

Maximal Unitarity at Two Loops

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Abstract

We show how to compute the coefficients of the double box basis integrals in a massless four-point amplitude in terms of tree amplitudes. We show how to choose suitable multidimensional contours for performing the required cuts, and derive consistency equations from the requirement that integrals of total derivatives vanish. Our formulæ for the coefficients can be used either analytically or numerically.

PACS numbers: 11.15.-q, 11.15.Bt, 11.55.Bq, 12.38.-t, 12.38.Bx

I. INTRODUCTION

The computation of higher-order corrections to amplitudes in gauge theories is important to searches for new physics at modern particle colliders. Next-to-leading order (NLO) corrections in quantum chromodynamics (QCD), in particular, play an important role in providing a reliable quantitative estimate of backgrounds to possible signals of new physics [1]. NLO corrections to differential cross sections require several ingredients beyond the tree-level amplitudes for the basic process under study: real-emission corrections, with an additional emitted gluon, or a gluon splitting into a quark–antiquark pair; and virtual one-loop corrections, with a virtual gluon or virtual quark in a closed loop. The required one-loop corrections are challenging with traditional Feynman-diagram methods, and become considerably more difficult as the number of final-state partons (gluons or quarks) grows.

The unitarity method [2–18], a new method which has emerged over the last decade and a half, has rendered such computations tractable. It has made possible a variety of computations of one-loop amplitudes, in particular of processes with many partons in the final state. In its most recent form, the method can be applied either analytically or purely numerically [19–29]. The numerical formalisms underly recent software libraries and programs that are being applied to LHC phenomenology. In the current formalism, the one-loop amplitude in QCD is written as a sum over a set of basis integrals, with coefficients that are rational in external spinors,

$$\text{Amplitude} = \sum_{j \in \text{Basis}} \text{coefficient}_j \text{Integral}_j + \text{Rational}. \quad (1.1)$$

The integral basis for amplitudes with massless internal lines contains box, triangle, and bubble integrals in addition to purely rational terms (dropping all terms of $\mathcal{O}(\epsilon)$ in the dimensional regulator). The coefficients are calculated from products of tree amplitudes, typically by performing contour integrals.

For NLO corrections to some processes, one-loop amplitudes do not suffice. This is the case for subprocesses whose leading-order amplitude begins at one loop. An example is the gluon fusion to diphoton subprocess, $gg \rightarrow \gamma\gamma$, which is an important background to searches for the Higgs boson at the LHC. Although this subprocess is nominally suppressed by a power of the strong coupling α_s , the large gluon parton density at smaller x can compensate for this additional power, giving rise to contributions to cross sections which are comparable to those

from tree-level quark-initiated subprocesses [30–32]. Other examples include production of electroweak boson pairs, $gg \rightarrow Z\gamma, ZZ, W^+W^-$. NLO corrections to such processes at the LHC require the computation of two-loop amplitudes [33].

Two-loop amplitudes are also required for any studies beyond NLO. Next-to-next-leading order (NNLO) fixed-order calculations form the next frontier. The only existing fully-exclusive NNLO jet calculations to date are for three-jet production in electron–positron annihilation [34]. These are necessary to determine α_s to 1% accuracy from jet data at LEP [35], competitively with other determinations. At the LHC, NNLO calculations will be useful for determining an honest theoretical uncertainty estimate on NLO calculations, for assessing scale stability in multi-scale processes such as W +multi-jet production, and will also be required for precision measurements of new physics once it is discovered.

The unitarity method has already been applied to higher-loop amplitudes. At one loop, there are different variants of the method. The basic unitarity approach forms a discontinuity out of the product of two tree amplitudes. Isolating the coefficients of specific basis integrals usually still requires performing symbolic algebra on the product of trees; this is not well-suited to a numerical approach, and also reduces efficiency of an analytic calculation. Basic unitarity corresponds to cutting two propagators in a one-loop amplitude. Generalized unitarity cuts more than two propagators at once, isolating fewer integrals. ‘Maximal’ generalized unitarity cuts as many propagators as possible; in combination with contour integrals over remaining degrees of freedom, this isolates individual integrals. At higher loops, ‘minimal’ generalized unitarity cuts the minimum number of propagators needed to break all loops into a product of trees. Each cut is again a product of tree amplitudes, but because not all possible propagators are cut, each generalized cut will correspond to several integrals and their coefficients, and algebra will again be required to isolate specific integrals and their coefficients. This approach does have the advantage of not requiring a basis of integrals. A number of calculations have been done this way, primarily in the $\mathcal{N} = 4$ supersymmetric gauge theory and $\mathcal{N} = 8$ supergravity [36–52, 80], but including several four-point calculations in QCD and supersymmetric theories with less-than-maximal supersymmetry [53–59].

In this paper, we take the first steps in developing the maximal generalized unitarity approach at two loops in a form suitable for both analytic and numerical calculation. We show how to extract the coefficient of the planar double box to leading order in the dimen-

sional regulator ϵ . Higher-loop amplitudes can be written in a similar form to those at one loop (1.1), as a sum over an integral basis [60], along with possible rational terms. At higher loops, however, the coefficients of the basis integrals are no longer functions of the external spinors alone, but will depend explicitly on ϵ . Just as at one loop, computing coefficients requires choosing contours for the unfrozen degrees of freedom. We use the equations relating generic tensor integrals to basis or master integrals in order to ensure the consistency and completeness of the choice of contours. The extraction of the double-box coefficient bears a superficial similarity to the procedure that would be followed in the leading-singularity approach [44, 61], but unlike the latter, manifestly ensures the consistency of the extraction with respect to terms that vanish after integration. Such terms inevitably arise when using the integration-by-parts (IBP) approach [62–69] in relating formally-irreducible tensor integrals to basis integrals. The extraction of higher-order terms in ϵ or the coefficients of integrals with fewer propagators, both of which we leave to future work, would also be different.

During the preparation of this manuscript, a preprint by Mastrolia and Ossola appeared [79], analyzing the two-loop integrand in a generalization of the formalism of Ossola, Papadopoulos and Pittau (OPP) [11], following a complementary approach to unitarity at two loops.

This paper is organized as follows. In Section II, we review maximal generalized unitarity at one loop, focusing on the computation of the coefficients of the box integral. In Section III, we give an outline of the two-loop formalism, and detail the solutions to the cut equations. In Section IV, we present the set of constraint equations, and their solutions. In Section V, we give the master formulæ for the double-box coefficients, and give some examples of their use in Section VI. We summarize in Section VII.

II. MAXIMAL UNITARITY AT ONE LOOP

We begin by reviewing the derivation of the formula for coefficients of one-loop boxes using quadruple cuts, originally written down by Britto, Cachazo, and Feng [6]. We adopt an approach and notation that generalize to our derivation for two-loop coefficients in following

sections. Our starting point is the formal diagrammatic expression for the amplitude,

$$\text{Amplitude} = \sum_{\substack{\text{Feynman} \\ \text{diagrams}_F}} \int \frac{d^D \ell}{(2\pi)^D} \text{Numerator}_F(\ell, \dots) \cdot \text{Propagators}_F(\ell, \dots), \quad (2.1)$$

where the ellipses represent dependence on external momenta, polarization vectors, and spinor strings. Although the whole point of the method is to avoid computing any Feynman diagrams explicitly, it is still convenient to refer to them in the abstract, as a means of providing the connection to field theory and to Feynman integrals.

Applying tensor and integral reductions [70], along with a Gram-determinant identity holding through $\mathcal{O}(\epsilon^0)$, we obtain the basic equation (1.1) without any reference to unitarity or on-shell conditions. (In a slight abuse of language, we will refer to integrals with no free indices, but numerator powers of the loop momentum contracted into external vectors, as “tensor integrals”.)

At one loop, it is sufficient for our purposes to concentrate on the four-dimensional components of the loop momentum. (The accompanying integrals must of course be evaluated keeping the full $(D = 4 - 2\epsilon)$ -dimensional dependence.) In order to derive formulæ for the coefficients of basis integrals, we apply cuts to both sides of eq. (2.1). In the basic unitarity method, we would replace two propagators, separated by a non-null sum of external momenta, by delta functions which freeze the loop momenta they carry to their on-shell values. In generalized unitarity [4, 6], we would like to apply additional delta functions to put additional momenta to on-shell values. However, once we put the momenta carried by more than two massless propagators to their on-shell values, the solutions to the on-shell equations are complex, and taken at face value, the delta functions would actually yield zero.

The same issue arose in the evaluation of the connected prescription [71] for amplitudes in Witten’s twistor string theory [72]; the solution is to use contour integrals instead of delta functions [45, 73, 74]. To do so, we think of complexifying the space in which the four-dimensional loop momenta live, from $\mathbb{R}^{1,3}$ to \mathbb{C}^4 , and taking the integrals on both sides of eq. (1.1) to be over a product of contours running along the real axis. We can imagine evaluating the loop integrals along other contours as well. New contours that will be useful for our purposes are those whose product encircles simultaneous poles in all four-dimensional components of the loop momentum. Performing the four-dimensional loop-momentum integral over each such contour will yield the residue at the corresponding

encircled joint or global pole. The residue extracts \square the terms in the integrand which contain each of the corresponding propagators, removes the denominators, divides by the appropriate Jacobian, and sets the components of the loop momentum to their values at the joint pole.

The Jacobian is a determinant which arises from the transformation to variables which express each denominator factor linearly in a different variable. Unlike a product of delta functions, which would produce a factor of the inverse of the absolute value of the Jacobian, the transformation here will produce a factor of the inverse of the Jacobian. This ensures that the factor is analytic in any variables on which it depends, so that further contour integrations can be carried out.

Notationally, it will still be convenient to use delta functions; to do so, define the product of delta functions to yield exactly this contour integral,

$$\int \frac{d^4\ell}{(2\pi)^4} \text{Numerator}_F(\ell, \dots) \delta(\ell^2) \delta((\ell - k_1)^2) \delta((\ell - k_1 - k_2)^2) \delta((\ell + k_4)^2) \equiv \oint_{T_Q} \frac{d^4\ell}{(2\pi)^4} \frac{\text{Numerator}_F(\ell, \dots)}{\ell^2(\ell - k_1)^2(\ell - k_1 - k_2)^2(\ell + k_4)^2}, \quad (2.2)$$

where we have divided out overall factors of $2\pi i$ associated with each delta function, and where T_Q is a four-torus encircling the solutions to the simultaneous equations,

$$\ell^2 = 0, \quad (\ell - k_1)^2 = 0, \quad (\ell - k_1 - k_2)^2 = 0, \quad (\ell + k_4)^2 = 0, \quad (2.3)$$

and where — in a non-standard bit of notation — we absorb a factor of $1/(2\pi i)$ into the definition of each contour integral, so that evaluating the four-fold contour integral yields a sum over residues with no additional factors of $2\pi i$.

In four-point amplitudes, the external momenta do not suffice to provide a basis for arbitrary external vectors; to three of them (say k_1 , k_2 , and k_4), we need to add another external vector, for example $v^\mu = \varepsilon(\mu, k_1, k_2, k_4)$. Then we can express all dot products of the loop momentum with external vectors in terms of four dot products: $\ell \cdot k_1$, $\ell \cdot k_2$, $\ell \cdot k_4$, and $\ell \cdot v$. In reducing integrals, odd powers of $v \cdot \ell$ will give rise to vanishing integrals because of parity; even powers can be re-expressed in terms of Gram determinants and thence in terms of the other dot products (up to terms involving the (-2ϵ) -dimensional components of the loop momentum). The remaining three dot products can be re-expressed as linear combinations of propagator denominators and external invariants, allowing integrals with insertions of them in the numerator to be simplified.

One would be tempted to believe that replacing the original contours running along the real axis with some other contour, such as T_Q , would leave the equality (1.1) undisturbed, but this is not quite right, because there are implicitly terms in the integrand of the left-hand side that are ‘total derivatives’, that is terms which integrate to zero. These terms arise during the integral reductions described above. They were made explicit in the decomposition of the integrand used by OPP [11]. For general contours, the reduction equation will then take the form,

$$\text{Amplitude} = \sum_{j \in \text{Basis}} c_j I_j + \text{Rat} + \sum_{j \in \text{Basis}} \sum_{t \in \substack{\text{Total} \\ \text{Derivatives}}} c'_{j,t} U_{j,t}, \quad (2.4)$$

where each $U_{j,t}$ is the integral of an expression $W_{j,t}$ which would vanish if taken over the real slice, for example $W_{1,1} = \varepsilon(\ell, k_1, k_2, k_4)$:

$$U_1 = \int \frac{d^D \ell}{(2\pi)^D} \frac{\varepsilon(\ell, k_1, k_2, k_4)}{\ell^2 (\ell - k_1)^2 (\ell - k_1 - k_2)^2 (\ell + k_4)^2}. \quad (2.5)$$

This integral will no longer necessarily vanish if we integrate over general contours in \mathbb{C}^4 . In this equation,

$$\varepsilon(q_1, q_2, q_3, q_4) \equiv \varepsilon_{\mu\nu\lambda\sigma} q_1^\mu q_2^\nu q_3^\lambda q_4^\sigma, \quad (2.6)$$

where $\varepsilon_{\mu\nu\lambda\sigma}$ is the standard antisymmetric Levi-Civita tensor. Another example is the cube of the Levi-Civita symbol, $W_{1,2} = W_{1,1}^3$, though as it turns out, this latter integrand does not contribute any additional equations below.

