

## A search for $AdS_5 \times S^2$ IIB supergravity solutions dual to $\mathcal{N} = 2$ SCFTs

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### Abstract

We present a systematic search for Type IIB supergravity solutions whose spacetimes include  $AdS_5$  and  $S^2$  factors, which would be candidate duals to  $\mathcal{N} = 2$  four-dimensional Superconformal field theories. The candidate solutions encode the  $SU(2)$  R-symmetry geometrically on the  $S^2$  and an additional Killing vector generates the  $U(1)$  R-symmetry. By analysing the Killing spinor equations we show that no such solutions exist. This suggests that if Type IIB backgrounds dual to  $\mathcal{N} = 2$  SCFTs exist, the  $SU(2)$  R-symmetry is realised non-geometrically. Finally, we also show that, in the context of both  $\mathcal{N} = 1$  and  $\mathcal{N} = 2$  Type IIB backgrounds with an  $AdS_5$  factor, the only candidate  $U(1)$  R-symmetry Killing vector directions are the ones that appear for generic values of the Killing spinors; no further Killing vectors exist for special values of the Killing spinors.

# 1 Introduction & Summary

Recently, there has been a renewed interest in  $\mathcal{N} = 2$  Superconformal field theories (SCFTs) coming from a number of directions. Wilson- and 't Hooft-loop computations in these theories have been performed using localization techniques in [1, 2] and matrix models [3]. Localization methods of [1] have also provided the basis for the AGT conjecture [4, 5] which relates supersymmetric quantities of four-dimensional  $\mathcal{N} = 2$  SCFTs to correlators in two-dimensional SCFTs [6, 7]. Investigations of S-duality properties of general  $\mathcal{N} = 2$  SCFTs have led to a better understanding of the moduli space of these theories [8, 9], and in particular to a conjecture about the existence of families of strongly coupled  $\mathcal{N} = 2$  SCFTs without a Lagrangian description [10]. Novel connections between integrable systems and  $\mathcal{N} = 2$  gauge theories have been discovered in [11, 12, 13, 14, 15, 16].

Within the context of the *AdS/CFT* correspondence, a proposal for the supergravity duals of  $\mathcal{N} = 2$  SCFTs has been made in M-theory [17] and in the Type IIA reduction [18]<sup>1</sup>. These spacetimes were found using an approach developed for the study of the Killing spinor equations (KSEs) of supergravity using spinor bilinears and the ansatz of [21].<sup>2</sup> Further, it has been shown in [24] that [21] is indeed the most general solution with the chosen spacetime ansatz. Given the existence of these M-theory and IIA solutions, a natural question is whether solutions dual to  $\mathcal{N} = 2$  SCFTs can be found in Type IIB supergravity. In this paper we perform a systematic search for Type IIB supergravity solutions that have  $\mathcal{N} = 2$  SCFT duals. We do this by considering spacetimes with  $AdS_5$  and  $S^2$  factors which realise the  $SO(2, 4)$  conformal and  $SU(2)$  R-symmetries geometrically. A detailed analysis of the KSEs and resulting bispinor relations reveals that no solutions beyond the maximally supersymmetric  $AdS_5 \times S^5$  solution exist. The earlier paper [25] showed that no IIB supergravity solutions with  $AdS_5 \times S^2 \times S^1$  and  $AdS_5 \times S^3$  factors (with each factor warped over a two-dimensional Riemann surface) exist. The first type of spacetime is a sub-class of the ansatz that we take in this paper,<sup>3</sup> and our results are consistent with the lack of solutions of this type.

At first sight, one might expect other ways of realising the  $SU(2)$  R-symmetry geometrically. However, in appendix F of [21] it is shown that the  $U(1)$  Killing direction cannot be fibred over the  $S^2$  associated with the  $SU(2)$  R-symmetry. Although this more general ansatz preserves  $SU(2)$  symmetry, if the Killing spinor is charged under translations in the  $U(1)$ -direction, in other words, if the  $U(1)$  is an R-symmetry, then the supercharges cannot form an  $SU(2)$  doublet. To see an example of this, one might consider a IIB spacetime containing an  $S^3$  squashed along its Hopf fibre. While this background preserves an  $SU(2) \times U(1)$  isometry the corresponding Killing spinors [26, 27] transform as singlets of

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<sup>1</sup>Singular solutions obtained via non-Abelian T-duality [19] from  $AdS_5 \times S^5$  appeared in [20].

<sup>2</sup>M2 and M5-brane probes in these backgrounds corresponding to loop and surface operators in the dual  $\mathcal{N} = 2$  SCFT have been studied in [22] and [23].

<sup>3</sup>We allow for the possibility of the  $U(1)$  R-symmetry direction to be fibred and warped.

$SU(2)$ .<sup>4</sup> As a result, the only way to realise the  $SU(2)$  R-symmetry geometrically in a way that is consistent with the superconformal algebra is by including a round  $S^2$  factor in the metric.

Since we do not find any IIB solutions with such an  $S^2$  factor beyond the maximally supersymmetric  $AdS_5 \times S^5$  solution we conclude that if IIB supergravity duals do exist for generic  $\mathcal{N} = 2$  SCFTs the  $SU(2)$  R-symmetry is realised non-geometrically. The possibility that the  $SU(2)$  R-symmetry is realised non-geometrically, was raised in the context of non-conformal  $\mathcal{N} = 2$  SYM theory in [28, 29]. It is possible that by including sources into the supergravity equations and realising the  $SU(2)$  on the corresponding branes one may realise the  $\mathcal{N} = 2$  superconformal algebra without an  $S^2$  factor in the spacetime. Or perhaps in the context of IIB the R-symmetry is only realised in the full string theory rather than the supergravity? A mild caveat to the above is that we take the internal space to be compact. Relaxing such a constraint, one still has the possibility of non-compact solutions with field content incorporating a constant five-form flux and non-constant harmonic axion and dilation, but only in the case when the complex three-form flux is zero [30].

The structure of this paper runs as follows. In section 2 we review the general reduction of IIB supergravity used in [30] for spacetimes with an  $AdS_5$  factor. In section 3 we reduce further on a round  $S^2$ , and write down the resulting algebraic and differential KSEs. Using these, in section 4 we look for potential Sasaki-Einstein type solutions. In section 5 we search for general solutions preserving our ansatz. Using some bispinor algebra we identify *two* putative Killing vector directions and find the conditions necessary for each of these to correspond to a global  $U(1)$  symmetry. In section 6 we show that the global  $U(1)$  symmetry constraints imply, that either the solution is the maximally supersymmetric  $AdS_5 \times S^5$  or the Killing vectors are zero. Finally, in section 7 we show that when these two Killing vectors are zero, the KSEs imply that the Killing spinors are also zero. This completes the demonstration that no solutions of the abovementioned form exist in IIB supergravity. In section 7 we also show that a similar argument holds more generally in the case of  $\mathcal{N} = 1$   $AdS_5$  backgrounds originally investigated in [30]. In that setting a unique Killing vector was identified (it was denoted as  $K_5$  in [30] and we will write it as  $K_5^{GMSW}$ ). One may want to ask what happens when we restrict the Killing spinors in a way that makes the vector equal to zero. Our analysis shows that in that case too, the KSEs imply that all the Killing spinors are zero, and hence, that no such solution exists.

## 2 Type IIB Review

In order to define some notation, in this section we review the construction of [30] for KSEs in Type IIB supergravity [31] with a spacetime containing an  $AdS_5$  factor. We will be essentially following the notation and conventions of [30]. Solutions of type IIB

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<sup>4</sup>We are grateful to Linda Uruchurtu for an explanation of this and related discussions.

supergravity in Einstein frame, preserve supersymmetry as long as the following variations vanish

$$\begin{aligned}\delta\psi_M &\simeq D_M\epsilon - \frac{1}{96}(\Gamma_M^{P_1P_2P_3}G_{P_1P_2P_3} - 9\Gamma^{P_1P_2}G_{MP_1P_2})\epsilon^c + \frac{i}{192}\Gamma^{P_1P_2P_3P_4}F_{MP_1P_2P_3P_4}\epsilon, \\ \delta\lambda &\simeq i\Gamma^M P_M\epsilon^c + \frac{i}{24}\Gamma^{P_1P_2P_3}G_{P_1P_2P_3}\epsilon,\end{aligned}\tag{2.1}$$

where  $F$  denotes the self-dual five-form flux,  $G$  the complex three-form flux and  $P$  the complex axion-dilaton. In terms of the conventional string-theory variables, these latter two may be further expressed as [32]

$$\begin{aligned}P &= -iQ + \frac{1}{2}d\phi = \frac{i}{2}e^\phi dC^{(0)} + \frac{1}{2}d\phi, \\ G &= ie^{\phi/2}(\tau dB - dC^{(2)}),\end{aligned}\tag{2.2}$$

where  $\tau \equiv C^{(0)} + ie^{-\phi}$ . In addition, there is a manifest  $SL(2, \mathbb{R})$  action transforming the constituents of  $P$  and  $G$  [32].

