}$, $\{x_f\}_{f=1\dots N}$ and unexcited twists at positions $\{x_t\}_{t=1\dots N}$. Then we choose N of untwisted states at the positions $\{x_f\}_{f=1\dots N}$ for which we take the limit $x_f \rightarrow x_t$. In order to get the desired amplitude with M untwisted and N excited twisted states we must choose in a proper way the $c_{(f)ni}$. This amounts not only to choose $c_{(f)ni}$ in (3.7) as a function of x_f as in eq. (4.9) but to introduce

a further normalization $\mathcal{R}(x_f)$ as in eq. (4.10) in such a way that we can “undo” the OPE and get a result which is a generating function for the twisted states

$$\exp \left\{ d_{(t)0}^2 x_0^{i=2(auxt)} - i\sqrt{2\alpha'} \cos \gamma_t \sum_{n=1}^{\infty} \left[d_{(t)n}(n-1)! \bar{\alpha}_{n-\epsilon_t}^{\dagger(auxt)} + \bar{d}_{(t_f)n}(n-1)! \alpha_{n-1+\epsilon_t}^{\dagger(auxt)} \right] \right\} |T_{(auxt)}\rangle \quad (4.1)$$

when expressed in a chart where $x_t = 0$. The “strange” normalization is chosen because it is the easiest map from operators to states, f.x. the twisted excited state which can be obtained by subtracting the divergences of the limit $y \rightarrow x_t^+$ of $[\partial_y^3 Z(y, y)]^2 \sigma_{\epsilon_t}(x_t, x_t)$ gives the state

$$\lim_{y \rightarrow 0^+} \left[\partial_y^2 [y^{1-\epsilon_t} \partial_y Z^{(auxt)}(y, y)] \right]^2 |T_{(auxt)}\rangle = \left(-i\sqrt{2\alpha'} \cos \gamma_t 2! \alpha_{2+\epsilon_t}^{\dagger(auxt)} \right)^2 |T_{(auxt)}\rangle \quad (4.2)$$

thus making contact between eq. (4.1) and eq. (4.5).

Let us start studying the OPE $\mathcal{S}(c, x_f) \sigma_t(0, 0)$. This can be studied in an auxiliary Hilbert space \mathcal{H}_{auxt} (not to be confused the the Hilbert space \mathcal{H}_t which we introduced in the first section and which is associated with coefficients $d_{(t)}$) where $\sigma_{\epsilon_t}(0, 0)$ is represented by the twisted vacuum $|T_{(auxt)}\rangle$ and the generating function $\mathcal{S}(c, x_f)$ as

$$\begin{aligned} \mathcal{S}_{(auxt)}(c_{(f)}, x_f) = & e^{-\frac{1}{2} c_{(f)0}^2 G_{bou, reg}^{zz} U(t_a)(x_f; x_f; \{x=0, x=\infty\}) - \frac{1}{2} \bar{c}_{(f)0}^2 G_{bou, reg}^{\bar{z}\bar{z}} U(t_a)(x_f; x_f; \{x=0, x=\infty\})} \\ & e^{-\frac{1}{2} \sum_{m,n=0}^{\infty} c_{(f)n} \bar{c}_{(f)m} \partial_x^n |_{x=x_f} \partial_y^m |_{y=x_f} G_{bou, reg}^{zz} U(t_a)(x; y; \{x=0, x=\infty\})} \\ & : e^{\sum_{n=0}^{\infty} \bar{c}_{(f)n} \partial_x^n |_{x=x_f} Z_{(auxt)}(x, \bar{x}) + c_{(f)n} \partial_x^n |_{x=x_f} \bar{Z}_{(auxt)}(x, \bar{x})} : \end{aligned} \quad (4.3)$$