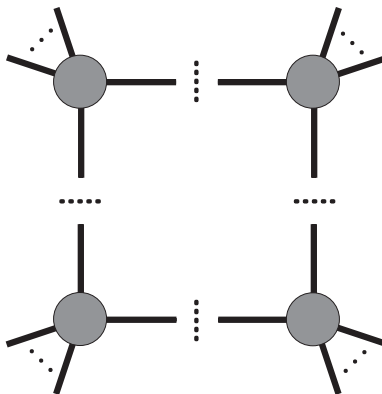


FIG. 1: The general quadruple cut in a one-loop box.

When we perform a quadruple cut, that is the integral over T_Q , we will restrict the set of Feynman diagrams in the expression for the amplitude to those containing all four

propagators; cut the propagators; and impose the on-shell conditions corresponding to the vanishing of the propagator denominators. If we imagine working in a physical gauge (such as light-cone gauge), this also restricts the cut lines to have physical polarizations. Each diagram then falls apart into a product of four diagrams, one corresponding to each corner of the box. The sum over diagrams factorizes into a four-fold sum over the tree diagrams at each corner, as shown in fig. 1. Each such sum will give an on-shell tree amplitude, with two cut loop momenta and the external legs attached to that corner as arguments,

$$\begin{aligned}
A^{1\text{-loop}} &= \sum_{\substack{\text{Feynman} \\ \text{diagrams}}}^F \int \frac{d^D \ell}{(2\pi)^D} \frac{N_F}{D_F} \\
&\rightarrow \sum_{\substack{\text{helicities} \\ \text{species}}} \sum_{\substack{\text{Feynman} \\ \text{diagrams}}}^A \frac{N_A}{D_A} \sum_{\substack{\text{Feynman} \\ \text{diagrams}}}^B \frac{N_B}{D_B} \sum_{\substack{\text{Feynman} \\ \text{diagrams}}}^C \frac{N_C}{D_C} \sum_{\substack{\text{Feynman} \\ \text{diagrams}}}^D \frac{N_D}{D_D} \\
&= \sum_{\substack{\text{helicities} \\ \text{species}}} A_A^{\text{tree}} A_B^{\text{tree}} A_C^{\text{tree}} A_D^{\text{tree}}.
\end{aligned} \tag{2.7}$$

(If we had not initially used a physical gauge, it is at this stage, summing over all diagrams, that we would recover the restriction to physical polarizations.) Integrating over T_Q in eq. (2.4) will give us the following equation,

$$\sum_{\substack{\text{helicities} \\ \text{species}}} A_A^{\text{tree}} A_B^{\text{tree}} A_C^{\text{tree}} A_D^{\text{tree}} = c_{\text{box}} + \sum_t c'_{\text{box},t} U_{\text{box},t} \tag{2.8}$$

where the Jacobian has canceled out of the equality, and the sum on the right-hand side runs over possible total derivatives with a box integrand. In order to solve for the desired coefficient c_{box} , or equivalently to ensure that the equality in eq. (1.1) is maintained, we must evaluate the integral over a linear combination of new contours such that all possible integrals of ‘total derivatives’ U_t are projected out. As we will show later, this requirement gives us constraints that determine the allowed combinations of contours, and in turn the equations for the coefficients of the box integrals.

In the case of the one-loop box integral, the joint-pole equations are given by eq. (2.3) or equivalently by,

$$\ell^2 = 0, \quad \ell \cdot k_1 = 0, \quad \ell \cdot k_2 = \frac{s}{2}, \quad \ell \cdot k_4 = 0, \tag{2.9}$$

which form makes it clear that there are two distinct solutions, and hence two distinct contours. The one ‘total derivative’ we must consider is the ε expression U_1 given above in eq. (2.5). It turns out that it evaluates to compensating values on the two solutions, so that

summing over them projects it out, and hence gives an equation for the coefficient of the box in terms of the product of tree amplitudes at each corner,

$$c_{\text{box}} = \frac{1}{2} \sum_{\substack{\text{helicities} \\ \text{species}}} A_A^{\text{tree}} A_B^{\text{tree}} A_C^{\text{tree}} A_D^{\text{tree}}. \quad (2.10)$$

As an example, study the coefficient of the one-mass box with $m_3^2 \neq 0$. Parametrize the four-dimensional part of the loop momentum as follows,

$$\ell^\mu = \alpha_1 k_1^\mu + \alpha_2 k_2^\mu + \frac{\alpha_3 s}{2 \langle 14 \rangle [42]} \langle 1^- | \gamma^\mu | 2^- \rangle + \frac{\alpha_4 s}{2 \langle 24 \rangle [41]} \langle 2^- | \gamma^\mu | 1^- \rangle. \quad (2.11)$$

A general contour integral for the four-dimensional part of the box integral then takes the form,

$$\frac{1}{s^4} \int_C d^4 \alpha_i J_\alpha \frac{1}{(\alpha_1 \alpha_2 - \omega \alpha_3 \alpha_4)(\alpha_1 \alpha_2 - \omega \alpha_3 \alpha_4 - \alpha_2)(\alpha_1 \alpha_2 - \omega \alpha_3 \alpha_4 - \alpha_2 - \alpha_1 + 1)} \times \frac{1}{(\alpha_1 \alpha_2 - \omega \alpha_3 \alpha_4 + \alpha_1 t/s + \alpha_2 u/s + \alpha_3 + \alpha_4)} \quad (2.12)$$

where $\omega = s^2/(tu)$. In this expression, J_α is the Jacobian that arises from changing variables from the ℓ^μ to the α_i . (We do not need its explicit form, only the knowledge that it is independent of the α_i , a consequence of the linearity of the ℓ^μ in the α_i .)

The cut equations (2.9) then take the form,

$$\begin{aligned} \left(\alpha_1 \alpha_2 - \frac{s^2}{tu} \alpha_3 \alpha_4 \right) s &= 0, \\ \alpha_2 s &= 0, \\ \alpha_1 s &= s, \\ \alpha_1 t + \alpha_2 u + \alpha_3 s + \alpha_4 s &= 0, \end{aligned} \quad (2.13)$$

which have two solutions,

$$\begin{aligned} \mathcal{S}_1 : \quad \alpha_1 &= 1, \quad \alpha_2 = 0, \quad \alpha_3 = -\frac{t}{s}, \quad \alpha_4 = 0; \\ \mathcal{S}_2 : \quad \alpha_1 &= 1, \quad \alpha_2 = 0, \quad \alpha_3 = 0, \quad \alpha_4 = -\frac{t}{s}. \end{aligned} \quad (2.14)$$

If we define $C_j(v)$ to be a small circle in the complex α_j -plane that encloses the point v , then the two contours we must consider are,

$$\begin{aligned} T_1 &= C_1(1) \times C_2(0) \times C_3(-t/s) \times C_4(0); \\ \text{and } T_2 &= C_1(1) \times C_2(0) \times C_3(0) \times C_4(-t/s). \end{aligned} \quad (2.15)$$

We can evaluate the four-fold integral (2.12) by ‘global residues’ [45, 75]. The sign of the result will depend on the orientation chosen for the contour; but this sign will drop out of final formulæ for integral coefficients so long as this orientation is chosen consistently throughout the calculation. To do so, we should first change to variables where each pole is in a different variable, and where the denominators are linear in that variable with unit coefficient. The Jacobian from this change of variables will take the form,

$$J_1 = \det_{i,j} \left(\frac{\partial f_j}{\partial \alpha_i} \right), \quad (2.16)$$

where

$$\begin{aligned} f_1 &= \alpha_1 \alpha_2 - \omega \alpha_3 \alpha_4, \\ f_2 &= \alpha_1 \alpha_2 - \omega \alpha_3 \alpha_4 - \alpha_2, \\ f_3 &= \alpha_1 \alpha_2 - \omega \alpha_3 \alpha_4 - \alpha_2 - \alpha_1 + 1, \end{aligned} \quad (2.17)$$

$$f_4 = \alpha_1 \alpha_2 - \omega \alpha_3 \alpha_4 + \alpha_1 t/s + \alpha_2 u/s + \alpha_3 + \alpha_4. \quad (2.18)$$

We find,

$$J_1 = \omega(\alpha_4 - \alpha_3). \quad (2.19)$$

Evaluating the box integral with a numerator $\text{Num}(\ell, \dots)$ along a contour given by a linear combination of the two T_i with weights a_i , we obtain,

$$s^{-4} J_\alpha \left(a_1 J_1^{-1} \text{Num}(\ell, \dots) \Big|_{\mathcal{S}_1} + a_2 J_1^{-1} \text{Num}(\ell, \dots) \Big|_{\mathcal{S}_2} \right). \quad (2.20)$$

Using the parametrization (2.11), we find the following expression for the Levi-Civita symbol we need,

$$\begin{aligned} \varepsilon(\ell, k_1, k_2, k_4) &= \alpha_3 s \varepsilon \left(\frac{\langle 1^- | \gamma^\mu | 2^- \rangle}{2(14)[42]}, k_1, k_2, k_4 \right) + \alpha_4 s \varepsilon \left(\frac{\langle 2^- | \gamma^\mu | 1^- \rangle}{2(24)[41]}, k_1, k_2, k_4 \right) \\ &= s(\alpha_3 - \alpha_4) \varepsilon \left(\frac{\langle 1^- | \gamma^\mu | 2^- \rangle}{2(14)[42]}, k_1, k_2, k_4 \right). \end{aligned} \quad (2.21)$$

The constraint that $U_1 = 0$ on the quadruple cut then implies that,

$$-s^{-4} J_\alpha \omega^{-1} (a_1 + a_2) = 0. \quad (2.22)$$

so that $a_2 = -a_1$. Higher odd powers of the Levi-Civita tensor lead to the same constraint.

If we evaluate both sides of eq. (1.1) on the linear combination of contours, we find,

$$s^{-4} J_\alpha \sum_{i=1}^2 a_i J_1^{-1} \prod_{j=1}^4 A_j^{\text{tree}} \Big|_{\mathcal{S}_i} = s^{-4} J_\alpha c \left(a_1 J_1^{-1} \Big|_{\mathcal{S}_1} + a_2 J_1^{-1} \Big|_{\mathcal{S}_2} \right), \quad (2.23)$$

where the product is over the tree amplitudes associated to each of the four vertices of the quadruple-cut box integral in fig. 1. Substituting in the solution to eq. (2.22), we find for the coefficient of the one-loop box,

$$c_{\text{box}} = \frac{1}{2} \sum_{i=1}^2 \prod_{j=1}^4 A_j^{\text{tree}} \Big|_{\mathcal{S}_i} \quad (2.24)$$

which is just eq. (2.10) when summed over possible helicity assignments and species circulating in the loop.

In the following sections, we show how to generalize these considerations to two loops.

III. MAXIMAL CUTS AT TWO LOOPS

Our basic approach to the planar double box at two loops will be similar to that reviewed above at one loop. We use a convenient parametrization of the loop momenta and choose new contours of integration to freeze the momenta flowing through all propagators. We choose those contours so that constraint equations arising from consistency conditions are satisfied. Unlike the procedure at one loop, cutting all seven propagators does not freeze all components of both loop momenta, so we must choose new contours for the remaining unfrozen degrees of freedom as well. In addition, we have a much larger and richer set of consistency conditions arising from IBP identities. Once we have solved the constraint equations, we will solve for the coefficients of specific basis integrals.

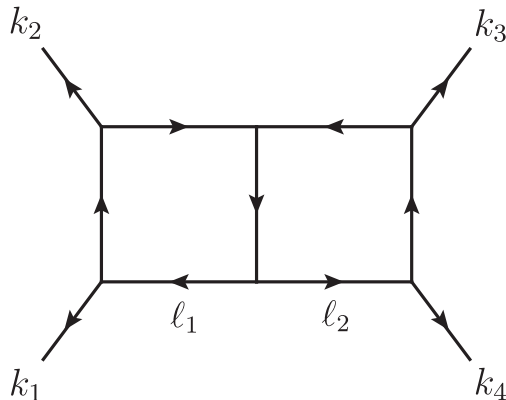


FIG. 2: The double box integral $P_{2,2}^{**}$.

In this section, we give a convenient parametrization of the loop momenta, and use it to

solve on-shell equations. We list these solutions below, along with the poles and possible contours for the remaining unfrozen degrees of freedom.

The two-loop double box integral, shown in fig. 2, is,

$$P_{2,2}^{**} = \int \frac{d^D \ell_1}{(2\pi)^D} \frac{d^D \ell_2}{(2\pi)^D} \frac{1}{\ell_1^2 (\ell_1 - k_1)^2 (\ell_1 - K_{12})^2 (\ell_1 + \ell_2)^2 \ell_2^2 (\ell_2 - k_4)^2 (\ell_2 - K_{34})^2}, \quad (3.1)$$

where $K_{i\dots j} \equiv k_i + \dots + k_j$, and the notation follows ref. [60].