One also has a local  $U(1)$  invariance associated to the gauge field  $Q_M$ , with the spinor  $\epsilon$ , fields  $P$  and  $G$  charged with charge  $\frac{1}{2}$ , 2 and 1 respectively.  $D$  above denotes the covariant derivative incorporating this local  $U(1)$  transformation, i.e.

$$D_M\epsilon = \left(\nabla_M - \frac{i}{2}Q_M\right)\epsilon.\tag{2.3}$$

Once the supersymmetry conditions are imposed, to ensure a genuine supergravity solution, one has to guarantee that the following field equations of motion,

$$\begin{aligned}F &= *_{10}F, \\ D *_{10}G &= P \wedge *_{10}G^* + iF \wedge G, \\ D *_{10}P &= -\frac{1}{4}G \wedge *_{10}G,\end{aligned}\tag{2.4}$$

and Einstein equation,

$$\begin{aligned}R_{MN} &= P_M P_N^* + P_N P_M^* + \frac{1}{96}F_{MP_1P_2P_3P_4}F_N^{P_1P_2P_3P_4} \\ &\quad + \frac{1}{8}\left(G_M^{P_1P_2}G_{NP_1P_2}^* + G_N^{P_1P_2}G_{MP_1P_2}^* - \frac{1}{6}g_{MN}G^{P_1P_2P_3}G_{P_1P_2P_3}^*\right),\end{aligned}\tag{2.5}$$

are satisfied. Lastly, one also needs to impose the Bianchi identities

$$\begin{aligned}dQ &= -iP \wedge P^*, \\ DP &= 0, \\ DG &= -P \wedge G^*, \\ dF &= \frac{i}{2}G \wedge G^*.\end{aligned}\tag{2.6}$$

We remark that the Bianchi for  $P$  is trivially satisfied.

However, not all these conditions on the geometry are independent. By examining integrability of the supersymmetry conditions, it was shown in appendix D of [30] that for spacetimes with  $AdS_5$  factors that the equations of motion are a consequence of supersymmetry. Our analysis will later make use of these equations of motion, so we recast them later in terms of the three-dimensional field content.

### 3 Supersymmetry conditions in lower dimensions

In this section we review the reduction of the supersymmetry conditions on  $AdS_5$  presented in [30], and extend those results by further decomposing on a round  $S^2$  to three-dimensions. We begin with the ansatz used in [30] for the ten-dimensional spacetime

$$\begin{aligned} ds_{10}^2 &= e^{2\Delta}[ds^2(AdS_5) + ds^2(M_5)], \\ F &= (vol_{AdS_5} + vol_{M_5})f. \end{aligned} \quad (3.1)$$

Above  $f$  constant, and  $Q, P$  and  $G$  all can take values on  $M_5$ . The ten-dimensional MW spinors are decomposed into Killing spinors on  $AdS_5$  and spinors on  $M_5$ ; the latter are denoted as  $\xi_i$ . As was shown in [30] this ansatz led to two differential,

$$D_m \xi_1 + \frac{i}{4} (e^{-4\Delta} f - 2m) \gamma_m \xi_1 + \frac{1}{8} e^{-2\Delta} G_{mnp} \gamma^{np} \xi_2 = 0 \quad (3.2)$$

$$\bar{D}_m \xi_2 - \frac{i}{4} (e^{-4\Delta} f + 2m) \gamma_m \xi_2 + \frac{1}{8} e^{-2\Delta} G_{mnp}^* \gamma^{np} \xi_1 = 0 \quad (3.3)$$

and four algebraic conditions,

$$\gamma^m \partial_m \Delta \xi_1 - \frac{1}{48} e^{-2\Delta} \gamma^{mnp} G_{mnp} \xi_2 - \frac{i}{4} (e^{-4\Delta} f - 4m) \xi_1 = 0 \quad (3.4)$$

$$\gamma^m \partial_m \Delta \xi_2 - \frac{1}{48} e^{-2\Delta} \gamma^{mnp} G_{mnp}^* \xi_1 + \frac{i}{4} (e^{-4\Delta} f + 4m) \xi_2 = 0 \quad (3.5)$$

$$\gamma^m P_m \xi_2 + \frac{1}{24} e^{-2\Delta} \gamma^{mnp} G_{mnp} \xi_1 = 0 \quad (3.6)$$

$$\gamma^m P_m^* \xi_1 + \frac{1}{24} e^{-2\Delta} \gamma^{mnp} G_{mnp}^* \xi_2 = 0. \quad (3.7)$$

In this paper, we consider  $M_5$  to contain an  $S^2$  factor. Similar two-sphere decompositions have appeared in [21, 24, 33], where the basic idea is to further split the spinors  $\xi_i$  by decomposing them as

$$\xi_i = \chi_+ \otimes \epsilon_{i+} + \chi_- \otimes \epsilon_{i-}, \quad (3.8)$$

where  $\chi_{\pm}$  denote solutions to the Killing spinor equation on  $S^2$

$$\nabla_{\alpha} \chi_{\pm} = \pm \frac{i}{2} \sigma_{\alpha} \chi_{\pm}. \quad (3.9)$$

Explicit expressions for  $\chi_{\pm}$  can be found in [35]. Without loss of generality we take  $\chi_- = \sigma_3 \chi_+$ . Out of the spinors  $\epsilon_{i\pm}$  it is possible to construct a number of spinor bilinears

that transform either as scalars or vectors on  $M_3$ . We refer the reader to the appendix for a complete list of these as well as our gamma matrix conventions.

To complete the task in the light of the LLM observation [21] that there should be no  $U(1)$  fibre over the  $S^2$ , we adopt the warped-product ansatz for the bosonic sector

$$\begin{aligned} ds^2(M_5) &= e^{2B} ds^2(S^2) + ds^2(M_3), \\ G &= \mathcal{A} \wedge \text{vol}_{S^2} + g \text{vol}_{M_3}. \end{aligned} \quad (3.10)$$

Furthermore, we use calligraphic notation to distinguish the  $D = 3$  fields  $\mathcal{P}, \mathcal{Q}$  from the  $D = 5$  fields  $P, Q$ .

After following the decomposition through, one may extract the three-dimensional supersymmetry conditions. We now have four differential,

$$0 = D_m \epsilon_{1\pm} + \frac{i}{4}(e^{-4\Delta} f - 2m) \sigma_m \epsilon_{1\mp} + \frac{i}{4} e^{-2\Delta} [g \sigma_m \epsilon_{2\pm} - e^{-2B} \mathcal{A}_m \epsilon_{2\mp}], \quad (3.11)$$

$$0 = \bar{D}_m \epsilon_{2\pm} - \frac{i}{4}(e^{-4\Delta} f + 2m) \sigma_m \epsilon_{2\mp} + \frac{i}{4} e^{-2\Delta} [g^* \sigma_m \epsilon_{1\pm} - e^{-2B} \mathcal{A}_m^* \epsilon_{1\mp}], \quad (3.12)$$

and twelve algebraic constraints

$$0 = [\pm i e^{-B} + \frac{i}{2}(e^{-4\Delta} f - 2m)] \epsilon_{1\pm} + \sigma^m \partial_m B \epsilon_{1\mp} - \frac{i}{2} e^{-2\Delta - 2B} \sigma^m \mathcal{A}_m \epsilon_{2\pm}, \quad (3.13)$$