In the previous equation the normal ordering is performed with respect to the operators entering the expansion of the quantum fields $Z_{(auxt)}(z, \bar{z})$ and $\bar{Z}_{(auxt)}(z, \bar{z})$ which act on \mathcal{H}_{auxt} . Then the OPE can be computed as

$$\begin{aligned} \mathcal{S}(c, x_f) \sigma_{\epsilon_t}(0, 0) \leftrightarrow & e^{-\frac{1}{2} c_{(f)0}^2 G_{bou, reg}^{zz} U(t_a)(x_f; x_f; \{x=0, x=\infty\}) - \frac{1}{2} \bar{c}_{(f)0}^2 G_{bou, reg}^{\bar{z}\bar{z}} U(t_a)(x_f; x_f; \{x=0, x=\infty\})} \\ & e^{-\frac{1}{2} \sum_{m,n=0}^{\infty} \partial_x^n |_{x=x_f} \partial_y^m |_{y=x_f} \left[c_{(f)n} \bar{c}_{(f)m} G_{bou, reg}^{zz} U(t_a)(x; y; \{x=0, x=\infty\}) \right]} \\ & e^{\sum_{n=0}^{\infty} \partial_x^n |_{x=x_f} \left[\bar{c}_{(f)n} Z_{(auxt)}^{(-)}(x, x) \right] + \partial_x^n |_{x=x_f} \left[c_{(f)n} \bar{Z}_{(auxt)}^{(-)}(x, x) \right]} |T_{(auxt)}\rangle \end{aligned} \quad (4.4)$$

which is similar to a rewriting of eq. (4.1) as

$$\begin{aligned} \lim_{x_f \rightarrow 0^+} e^{\bar{d}_{(t)0} \left[Z_{(auxt)}^{(-)}(x_f, x_f) \right] + d_{(t)0} \left[\bar{Z}_{(auxt)}^{(-)}(x_f, x_f) \right]} \\ e^{\sum_{n=1}^{\infty} \bar{d}_{(t)n} \partial_x^{n-1} |_{x=x_f} \left[x^{1-\epsilon_t} \partial_x Z_{(auxt)}^{(-)}(x, x) \right] + d_{(t)n} \partial_x^{n-1} |_{x=x_f} \left[x^{\epsilon_t} \partial_x \bar{Z}_{(auxt)}^{(-)}(x, x) \right]} |T_{(auxt)}\rangle \end{aligned} \quad (4.5)$$

where in the second line we have written ∂Z since we want to get rid of zero modes and in the limit it is necessary to write $x_f \rightarrow 0^+$ since the the behavior of ∂Z changes by an overall normalization when $x < 0$.

Comparison between the two previous expressions suggests to consider then the operator acting on the Hilbert space $\mathcal{H}_{(aux\ t)}$

$$\begin{aligned} \mathcal{T}_{(aux\ t)}(d_{(t)}, x_f) &= \mathcal{N}(d_{(t)}, x_f, x_t) e^{\bar{d}_{(t)0}} [Z_{(aux\ t, reg)}(x_f, x_f)] + d_{(t)0} [\bar{Z}_{(aux\ t, reg)}(x_f, x_f)] \\ &\quad e^{\sum_{n=1}^{\infty} \bar{d}_{(t)n} \partial_x^{n-1} [x^{1-\epsilon t} \partial_x Z_{(aux\ t, reg)}(x, x)] + d_{(t)n} \partial_x^{n-1} [x^{\epsilon t} \partial_x \bar{Z}_{(aux\ t, reg)}(x, x)]} \Big|_{x=x_f^+} \end{aligned} \quad (4.6)$$

where $Z_{(aux\ t, reg)}(x_f, x_f)$ is point split regularized of $Z_{(aux\ t)}(x_f, x_f)$ defined as

$$Z_{(aux\ t, reg)}(x_f, x_f) = Z_{(aux\ t)}^{(-)}(x_f e^{-\eta}, x_f e^{-\eta}) + Z_{(aux\ t)}^{(+)}(x_f, x_f),$$