We will focus in this paper on extracting coefficients of basis integrals only to leading order in the dimensional regulator ϵ , for which it suffices to consider the four-dimensional components of the loop momentum as far as cuts are concerned. The double box has seven propagators; if we cut all of them, that is put all of the momenta they are carrying to on-shell values, we will be left with one additional degree of freedom. To cut the momenta in this way, we must shift the contours of integration for the components of the two loop momenta ℓ_1 and ℓ_2 to encircle the joint solutions to the on-shell equations,

$$\begin{aligned} \ell_1^2 = 0, \quad (\ell_1 - k_1)^2 = 0, \quad (\ell_1 - K_{12})^2 = 0, \quad \ell_2^2 = 0, \quad (\ell_2 - k_4)^2 = 0, \\ (\ell_2 - K_{34})^2 = 0, \quad (\ell_1 + \ell_2)^2 = 0. \end{aligned} \quad (3.2)$$

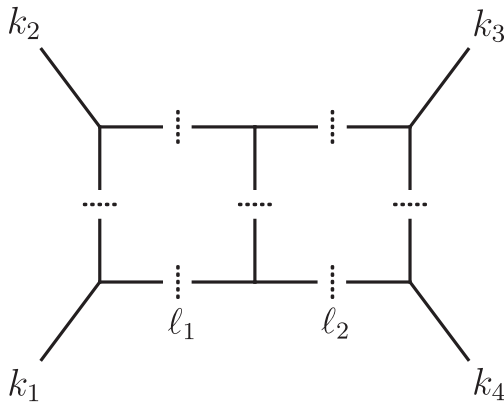


FIG. 3: The heptacut double box.

As explained in Section II, we can write the four-dimensional heptacut integral symbolically as,

$$\begin{aligned} \int \frac{d^4 \ell_1}{(2\pi)^4} \frac{d^4 \ell_2}{(2\pi)^4} \delta(\ell_1^2) \delta((\ell_1 - k_1)^2) \delta((\ell_1 - K_{12})^2) \delta((\ell_1 + \ell_2)^2) \\ \times \delta(\ell_2^2) \delta((\ell_2 - k_4)^2) \delta((\ell_2 - K_{34})^2), \end{aligned} \quad (3.3)$$

again dropping overall factors of $2\pi i$ associated with the delta functions. This heptacut is depicted in fig. 3.

To solve the on-shell equations, we use the following parametrization of the loop momenta,

$$\begin{aligned}\ell_1^\mu &= \alpha_1 k_1^\mu + \alpha_2 k_2^\mu + \frac{s_{12}\alpha_3}{2\langle 14\rangle[42]} \langle 1^- | \gamma^\mu | 2^- \rangle + \frac{s_{12}\alpha_4}{2\langle 24\rangle[41]} \langle 2^- | \gamma^\mu | 1^- \rangle, \\ \ell_2^\mu &= \beta_1 k_3^\mu + \beta_2 k_4^\mu + \frac{s_{12}\beta_3}{2\langle 31\rangle[14]} \langle 3^- | \gamma^\mu | 4^- \rangle + \frac{s_{12}\beta_4}{2\langle 41\rangle[13]} \langle 4^- | \gamma^\mu | 3^- \rangle.\end{aligned}\quad (3.4)$$

Using this parametrization, the six corresponding heptacut equations involving only one loop momentum are,

$$\begin{aligned}\ell_1^2 &= s_{12} \left(\alpha_1 \alpha_2 + \frac{\alpha_3 \alpha_4}{\chi(\chi+1)} \right) = 0, \\ (\ell_1 - k_1)^2 &= s_{12} \left((\alpha_1 - 1) \alpha_2 + \frac{\alpha_3 \alpha_4}{\chi(\chi+1)} \right) = 0, \\ (\ell_1 - K_{12})^2 &= s_{12} \left((\alpha_1 - 1)(\alpha_2 - 1) + \frac{\alpha_3 \alpha_4}{\chi(\chi+1)} \right) = 0, \\ \ell_2^2 &= s_{12} \left(\beta_1 \beta_2 + \frac{\beta_3 \beta_4}{\chi(\chi+1)} \right) = 0, \\ (\ell_2 - k_4)^2 &= s_{12} \left(\beta_1(\beta_2 - 1) + \frac{\beta_3 \beta_4}{\chi(\chi+1)} \right) = 0, \\ (\ell_2 - K_{34})^2 &= s_{12} \left((\beta_1 - 1)(\beta_2 - 1) + \frac{\beta_3 \beta_4}{\chi(\chi+1)} \right) = 0,\end{aligned}\quad (3.5)$$

where

$$\chi \equiv \frac{s_{14}}{s_{12}}.\quad (3.6)$$

We can simplify these equations, obtaining

$$\begin{aligned}\alpha_1 &= 1, & \alpha_2 &= 0, & \alpha_3 \alpha_4 &= 0, \\ \beta_1 &= 0, & \beta_2 &= 1, & \beta_3 \beta_4 &= 0.\end{aligned}\quad (3.7)$$

These equations have four distinct solutions. If we substitute these values into eq. (3.2), we find for the last equation,

$$\begin{aligned}0 &= (\ell_1 + \ell_2)^2 = 2\ell_1 \cdot \ell_2 = \\ &2 \left(k_1^\mu + \frac{s_{12}\alpha_3}{2\langle 14\rangle[42]} \langle 1^- | \gamma^\mu | 2^- \rangle + \frac{s_{12}\alpha_4}{2\langle 24\rangle[41]} \langle 2^- | \gamma^\mu | 1^- \rangle \right) \\ &\quad \times \left(k_{4\mu} + \frac{s_{12}\beta_3}{2\langle 31\rangle[14]} \langle 3^- | \gamma_\mu | 4^- \rangle + \frac{s_{12}\beta_4}{2\langle 41\rangle[13]} \langle 4^- | \gamma_\mu | 3^- \rangle \right).\end{aligned}\quad (3.8)$$

For two of the four solutions to eqs. (3.7), this equation has two solutions, so that overall we find six solutions to the heptacut equations (3.5, 3.8). To each of the six solutions \mathcal{S}_j , we can associate a seven-torus in the parameters α_i and β_i that encircles the solution.

For the solution $\alpha_4 = 0 = \beta_4$, the last equation (3.8) simplifies to,

$$0 = \left([4\ 1] + \frac{s_{12}\alpha_3}{\langle 1\ 4 \rangle} \right) \left(\langle 1\ 4 \rangle - \frac{s_{12}\beta_3}{[1\ 4]} \right), \quad (3.9)$$

which has two distinct solutions,

$$\begin{aligned} \mathcal{S}_1 : \quad & \alpha_3 = -\chi, \quad \beta_3 \text{ arbitrary}; \\ \mathcal{S}_2 : \quad & \beta_3 = -\chi, \quad \alpha_3 \text{ arbitrary}. \end{aligned} \quad (3.10)$$

In all solutions, we will change variables so that the remaining degree of freedom is called z .

Likewise, the solution $\alpha_3 = 0 = \beta_3$ also yields two solutions to eq. (3.8),

$$\begin{aligned} \mathcal{S}_3 : \quad & \alpha_4 = -\chi, \quad \beta_4 = z; \\ \mathcal{S}_4 : \quad & \beta_4 = -\chi, \quad \alpha_4 = z. \end{aligned} \quad (3.11)$$

For the remaining two solutions, the last equation (3.8) does not factorize, and we obtain only one solution; for $\alpha_3 = 0 = \beta_4$,

$$\mathcal{S}_5 : \quad \alpha_4 = z, \quad \beta_3 = -(\chi + 1) \frac{z + \chi}{z + \chi + 1}; \quad (3.12)$$

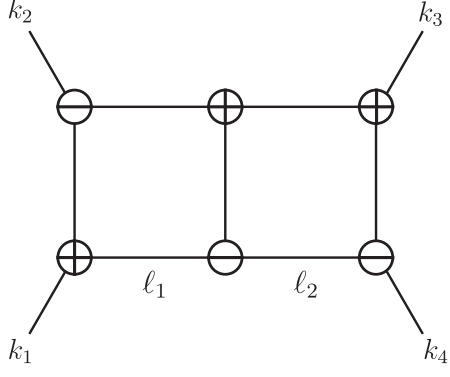
and for $\alpha_4 = 0 = \beta_3$,

$$\mathcal{S}_6 : \quad \alpha_3 = z, \quad \beta_4 = -(\chi + 1) \frac{z + \chi}{z + \chi + 1}. \quad (3.13)$$

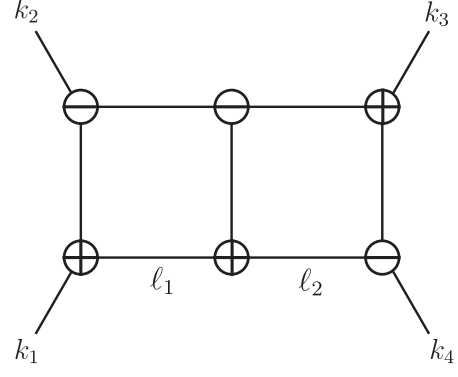
In the last two solutions, we could equally well have chosen a different parametrization, where β_3 or β_4 respectively are set to z . This just amounts to a change of variables, of course, but does break the manifest symmetry between the two loops.

The existence of six kinematic solutions can also be understood from holomorphicity considerations of the cuts. When we cut all propagators, each of the six vertices in the double box has three massless momenta attached. We can write these momenta in terms of spinors, $k^\mu = \lambda^\alpha \sigma_{\alpha\dot{\alpha}}^\mu \tilde{\lambda}^{\dot{\alpha}}$. Momentum conservation at each vertex [72] then implies that either,

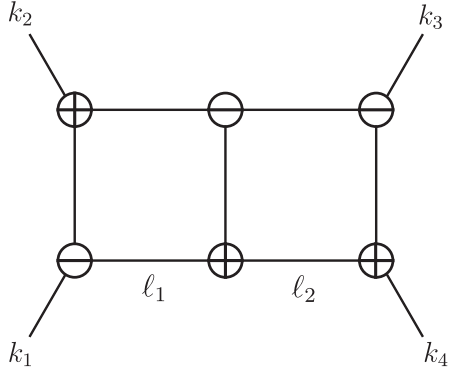
- 1) the holomorphic spinors λ of the momenta are collinear (proportional), $\lambda_a \propto \lambda_b \propto \lambda_c$. We will depict such a vertex using a circled plus (\oplus). Such a vertex would allow only an $\overline{\text{MHV}}$ tree amplitude to be attached (of course the holomorphicity properties of the cut are independent of any tree amplitude).



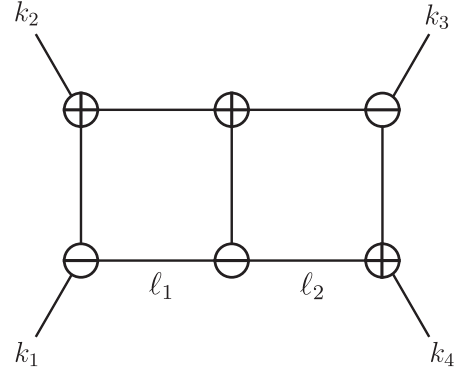
Solution \mathcal{S}_1 , obtained by setting
 $\alpha_3 = -\chi$, $\beta_3 = z$,
 $\alpha_4 = 0$, $\beta_4 = 0$.



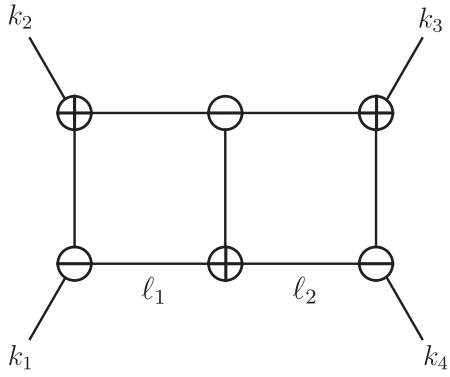
Solution \mathcal{S}_2 , obtained by setting
 $\alpha_3 = z$, $\beta_3 = -\chi$,
 $\alpha_4 = 0$, $\beta_4 = 0$.



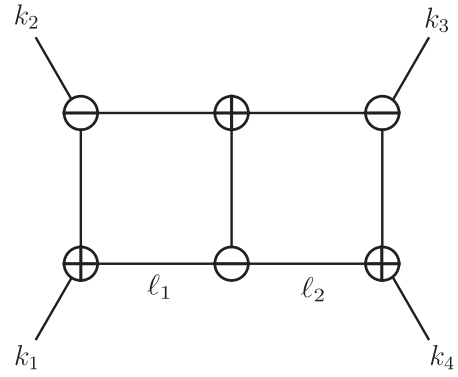
Solution \mathcal{S}_3 , obtained by setting
 $\alpha_3 = 0$, $\beta_3 = 0$,
 $\alpha_4 = -\chi$, $\beta_4 = z$.



Solution \mathcal{S}_4 , obtained by setting
 $\alpha_3 = 0$, $\beta_3 = 0$,
 $\alpha_4 = z$, $\beta_4 = -\chi$.