$$0 = [\pm i e^{-B} - \frac{i}{2}(e^{-4\Delta} f + 2m)] \epsilon_{2\pm} + \sigma^m \partial_m B \epsilon_{2\mp} - \frac{i}{2} e^{-2\Delta - 2B} \sigma^m \mathcal{A}_m^* \epsilon_{1\pm}, \quad (3.14)$$

$$0 = \sigma^m \partial_m \Delta \epsilon_{1\pm} - \frac{i}{8} e^{-2\Delta} [g \epsilon_{2\pm} - e^{-2B} \sigma^m \mathcal{A}_m \epsilon_{2\mp}] - \frac{i}{4} (e^{-4\Delta} f - 4m) \epsilon_{1\mp}, \quad (3.15)$$

$$0 = \sigma^m \partial_m \Delta \epsilon_{2\pm} - \frac{i}{8} e^{-2\Delta} [g^* \epsilon_{1\pm} - e^{-2B} \sigma^m \mathcal{A}_m^* \epsilon_{1\mp}] + \frac{i}{4} (e^{-4\Delta} f + 4m) \epsilon_{2\mp}, \quad (3.16)$$

$$0 = \sigma^m \mathcal{P}_m \epsilon_{2\pm} + \frac{i}{4} e^{-2\Delta} [g \epsilon_{1\pm} - e^{-2B} \sigma^m \mathcal{A}_m \epsilon_{1\mp}], \quad (3.17)$$

$$0 = \sigma^m \mathcal{P}_m^* \epsilon_{1\pm} + \frac{i}{4} e^{-2\Delta} [g^* \epsilon_{2\pm} - e^{-2B} \sigma^m \mathcal{A}_m^* \epsilon_{2\mp}], \quad (3.18)$$

By combining some of the above algebraic constraints, one may also show that

$$0 = [\pm i e^{-B} - \frac{i}{2} e^{-4\Delta} f + 3im] \epsilon_{1\pm} + \sigma^m \partial_m (4\Delta + B) \epsilon_{1\mp} - \frac{i}{2} e^{-2\Delta} g \epsilon_{2\mp}, \quad (3.19)$$

$$0 = [\pm i e^{-B} + \frac{i}{2} e^{-4\Delta} f + 3im] \epsilon_{2\pm} + \sigma^m \partial_m (4\Delta + B) \epsilon_{2\mp} - \frac{i}{2} e^{-2\Delta} g^* \epsilon_{1\mp}. \quad (3.20)$$

The field equations of motion for our ansatz become

$$\begin{aligned} D(e^{4\Delta - 2B} * \mathcal{A}) &= e^{4\Delta - 2B} \mathcal{P} \wedge * \mathcal{A}^* - i f g \text{vol}(M_3), \\ D(e^{4\Delta + 2B} g) &= e^{4\Delta + 2B} g^* \mathcal{P} - i f \mathcal{A}, \\ D(e^{8\Delta + 2B} * \mathcal{P}) &= -\frac{1}{4} e^{4\Delta} [e^{-2B} \mathcal{A} \wedge * \mathcal{A} + e^{2B} g^2 \text{vol}(M_3)], \end{aligned} \quad (3.21)$$

while the Bianchi identities may be expressed as

$$\begin{aligned} d\mathcal{P} &= 2iQ \wedge \mathcal{P}, \\ d\mathcal{A} &= iQ \wedge \mathcal{A} - \mathcal{P} \wedge \mathcal{A}^*. \end{aligned} \quad (3.22)$$

Observe that  $g$  drops out from the Bianchi identities and only its derivative enters the equations of motion.

Finally, in orthonormal frame the Einstein equations may be written

$$\begin{aligned} & \eta_{\mu\nu} \left[ -4m^2 - \nabla_m \nabla^m \Delta - 2(4\partial_m \Delta + \partial_m B) \partial^m \Delta \right] \\ & = \eta_{\mu\nu} \left[ -\frac{1}{4} f^2 e^{-8\Delta} - \frac{1}{8} e^{-4\Delta} (|g|^2 + e^{-4B} \mathcal{A}_m \mathcal{A}^{*m}) \right] \end{aligned} \quad (3.23)$$

$$\begin{aligned} & \delta_{\alpha\beta} \left[ e^{-2B} - \nabla_m \nabla^m (\Delta + B) - 2(4\partial_m \Delta + \partial_m B) (\partial^m \Delta + \partial^m B) \right], \\ & = \delta_{\alpha\beta} \left[ \frac{1}{4} f^2 e^{-8\Delta} + \frac{3}{8} e^{-4\Delta-4B} \mathcal{A}_m \mathcal{A}^{*m} - \frac{1}{8} e^{-4\Delta} |g|^2 \right] \end{aligned} \quad (3.24)$$

$$\begin{aligned} & R_{mn} - 2\nabla_n \nabla_m (4\Delta + B) + 8\partial_n \Delta \partial_m \Delta - 2\partial_n B \partial_m B - \delta_{mn} [\nabla^p \nabla_p \Delta + 2\partial_p (4\Delta + B) \partial^p \Delta], \\ & = \mathcal{P}_m \mathcal{P}_n^* + \mathcal{P}_n \mathcal{P}_m^* + \frac{1}{4} f^2 e^{-8\Delta} \delta_{mn} + \frac{3}{8} |g|^2 e^{-4\Delta} \delta_{mn} \\ & + \frac{1}{8} e^{-4\Delta-4B} [2\mathcal{A}_m \mathcal{A}_n^* + 2\mathcal{A}_n \mathcal{A}_m^* - \delta_{mn} \mathcal{A}_p \mathcal{A}^{*p}]. \end{aligned} \quad (3.25)$$

Here we have used  $\mu, \nu$  for  $AdS_5$ ,  $\alpha, \beta$  for  $S^2$  and finally  $m, n$  for the remaining directions. We have taken the sphere's radius to be unity while the radius of  $AdS_5$  is  $m^{-1}$ .

Having made the conditions on any supersymmetric three-dimensional geometry explicit in this section, we turn our attention in the next section to the example of Sasaki-Einstein where there are only two independent spinors, and not four, as in the general case.

## 4 $M_5$ Sasaki-Einstein

In this section, by way of a warm-up, we address what happens when we set one of the  $D = 5$  spinors  $\xi_i$  to zero and the  $M_5$  geometry satisfies the Sasaki-Einstein Killing spinor equation. We will see from the spinor bilinear analysis below that one encounters multiple Killing directions, where the warp factor  $B$  depends on some of the Killing directions. From the Killing spinor equation on  $M_3$  we show that  $M_3$  is isomorphic to  $S^3$  and proceed to construct explicitly the  $S^3$  Killing vectors from the vector bilinears. Back in five-dimensions, this translates into  $M_5$  being simply  $S^5$  or some quotient (for example [34]).

We begin by recalling the observation in [30] that the Sasaki-Einstein geometries correspond to choosing  $\xi_2 = 0$  in  $D = 5$ , which, in turn, sets  $\epsilon_{2\pm} = 0$  in our  $D = 3$  notation. In this setting, we can also use arguments presented in [30] showing that the three-form flux,  $G$ , is zero, and the axion and dilaton are simply constants when  $M_5$  is compact<sup>5</sup>. The only remaining non-constant scalar is then the warp factor,  $B$ , with the overall ten-dimensional warp factor,  $\Delta$  becoming a constant

$$e^{-4\Delta} f = 4m. \quad (4.1)$$

As an aside, observe that, as  $d\Delta = 0$ , (4.1) is consistent with the Einstein equation in the  $AdS_5$  directions (3.23).

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<sup>5</sup>Harmonic functions  $f$  satisfy the maximum principle: if  $K$  is any compact subset of a connected set  $U$ , then  $f$ , restricted to  $K$ , is a constant.

With these simplifications, the three-dimensional supersymmetry conditions reduce to

$$\nabla_m \epsilon_{\pm} + \frac{i}{2} m \sigma_m \epsilon_{\mp} = 0, \quad (4.2)$$

$$[\pm i e^{-B} + im] \epsilon_{\pm} + \sigma^m \partial_m B \epsilon_{\mp} = 0, \quad (4.3)$$

where we have dropped redundant subscripts and replaced  $D$  with  $\nabla$  as now  $\mathcal{Q} = 0$ . The familiar reader will identify above the Killing spinor equation on  $S^3$  [35], a fact that we will return to soon.