no normal ordering is performed and the normalization factor is given by

$$\begin{aligned} \mathcal{N}^{-1}(d_{(t)}, x_f, x_t) &= \left\{ e^{\frac{1}{2} \bar{d}_{(t)0}^2 G_{T(t)\ bou}^{zz}(x; y) + \frac{1}{2} d_{(t)0}^2 G_{T(t)\ bou}^{\bar{z}\bar{z}}(x; y) + \bar{d}_{(t)0} d_{(t)0} \frac{G_{T(t)\ bou}^{z\bar{z}}(x; y) + G_{T(t)\ bou}^{\bar{z}z}(x; y)}{2}} \right. \\ &\quad e^{\frac{1}{2} \sum_{n=1}^{\infty} \bar{d}_{(t)0} d_{(t)n} [\partial_x^{n-1} (x^{\epsilon t} \partial_x G_{T(t)\ bou}^{\bar{z}\bar{z}}(x; y)) + \partial_y^{n-1} (y^{\epsilon t} \partial_y G_{T(t)\ bou}^{zz}(x; y))] } \\ &\quad e^{\frac{1}{2} \sum_{n=1}^{\infty} d_{(t)0} \bar{d}_{(t)n} [\partial_x^{n-1} (x^{1-\epsilon t} \partial_x G_{T(t)\ bou}^{z\bar{z}}(x; y)) + \partial_y^{n-1} (y^{1-\epsilon t} \partial_y G_{T(t)\ bou}^{\bar{z}z}(x; y))] } \\ &\quad e^{\frac{1}{2} \sum_{n, l=1}^{\infty} d_{(t)l} \bar{d}_{(t)n} [\partial_y^{l-1} \partial_x^{n-1} (y^{\epsilon t} x^{1-\epsilon t} \partial_x \partial_y G_{T(t)\ bou}^{z\bar{z}}(x; y)) + \partial_x^{l-1} \partial_y^{n-1} (x^{\epsilon t} y^{1-\epsilon t} \partial_x \partial_y G_{T(t)\ bou}^{\bar{z}z}(x; y))] } \\ &\quad \left. \right\} \Big|_{x=x_f; y=x_f e^{-\eta}} \end{aligned} \quad (4.7)$$

where $G_{T(t)\ bou}^{ij}(y; x) = G_{T(t)\ bou}^{ji}(x; y) = G_{bou}^{ij}(y; x; \{x_1 = 0, x_2 = \infty\})$ are the (analytic continuation of the) boundary Green functions defined in eq.s (1.16).

The reason why we have written the previous expression in a non normal ordered way is to understand the expression of the regularization factor of the corresponding ‘‘classical’’ vertex (4.11) which we want to insert in the path integral.

The previous operator can also be written in a way to make its connection with the idea of undoing the OPE clearer as

$$\begin{aligned} \mathcal{T}_{(aux\ t)}(d_{(t)}, x_f) &= \mathcal{R}(d_{(t)}, x_f) \mathcal{S}_{(aux\ t)}(c_{(f)}(d_{(t)}, x_f), x_f) \\ &= \mathcal{R}(d_{(t)}, x_f) e^{\sum_{n=0}^{\infty} \bar{d}_{(t)n} \partial_x^n |_{x=x_f} [x^{-\epsilon t} Z_{(aux\ t)}(x, x)] + d_{(f)n} \partial_x^n |_{x=x_f} [x^{-(1-\epsilon t)} \bar{Z}_{(aux\ t)}(x, x)]} \end{aligned} \quad (4.8)$$

where

$$\begin{aligned} c_{(f)0}(d_{(t)}, x_f) &= d_{(t)0} \quad \bar{c}_{(f)0}(d_{(t)}, x_f) = \bar{d}_{(t)0} \\ c_{(f)n}(d_{(t)}, x_f) &= \sum_{k=n}^{\infty} \binom{k-1}{n-1} d_{(t)k} \partial^{k-n} x_f^{-(1-\epsilon t)} \\ \bar{c}_{(f)n}(d_{(t)}, x_f) &= \sum_{k=n}^{\infty} \binom{k-1}{n-1} \bar{d}_{(t)k} \partial^{k-n} x_f^{\epsilon t} \end{aligned} \quad (4.9)$$

and the normalization factor is

$$\mathcal{R}(d_{(t)}, x_f) = e^{-\sum_{m,n=0}^{\infty} d_{(t)n} \bar{d}_{(t)m} \partial_x^{n-1} \partial_y^{m-1} \left[x^{1-\epsilon_t} y^{\epsilon_t} \partial_x \partial_y G_{bou, reg}^{z\bar{z}}(x; y; \{x=0, x=\infty\}) \right]} \Big|_{x=y=x_f} \quad (4.10)$$

in order to undo the OPE and get the desired result as in eq. (4.5).