Solution \mathcal{S}_5 , obtained by setting
 $\alpha_3 = 0$, $\beta_3 = -(\chi + 1) \frac{z+\chi}{z+\chi+1}$,
 $\alpha_4 = z$, $\beta_4 = 0$.



Solution \mathcal{S}_6 , obtained by setting
 $\alpha_3 = z$, $\beta_3 = 0$,
 $\alpha_4 = 0$, $\beta_4 = -(\chi + 1) \frac{z+\chi}{z+\chi+1}$.

FIG. 4: The six solutions to the heptacut equations for the two-loop planar double box.

- 2) the antiholomorphic spinors $\tilde{\lambda}$ of the momenta are collinear, $\tilde{\lambda}_a \propto \tilde{\lambda}_b \propto \tilde{\lambda}_c$. We will depict such a vertex using a circled minus (\ominus). Such a vertex would allow only an MHV tree amplitude to be attached.

For general kinematics, neither the external holomorphic spinors λ_j nor the external antiholomorphic spinors $\tilde{\lambda}_j$ are collinear. A configuration with an uninterrupted chain of either \oplus or \ominus vertices connecting any two external legs is thus disallowed. There are exactly six ways of assigning these two labelings to vertices avoiding such chains, hence six solutions. The six solutions are shown diagrammatically in fig. 4. (The labeling of holomorphically-collinear vertices as \oplus , and of antiholomorphically-collinear ones as \ominus is not uniform in the literature.)

In evaluating the contour integrals represented by the delta functions in eq. (3.3), we encounter two Jacobians: one from changing variables from the components of ℓ_j to the α_i and β_i ; and one from actually performing the contour integrals in the latter variables. It is the latter Jacobian that is important for our purposes. The former Jacobian is equal to $J_\alpha J_\beta$, where

$$J_\alpha = \det_{\mu,i} \frac{\partial \ell_1^\mu}{\partial \alpha_i} = -\frac{is_{12}^2}{4\chi(\chi+1)}, \quad J_\beta = \det_{\mu,i} \frac{\partial \ell_2^\mu}{\partial \beta_i} = -\frac{is_{12}^2}{4\chi(\chi+1)}. \quad (3.14)$$

To evaluate the latter Jacobian, we may note that three of the delta functions (or equivalently three of the contour integrals) involve only α variables, and three involve only β variables. We can thus split up the problem into three steps: computing the Jacobian associated with ℓ_1 , that is with the α variables alone; computing the Jacobian associated with ℓ_2 , that is with the β variables alone; and finally, computing the Jacobian associated with the middle propagator, involving both ℓ_1 and ℓ_2 .

For each of the six solutions, we must compute the Jacobian independently. As an example, consider the second solution \mathcal{S}_2 . The first Jacobian arises from considering the integral,

$$\int d\alpha_1 d\alpha_2 d\alpha_4 \delta \left[s_{12} \left(\alpha_1 \alpha_2 + \frac{\alpha_3 \alpha_4}{\chi(\chi+1)} \right) \right] \delta \left[s_{12} \left((\alpha_1 - 1) \alpha_2 + \frac{\alpha_3 \alpha_4}{\chi(\chi+1)} \right) \right] \\ \times \delta \left[s_{12} \left((\alpha_1 - 1) (\alpha_2 - 1) + \frac{\alpha_3 \alpha_4}{\chi(\chi+1)} \right) \right], \quad (3.15)$$

associated with the ℓ_1 loop. Define

$$\begin{pmatrix} g_1(\alpha_1, \alpha_2, \alpha_4) \\ g_2(\alpha_1, \alpha_2, \alpha_4) \\ g_3(\alpha_1, \alpha_2, \alpha_4) \end{pmatrix} = \begin{pmatrix} s_{12} \left(\alpha_1 \alpha_2 + \frac{\alpha_3 \alpha_4}{\chi(\chi+1)} \right) \\ s_{12} \left((\alpha_1 - 1) \alpha_2 + \frac{\alpha_3 \alpha_4}{\chi(\chi+1)} \right) \\ s_{12} \left((\alpha_1 - 1)(\alpha_2 - 1) + \frac{\alpha_3 \alpha_4}{\chi(\chi+1)} \right) \end{pmatrix}. \quad (3.16)$$

The first Jacobian is then,

$$J_1 = \det_{i,j} \frac{\partial g_i}{\partial \alpha_j} = s_{12}^3 \det \begin{pmatrix} \alpha_2 & \alpha_1 & \frac{\alpha_3}{\chi(\chi+1)} \\ \alpha_2 & \alpha_1 - 1 & \frac{\alpha_3}{\chi(\chi+1)} \\ \alpha_2 - 1 & \alpha_1 - 1 & \frac{\alpha_3}{\chi(\chi+1)} \end{pmatrix} = -\frac{s_{12}^3}{\chi(\chi+1)} \alpha_3 \quad (3.17)$$

(As explained in Section II, the Jacobians will appear in the denominator as determinants rather than as absolute values of determinants.) Similarly, the second Jacobian arises from considering the integral,

$$\begin{aligned} \int d\beta_1 d\beta_2 d\beta_4 \delta \left[s_{12} \left(\beta_1 \beta_2 + \frac{\beta_3 \beta_4}{\chi(\chi+1)} \right) \right] \delta \left[s_{12} \left(\beta_1 (\beta_2 - 1) + \frac{\beta_3 \beta_4}{\chi(\chi+1)} \right) \right] \\ \times \delta \left[s_{12} \left((\beta_1 - 1)(\beta_2 - 1) + \frac{\beta_3 \beta_4}{\chi(\chi+1)} \right) \right], \end{aligned} \quad (3.18)$$

associated with the ℓ_2 loop. Define

$$\begin{pmatrix} h_1(\beta_1, \beta_2, \beta_4) \\ h_2(\beta_1, \beta_2, \beta_4) \\ h_3(\beta_1, \beta_2, \beta_4) \end{pmatrix} = \begin{pmatrix} s_{12} \left(\beta_1 \beta_2 + \frac{\beta_3 \beta_4}{\chi(\chi+1)} \right) \\ s_{12} \left(\beta_1 (\beta_2 - 1) + \frac{\beta_3 \beta_4}{\chi(\chi+1)} \right) \\ s_{12} \left((\beta_1 - 1)(\beta_2 - 1) + \frac{\beta_3 \beta_4}{\chi(\chi+1)} \right) \end{pmatrix}. \quad (3.19)$$

The second Jacobian is then,

$$J_2 = \det_{i,j} \frac{\partial h_i}{\partial \beta_j} = s_{12}^3 \det \begin{pmatrix} \beta_2 & \beta_1 & \frac{\beta_3}{\chi(\chi+1)} \\ \beta_2 - 1 & \beta_1 & \frac{\beta_3}{\chi(\chi+1)} \\ \beta_2 - 1 & \beta_1 - 1 & \frac{\beta_3}{\chi(\chi+1)} \end{pmatrix} = \frac{s_{12}^3}{\chi(\chi+1)} \beta_3. \quad (3.20)$$

The remaining integration we must consider is over α_3 and β_3 ,

$$\begin{aligned} \frac{1}{2} \int d\alpha_3 d\beta_3 \frac{J_\alpha J_\beta}{J_1 J_2} \delta \left[\frac{s_{12}}{2\chi} (\alpha_3 + \chi)(\beta_3 + \chi) \right] = \\ \frac{1}{32s_{12}^2} \int \frac{d\alpha_3 d\beta_3}{\alpha_3 \beta_3} \delta \left[\frac{s_{12}}{2\chi} (\alpha_3 + \chi)(\beta_3 + \chi) \right], \end{aligned} \quad (3.21)$$

which leaves a remaining contour integration over z (*i.e.* α_3), along with the overall inverse Jacobian,

$$J^{-1}(z) = -\frac{1}{16s_{12}^3 z(z+\chi)}. \quad (3.22)$$

The computation for the other five solutions is similar; it turns out that we obtain the same overall Jacobian for all solutions. The contour for the z integration remains to be chosen; for this solution, there are two possible non-trivial contours, one encircling $z = 0$, and the other, encircling $z = -\chi$. (We set aside a possible non-trivial contour encircling $z = \infty$, as its contribution when integrating an arbitrary multiplying function $f(z)$ sums to zero when combined with the contributions of these two contours.) The pole at $z = -\chi$ is the eighth pole in the octacut of ref. [61]. In addition, for solutions $\mathcal{S}_{5,6}$, the denominator of $\beta_{3,4}$ (eqs. (3.12) and (3.13)) can give rise to additional poles at $z = -\chi - 1$ in tensor integrals. (As noted in Section II, in a slight abuse of language, we refer to integrals with no free indices, but numerator powers of the loop momenta contracted into external vectors, as “tensor integrals”.)

Collecting the information above, we have the following contours we can utilize in seeking equations for integral coefficients,

$$\begin{aligned}
T_{1,1} &= T_0 \times C_{\alpha_3}(-\chi) \times C_{\alpha_4}(0) \times C_{\beta_3=z}(0) \times C_{\beta_4}(0), \\
T_{1,2} &= T_0 \times C_{\alpha_3}(-\chi) \times C_{\alpha_4}(0) \times C_{\beta_3=z}(-\chi) \times C_{\beta_4}(0), \\
T_{2,1} &= T_0 \times C_{\alpha_3=z}(0) \times C_{\alpha_4}(0) \times C_{\beta_3}(-\chi) \times C_{\beta_4}(0), \\
T_{2,2} &= T_0 \times C_{\alpha_3=z}(-\chi) \times C_{\alpha_4}(0) \times C_{\beta_3}(-\chi) \times C_{\beta_4}(0), \\
T_{3,1} &= T_0 \times C_{\alpha_3}(0) \times C_{\alpha_4}(-\chi) \times C_{\beta_3}(0) \times C_{\beta_4=z}(0), \\
T_{3,2} &= T_0 \times C_{\alpha_3}(0) \times C_{\alpha_4}(-\chi) \times C_{\beta_3}(0) \times C_{\beta_4=z}(-\chi), \\
T_{4,1} &= T_0 \times C_{\alpha_3}(0) \times C_{\alpha_4=z}(0) \times C_{\beta_3}(0) \times C_{\beta_4}(-\chi), \\
T_{4,2} &= T_0 \times C_{\alpha_3}(0) \times C_{\alpha_4=z}(-\chi) \times C_{\beta_3}(0) \times C_{\beta_4}(-\chi), \\
T_{5,1} &= T_0 \times C_{\alpha_3}(0) \times C_{\alpha_4=z}(0) \times C_{\beta_3} \left(-\frac{(1+\chi)(z+\chi)}{z+\chi+1} \right) \times C_{\beta_4}(0), \\
T_{5,2} &= T_0 \times C_{\alpha_3}(0) \times C_{\alpha_4=z}(-\chi) \times C_{\beta_3} \left(-\frac{(1+\chi)(z+\chi)}{z+\chi+1} \right) \times C_{\beta_4}(0), \\
T_{5,3} &= T_0 \times C_{\alpha_3}(0) \times C_{\alpha_4=z}(-\chi-1) \times C_{\beta_3} \left(-\frac{(1+\chi)(z+\chi)}{z+\chi+1} \right) \times C_{\beta_4}(0), \\
T_{6,1} &= T_0 \times C_{\alpha_3=z}(0) \times C_{\alpha_4}(0) \times C_{\beta_3}(0) \times C_{\beta_4} \left(-\frac{(1+\chi)(z+\chi)}{z+\chi+1} \right), \\
T_{6,2} &= T_0 \times C_{\alpha_3=z}(-\chi) \times C_{\alpha_4}(0) \times C_{\beta_3}(0) \times C_{\beta_4} \left(-\frac{(1+\chi)(z+\chi)}{z+\chi+1} \right), \\
T_{6,3} &= T_0 \times C_{\alpha_3=z}(-\chi-1) \times C_{\alpha_4}(0) \times C_{\beta_3}(0) \times C_{\beta_4} \left(-\frac{(1+\chi)(z+\chi)}{z+\chi+1} \right),
\end{aligned} \tag{3.23}$$

where each subscript denotes the variable in whose plane the circle lies, and where

$$T_0 = C_{\alpha_1}(1) \times C_{\alpha_2}(0) \times C_{\beta_1}(0) \times C_{\beta_2}(1), \quad (3.24)$$

corresponding to the on-shell values in eq. (3.7). We will call the complete contours, including a choice of contour for z , the ‘augmented heptacut’.

Naively, we could deform the original contour of integration for the double box (3.1), along the product of real axes for all components of ℓ_1 and ℓ_2 , to any linear combination of contours in eq. (3.23) that we wish. However, an arbitrary deformation will not preserve the vanishing of total derivatives, analogs to U_1 given in eq. (2.5). In order to ensure that such objects vanish as they must, we impose constraints on the contours. We derive these in the next section.

IV. CONSTRAINT EQUATIONS FOR CONTOURS

Integral reductions are implicitly part of the simplifications applied to a sum over Feynman diagrams in order to obtain the basic equation at either one loop (1.1), or at two loops,

$$\text{Amplitude} = \sum_{j \in \text{Two-Loop Basis}} \text{coefficient}_j \text{Integral}_j + \text{Rational}. \quad (4.1)$$

The basis at two loops will contain integrals with up to eight propagators in the planar case [60], though a specific complete and independent choice of integrals for a general amplitude has not yet been written down. (The same restriction to eight propagators or fewer presumably applies in the non-planar case as well, using arguments along the same lines as given in ref. [60].)