In the present case, the relevant scalar spinor bilinears are  $S_1, S_2, T_1$  and  $U_1$ , while the vector bilinears are  $K^1, K^2, L^1, M^1, M^2$  and  $N^1$  (*cf.* equations (B.1) and (B.2) in the appendix). Using equation (4.2) it is possible to determine the scalar bilinear differential conditions

$$dS_1 = 0, \quad (4.4)$$

$$dS_2 = m \Im(L_1), \quad (4.5)$$

$$dT_1 = -imK^2, \quad (4.6)$$

$$dU_1 = -imM^2. \quad (4.7)$$

We can also determine the following algebraic expressions from (4.3)

$$S_2 = -me^B S_1, \quad (4.8)$$

$$\Re(T_1) = 0. \quad (4.9)$$

Above  $\Re$  and  $\Im$  denote the real and imaginary parts of a complex number. Using the algebraic and differential KSEs one can show that  $K^1, \Re(L^1), M^1$  and  $N^1$  are Killing directions. For the moment, we postpone any attempt to determine the relationship between these Killing directions. In general, we have 6 Killing vectors as  $M^1$  and  $N^1$  are complex, however we stress that some of these Killing vectors may be trivially zero for a particular choice of the spinors, while it is also possible that they align as noted in [24]. To have a geometry dual to  $\mathcal{N} = 2$  SCFTs, we require the existence of a single overall  $U(1)$  isometry direction from the above list.

These directions will generate symmetries of the overall spacetime, presenting us with candidate  $U(1)$  R-symmetry directions, provided the warp factor,  $B$ , is independent of the Killing directions. From (4.3), we may determine the following relationships

$$\begin{aligned} i_{K^1} dB &= -e^{-B} \Im(T_1), \\ i_{\Re(L^1)} dB &= 0, \\ i_{M^1} dB &= ie^{-B} U_1, \\ i_{N^1} dB &= 0. \end{aligned} \quad (4.10)$$

If we demand that these directions correspond to R-symmetries (global  $U(1)$ 's), we may show that there is no geometry. To see this, note that from (4.9) and (4.10), we have



$T_1 = 0$  if  $K^1$  is either zero or an R-symmetry direction. The derivative of  $T_1$ ,

$$dT_1 = -imK^2, \quad (4.11)$$

then tells us that  $K^2$  is zero. Finally, we can see that  $K^1$  is also zero from (B.12). Together  $K^1 = K^2 = 0$  tell us that  $\epsilon_+ = \epsilon_- = 0$ , thus ruling out a solution. A similar conclusion may be reached working with  $U_1$ ,  $M^1$  and  $M^2$ .

In fact, one may show directly that the  $D = 3$  KSEs imply that the space  $M_3$  is a three-sphere. Evoking the integrability relationship

$$\nabla_{[m} \nabla_{n]} \epsilon_{\pm} = \frac{1}{8} R_{mnpq} \sigma^{pq} \epsilon_{\pm}, \quad (4.12)$$

one can show that

$$m^2 \epsilon_{mnp} \sigma^p \epsilon_{\pm} = \frac{1}{2} R_{mnq_1 q_2} \epsilon^{q_1 q_2} \sigma^{q_3} \epsilon_{\pm}. \quad (4.13)$$

This implies that  $M_3$  is a manifold of constant curvature. Then, it is well known [36], that for any  $n$ -dimensional connected, complete Riemannian manifold  $M$  of constant curvature  $\frac{1}{a^2}$ , the universal covering manifold of  $M$  is isomorphic to a sphere of radius  $a$ ,  $M = S^n/G$ , where  $G$  denotes some finite subgroup of  $O(n+1)$  that acts freely.

Having confirmed that there is one smooth (maximally) supersymmetric geometry that satisfies the above conditions on the spinor bilinears, it is an instructive exercise to recover  $M_3 = S^3$  from the conditions on the spinor bilinears directly. We start by dropping the requirement  $\Im(T_1) = 0$ , thus allowing the warp factor  $B$  to depend on the Killing directions on  $M_3$ , while for simplicity also setting  $m = 1$ . Observe that the 6 Killing vectors noted earlier now may be interpreted as the generators of  $SO(4)$ , the symmetry of the three-sphere. We will now show that orthonormal frame  $e^i$ , which may be read off from the vector bilinears  $\bar{\epsilon} \sigma_i \epsilon^i$ , is simply that corresponding to a three-sphere.

We begin by rotating  $K^1$  and  $K^2$  into the  $e^1, e^2$  plane. Since these are just two vectors, it is always valid to do this. After rotation, the conditions  $\bar{\epsilon}_{\pm} \sigma_3 \epsilon_{\pm} = 0$  may be satisfied by writing

$$\epsilon_+ = r_1 \begin{pmatrix} e^{i\phi_1} \\ e^{i\phi_2} \end{pmatrix}, \quad \epsilon_- = r_2 \begin{pmatrix} e^{i\psi_1} \\ e^{i\psi_2} \end{pmatrix}, \quad (4.14)$$

where  $r_i$  correspond to the norms of the spinors. The norms on the spinors may be determined from (4.8) and constant  $S_1 = 1$ :

$$r_1 = \sin \frac{\theta}{2}, \quad r_2 = \cos \frac{\theta}{2}. \quad (4.15)$$

It is worth noting that with this choice  $e^B = \cos \theta$ , so that the overall  $S^5$  is the following fibration

$$ds^2(S^5) = \cos^2 \theta ds^2(S^2) + d\theta^2 + \sin^2 \theta ds^2(S^2). \quad (4.16)$$

As  $B$  is just a function of  $\theta$ , we align the  $\theta$ -direction with  $e^1$ . By combining (4.10) and the knowledge that  $\Re(L_1)$  and  $N_1$  are not along  $e^1$ ,  $K^1 \cdot \Re(L^1) = 0$  from the Fierz

identity, and  $d(e^B) = -\mathfrak{S}(L^1)$  from (4.5) and (4.8), one finds consistent conditions that whittle down the angles to two. The form of the spinors is then

$$\epsilon_+ = \sin \frac{\theta}{2} \begin{pmatrix} e^{i\phi_1} \\ e^{i\phi_2} \end{pmatrix}, \quad \epsilon_- = -i \cos \frac{\theta}{2} \begin{pmatrix} e^{i\phi_2} \\ e^{i\phi_1} \end{pmatrix}. \quad (4.17)$$

The  $e^i$  labeling the orthonormal frame may then be read off from  $d(e^B) = -\mathfrak{S}(L^1)$ , (4.6) and (4.7), leading to the following

$$e^1 = -d\theta, \quad e^2 = \sin \theta (d\phi_1 - d\phi_2), \quad e^3 = \sin \theta \sin(\phi_1 - \phi_2) (d\phi_1 + d\phi_2). \quad (4.18)$$

A redefinition

$$\phi = \phi_1 - \phi_2, \quad \psi = \phi_1 + \phi_2, \quad (4.19)$$

recovers the metric on  $S^3$  with unit radius. As another consistency check we work out the Killing vectors for the space:

$$\begin{aligned} K^1 &\rightarrow -\cos \phi \partial_\theta + \sin \phi \cot \theta \partial_\phi \\ \mathfrak{R}(L^1) &\rightarrow -\partial_\psi \\ M^1 &\rightarrow -\sin \phi \cos \psi \partial_\theta - \cot \theta (\cos \phi \cos \psi \partial_\phi - \frac{\sin \psi}{\sin \phi} \partial_\psi), \\ &\rightarrow -\sin \phi \sin \psi \partial_\theta - \cot \theta (\cos \phi \sin \psi \partial_\phi + \frac{\cos \psi}{\sin \phi} \partial_\psi), \\ N^1 &\rightarrow \sin \psi \partial_\phi + \cos \psi \cot \phi \partial_\psi, \quad -\cos \psi \partial_\phi + \sin \psi \cot \phi \partial_\psi. \end{aligned} \quad (4.20)$$

These are simply the generators of  $SO(4)$ . Note neither  $\mathfrak{R}(L^1)$  nor  $N^1$  have components along the  $\theta$  direction, a fact that is consistent with (4.10). Observe also that  $K^1$  and  $\mathfrak{R}(L^1)$  are commuting. We will see this is the case later when we turn to the more general case.