We shall now translate the previous operator (4.6) into an abstract operator we can insert in the path integral at an arbitrary point x_t , therefore we move it from $x_t = 0$ to a generic x_t and we consider a generating vertex as

$$\begin{aligned} \mathcal{T}(d_{(t)}, x_t) &= \lim_{x \rightarrow x_t^+} \mathcal{N}_{\mathcal{T}}(d_{(t)}, x, x_t) e^{\bar{d}_{(t)0} \langle Z(x, \bar{x}) \rangle + d_{(t)0} \langle \bar{Z}(x, \bar{x}) \rangle} \\ &e^{\sum_{n=1}^{\infty} \bar{d}_{(t)n} \partial_x^{n-1} \langle (x-x_t)^{1-\epsilon_t} \partial_x Z(x, \bar{x}) \rangle + d_{(t)n} \partial_x^{n-1} \langle (x-x_t)^{\epsilon_t} \partial_x \bar{Z}(x, \bar{x}) \rangle} \end{aligned} \quad (4.11)$$

where $x \rightarrow x_t$ has to be understood as taking the limit after the path integral has been computed. We have defined the averaged fields such as

$$\langle (x-x_t)^{1-\epsilon_t} \partial_x Z(x, \bar{x}) \rangle = \int_{\partial H} dy \delta_{reg}(x-y) (y-x_t)^{1-\epsilon_t} \partial_y Z(y, \bar{y}) \quad (4.12)$$

because we want a well defined regulated expression after performing the path integral and introduced the normalization factor

$$\begin{aligned} \mathcal{N}_{\mathcal{T}}(d_{(t)}, x, x_t) &= e^{-\frac{1}{2} \bar{d}_{(t)0}^2 \langle \langle G_{N=2}^{zz} \text{ bou}(x; x) \rangle \rangle - \frac{1}{2} d_{(t)0}^2 \langle \langle G_{N=2}^{\bar{z}\bar{z}} \text{ bou}(x; x) \rangle \rangle - \bar{d}_{(t)0} d_{(t)0} \langle \langle G_{N=2}^{z\bar{z}} \text{ bou}(x; x) \rangle \rangle} \\ &e^{-\frac{1}{2} \sum_{n=1}^{\infty} \bar{d}_{(t)n} d_{(t)n} \partial_x^{n-1} \langle \langle (x-x_t)^{\epsilon_t} \partial_x G_{N=2}^{\bar{z}\bar{z}} \text{ bou}(x; y) \rangle \rangle} \\ &e^{-\frac{1}{2} \sum_{n=1}^{\infty} d_{(t)n} \bar{d}_{(t)n} \partial_x^{n-1} \langle \langle (x-x_t)^{1-\epsilon_t} \partial_x G_{N=2}^{zz} \text{ bou}(x; y) \rangle \rangle} \\ &e^{-\frac{1}{2} \sum_{n,l=1}^{\infty} d_{(t)l} \bar{d}_{(t)n} \partial_y^{l-1} \partial_x^{n-1} \langle \langle (y-x_t)^{\epsilon_t} (x-x_t)^{1-\epsilon_t} \partial_x \partial_y G_{N=2}^{z\bar{z}} \text{ bou}(x; y) \rangle \rangle} \Big|_{y=x} \end{aligned} \quad (4.13)$$