As we saw in Section II, integral reductions at one loop involve only rewriting dot products of the loop momentum in terms of linear combinations of propagators and external invariants, along with the use of Lorentz invariance and parity to eliminate some integrals. For the box integral, in particular, the only non-trivial constraint arises from the use of parity, which requires that

$$\int \frac{d^D \ell}{(2\pi)^D} \frac{\varepsilon(\ell, k_1, k_2, k_4)}{\ell^2(\ell - k_1)^2(\ell - k_1 - k_2)^2(\ell + k_4)^2} = 0. \quad (4.2)$$

This constraint must be respected by the unitarity procedure; otherwise, applying a cut to the original integral and to the integral after reduction would yield different, and hence

inconsistent, answers. At one loop, it gives rise to one constraint equation, which fixes the relative normalization of the contours encircling the two solutions to the on-shell equations.

Similar constraints arise at two loops, though we have a greater variety of Levi-Civita symbols to consider. Denoting the insertion of the function $f(\ell_1, \ell_2)$ in the numerator of the double box by

$$P_{2,2}^{**}[f(\ell_1, \ell_2)] = \int \frac{d^D \ell_1}{(2\pi)^D} \frac{d^D \ell_2}{(2\pi)^D} \frac{f(\ell_1, \ell_2)}{\ell_1^2(\ell_1 - k_1)^2(\ell_1 - K_{12})^2(\ell_1 + \ell_2)^2\ell_2^2(\ell_2 - k_4)^2(\ell_2 - K_{34})^2}, \quad (4.3)$$

we must require that the vanishing of the following integrals,

$$\begin{aligned} P_{2,2}^{**}[\varepsilon(\ell_1, k_2, k_3, k_4)], \quad P_{2,2}^{**}[\varepsilon(\ell_2, k_2, k_3, k_4)], \quad P_{2,2}^{**}[\varepsilon(\ell_1, \ell_2, k_1, k_2)], \\ P_{2,2}^{**}[\varepsilon(\ell_1, \ell_2, k_1, k_3)], \quad \text{and} \quad P_{2,2}^{**}[\varepsilon(\ell_1, \ell_2, k_2, k_3)], \end{aligned} \quad (4.4)$$

continues to hold for integration over our chosen linear combination of contours. This is the complete set of Levi-Civita symbols that arises during integral reduction, after using momentum conservation.

At two loops, additional reductions are required in order to arrive at a linearly-independent set of basis integrals. These are usually obtained through IBP relations [62, 63, 66–68]; that is, they correspond to adding a total derivative to the original integrand. Each such total derivative, or equivalently each non-trivial reduction identity, gives rise to a constraint requiring that the unitarity procedure give vanishing coefficients for the additional terms; or equivalently, that the unitarity procedure respect the reduction equations. This is not automatically true contour-by-contour, and hence gives rise to non-trivial constraints on the choice of contours, and the weighting of different solutions.

In two-loop four-point amplitudes, we can express all dot products of loop momenta with external vectors in terms of eight dot products: $\ell_j \cdot k_1$, $\ell_j \cdot k_2$, $\ell_j \cdot k_4$, and $\ell_j \cdot v$, where $v^\mu = \varepsilon(\mu, k_1, k_2, k_4)$. Just as at one loop, odd powers of v will give rise to vanishing integrals, as expressed in the Levi-Civita constraints discussed above. Even powers can again be re-expressed in terms of the other dot products (up to terms involving the (-2ϵ) -dimensional components of the loop momentum). All integrals can then be rewritten in terms of the six dot products of the loop momenta with the external momenta.

Of these six dot products, three of them — $\ell_1 \cdot k_1$, $\ell_1 \cdot k_2$, $\ell_2 \cdot k_4$ — can be rewritten as linear combinations of the propagator denominators and external invariants. One additional

dot product of ℓ_2 — say $\ell_2 \cdot k_2$ — can be rewritten in terms of the remaining two ($\ell_1 \cdot k_4$ and $\ell_2 \cdot k_1$), propagator denominators, and external invariants. The remaining two dot products are called irreducible. At a first stage, then, before using IBP identities, we can reduce an arbitrary double-box integral appearing in a gauge-theory amplitude to a linear combination of the 22 different integrals that can arise with powers of the two irreducible numerators.

We have the following naively-irreducible integrals,

$$\begin{aligned}
& P_{2,2}^{**}[1], P_{2,2}^{**}[\ell_2 \cdot k_1], P_{2,2}^{**}[(\ell_2 \cdot k_1)^2], P_{2,2}^{**}[(\ell_2 \cdot k_1)^3], P_{2,2}^{**}[\ell_1 \cdot k_4], \\
& P_{2,2}^{**}[(\ell_2 \cdot k_1)(\ell_1 \cdot k_4)], P_{2,2}^{**}[(\ell_2 \cdot k_1)^2(\ell_1 \cdot k_4)], P_{2,2}^{**}[(\ell_2 \cdot k_1)^3(\ell_1 \cdot k_4)], \\
& P_{2,2}^{**}[(\ell_1 \cdot k_4)^2], P_{2,2}^{**}[(\ell_2 \cdot k_1)(\ell_1 \cdot k_4)^2], P_{2,2}^{**}[(\ell_2 \cdot k_1)^2(\ell_1 \cdot k_4)^2], \\
& P_{2,2}^{**}[(\ell_2 \cdot k_1)^3(\ell_1 \cdot k_4)^2], P_{2,2}^{**}[(\ell_1 \cdot k_4)^3], P_{2,2}^{**}[(\ell_2 \cdot k_1)(\ell_1 \cdot k_4)^3], \\
& P_{2,2}^{**}[(\ell_2 \cdot k_1)^2(\ell_1 \cdot k_4)^3], P_{2,2}^{**}[(\ell_2 \cdot k_1)^3(\ell_1 \cdot k_4)^3], P_{2,2}^{**}[(\ell_1 \cdot k_4)^4], \\
& P_{2,2}^{**}[(\ell_2 \cdot k_1)(\ell_1 \cdot k_4)^4], P_{2,2}^{**}[(\ell_2 \cdot k_1)^2(\ell_1 \cdot k_4)^4], P_{2,2}^{**}[(\ell_2 \cdot k_1)^4], \\
& P_{2,2}^{**}[(\ell_2 \cdot k_1)^4(\ell_1 \cdot k_4)], P_{2,2}^{**}[(\ell_2 \cdot k_1)^4(\ell_1 \cdot k_4)^2].
\end{aligned} \tag{4.5}$$

In the massless case, it turns out that there are 20 IBP relations between these integrals, which allow further reductions. These reductions allow us to pick certain pairs, for example,

$$P_{2,2}^{**}[1] \quad \text{and} \quad P_{2,2}^{**}[\ell_1 \cdot k_4], \tag{4.6}$$

as master integrals for the set in eq. (4.5), and thus also as basis integrals for an amplitude.

The remaining integrals are given in terms of these two by linear equations, for example

$$\begin{aligned}
P_{2,2}^{**}[\ell_2 \cdot k_1] &= P_{2,2}^{**}[\ell_1 \cdot k_4], \\
P_{2,2}^{**}[(\ell_1 \cdot k_4)(\ell_2 \cdot k_1)] &= \frac{1}{8}\chi s_{12}^2 P_{2,2}^{**}[1] - \frac{3}{4}s_{12} P_{2,2}^{**}[\ell_1 \cdot k_4] + \dots, \\
P_{2,2}^{**}[(\ell_2 \cdot k_1)^2] &= -\frac{\epsilon\chi s_{12}^2}{4(1-2\epsilon)} P_{2,2}^{**}[1] + \frac{(\chi+3\epsilon)s_{12}}{2(1-2\epsilon)} P_{2,2}^{**}[\ell_1 \cdot k_4] + \dots, \\
P_{2,2}^{**}[(\ell_1 \cdot k_4)^2] &= -\frac{\epsilon\chi s_{12}^2}{4(1-2\epsilon)} P_{2,2}^{**}[1] + \frac{(\chi+3\epsilon)s_{12}}{2(1-2\epsilon)} P_{2,2}^{**}[\ell_1 \cdot k_4] + \dots, \\
P_{2,2}^{**}[(\ell_1 \cdot k_4)(\ell_2 \cdot k_1)^2] &= -\frac{(1-3\epsilon)\chi s_{12}^3}{16(1-2\epsilon)} P_{2,2}^{**}[1] + \frac{(3-9\epsilon-2\epsilon\chi)s_{12}^2}{8(1-2\epsilon)} P_{2,2}^{**}[\ell_1 \cdot k_4] + \dots, \\
P_{2,2}^{**}[(\ell_1 \cdot k_4)^2(\ell_2 \cdot k_1)] &= -\frac{(1-3\epsilon)\chi s_{12}^3}{16(1-2\epsilon)} P_{2,2}^{**}[1] + \frac{(3-9\epsilon-2\epsilon\chi)s_{12}^2}{8(1-2\epsilon)} P_{2,2}^{**}[\ell_1 \cdot k_4] + \dots, \\
P_{2,2}^{**}[(\ell_2 \cdot k_1)^3] &= \frac{\epsilon\chi(1-2\chi-3\epsilon)s_{12}^3}{16(1-\epsilon)(1-2\epsilon)} P_{2,2}^{**}[1]
\end{aligned} \tag{4.7}$$

$$\begin{aligned}
& + \frac{(2\chi^2 - 3\epsilon(1 - 2\chi) + \epsilon^2(9 + 2\chi))s_{12}^2}{8(1 - \epsilon)(1 - 2\epsilon)} P_{2,2}^{**}[\ell_1 \cdot k_4] + \dots, \\
P_{2,2}^{**}[(\ell_1 \cdot k_4)^3] = & \frac{\epsilon\chi(1 - 2\chi - 3\epsilon)s_{12}^3}{16(1 - \epsilon)(1 - 2\epsilon)} P_{2,2}^{**}[1] \\
& + \frac{(2\chi^2 - 3\epsilon(1 - 2\chi) + \epsilon^2(9 + 2\chi))s_{12}^2}{8(1 - \epsilon)(1 - 2\epsilon)} P_{2,2}^{**}[\ell_1 \cdot k_4] + \dots,
\end{aligned}$$

where the ellipses denote additional integrals with fewer propagators. We must require that these equations (and the other 12 we do not display explicitly) are preserved by the choice of contours. The contour integrals which implement the augmented heptacut will yield vanishing results for the integrals with fewer propagators, so they do not enter the constraint equations. As we are considering only four-dimensional cuts, the augmented heptacuts are effectively four-dimensional.

In order to find the explicit form of the constraint equations, denote the weight of contour $T_{j,r}$ by $a_{r,j}$,

$$\begin{aligned}
a_{1,j} & \longrightarrow \text{encircling } z = 0 \text{ for solution } \mathcal{S}_j, \\
a_{2,j} & \longrightarrow \text{encircling } z = -\chi \text{ for solution } \mathcal{S}_j, \\
a_{3,j} & \longrightarrow \text{encircling } z = -\chi - 1 \text{ for solution } \mathcal{S}_j.
\end{aligned} \tag{4.8}$$

For a numerator insertion of $f(\ell_1, \ell_2)$ in the numerator of the double box, the augmented heptacut is then,

$$\begin{aligned}
& \sum_{j=1}^4 \sum_{r=1}^2 a_{r,j} \oint_{T_{j,r}} d^4\alpha_i d^4\beta_i f(\ell_1, \ell_2) \times \text{Propagators}(\ell_1, \ell_2) \Big|_{\text{param}} \\
& + \sum_{j=5}^6 \sum_{r=1}^3 a_{r,j} \oint_{T_{j,r}} d^4\alpha_i d^4\beta_i f(\ell_1, \ell_2) \times \text{Propagators}(\ell_1, \ell_2) \Big|_{\text{param}}, \tag{4.9}
\end{aligned}$$

where the notation $|_{\text{param}}$ indicates that we use the parametrization of ℓ_1 and ℓ_2 given in eq. (3.4). The signs in front of each coefficient $a_{r,j}$ in the result will depend on the orientation chosen for the corresponding contour; but this sign will drop out of final formulæ for integral coefficients so long as this orientation is chosen consistently throughout the calculation.