Finally, before moving onto the general case, we remark that multiple Killing directions also appeared in [33], and, in particular, the Lie derivative of a similar  $S^2$  warp-factor with respect to one Killing direction was found to be non-zero. Though this Killing direction was consistently removed from the supersymmetry conditions and subsequent analysis led to a class of known geometries [37, 38], in the light of observations here, it would be interesting to revisit that example and retain all the Killing vectors.

## 5 A tale of two Killing vectors

In this section we consider the general form of our ansatz and start by identifying the candidate Killing directions. Recall that a  $U(1)$  Killing direction (denoted  $K_5^{GMSW}$ ) exists in  $D = 5$  [30] for a general  $M_5$  and generic values of all fields. As a result, we naturally expect this  $U(1)$  direction to descend from  $D = 5$  to give a solution to the Killing equation

in  $D = 3$ . In fact, it is possible to identify the combination  $X = K^1 + K^3$  and the combination  $Y = \Re(L^1 + L^6)$  (see appendix for the explicit definitions of the spinor bilinears) as solutions to the Killing equation for all values of the spacetime fields. The  $D = 5$  Killing direction  $K_5^{GMSW} = \frac{1}{2}(\bar{\xi}_1\gamma^m\xi_1 + \bar{\xi}_2\gamma^m\xi_2)$  can be decomposed using our ansatz our ansatz (3.8) to give

$$K_5^{GMSW} = (\bar{\chi}_+\sigma_3\chi_+)X + (\bar{\chi}_+\chi_+)Y. \quad (5.1)$$

In other words,  $K_5^{GMSW}$  is a particular linear combination of the two Killing directions we found in  $D = 3$ ; note that the coefficients in the linear combinations are independent of  $M_3$ .

Our analysis so far shows that  $X$  and  $Y$  are Killing directions on  $M_3$ . Since we only expect one  $U(1)$  R-symmetry, one possible interpretation might be that either  $X$  or  $Y$  generates the R-symmetry, with the other vector simply corresponding to a  $U(1)$  isometry direction unrelated to the R-symmetry. Such an example exists in the uplift of the  $Y^{p,q}$  spaces [39] to M-theory [40] where the Reeb vector associated with the R-symmetry combines with another  $U(1)$  from  $Y^{p,q}$  to give the R-symmetry direction in M-theory. See appendix C of [41] for further discussion on this subject.

Let us first check then, under what conditions  $X$  may be promoted to a  $U(1)$  isometry of the overall solution.  $X$  corresponding to an overall  $U(1)$  is an important prerequisite for it being identified as an R-symmetry direction. We proceed to calculate the Lie derivative of the various fields and warp factors with respect to  $X$ . After some arithmetic we find

$$\begin{aligned} \mathcal{L}_X\Delta &= 0, \\ \mathcal{L}_XB &= -2e^{-B}\Im(T_1 + T_6), \\ \mathcal{L}_X\mathcal{P} &= 0, \quad \Rightarrow \mathcal{L}_X\mathcal{Q} = 0, \\ \mathcal{L}_X\mathcal{A} &= 0, \\ i_X dg &= 0. \end{aligned} \quad (5.2)$$

To derive these equations we use the algebraic conditions (3.13) - (3.18), equation (B.3), equation of motion for  $g$  (3.21), the Bianchi identity (3.22) as well as  $i_X\mathcal{P} = 0$ . As a result we conclude that the Killing direction,  $X$ , can be promoted to a symmetry of the full solution provided

$$\Im(T_1 + T_6) = 0. \quad (5.3)$$

We now show that this condition implies either  $f = 0$  or  $g = 0$  or  $X = 0$ . At each step we will assume that both  $f$  and  $g$  are non-zero and will ultimately show that this implies  $X = 0$ . In section 6 below we will then consider the case  $X \neq 0$  and  $f = g = 0$ , while in section 7 below we will consider the case  $X = 0$ . Firstly, equations (5.3), (B.6) and (B.9) imply  $T_2 = T_5 = 0$ . Combining this result with equations (B.11), (B.17) and (B.18) one finds that  $L^3 = L^4$  and  $K^2 = K^4$ . Using the latter relation, together with equation (5.3),

and  $T_2 = T_5 = 0$  in equations (B.12) and (B.13) we find that  $X = K^1 + K^3 = 0$ . To sum up we have shown that  $X$  is an isometry of the full solution only when  $f$  or  $g$  is zero.

We now turn to the analysis of  $Y$ . One can show that

$$\mathcal{L}_Y \Delta = 0, \quad (5.4)$$

$$\mathcal{L}_Y B = 0, \quad (5.5)$$

$$\mathcal{L}_Y \mathcal{P} = 0, \quad (5.6)$$

$$\mathcal{L}_Y g = fg(S_1 + S_3) = 0, \quad (5.7)$$

$$\mathcal{L}_Y \mathcal{A} = 2e^{4\Delta+2B}(g^* \mathcal{P} - 2d\Delta g) - if\mathcal{A} = 0. \quad (5.8)$$

In deriving these identities we have used equations (3.11) and (3.12), the algebraic conditions (3.13) - (3.18), the Bianchi identity (3.22), equations of motion (3.21), equations (B.15) and (B.16), as well as reality properties of the spinor bilinears.<sup>6</sup>

It is worth noting that (5.7) is a particularly strong constraint.  $S_1 + S_3$  is a constant related to the norm of the spinors and cannot be zero, so either  $f = 0$  or  $g = 0$ <sup>7</sup>.

Our two cases have now coalesced and we proceed to show that *both*  $f$  and  $g$  are zero. Now if one adopts  $f = 0$ , one sees from (B.4) and (B.5) that  $S_2 = S_4$ , while (B.7) and (B.8) in turn confirm that  $g$  is also zero. Therefore,  $f = 0 \Rightarrow g = 0$ .

From the relations derived in the appendix, it is not obvious that  $g = 0$  implies  $f = 0$ . However, as is clear from the equations of motion (3.21), when  $g = 0$ , either  $f = 0$  or  $\mathcal{A} = 0$ . When  $\mathcal{A} = 0$ , the type IIB three-form flux vanishes and it is possible to show that both the axion and dilaton are harmonic. Then, if  $M_3$  is compact, as in the case of most interest to AdS/CFT, one deduces that both are constants using the maximum principle of harmonic forms.

Returning to five-dimensions, we are now on the cusp of declaring that the five-dimensional geometry is Sasaki-Einstein when  $\mathcal{A} = g = 0$  i.e. no three-form flux. However, before making such a statement, we require one of the spinors  $\xi_i$  appearing in (3.2) to (3.7) to vanish. Recall that it was shown in [30] that  $\xi_2 = 0$  implies the five-dimensional space is Sasaki-Einstein. Indeed, when the three-form flux is zero, it is a simple exercise to show that (3.4) and (3.5) together imply  $d\Delta(\bar{\xi}_1 \xi_1 + \bar{\xi}_2 \xi_2) = 0$ . In other words  $\Delta$  is a constant and the Killing spinor equations can only be satisfied if one of the  $\xi_i$  is zero. The property that the five-dimensional space is then Sasaki-Einstein follows and the analysis reverts to that in the previous section.

<sup>6</sup>For example, the fact that  $S_1, \dots, S_4$  are real.

<sup>7</sup>Note we have derived (5.7) on the assumption that  $dg \neq 0$  and this constraint on  $f$  and  $g$  disappears once  $g$  is a constant. However, when  $g$  is constant it is possible to derive a new version of (5.8) stating

$$\mathcal{L}_Y \mathcal{A} = 0 \Rightarrow 2dBg - i\mathcal{Q}g + \mathcal{P}g^* = 0. \quad (5.9)$$

Observe now that the warp factor  $B$  is charged under  $X$  whereas  $\mathcal{P}$  and  $\mathcal{Q}$  are not. This expression can only be consistent if either  $g = 0$ , or  $\mathcal{L}_X B = 0$ , which we have seen from the last section also implies either  $f = 0$  or  $g = 0$ . So, the implication of one of  $f$  or  $g$  being zero does not change when  $g$  is a constant.