where the doubly regularized Green functions are defined such as

$$\begin{aligned} \langle \langle G_{N=2}^{zz} \text{ bou}(x; x) \rangle \rangle &= \int dy_1 \int dy_2 \delta_{reg}(x-y_1) \delta_{reg}(x-y_2) \\ &G_{bou}^{zz}(y_1; y_2; \{x_1 = x_t, x_2 = \infty\}) \\ \langle \langle (y-x_t)^{\epsilon_t} (x-x_t)^{1-\epsilon_t} \partial_x \partial_y G_{N=2}^{z\bar{z}} \text{ bou}(x; y) \rangle \rangle &= \int dy_1 \int dy_2 \delta_{reg}(x-y_1) \delta_{reg}(y-y_2) \\ &(y_2-x_t)^{\epsilon_t} (y_1-x_t)^{1-\epsilon_t} \partial_1 \partial_2 G_{bou}^{z\bar{z}}(y_1; y_2; \{x_1 = x_t, x_2 = \infty\}) \end{aligned} \quad (4.14)$$

Since the previous expression is in nuce the same as for the untwisted matter, we can immediately deduce from (3.16) the result of inserting and integrating over the X to be eq. (1.30).

In analogy with what done for the untwisted states we can realize the algebra

$$\left[d_{(t)n}, \frac{\overleftarrow{\partial}}{\partial d_{(u)m}} \right] = \left[\bar{d}_{(t)n}, \frac{\overleftarrow{\partial}}{\partial \bar{d}_{(u)m}} \right] = \delta_{m,n} \delta_{u,t} \quad (4.15)$$

with operators acting on the twisted scalar (discharged string) auxiliary Hilbert spaces \mathcal{H}_t as

$$\begin{aligned}
& 1 \rightarrow \langle T_{\epsilon_t}, x_{(t)0}^\dagger = 0 | \\
& \bar{d}_{(t)n} \rightarrow \frac{i}{\sqrt{2\alpha'}} \frac{\alpha_{(t)n-1+\epsilon_t}}{\cos \gamma_t (n-1)! (n-1+\epsilon_t)}, \quad d_{(t)n} \rightarrow \frac{i}{\sqrt{2\alpha'}} \frac{\bar{\alpha}_{(t)n-\epsilon_t}}{\cos \gamma_t (n-1)! (n-\epsilon_t)} \quad n > 0 \\
& \frac{\overleftarrow{\partial}}{\partial \bar{d}_{(t)m}} \rightarrow -i\sqrt{2\alpha'} (m-1)! \cos \gamma_t \alpha_{(t)m-1+\epsilon_t}^\dagger, \quad \frac{\overleftarrow{\partial}}{\partial d_{(t)m}} \rightarrow -i\sqrt{2\alpha'} (m-1)! \cos \gamma_t \bar{\alpha}_{(t)m-\epsilon_t}^\dagger \quad m > 0
\end{aligned} \tag{4.16}$$

which gives eq. (1.29) when substituted into eq. (1.30).

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A. Check of the $N = 2$ amplitudes

We would now check that the operatorial amplitudes with $N = 2$ and the path integral approach give the same result, phases included. Let us consider the tachyonic amplitude

$$\langle \sigma_{-\epsilon, \lambda}(x_\infty, x_\infty) \sigma_{\epsilon, \kappa}(x_0, x_0) V_T(x_1; k_{(1)}) \dots V_T(x_M; k_{(M)}) \rangle \tag{A.1}$$

with $x_{t=1} = x_0$, $x_{t=2} = x_\infty$ and γ_0 and γ_1 arbitrary but $\gamma_2 = \gamma_0$ so that $\pi\epsilon_1 = -\pi\epsilon_2$. Using the results from ([12]) we can compute it in the limit $x_0 \rightarrow 0$ and $x_\infty \rightarrow \infty$, for $x_1 > \dots x_M > 0$ and when multiplied by the appropriate power of x_∞ as