We can write down a compact expression for the augmented heptacut of the general tensor integral,

$$P_{2,2}^{**}[(\ell_1 \cdot k_4)^m (\ell_2 \cdot k_1)^n] = -\frac{1}{128} \left[\delta_{m,0} \left(\frac{s_{12}}{2} \right)^{n-3} \oint_{\Gamma_1} \frac{dz}{z} (z + \chi)^{n-1} \right]$$

$$\begin{aligned}
& + \delta_{n,0} \left(\frac{s_{12}}{2} \right)^{m-3} \oint_{\Gamma_2} \frac{dz}{z} (z + \chi)^{m-1} \\
& + \delta_{m,0} \left(\frac{s_{12}}{2} \right)^{n-3} \oint_{\Gamma_3} \frac{dz}{z} (z + \chi)^{n-1} \\
& + \delta_{n,0} \left(\frac{s_{12}}{2} \right)^{m-3} \oint_{\Gamma_4} \frac{dz}{z} (z + \chi)^{m-1} \\
& + \left(\frac{s_{12}}{2} \right)^{m+n-3} \oint_{\Gamma_5} \frac{dz}{z} (z + \chi)^{m-1} \left(-\frac{z}{z + \chi + 1} \right)^n \\
& + \left(\frac{s_{12}}{2} \right)^{m+n-3} \oint_{\Gamma_6} \frac{dz}{z} (z + \chi)^{m-1} \left(-\frac{z}{z + \chi + 1} \right)^n \Big], \quad (4.10)
\end{aligned}$$

where Γ_j denotes the z component of $\sum_r a_{r,j} T_{j,r}$, and where in our notation, the contour integral implicitly includes a factor of $1/(2\pi i)$, as noted in Section II.

We can evaluate this expression using the contours as weighted in eq. (4.9); we find,

$$P_{2,2}^{**}[1] = -\frac{1}{16\chi s_{12}^3} \sum_{j=1}^6 (a_{1,j} - a_{2,j}), \quad (4.11)$$

$$P_{2,2}^{**}[(\ell_1 \cdot k_4)^m] = -\frac{1}{32s_{12}^2} \left(\frac{\chi s_{12}}{2} \right)^{m-1} \sum_{j \neq 1,3} a_{1,j}, \quad (4.12)$$

$$P_{2,2}^{**}[(\ell_2 \cdot k_1)^n] = -\frac{1}{32s_{12}^2} \left(\frac{\chi s_{12}}{2} \right)^{n-1} (-a_{2,6} + a_{3,6} - a_{2,5} + a_{3,5} + a_{1,1} + a_{1,3}), \quad (4.13)$$

$$\begin{aligned}
P_{2,2}^{**}[(\ell_1 \cdot k_4)^m (\ell_2 \cdot k_1)^n] &= \frac{1}{64s_{12}} \left(-\frac{s_{12}}{2} \right)^{m+n-2} \left[\Theta \left(\min(m, n) - \frac{5}{2} \right) (\chi + 1)(\chi + 2) \right. \\
&\quad \left. + \Theta \left(\min(m, n) - \frac{3}{2} \right) (m + n - 3)(\chi + 1) + \Theta \left(\min(m, n) - \frac{1}{2} \right) \right] \\
&\quad \times (a_{3,6} + a_{3,5}), \quad (4.14)
\end{aligned}$$

where $m, n \geq 1$ and the last result is valid only for $0 \leq m + n \leq 6$ and $0 \leq m, n \leq 4$ (corresponding to the numerator insertions allowed in gauge theory in $D = 4 - 2\epsilon$ dimensions).

With these expressions, we now turn to the constraint equations. Let us begin with the equations arising from the insertion of Levi-Civita tensors (4.4). Start with $\varepsilon(\ell_1, k_2, k_3, k_4)$,

$$\begin{aligned}
0 &= P_{2,2}^{**}[\varepsilon(\ell_1, k_2, k_3, k_4)] \implies \\
0 &= -\frac{1}{16s_{12}^3} \left(\oint_{\Gamma_1} \frac{dz}{z} \frac{\varepsilon \left(k_1^\mu - \frac{s_{12}}{2} \frac{\langle 1^- | \gamma^\mu | 2^- \rangle}{\langle 1^- | 4 | 2^- \rangle} \chi, k_2, k_3, k_4 \right)}{z + \chi} + \oint_{\Gamma_2} \frac{dz}{z} \frac{\varepsilon \left(k_1^\mu + \frac{s_{12}}{2} \frac{\langle 1^- | \gamma^\mu | 2^- \rangle}{\langle 1^- | 4 | 2^- \rangle} z, k_2, k_3, k_4 \right)}{z + \chi} \right. \\
&\quad + \oint_{\Gamma_3} \frac{dz}{z} \frac{\varepsilon \left(k_1^\mu - \frac{s_{12}}{2} \frac{\langle 2^- | \gamma^\mu | 1^- \rangle}{\langle 2^- | 4 | 1^- \rangle} \chi, k_2, k_3, k_4 \right)}{z + \chi} + \oint_{\Gamma_4} \frac{dz}{z} \frac{\varepsilon \left(k_1^\mu + \frac{s_{12}}{2} \frac{\langle 2^- | \gamma^\mu | 1^- \rangle}{\langle 2^- | 4 | 1^- \rangle} z, k_2, k_3, k_4 \right)}{z + \chi} \\
&\quad \left. + \oint_{\Gamma_5} \frac{dz}{z} \frac{\varepsilon \left(k_1^\mu + \frac{s_{12}}{2} \frac{\langle 2^- | \gamma^\mu | 1^- \rangle}{\langle 2^- | 4 | 1^- \rangle} z, k_2, k_3, k_4 \right)}{z + \chi} + \oint_{\Gamma_6} \frac{dz}{z} \frac{\varepsilon \left(k_1^\mu + \frac{s_{12}}{2} \frac{\langle 1^- | \gamma^\mu | 2^- \rangle}{\langle 1^- | 4 | 2^- \rangle} z, k_2, k_3, k_4 \right)}{z + \chi} \right).
\end{aligned}$$

(4.15)

Evaluating this expression on the augmented heptacut (4.9), we obtain,

$$\begin{aligned}
& \frac{1}{32s_{12}^2} \left[(a_{2,2} + a_{2,6} - a_{1,1} + a_{2,1}) \varepsilon \left(\frac{\langle 1^- | \gamma^\mu | 2^- \rangle}{\langle 1^- | 4 | 2^- \rangle}, k_2, k_3, k_4 \right) \right. \\
& \quad \left. + (a_{2,5} + a_{2,4} - a_{1,3} + a_{2,3}) \varepsilon \left(\frac{\langle 2^- | \gamma^\mu | 1^- \rangle}{\langle 2^- | 4 | 1^- \rangle}, k_2, k_3, k_4 \right) \right] \\
& = \frac{1}{32s_{12}^2} (a_{2,2} + a_{2,6} - a_{2,5} - a_{2,4} - a_{1,1} + a_{2,1} + a_{1,3} - a_{2,3}) \varepsilon \left(\frac{\langle 1^- | \gamma^\mu | 2^- \rangle}{\langle 1^- | 4 | 2^- \rangle}, k_2, k_3, k_4 \right),
\end{aligned} \tag{4.16}$$

where the last line follows from the fact that the two Levi-Civita symbols appearing on the first line are equal but opposite in value.

Similarly, from the insertion of $\varepsilon(\ell_2, k_2, k_3, k_4)$ one finds

$$\begin{aligned}
0 & = P_{2,2}^{**}[\varepsilon(\ell_2, k_2, k_3, k_4)] \implies \\
0 & = -\frac{1}{32s_{12}^2} (-a_{1,2} + a_{2,2} + a_{1,6} - a_{3,6} - a_{1,5} + a_{3,5} + a_{1,4} - a_{2,4} + a_{2,1} - a_{2,3}) \\
& \quad \times \varepsilon \left(\frac{\langle 3^- | \gamma^\mu | 4^- \rangle}{\langle 3^- | 1 | 4^- \rangle}, k_2, k_3, k_4 \right),
\end{aligned} \tag{4.17}$$

and from the insertion of $\varepsilon(\ell_1, \ell_2, k_i, k_j)$ with $(i, j) \in \{(1, 2), (1, 3), (2, 3)\}$ one finds

$$\begin{aligned}
0 & = P_{2,2}^{**}[\varepsilon(\ell_1, \ell_2, k_1, k_2)] \implies \\
0 & = -\frac{1}{32s_{12}^2} (a_{2,6} - a_{3,6} - a_{2,5} + a_{3,5} - a_{1,1} + a_{1,3}) \varepsilon \left(\frac{\langle 1^- | \gamma^\mu | 2^- \rangle}{\langle 1^- | 4 | 2^- \rangle}, k_2, k_3, k_4 \right), \\
0 & = P_{2,2}^{**}[\varepsilon(\ell_1, \ell_2, k_1, k_3)] \implies \\
0 & = \frac{1}{32s_{12}^2} (a_{2,6} - a_{2,5} - a_{1,1} + a_{1,3}) \varepsilon \left(\frac{\langle 1^- | \gamma^\mu | 2^- \rangle}{\langle 1^- | 4 | 2^- \rangle}, k_2, k_3, k_4 \right), \\
0 & = P_{2,2}^{**}[\varepsilon(\ell_1, \ell_2, k_2, k_3)] \implies \\
0 & = -\frac{1}{32s_{12}^2} (a_{1,2} - a_{1,6} + a_{2,6} + a_{1,5} - a_{2,5} - a_{1,4} - a_{1,1} + a_{1,3}) \\
& \quad \times \varepsilon \left(\frac{\langle 1^- | \gamma^\mu | 2^- \rangle}{\langle 1^- | 4 | 2^- \rangle}, k_2, k_3, k_4 \right).
\end{aligned} \tag{4.18}$$

These results combine to give the constraints,

$$\begin{aligned}
a_{2,2} + a_{2,6} - a_{2,5} - a_{2,4} - a_{1,1} + a_{2,1} + a_{1,3} - a_{2,3} & = 0, \\
a_{1,2} - a_{2,2} - a_{1,6} + a_{3,6} + a_{1,5} - a_{3,5} - a_{1,4} + a_{2,4} - a_{2,1} + a_{2,3} & = 0, \\
a_{2,6} - a_{3,6} - a_{2,5} + a_{3,5} - a_{1,1} + a_{1,3} & = 0, \\
a_{2,6} - a_{2,5} - a_{1,1} + a_{1,3} & = 0, \\
a_{1,2} - a_{1,6} + a_{2,6} + a_{1,5} - a_{2,5} - a_{1,4} - a_{1,1} + a_{1,3} & = 0,
\end{aligned} \tag{4.19}$$

or equivalently,

$$\begin{aligned}
a_{1,2} - a_{1,6} + a_{1,5} - a_{1,4} &= 0, \\
a_{2,2} - a_{2,4} + a_{2,1} - a_{2,3} &= 0, \\
a_{2,6} - a_{2,5} - a_{1,1} + a_{1,3} &= 0, \\
a_{3,6} - a_{3,5} &= 0.
\end{aligned} \tag{4.20}$$

This set has one equation less: not all the equations from the Levi-Civita symbols are independent. We see that these equations are solved by insisting that the contours for complex-conjugate pairs of solutions ($\mathcal{S}_1 \leftrightarrow \mathcal{S}_3$, $\mathcal{S}_2 \leftrightarrow \mathcal{S}_4$, and $\mathcal{S}_5 \leftrightarrow \mathcal{S}_6$) carry equal weights. This nicely generalizes the one-loop constraint on contours. However, these are not the only possible solutions; solutions which do not insist complex-conjugate pairs carry equal weight are also possible.

We next impose the constraints following from the IBP reductions. Evaluating both sides of equations (4.7) along with the remaining 12 reduction equations not displayed above, and setting $\epsilon = 0$, we find two additional constraint equations,

$$\begin{aligned}
a_{1,2} + a_{1,6} + a_{1,5} + a_{1,4} &= -a_{2,6} + a_{3,6} - a_{2,5} + a_{3,5} + a_{1,1} + a_{1,3}, \\
a_{3,6} + a_{3,5} &= -\frac{1}{2} \sum_{j=1}^6 (a_{1,j} - a_{2,j}) + \frac{3}{2} \sum_{j \neq 1,3} a_{1,j}.
\end{aligned} \tag{4.21}$$

In principle, one might expect 18 additional equations from the remaining reduction identities; but these all turn out to be automatically satisfied on the solutions of this pair of equations.