In summary, in this section we have shown that requiring either of the Killing directions  $X$  or  $Y$  to correspond to a global  $U(1)$  symmetry corresponding to the R-symmetry, implies that  $f = g = 0$ . We turn our focus to the analysis of these spacetimes in the next section.

## 6 Geometries with $f = g = 0$

In the last section we showed that the Killing vectors  $X$  and  $Y$  on  $M_3$  could be promoted to global  $U(1)$  isometries provided we restrict our ansatz by setting  $f = g = 0$ . In this section we restrict our ansatz to  $f = g = 0$ .<sup>8</sup> We show that, for such backgrounds, requiring  $X$  or  $Y$  to be a global  $U(1)$  symmetry results in  $X$  and  $Y$  being equal to zero. The same result holds also if one entertains the idea that a linear combination of  $X$  and  $Y$  correspond to the R-symmetry direction. In the next section we will show that the condition  $X = Y = 0$  together with the KSEs in fact sets the Killing spinors to zero, leaving us with no solutions.

We start, by further examining the Killing directions when  $f = g = 0$ . As noted in section 4 above, when some of the field content is removed, extra Killing directions may emerge. After re-examining the KSEs with  $f = g = 0$ , we find that in the present case there are no new Killing directions. We now examine the relationship between  $X$  and  $Y$  by checking to see if the Killing vectors commute. This may be done by calculating the Lie derivative  $\mathcal{L}_X Y$ . Making use of the Fierz identity and the identities  $\sigma_{mp}\sigma_q\sigma^m = -\sigma^m\sigma_q\sigma_{mp} = 2\delta_{pq}$ , one can show indeed that  $i_X dY = i_Y dX = 0$  and that the vectors commute

$$\mathcal{L}_X Y = [X, Y] = 0. \tag{6.1}$$

Recall that in the LLM geometry [21], which has a similar ansatz, two Killing directions align to give the R-symmetry [24]. In the IIB geometry we are presently considering, it is possible to use the Fierz identity to show that  $X$  and  $Y$  are in fact orthogonal<sup>9</sup>

$$X \cdot Y = 0. \tag{6.2}$$

The algebraic constraints (3.13) - (3.20), together with  $f = g = 0$  can be used to

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<sup>8</sup>All equations in this section are derived with  $f = g = 0$ .

<sup>9</sup>Firstly, one shows that  $K^1 \cdot \Re(L^1) = K^3 \cdot \Re(L^6) = 0$  using (6.6) below, while the cross-terms can be confirmed to vanish by using two iterations of the Fierz identity, (6.6) again and (6.4) below.

obtain the following spinor bilinear relations

$$S_1 = S_3, \quad S_2 = S_4, \quad (6.3)$$

$$T_3 + T_4 = 0, \quad (6.4)$$

$$U_3 + U_4 = 0, \quad (6.5)$$

$$\Re(T_1) = \Re(T_6) = 0, \quad (6.6)$$

$$U_2 - U_5 = -3me^B(U_2 + U_5), \quad (6.7)$$

$$T_2 - T_5 = -3me^B(T_2 + T_5), \quad (6.8)$$

$$S_2 = -3me^B S_1. \quad (6.9)$$

From the vector bilinear differential conditions, we note that the two Killing vectors  $X$  and  $Y$  satisfy

$$dX = 2m * Y, \quad dY = 2m * X. \quad (6.10)$$

As an aside, observe that  $d * X = d * Y = 0$ , so that the Laplacian acting on  $X$  or  $Y$  becomes  $*d*dX = 4m^2 X$ . Interestingly, if  $M_3$  is a three-dimensional compact Riemannian Einstein space normalised such that  $R_{mn} = 2m^2 g_{mn}$ , the Laplacian eigenvalues  $\kappa$  satisfy  $\kappa \geq 4m^2$  with saturation happening when the one-form is Killing [42]. This precisely what one notes above for  $X$  and  $Y$ .

In addition, we also see from the vector torsion conditions that  $X$  satisfies

$$dX = 2dB \wedge X. \quad (6.11)$$

So what have we learned about  $M_3$ ? We have seen that  $M_3$  has two orthogonal Killing directions, so we can think of  $M_3$  having three directions:  $X$ ,  $Y$  and an additional direction parameterised by, say,  $\theta$ . Observe that  $X$  and  $Y$  are coupled, so we cannot set one of them to zero without also setting the other to zero. Presently, we explore the possibility that  $X \neq 0 \neq Y$  and that either  $X$  or  $Y$  corresponds to the  $U(1)$  R-symmetry (and so a global  $U(1)$  symmetry) while the other is simply a generic isometry direction.

#### **$X$ is a global $U(1)$ symmetry direction**

Recall from section 5 that  $X$  can be a global  $U(1)$  symmetry if  $\Im(T_1 + T_6) = 0$ . As  $T_1$  and  $T_6$  are both pure imaginary, combining (6.8), (B.19), (B.20), (B.12) and (B.13), together with  $f = g = 0$  one can show that

$$X = K^2 + K^4 = AT_2 = AT_5 = 0. \quad (6.12)$$

But equation (6.10) then implies that  $Y = 0$ . So requiring that  $X$  is a global  $U(1)$  symmetry implies that  $X = Y = 0$ .

#### **$Y$ is a global $U(1)$ symmetry direction**

Instead let us see what happens when we require that  $Y$  be a global  $U(1)$  symmetry. Recall that presently  $M_3$  is parametrised by three directions  $X$ ,  $Y$  and  $\theta$ . Equations (5.2)

and (6.11) imply that  $B$  depends on  $X$  and  $\theta$ , but is independent of the  $Y$ -direction. We also recall, from the results in section 5 that both  $\mathcal{A}$  and  $d\Delta$  have only components along the  $\theta$ -direction. Then from equation (B.14) one can see that  $\Im(L^1 + L^6)$  has no component along the  $X$  or  $Y$  directions. This last fact, in conjunction with a similar observation for  $d\Delta$ , implies, via equation (B.10), that  $B$  has no  $X$ -dependence after all. As a result

$$\Im(T_1 + T_6) = 0, \quad (6.13)$$

and so  $X = Y = 0$ .<sup>10</sup>

To summarise, in this section we have shown that one cannot require  $X$  or  $Y$  or any linear combination to be a global  $U(1)$  symmetry direction. In the next section we will investigate the possibility that both  $X$  and  $Y$  are zero.

## 7 New R-symmetry directions?

In the last section we saw that the KSEs do not allow for  $X$  or  $Y$  to be global  $U(1)$  symmetry directions, and, hence, they cannot be  $U(1)$  R-symmetry directions either. We are left with the possibility that  $X = Y = 0$ . It is easy to show that these conditions alone do not force all the Killing spinors to be zero. Rather, they merely reduce the number of independent Killing spinor components. As such, *a priori* it appears possible to consider the KSEs with this restricted choice of Killing spinors, and to re-start a search for solutions. Such a possibility appears already in the  $\mathcal{N} = 1$  setting discussed in [30]. There a unique Killing vector,  $K_5^{GMSW}$ , was found;<sup>11</sup> and one may wonder whether solutions to the KSEs exist for which  $K_5^{GMSW} = 0$ , and instead a new  $U(1)$  R-symmetry Killing vector emerges in this restricted setting.<sup>12</sup> In this section we will show that this apparent loophole in fact does not lead to any new solutions in the  $\mathcal{N} = 1$  case. In particular we will show that setting  $K_5^{GMSW} = 0$ , together with the KSEs imply that the  $D = 5$  Killing spinors  $\xi_i$  have to be identically equal to zero. Since the  $\mathcal{N} = 2$  case we are considering here is a special case of the  $\mathcal{N} = 1$  geometries studied in [30], and since  $X = Y = 0$  implies that  $K_5^{GMSW} = 0$ , the argument presented in this section will also imply that no solutions exist when  $X$  and  $Y$  are set to zero.