$$\begin{aligned}
& \langle T_\epsilon, -\lambda | e^{-\frac{1}{2}R^2(\epsilon)} \Delta(k_{(1)}) x_1^{-\Delta(k_{(1)})} e^{i(\bar{k}_{(1)}z_0 + k_{(1)}\bar{z}_0)} e^{i \cos \gamma_1 [\bar{k}_{(1)}Z_{nzm}(x_1, x_1) + k_{(1)}\bar{Z}_{nzm}(x_1, x_1)]} \dots | T_\epsilon, \kappa \rangle \\
& = \delta(\kappa + \kappa + \sum_a k_{(a)2}) \\
& \prod_a e^{\frac{1}{2}\pi\alpha' \frac{1}{\tan \gamma_1 - \tan \gamma_0} (k_{(a)}^2 - \bar{k}_{(a)}^2)} \\
& \prod_a \left[e^{-\frac{1}{2}R^2(\epsilon)} \Delta(k_{(a)}) x_a^{-\Delta(k_{(a)})} \right] \\
& \prod_a e^{-\pi\alpha' \frac{1}{\tan \gamma_1 - \tan \gamma_0} \frac{\kappa}{\sqrt{2}} (k_{(a)} - \bar{k}_{(a)})} \\
& \prod_{a < b} \left[e^{\pi\alpha' \frac{1}{\tan \gamma_1 - \tan \gamma_0} (k_{(a)} - \bar{k}_{(a)}) (k_{(b)} + \bar{k}_{(b)})} e^{\alpha' \cos^2 \gamma_1 (k_{(a)} \bar{k}_{(b)} g_{1-\epsilon} \left(\frac{x_b}{x_a} \right) + \bar{k}_{(a)} k_{(b)} g_\epsilon \left(\frac{x_b}{x_a} \right))} \right] \tag{A.2}
\end{aligned}$$

where we have used the commutation relations (1.22), $R^2(\epsilon) = \lim_{u \rightarrow 1^-} [g_\epsilon(u) + g_{1-\epsilon}(u) - 2 \log(1-u)] = -(\psi(\epsilon) + \psi(1-\epsilon) - 2\psi(1))$ and $\Delta(k_{(a)}) = 2\alpha' \cos^2 \gamma k_{(a)} \bar{k}_{(a)}$ is the conformal dimension of the tachyonic vertex.

We can now compare with the general expression (1.30) with the identifications $c_{(a)0} \rightarrow i k_{(a)}$, $\bar{c}_{(a)0} \rightarrow i \bar{k}_{(a)}$, $d_{(1)0} \rightarrow i \frac{\kappa}{\sqrt{2}}$, $\bar{d}_{(1)0} \rightarrow i \frac{\kappa}{\sqrt{2}}$, $d_{(2)0} \rightarrow i \frac{\lambda}{\sqrt{2}}$ and $\bar{d}_{(2)0} \rightarrow i \frac{\lambda}{\sqrt{2}}$. We can also compare with the expression (1.29) upon the product with the state...

in order to understand where the different terms come from the path integral point of view. In matching these terms is important to be careful in rewriting all the Green functions $G(x; y)$ in such a way that $x > y$ by using the symmetry (1.2) since this is the natural way they appear from the operatorial formalism. We recognize that

- the factor $\exp\{\frac{1}{2}\pi\alpha' \frac{1}{\tan\gamma_1 - \tan\gamma_0} (k_{(a)}^2 - \bar{k}_{(a)}^2)\}$ come from the $G_{bou\ reg\ U(t_a)}^{zz}$ and $G_{bou\ reg\ U(t_a)}^{z\bar{z}}$ terms.
- The factors $\prod_a \left[e^{-\frac{1}{2}R^2(\epsilon) \Delta(k_{(a)}) x_a^{-\Delta(k_{(a)})}} \right]$ come from the $G_{bou\ reg\ U(t_a)}^{z\bar{z}}$ terms. In particular the result follows from the following steps

$$\begin{aligned}
G_{bou\ reg\ U(t_a)}^{z\bar{z}}(x_a^+, x_a) &= \frac{\pi\alpha'}{\tan\gamma_1 - \tan\gamma_0} - \pi\alpha' \sin\gamma_1 \cos\gamma_1 + 2\alpha' \cos^2\gamma_1 \ln|x_a| \\
&\quad - 2\alpha' \cos^2\gamma_1 \left(g_\epsilon \left(\frac{x_a}{x_a^+} \right) - \log \left(1 - \frac{x_a}{x_a^+} \right) \right) \\
&= \frac{\pi\alpha'}{\tan(\gamma_1 - \gamma_0)} \cos^2\gamma_1 + 2\alpha' \cos^2\gamma_1 \ln|x_a| \\
&\quad - 2\alpha' \cos^2\gamma_1 (\psi(1 - \epsilon) - \psi(1)) \\
&= 2\alpha' \cos^2\gamma_1 \ln|x_a| - \alpha' \cos^2\gamma_1 (\psi(\epsilon) + \psi(1 - \epsilon) - 2\psi(1))
\end{aligned} \tag{A.3}$$