Beyond ensuring that all the reduction identities are valid, we ultimately want to determine the coefficients of the two basis integrals (4.6). Because the system of equations leaves many undetermined weights $a_{r,j}$, we have the freedom to choose values which also kill one or the other of the basis integrals. That is, we can choose contours for which one or the other of the basis integrals has vanishing augmented heptacut. To project out the second basis integral, $P_{2,2}^{**}[\ell_1 \cdot k_4]$, we should also require that eq. (4.12) with $m = 1$ vanish,

$$\sum_{j \neq 1,3} a_{1,j} = 0. \tag{4.22}$$

To project out the first basis integral, $P_{2,2}^{**}[1]$, we should require that eq. (4.11) vanish,

$$\sum_{j=1}^6 (a_{1,j} - a_{2,j}) = 0. \tag{4.23}$$

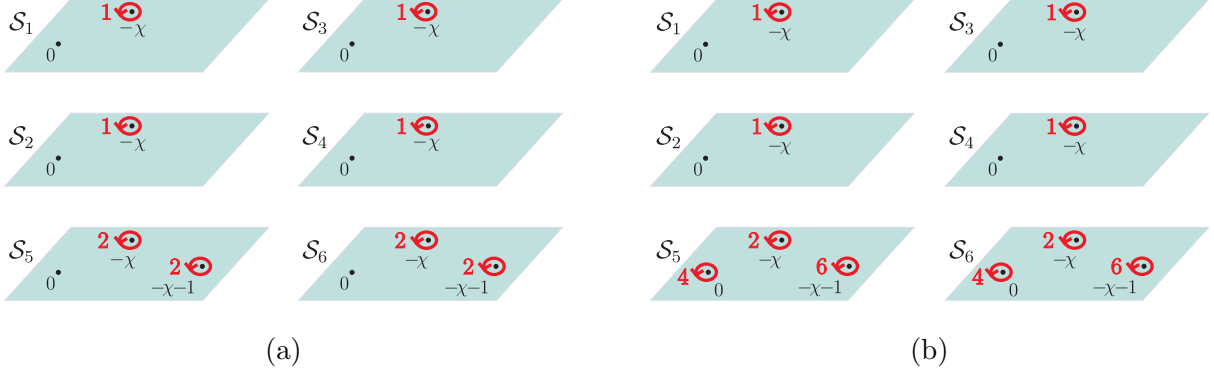


FIG. 5: Schematic representation of contours for the coefficients of the two basis double boxes: (a) the scalar double box, $P_{2,2}^{**}[1]$ (b) the double box with an irreducible numerator insertion, $P_{2,2}^{**}[\ell_1 \cdot k_4]$. The contours encircle the global poles distributed across the six kinematical solutions; the integers next to the contours indicate the winding number. Both representations are for the choice $u = \frac{1}{2}$ and $v = 1$ in eqs. (4.24) and (4.25).

The following values,

$$\begin{aligned}
a_{1,1} &= -2u + v, & a_{2,1} &= u, \\
a_{1,2} &= -2u + v, & a_{2,2} &= u, \\
a_{1,3} &= -2u + v, & a_{2,3} &= u, \\
a_{1,4} &= -2u + v, & a_{2,4} &= u, \\
a_{1,5} &= 2u - v, & a_{2,5} &= v, & a_{3,5} &= 2u, \\
a_{1,6} &= 2u - v, & a_{2,6} &= v, & a_{3,6} &= 2u,
\end{aligned} \tag{4.24}$$

(where u, v are real parameters) solve all the constraint equations (4.20, 4.21), and also set the heptacut of the basis integral $P_{2,2}^{**}[\ell_1 \cdot k_4]$ to zero, thereby allowing us to extract the coefficient of the first basis integral, $P_{2,2}^{**}[1]$. We will call a specific choice of contours weighted by these values P_1 , leaving the dependence on u and v implicit. A particularly simple solution is given by $u = \frac{1}{2}$ and $v = 1$. This choice is illustrated schematically in fig. 5(a).

Similarly, the following values,

$$\begin{aligned}
a_{1,1} &= -2u + v, & a_{2,1} &= u, \\
a_{1,2} &= -2u + v, & a_{2,2} &= u, \\
a_{1,3} &= -2u + v, & a_{2,3} &= u, \\
a_{1,4} &= -2u + v, & a_{2,4} &= u, \\
a_{1,5} &= 6u - v, & a_{2,5} &= v, & a_{3,5} &= 6u, \\
a_{1,6} &= 6u - v, & a_{2,6} &= v, & a_{3,6} &= 6u,
\end{aligned} \tag{4.25}$$

(where again u, v are real parameters) solve all the constraint equations (4.20, 4.21), sets to zero the heptacut of the basis integral $P_{2,2}^{**}[1]$, and thereby extracts the coefficient of $P_{2,2}^{**}[\ell_1 \cdot k_4]$. We will call a specific choice of contours weighted by these values P_2 , again leaving the dependence on u and v implicit. The choice $u = \frac{1}{2}$ and $v = 1$ again gives a particularly simple solution. It is illustrated schematically in fig. 5(b).

Before turning to the extraction procedure, we may observe that the four-dimensional heptacuts do not suffice to extract information about the coefficients beyond $\mathcal{O}(\epsilon^0)$. The problem is that we can find non-vanishing linear combinations of tensor integrals whose heptacut integrand vanishes identically for all six solutions. As a result, not only do integrals over all contours $T_{j,a}$ vanish, but even integrals constructed by multiplying the integrand by an arbitrary function of the remaining degree of freedom z would vanish. We call such linear combinations *magic*. Examples of magic combinations include,

$$M_1 = P_{2,2}^{**}[2, 2] + \frac{s_{12}}{2} P_{2,2}^{**}[2, 1] + \frac{s_{12}}{2} P_{2,2}^{**}[1, 2] - \chi \left(\frac{s_{12}}{2} \right)^2 P_{2,2}^{**}[1, 1], \tag{4.26}$$

$$M_2 = P_{2,2}^{**}[3, 2] + \frac{s_{12}}{2} P_{2,2}^{**}[3, 1] + \frac{s_{12}}{2} P_{2,2}^{**}[2, 2] - \chi \left(\frac{s_{12}}{2} \right)^2 P_{2,2}^{**}[2, 1], \tag{4.27}$$

where the abbreviated notation $P_{2,2}^{**}[m, n]$ is defined by,

$$P_{2,2}^{**}[m, n] \equiv P_{2,2}^{**}[(\ell_1 \cdot k_4)^m (\ell_2 \cdot k_1)^n]. \tag{4.28}$$

The magic combinations do not vanish, but both coefficients of master integrals are of $\mathcal{O}(\epsilon)$ after use of IBP reduction equations.

V. INTEGRAL COEFFICIENTS

With solutions to the constraint equations that also isolate specific basis integrals in hand, we can write down a procedure for computing the coefficients of the integrals in the master

equation (4.1). To do so, we apply the augmented heptacuts to the left-hand side of the master equation. The basic heptacut will break apart the two-loop amplitude into a product of six on-shell tree amplitudes, one for each vertex in the double box. We will be left with the integral over the z contour. On the right-hand side, we have the two basis integrals (4.6) chosen earlier. Here, apply the augmented heptacut, and perform all integrations. This gives us the relation,

$$\frac{1}{128} \left(\frac{2}{s_{12}} \right)^3 \sum_{i=1}^6 \oint_{\Gamma_i} \frac{dz}{z(z+\chi)} (-i) \prod_{j=1}^6 A_j^{\text{tree}}(z) = \frac{c_1}{16\chi s_{12}^3} \sum_{j=1}^6 (a_{1,j} - a_{2,j}) + \frac{c_2}{32s_{12}^2} \sum_{j \neq 1,3} a_{1,j}. \quad (5.1)$$

In this equation, the product of amplitudes arises from a factor of a tree-level amplitude at each vertex of the double box with all seven propagators cut.

As explained in the previous section, through a judicious choice of contours, we can make the coefficient of c_2 in this equation vanish, or alternatively the coefficient of c_1 vanish. This would then allow us to solve for c_1 and c_2 , respectively. We gave such choices in eqs. (4.24) and (4.25). Using them, we can write an expression for c_1 ,

$$c_1 = \frac{i\chi}{8u} \oint_{P_1} \frac{dz}{z(z+\chi)} \prod_{j=1}^6 A_j^{\text{tree}}(z), \quad (5.2)$$

and for c_2 ,

$$c_2 = -\frac{i}{4s_{12}u} \oint_{P_2} \frac{dz}{z(z+\chi)} \prod_{j=1}^6 A_j^{\text{tree}}(z). \quad (5.3)$$

The right-hand sides of these equations must be summed over possible helicity and particle-species assignments. The explicit integration is understood to be over the z component of P_1 and P_2 respectively, with the integrations over the other α_i and β_i implicit in the solutions \mathcal{S}_j , and with the dependence of P_j on the parameters u and v left implicit. The formulæ (5.2) and (5.3) represent the central result of this paper. They are valid for any gauge theory, and indeed for any amplitude satisfying the power-counting rules of gauge theory. With the notation,

$$V|_{\mathcal{S}_1+\mathcal{S}_2-\mathcal{S}_3} \equiv V|_{\mathcal{S}_1} + V|_{\mathcal{S}_2} - V|_{\mathcal{S}_3}, \quad (5.4)$$

we can write out these formulæ more explicitly,

$$\begin{aligned}
c_1 &= \frac{(v-2u)i}{8u} \operatorname{Res}_{z=0} \frac{1}{z} \prod_{j=1}^6 A_j^{\text{tree}}(z) \Big|_{\mathcal{S}_1+\mathcal{S}_2+\mathcal{S}_3+\mathcal{S}_4-\mathcal{S}_5-\mathcal{S}_6} - \frac{iv}{8u} \operatorname{Res}_{z=-\chi} \frac{1}{z+\chi} \prod_{j=1}^6 A_j^{\text{tree}}(z) \Big|_{\mathcal{S}_5+\mathcal{S}_6} \\
&\quad - \frac{i}{8} \operatorname{Res}_{z=-\chi} \frac{1}{z+\chi} \prod_{j=1}^6 A_j^{\text{tree}}(z) \Big|_{\mathcal{S}_1+\mathcal{S}_2+\mathcal{S}_3+\mathcal{S}_4} + \frac{i\chi}{4(1+\chi)} \operatorname{Res}_{z=-\chi-1} \prod_{j=1}^6 A_j^{\text{tree}}(z) \Big|_{\mathcal{S}_5+\mathcal{S}_6}, \\
c_2 &= -\frac{(v-2u)i}{4s_{12}u\chi} \operatorname{Res}_{z=0} \frac{1}{z} \prod_{j=1}^6 A_j^{\text{tree}}(z) \Big|_{\mathcal{S}_1+\mathcal{S}_2+\mathcal{S}_3+\mathcal{S}_4} - \frac{(6u-v)i}{4s_{12}u\chi} \operatorname{Res}_{z=0} \frac{1}{z} \prod_{j=1}^6 A_j^{\text{tree}}(z) \Big|_{\mathcal{S}_5+\mathcal{S}_6} \\
&\quad + \frac{i}{4s_{12}\chi} \operatorname{Res}_{z=-\chi} \frac{1}{z+\chi} \prod_{j=1}^6 A_j^{\text{tree}}(z) \Big|_{\mathcal{S}_1+\mathcal{S}_2+\mathcal{S}_3+\mathcal{S}_4} + \frac{iv}{4s_{12}u\chi} \operatorname{Res}_{z=-\chi} \frac{1}{z+\chi} \prod_{j=1}^6 A_j^{\text{tree}}(z) \Big|_{\mathcal{S}_5+\mathcal{S}_6} \\
&\quad - \frac{3i}{2s_{12}(1+\chi)} \operatorname{Res}_{z=-\chi-1} \prod_{j=1}^6 A_j^{\text{tree}}(z) \Big|_{\mathcal{S}_5+\mathcal{S}_6}. \tag{5.5}
\end{aligned}$$

These formulæ are not manifestly independent of the choice of contour, but the constraint equations ensure that they are. We will see explicit examples in the next section. Of course, the independence of the final result of the choice of contour does not mean that the results at intermediate steps are independent; certain choices of contour may in fact simplify analytic or numerical calculations. We have already seen hints of this in the choices of P_1 and P_2 , where some values of u and v will require evaluation of fewer contours, and hence possibly fewer numerical evaluations if the formulæ (5.2) and (5.3) are used in a numerical setting.

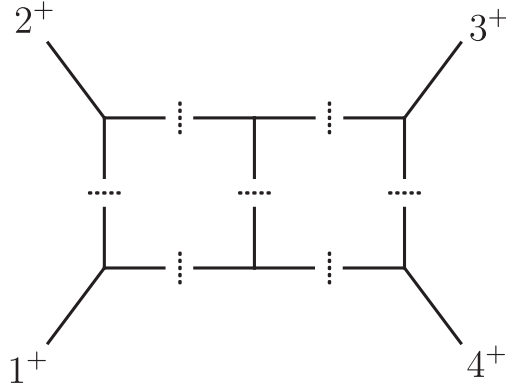


FIG. 6: Graphical representation of $\prod_{j=1}^6 A_j^{\text{tree}}(z) \Big|_{D=4}$ for the all-plus QCD amplitude.

At one loop, one can choose a basis so that integral coefficients are independent of the dimensional regulator ϵ , and four-dimensional cuts suffice to compute all of them. (Computing the rational terms requires use of D -dimensional cuts.) At two loops, the coefficients

of integral reductions, and hence generally of integrals in eq. (4.1), will depend explicitly on ϵ . In particular, c_1 and c_2 above will depend explicitly on ϵ . In general, this dependence cannot be extracted from four-dimensional heptacuts alone, because of the vanishing of magic combinations discussed in Section IV. We can also see the need for cuts beyond four dimensions, or considerably relaxing some of the heptacut conditions, by considering the two-loop all-plus amplitude, $A_4^{2\text{-loop}}(++++)$, computed in ref. [53]. In this case, the product of tree amplitudes in eqs. (5.2) and (5.3) will necessarily vanish in four dimensions, because there is no assignment of internal helicities in fig. 6 that will leave all three-point amplitudes non-vanishing. The same observation still holds if we relax some of the cut conditions, examining hexacuts or pentacuts.