From [30] we see that the norm of  $K_5^{GMSW}$  is

$$|K_5^{GMSW}|^2 = \sin^2 \zeta^{GMSW} + |S^{GMSW}|^2, \quad (7.1)$$

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<sup>10</sup>To get around this argument, one may attempt to set  $\Im(L^1 + L^6) = 0$ . However, this is catastrophic as  $d(4\Delta + B) = 0$  with  $f = g = 0$  means from equations (3.19) and (3.20) that  $B$  is also a constant,  $e^{-B} = \pm 3m$  and that two spinors are zero.

<sup>11</sup>Throughout this section we shall use the superscript GMSW to denote notation from [30].

<sup>12</sup>We are grateful to Dario Martelli and James Sparks for a number of illuminating discussions during the Benasque Strings 2011 workshop on this subject.

thus implying that both  $S^{GMSW}$  and  $\sin \zeta^{GMSW}$  are zero. Then from equation (3.15) of the same paper we have that the complex vector  $K^{GMSW}$  is zero. Furthermore, using equation (3.22) of [30], it is possible to show that the two-form bilinear  $U^{GMSW}$  is also zero. These extra constraints on the spinors may be summarised as

$$\begin{aligned}
\sin \zeta^{GMSW} &= \frac{1}{2}(\bar{\xi}_1 \xi_1 - \bar{\xi}_2 \xi_2) = 0, \\
S^{GMSW} &= \bar{\xi}_2^c \xi_1 = 0, \\
K_5^{mGMSW} &= \frac{1}{2}(\bar{\xi}_1 \gamma^m \xi_1 + \bar{\xi}_2 \gamma^m \xi_2) = 0, \\
K_m^{GMSW} &= \bar{\xi}_1^c \gamma_m \xi_2 = 0, \\
iU_{mn}^{GMSW} &= \frac{1}{2}(\bar{\xi}_1 \gamma^{mn} \xi_1 + \bar{\xi}_2 \gamma^{mn} \xi_2) = 0.
\end{aligned} \tag{7.2}$$

We now show that these conditions imply  $\xi_1 = \xi_2 = 0$ . With the explicit gamma matrix representation given in [30] we have

$$C_5 = \mathbf{1} \otimes i\sigma^2. \tag{7.3}$$

One can now decompose  $\xi_1$  and  $\xi_2$  as

$$\xi_1 = \begin{pmatrix} r \\ s \\ t \\ u \end{pmatrix}, \quad \xi_2 = \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix}, \tag{7.4}$$

where the components are in general complex. Now, by combining  $S^{GMSW} = K^{GMSW} = 0$ , one establishes the following relations

$$\begin{aligned}
tz &= uy, & rx &= sw, & tx &= sy, \\
uw &= rz, & ux &= sz, & tw &= ry.
\end{aligned} \tag{7.5}$$

Using the remaining relationships it is easy to find that one component of  $\xi_1$  or  $\xi_2$  is zero. For example, using  $(K_5^3)^{GMSW} = (K_5^4)^{GMSW} = U_{13}^{GMSW} = U_{14}^{GMSW} = 0$ , it is possible to infer that

$$z(|r|^2 + |w|^2) = 0, \tag{7.6}$$

so that either  $z = 0$  or  $r = w = 0$  is zero. Once one of the components in  $\xi_1$  or  $\xi_2$  can be shown to be zero, the result that  $\xi_1 = \xi_2 = 0$  is immediate from the remaining relationships coming from  $K_5^{GMSW} = U^{GMSW} = 0$ . Therefore, we conclude that setting  $K_5^{GMSW} = 0$ , together with the KSEs, implies that the Killing spinors are zero.

In fact a similar argument can also be seen to hold for the  $\mathcal{N} = 2$  case that has been the focus of this paper. When  $X = Y = 0$ , we also have  $K_5^{GMSW} = 0$  (*cf.* equation (5.1)). We can then re-write the conditions (7.2), when reduced using the ansatz (3.8). In particular, the  $U_{mn}^{GMSW}$  condition, with  $m$  and  $n$  in the directions of  $S^2$  is particularly strong (using



the gamma matrix basis given in the appendix) and implies that the sum of the norms of  $\epsilon_{i\pm}$  has to be zero, and so the Killing spinors  $\epsilon_{i\pm}$  have to be zero themselves.

This concludes our analysis. We have used the KSEs to search for Type IIB solutions with  $AdS_5$  and  $S^2$  factors and R-symmetry  $SU(2) \times U(1)$ . We have found that no non-trivial solutions of such a type exist. Our result suggests that if one can find Type IIB solutions which are holographic duals of four-dimensional  $\mathcal{N} = 2$  SCFTs, the  $SU(2)$  R-symmetry will have to be realised in some non-geometric way which does not lead to the presence of  $SU(2)$  Killing vectors in the spacetime.

## Acknowledgements

We would like to thank Jerome Gauntlett, Chris Hull, Oleg Lunin, Juan Maldacena, Hiroaki Nakajima, Dimitri Skliros, Linda Uruchurtu, Oscar Varela, Hossein Yavartanoo, and especially Dario Martelli and James Sparks for interesting discussions and sharing their insights with us. We are grateful to the organisers and participants of the Benasque Strings 2011 workshop, and the *Centro de Ciencias de Benasque Pedro Pascual* for providing a stimulating and productive atmosphere for the final stages of this project. EÓC also expresses gratitude to the *Simons Center for Geometry and Physics* and the organisers of the Simons Summer Workshop on Geometry and Physics 2011 for generous hospitality while this draft received some final tweaks. The work of BS is supported by an EPSRC Advanced Research Fellowship.

## A Spinor and $\gamma$ -matrix conventions

We will use the spinor conventions of [30] (see also [43]) under the understanding that we are in addition decomposing the internal  $M_5$  space into a direct product of  $S^2$  and  $M_3$ . We begin by recalling that the  $M_5$  gamma matrices  $\gamma_i$   $i = 1, \dots, 5$  satisfy the following:

$$\begin{aligned}\gamma_i &= \gamma_i^\dagger, \\ C_5^{-1} \gamma_i C_5 &= \gamma_i^T,\end{aligned}\tag{A.1}$$

where  $\tilde{D}_5 = C_5$  and  $C_5^* = -C_5^{-1}$ ,  $C_5 = -C_5^T$ . In addition,  $\gamma_{12345} = 1$ . A five-dimensional spinor  $\chi$ , where  $\chi^c = C_5 \chi^*$  satisfied  $\chi^{cc} = -\chi$ .

Then adopting the following choice for the decomposition of  $\gamma_i$

$$\begin{aligned}\gamma_\alpha &= |\epsilon_{\alpha\beta}| \sigma_\beta \otimes 1, \\ \gamma_{m+2} &= \sigma_3 \otimes \sigma_m,\end{aligned}\tag{A.2}$$

where  $\alpha = 1, 2$ ,  $m = 1, 2, 3$  and we have introduced somewhat awkward ordering so that  $\gamma_{12345} = 1$  as in [30]. One sees that a natural choice is

$$C_5 = \sigma_1 \times \sigma_2.\tag{A.3}$$

Here we have taken  $C_3 = \sigma_2$  so that  $C_3^{-1}\sigma_m C_3 = -\sigma_m^T$  consistent with [43]. With this choice if  $\chi$  is a Killing spinor on  $S^2$ , i.e. a solution to

$$\nabla_\alpha \chi = i \frac{\sigma_\alpha}{2} \chi, \quad (\text{A.4})$$

then it is easy to see that the conjugate  $\chi^c = C_2 \chi^* = \sigma_1 \chi^*$  satisfies the same equation with the opposite sign.