where we have used the expression for the Green function given in eq. (1.16) since we have chosen $x = x_a^+$ and in the last line we have used the digamma property $\psi(1 - \epsilon) = \psi(\epsilon) + \pi \cot(\pi\epsilon)$. It is also worth stressing that the result is independent on setting x_a^+ in the first argument since the function $G_{bou\ reg\ U(t_a)}^{z\bar{z}}$ is continued analytically at $x = y = x_a$ in such a way that $G^{ij}(x; y) = G^{ji}(y; x)$ so that would we have chosen the first argument to be x_a^- we would have got the same result computing $G_{bou\ reg\ U(t_a)}^{z\bar{z}}(x_a, x_a^-)$.

- The terms $e^{-\pi\alpha' \frac{1}{\tan\gamma_1 - \tan\gamma_0} \frac{\kappa}{\sqrt{2}} (k_{(a)} - \bar{k}_{(a)})}$ arise from the $\prod_{t,a}$ terms. While rewriting the Green functions $G(x; y)$ in such a way that $x > y$ we see that the terms with $t = 2$ (at x_∞) cancel while those from $t = 1$ (at x_0) do not and reproduce the operatorial result.
- The terms $\prod_{a < b}$ come trivially from the corresponding ones in eq. (1.30).
- The terms \prod_t in (1.30) give a trivial result in a non trivial way. It is immediate to find that $G_{bou, reg}^{zz}(t) = G_{bou, reg}^{z\bar{z}}(t) = 0$. On the other side we get for $0 < \frac{y-x_0}{x-x_0} < 1$ and $0 < \omega = \frac{y-x_0}{x-x_0} \frac{x-x_\infty}{y-x_\infty} < 1$

$$G_{bou, reg}^{z\bar{z}}(t)(x; y; \{x_0, x_\infty\}) = -2\alpha' \cos^2\gamma \left[g_\epsilon(\omega) - g_\epsilon\left(\frac{y-x_0}{x-x_0}\right) \right] \tag{A.4}$$

Now we can write $x = x_0 + \alpha$, $y = x_0 + \alpha(1 - \delta)$ with $\alpha > 0$ and $0 < \delta < 1$ and expand in δ to get

$$G_{bou, reg}^{z\bar{z}}(t)(x; y; \{x_0, x_\infty\}) = -2\alpha' \cos^2\gamma \left[-\log \left(1 + \frac{\alpha}{x_\infty - x_0} \right) + \frac{(\epsilon - 1)\alpha}{x_\infty - x_0 + \alpha} \delta + O(\delta^2) \right] \tag{A.5}$$

which vanishes when $\alpha = \delta = 0$. In a similar way can be treated the case when $x, y \rightarrow x_\infty$.

- The terms $\prod_{t < u}$ in (1.30) give also a trivial result in a not completely trivial fashion. We have to evaluate the Green functions for $\omega = \frac{y-x_0}{x-x_0} \frac{x-x_\infty}{y-x_\infty}$ when $x = x_0$ and $y = x_\infty$ so $\omega = 0$ and we are left with only the constant terms, as we expect from the general asymptotic (1.31) hence $\prod_{t < u} = \exp\left\{\frac{\pi\alpha'}{\tan\gamma_1 - \tan\gamma_0} [-d_{(1)0}d_{(2)0} + \bar{d}_{(1)0}\bar{d}_{(2)0} - d_{(1)0}\bar{d}_{(2)0} + \bar{d}_{(1)0}d_{(2)0}]\right\}$ which vanishes when evaluated with the previously stated substitutions for which $d_{(t)0} = \bar{d}_{(t)0}$.