## B Bispinor relations

Here we record the bilinears that appear in the analysis. We label the scalars constructed out of our spinors in the following fashion,

$$\begin{aligned} S_1 &= \frac{1}{2}(\bar{\epsilon}_{1+}\epsilon_{1+} + \bar{\epsilon}_{1-}\epsilon_{1-}), & S_2 &= \frac{1}{2}(\bar{\epsilon}_{1+}\epsilon_{1+} - \bar{\epsilon}_{1-}\epsilon_{1-}), \\ S_3 &= \frac{1}{2}(\bar{\epsilon}_{2+}\epsilon_{2+} + \bar{\epsilon}_{2-}\epsilon_{2-}), & S_4 &= \frac{1}{2}(\bar{\epsilon}_{2+}\epsilon_{2+} - \bar{\epsilon}_{2-}\epsilon_{2-}), \\ T_1 &= \bar{\epsilon}_{1+}\epsilon_{1-}, & T_2 &= \bar{\epsilon}_{1+}\epsilon_{2+}, & T_3 &= \bar{\epsilon}_{1+}\epsilon_{2-}, \\ T_4 &= \bar{\epsilon}_{1-}\epsilon_{2+}, & T_5 &= \bar{\epsilon}_{1-}\epsilon_{2-}, & T_6 &= \bar{\epsilon}_{2+}\epsilon_{2-}, \\ U_1 &= \bar{\epsilon}_{1+}^c\epsilon_{1-}, & U_2 &= \bar{\epsilon}_{1+}^c\epsilon_{2+}, & U_3 &= \bar{\epsilon}_{1+}^c\epsilon_{2-}, \\ U_4 &= \bar{\epsilon}_{1-}^c\epsilon_{2+}, & U_5 &= \bar{\epsilon}_{1-}^c\epsilon_{2-}, & U_6 &= \bar{\epsilon}_{2+}^c\epsilon_{2-}, \end{aligned} \quad (\text{B.1})$$

and the vectors thus:

$$\begin{aligned} K_m^1 &= \frac{1}{2}(\bar{\epsilon}_{1+}\sigma_m\epsilon_{1+} + \bar{\epsilon}_{1-}\sigma_m\epsilon_{1-}), & K_m^2 &= \frac{1}{2}(\bar{\epsilon}_{1+}\sigma_m\epsilon_{1+} - \bar{\epsilon}_{1-}\sigma_m\epsilon_{1-}), \\ K_m^3 &= \frac{1}{2}(\bar{\epsilon}_{2+}\sigma_m\epsilon_{2+} + \bar{\epsilon}_{2-}\sigma_m\epsilon_{2-}), & K_m^4 &= \frac{1}{2}(\bar{\epsilon}_{2+}\sigma_m\epsilon_{2+} - \bar{\epsilon}_{2-}\sigma_m\epsilon_{2-}), \\ L_m^1 &= \bar{\epsilon}_{1+}\sigma_m\epsilon_{1-}, & L_m^2 &= \bar{\epsilon}_{1+}\sigma_m\epsilon_{2+}, & L_m^3 &= \bar{\epsilon}_{1+}\sigma_m\epsilon_{2-}, \\ L_m^4 &= \bar{\epsilon}_{1-}\sigma_m\epsilon_{2+}, & L_m^5 &= \bar{\epsilon}_{1-}\sigma_m\epsilon_{2-}, & L_m^6 &= \bar{\epsilon}_{2+}\sigma_m\epsilon_{2-}, \\ M_m^1 &= \frac{1}{2}(\bar{\epsilon}_{1+}^c\sigma_m\epsilon_{1+} + \bar{\epsilon}_{1-}^c\sigma_m\epsilon_{1-}), & M_m^2 &= \frac{1}{2}(\bar{\epsilon}_{1+}^c\sigma_m\epsilon_{1+} - \bar{\epsilon}_{1-}^c\sigma_m\epsilon_{1-}), \\ M_m^3 &= \frac{1}{2}(\bar{\epsilon}_{2+}^c\sigma_m\epsilon_{2+} + \bar{\epsilon}_{2-}^c\sigma_m\epsilon_{2-}), & M_m^4 &= \frac{1}{2}(\bar{\epsilon}_{2+}^c\sigma_m\epsilon_{2+} - \bar{\epsilon}_{2-}^c\sigma_m\epsilon_{2-}), \\ N_m^1 &= \bar{\epsilon}_{1+}^c\sigma_m\epsilon_{1-}, & N_m^2 &= \bar{\epsilon}_{1+}^c\sigma_m\epsilon_{2+}, & N_m^3 &= \bar{\epsilon}_{1+}^c\sigma_m\epsilon_{2-}, \\ N_m^4 &= \bar{\epsilon}_{1-}^c\sigma_m\epsilon_{2+}, & N_m^5 &= \bar{\epsilon}_{1-}^c\sigma_m\epsilon_{2-}, & N_m^6 &= \bar{\epsilon}_{2+}^c\sigma_m\epsilon_{2-}. \end{aligned} \quad (\text{B.2})$$

The relationships presented here may all be derived through either Fierz identities, or direct manipulation of the algebraic  $D = 3$  supersymmetry conditions (3.13) - (3.20). We

break them down into expressions involving scalars

$$\Re(T_1) = \Re(T_6) = 0, \quad (\text{B.3})$$

$$e^{-B}(S_2 - S_4) = m(S_1 - S_3) - \frac{1}{2}e^{-4\Delta}f(S_1 + S_3), \quad (\text{B.4})$$

$$e^{-4\Delta}f(S_1 + S_3) = 4m(S_1 - S_3), \quad (\text{B.5})$$

$$e^{-2\Delta}f(T_2 - T_5) = ig^*\Im(T_1 + T_6), \quad (\text{B.6})$$

$$6m(T_3 + T_4) = e^{-2\Delta}g^*(S_1 + S_3), \quad (\text{B.7})$$

$$-2e^{-B}(T_3 + T_4) + e^{-4\Delta}f(T_3 - T_4) = e^{-2\Delta}g^*(S_2 - S_4), \quad (\text{B.8})$$

$$3m(T_2 + T_5) = -e^{-B}(T_2 - T_5). \quad (\text{B.9})$$

and expressions involving vectors

$$(S_1 + S_3)d(4\Delta + B) = -e^{-B}\Im(L^1 + L^6), \quad (\text{B.10})$$

$$2(T_5 - T_2)d(4\Delta + B) = 6im(L^3 - L^4) - ie^{-2\Delta}g^*(K^2 - K^4), \quad (\text{B.11})$$

$$\begin{aligned} 2(T_1 + T_6)d(4\Delta + B) &= -6im(K^2 + K^4) + ie^{-4\Delta}f(K^2 - K^4) \\ &\quad - 2ie^{-B}(K^1 + K^3). \end{aligned} \quad (\text{B.12})$$

From (3.13) and (3.14) one finds

$$\begin{aligned} 2dB\Im(T_1 + T_6) &= 2e^{-B}(K^1 + K^3) + e^{-4\Delta}f(K^2 - K^4) - 2m(K^2 + K^4) \\ &\quad - e^{-2\Delta-2B}\Re(\mathcal{A}(T_2 - T_5)). \end{aligned} \quad (\text{B.13})$$

From (3.15) and (3.16) we get:

$$\begin{aligned} 2d\Delta(S_2 + S_4) &= \frac{1}{4}e^{-2\Delta-2B}\Im(\mathcal{A}(T_3 - T_4)) - \frac{1}{2}e^{-4\Delta}f\Im(L^1 - L^6) \\ &\quad + 2m\Im(L^1 + L^6). \end{aligned} \quad (\text{B.14})$$

Finally, we also find useful the following torsion conditions involving scalars:

$$e^{-2\Delta}d(e^{2\Delta}S_1) = 2S_3d\Delta, \quad (\text{B.15})$$

$$e^{-2\Delta}d(e^{2\Delta}S_3) = 2S_1d\Delta, \quad (\text{B.16})$$

$$D(e^{4\Delta+B}T_2) = \frac{i}{4}e^{2\Delta-B}\mathcal{A}^*(T_1 - T_6^*) + \frac{i}{2}e^{4\Delta}(1 - 2me^B)(L^3 - L^4), \quad (\text{B.17})$$

$$D(e^{4\Delta+B}T_5) = \frac{i}{4}e^{2\Delta-B}\mathcal{A}^*(T_1^* - T_6) + \frac{i}{2}e^{4\Delta}(1 + 2me^B)(L^3 - L^4). \quad (\text{B.18})$$

In section 6 we employ the following bispinor relations which are valid when  $f = g = 0$ .

$$dT_1 = imK^2 + \frac{i}{4}e^{-2\Delta-2B}[\mathcal{A}T_2 - \mathcal{A}^*T_5^*], \quad (\text{B.19})$$

$$dT_6 = imK^4 + \frac{i}{4}e^{-2\Delta-2B}[\mathcal{A}^*T_2^* - \mathcal{A}T_5]. \quad (\text{B.20})$$

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