B. Behavior of the Green function when $x, y \rightarrow x_t$

In this section we follow and adapt the computation done in ([6]). We start considering the derivative of Green function of the left moving part defined as

$$\partial_z \partial_w G_{LL}^{z\bar{z}}(z; w; \{x_t\}_{t=1\dots N}) = \frac{\langle \partial_z Z_L(z) \partial_w \bar{Z}_L(w) \sigma_{\epsilon_1, \kappa_1}(x_1, \bar{x}_1) \dots \sigma_{\epsilon_N, \kappa_N}(x_N, \bar{x}_N) \rangle_{disk}}{\langle \sigma_{\epsilon_1, \kappa_1}(x_1, \bar{x}_1) \dots \sigma_{\epsilon_N, \kappa_N}(x_N, \bar{x}_N) \rangle_{disk}} \quad (\text{B.1})$$

which has asymptotics

$$\begin{aligned} -\frac{1}{2\alpha'} \partial_z \partial_w G_{LL}^{z\bar{z}}(z; w; \{x_t\}_{t=1\dots N}) &\sim_{z \rightarrow w} \frac{1}{(z-w)^2} + O(1) \\ &\sim_{z \rightarrow x_t} (z-x_t)^{\epsilon_t-1} \\ &\sim_{w \rightarrow x_t} (w-x_t)^{-\epsilon_t} \end{aligned} \quad (\text{B.2})$$

then we can write

$$\begin{aligned} -\frac{1}{2\alpha'} \partial_z \partial_w G_{LL}^{z\bar{z}}(z; w; \{x_t\}_{t=1\dots N}) &= \prod_u \left(\frac{w-x_u}{z-x_u} \right)^{1-\epsilon_u} \left[\frac{1}{(z-w)^2} \sum_{u < v} a_{uv} \frac{(z-x_u)(z-x_v)}{(w-x_u)(w-x_v)} \right. \\ &\quad \left. + \sum_{u_1 < u_2 < u_3 < u_4} \frac{b_{u_1 u_2 u_3 u_4}}{(w-x_{u_1})(w-x_{u_2})(w-x_{u_3})(w-x_{u_4})} \right] \end{aligned} \quad (\text{B.3})$$

Now we can study the behavior $x, y \rightarrow x_1$ by setting

$$z = x_1 + \alpha, \quad w = x_1 + \alpha(1 - \delta) \quad (\text{B.4})$$

and letting $\alpha, \delta \rightarrow 0^+$. A simple computation gives

$$-\frac{1}{2\alpha'} \partial_z \partial_w G_{LL}^{z\bar{z}}(z; w; \{x_t\}_{t=1\dots N}) \sim \frac{\sum_{u < v} a_{uv}}{\delta^2} + \frac{1}{\alpha} \frac{1}{1-\delta} \sum_{1 < u_2 < u_3 < u_4} \frac{b_{1u_2u_3u_4}}{(x_1-x_{u_2})(x_1-x_{u_3})(x_1-x_{u_4})} + O(1) \quad (\text{B.5})$$

from which we deduce the constraint

$$\sum_{u < v} a_{uv} = 1. \quad (\text{B.6})$$

Then the regularized Green function which is obtained by subtracting the corresponding Green function with only two twist, one of which in x_1 and the other at ∞ is of the form

$$\partial_z \partial_w G_{LL, reg}^{z\bar{z}}(z; w; \{x_t\}_{t=1\dots N}) \sim \frac{B_1(x_u)}{\alpha} \frac{1}{1-\delta} + O(1) \quad (\text{B.7})$$

hence the terms in \prod_t are well defined since

$$(z - x_1)^{1-\epsilon_1} (w - x_1)^{\epsilon_1} \partial_z \partial_w G_{LL, reg}^{z\bar{z}}(z; w; \{x_t\}_{t=1\dots N}) \sim \alpha (1 - \delta)^{\epsilon_1} \left[\frac{B_1(x_u)}{\alpha} \frac{1}{1-\delta} + O(1) \right]. \quad (\text{B.8})$$

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