VI. EXAMPLES

In this section, we apply the formalism developed in previous sections to several examples of two-loop four-point amplitudes. We use the master formulæ (5.2) and (5.3) to compute the coefficients to $\mathcal{O}(\epsilon^0)$ of the two double box basis integrals, $P_{2,2}^{**}[1]$ and $P_{2,2}^{**}[\ell_1 \cdot k_4]$. We consider three different contributions to four-gluon amplitudes in supersymmetric theories with $\mathcal{N} = 4, 2, 1$ supersymmetries: the s - and t -channel contributions to $A_4^{2\text{-loop}}(1^-, 2^-, 3^+, 4^+)$, and the s -channel contributions to $A_4^{2\text{-loop}}(1^-, 2^+, 3^-, 4^+)$. (The t -channel contributions to the latter amplitudes can be obtained by relabeling the arguments of the s -channel contribution.)

We will express the results as multiples of the tree-level four-point amplitudes,

$$A_{--++}^{\text{tree}} = \frac{i\langle 12 \rangle^3}{\langle 23 \rangle \langle 34 \rangle \langle 41 \rangle}, \quad (6.1)$$

and

$$A_{-+--}^{\text{tree}} = \frac{i\langle 13 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle}. \quad (6.2)$$

In this section, it will be convenient to have a label for each cut propagator in the double box. Accordingly, we adopt a different labeling from previous sections. It is displayed in fig. 7.

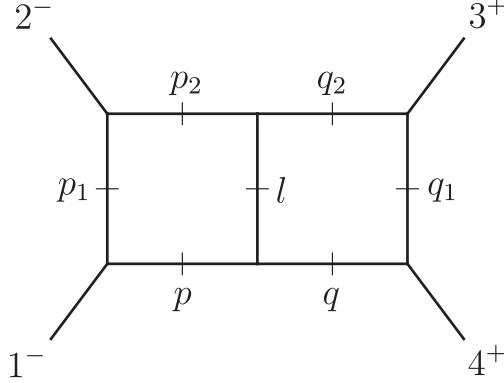


FIG. 7: The labeling of internal momenta used in Section VI, here shown for the s -channel contribution to $A_4^{2\text{-loop}}(1^-, 2^-, 3^+, 4^+)$.

A. The s -channel contribution to $A_4^{2\text{-loop}}(1^-, 2^-, 3^+, 4^+)$

For this contribution, shown above in fig. 7, the helicities of the external states allow only gluons to propagate in either loop. For this reason, we will get the same result independent of the number of supersymmetries. We find that for all six solutions to the on-shell equations,

$$\prod_{j=1}^6 A_j^{\text{tree}} = -i s_{12}^2 s_{23} A_{--++}^{\text{tree}}. \quad (6.3)$$

We can then use eq. (5.2) (or equivalently the first equation in eq. (5.5)) to obtain,

$$\begin{aligned} c_1 &= -i s_{12}^2 s_{23} A_{--++}^{\text{tree}} \left(\frac{(v-2u)i}{4u} - \frac{iv}{4u} - \frac{i}{2} \right) \\ &= -s_{12}^2 s_{23} A_{--++}^{\text{tree}}; \end{aligned} \quad (6.4)$$

and eq. (5.3) (or equivalently the second equation in eq. (5.5)) to obtain,

$$\begin{aligned} c_2 &= -i s_{12}^2 s_{23} A_{--++}^{\text{tree}} \left(-\frac{(v-2u)i}{s_{12}u\chi} - \frac{(6u-v)i}{2s_{12}u\chi} + \frac{i}{s_{12}\chi} + \frac{iv}{2s_{12}u\chi} \right) \\ &= 0. \end{aligned} \quad (6.5)$$

We see that the dependence on the parameters u and v has disappeared, as expected. In the $\mathcal{N} = 4$ theory, these turn out to be the exact coefficients; in theories with fewer supersymmetries, there are additional terms of $\mathcal{O}(\epsilon)$ in these coefficients.

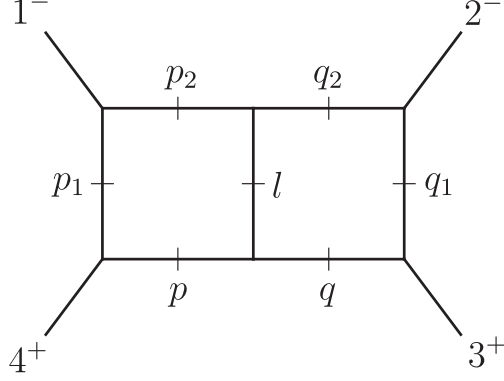


FIG. 8: The heptacut for the t -channel contribution to $A_4^{2\text{-loop}}(1^-, 2^-, 3^+, 4^+)$.

B. The t -channel contribution to $A_4^{2\text{-loop}}(1^-, 2^-, 3^+, 4^+)$

We turn next to the computation of the coefficients in the t -channel contribution to the same amplitude considered in the previous section. The heptacut for this contribution is shown in fig. 8. In applying the formulæ for the coefficients, we have cyclicly permuted the external momentum arguments, $(1, 2, 3, 4) \rightarrow (4, 1, 2, 3)$, so that we must replace $\chi \rightarrow \chi^{-1}$. Otherwise, they are of course unchanged.

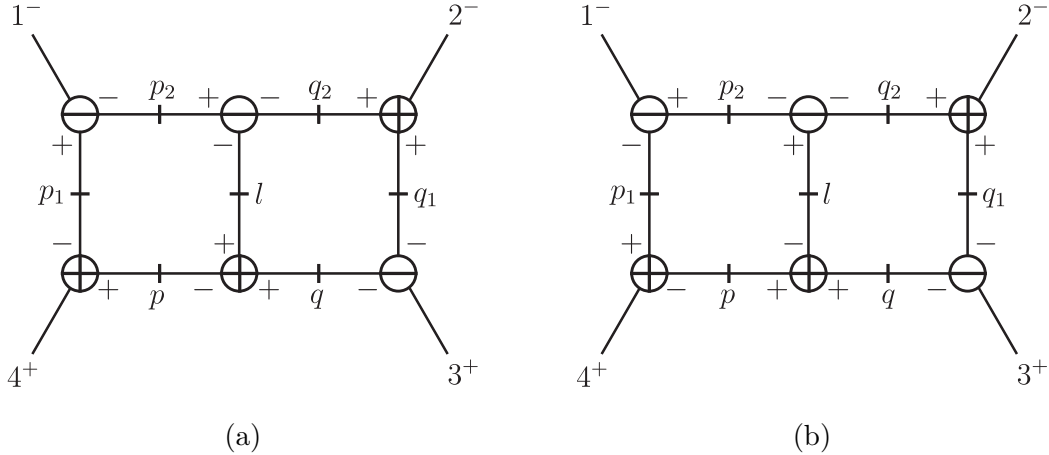


FIG. 9: The two distinct assignments of internal helicities in solution \mathcal{S}_2 for the t -channel double-box contributions to $A_4^{2\text{-loop}}(1^-, 2^-, 3^+, 4^+)$: (a) configuration A and (b) configuration B.

In this contribution, computing the required products of tree amplitudes is more involved, and the computation also requires sums over supermultiplets of states propagating in the loops. As an example, we work through the computation of the product in solution \mathcal{S}_2 .

We have two possible helicity assignments for the internal lines, shown in fig. 9. For gluon internal lines, we multiply the amplitudes at the six vertices to obtain,

$$\prod_{j=1}^6 A_j^{\text{tree, gluon}} = -\frac{1}{\Delta} \times \begin{cases} A^4 & \text{for configuration A,} \\ B^4 & \text{for configuration B,} \end{cases} \quad (6.6)$$

where

$$\begin{aligned} A^4 &= ([p\ 4] \langle 1\ p_2 \rangle \langle q_2\ l \rangle [l\ q] \langle q\ q_1 \rangle [q_1\ q_2])^4, \\ B^4 &= ([4\ p_1] \langle p_1\ 1 \rangle \langle p_2\ q_2 \rangle [q\ p] \langle q\ q_1 \rangle [q_1\ q_2])^4, \\ \Delta &= [p\ 4] [4\ p_1] [p_1\ p] \langle p_1\ 1 \rangle \langle 1\ p_2 \rangle \langle p_2\ p_1 \rangle \langle p_2\ q_2 \rangle \langle q_2\ l \rangle \langle l\ p_2 \rangle [l\ q] [q\ p] [p\ l] \\ &\quad \times \langle q\ q_1 \rangle \langle q_1\ 3 \rangle \langle 3\ q \rangle [q_1\ q_2] [q_2\ 2] [2\ q_1], \end{aligned} \quad (6.7)$$

and the minus sign in eq. (6.6) comes from the factor of i in each $A_j^{\text{tree, gluon}}$.

The helicity assignments of the internal lines allow only gluons to propagate in the right (q) loop, whereas the entire supersymmetric multiplet of states can propagate in the left (p) loop. For $\mathcal{N} = 4$ super Yang-Mills, the sum over states yields,

$$\sum_{\substack{\mathcal{N}=4 \\ \text{multiplet}}} \prod_{j=1}^6 A_j^{\text{tree}} \Big|_{\mathcal{S}_2} = -\frac{(A+B)^4}{\Delta}. \quad (6.8)$$

On the other hand, from refs. [36] and [61] we know that in the $\mathcal{N} = 4$ theory,

$$\sum_{\substack{\mathcal{N}=4 \\ \text{multiplet}}} \prod_{j=1}^6 A_j^{\text{tree}} \Big|_{\mathcal{S}_2} = -i s_{12} s_{23}^2 A_{--++}^{\text{tree}}. \quad (6.9)$$

As a calculational shortcut, we use the equality of the expressions in eqs. (6.8) and (6.9) to fix the relative sign of A and B in eq. (6.7). (Of course, the relative signs can also be determined *a priori*, without reference to results in the literature, by carefully tracking the direction — incoming or outgoing — of the momenta at a given vertex and using the analytic continuation rule that changing the sign of a momentum, $p_i \rightarrow -p_i$, is effected by changing the sign of the holomorphic spinor [76]: $\lambda_i^\alpha \rightarrow -\lambda_i^\alpha$ while $\tilde{\lambda}_i^{\dot{\alpha}} \rightarrow \tilde{\lambda}_i^{\dot{\alpha}}$.) One finds,

$$\begin{aligned} A &= [p\ 4] \langle 1\ p_2 \rangle \langle q_2\ l \rangle [l\ q] \langle q\ q_1 \rangle [q_1\ q_2], \\ B &= -[4\ p_1] \langle p_1\ 1 \rangle \langle p_2\ q_2 \rangle [q\ p] \langle q\ q_1 \rangle [q_1\ q_2]. \end{aligned} \quad (6.10)$$

Ref. [77] teaches us that the sum over the $\mathcal{N} = 4, 2, 1, 0$ multiplet of states is related to the $\mathcal{N} = 4$ state sum via

$$\sum_{\substack{\text{SUSY} \\ \text{multiplet}}} \prod_{j=1}^6 A_j^{\text{tree}} = \frac{(A+B)^{\mathcal{N}} (A^{4-\mathcal{N}} + B^{4-\mathcal{N}})}{(A+B)^4} \left(1 - \frac{1}{2} \delta_{\mathcal{N},4}\right) \sum_{\substack{\mathcal{N}=4 \\ \text{multiplet}}} \prod_{j=1}^6 A_j^{\text{tree}}, \quad (6.11)$$

so that the sum over the supersymmetric multiplet of states can be calculated from the gluonic contributions alone (indeed, recall that A and B in eq. (6.7) were obtained from the product of purely gluonic amplitudes corresponding to configurations A and B, respectively).

We can simplify the expression for the ratio between the supersymmetric state sums in eq. (6.11) by factoring out as many common factors of A and B as possible (exploiting momentum conservation fully). Setting $A = \alpha F$ and $B = \beta F$, for $\mathcal{N} = 4, 2, 1$ the ratio appearing in eq. (6.11) simplifies to

$$R = \frac{(\alpha + \beta)^{\mathcal{N}}(\alpha^{4-\mathcal{N}} + \beta^{4-\mathcal{N}})}{(\alpha + \beta)^4} \left(1 - \frac{1}{2}\delta_{\mathcal{N},4}\right) = \frac{(\alpha^{4-\mathcal{N}} + \beta^{4-\mathcal{N}}) \left(1 - \frac{1}{2}\delta_{\mathcal{N},4}\right)}{(\alpha + \beta)^{4-\mathcal{N}}} \quad (6.12)$$

$$= 1 - (4 - \mathcal{N}) \left(\frac{\alpha}{\alpha + \beta}\right) + (4 - \mathcal{N}) \left(\frac{\alpha}{\alpha + \beta}\right)^2. \quad (6.13)$$

where the last equality holds only for $\mathcal{N} = 4, 2, 1$; it can be obtained by expanding the numerator $(\alpha^{4-\mathcal{N}} + \beta^{4-\mathcal{N}}) \left(1 - \frac{1}{2}\delta_{\mathcal{N},4}\right)$ in eq. (6.12) in β around $-\alpha$.

In the case at hand, we can use momentum conservation ($l = p_2 + q_2$ and $p_1 = p - k_4$) to rewrite A and B as follows,

$$\begin{aligned} A &= [p 4] \langle 1 p_2 \rangle \langle q_2 p_2 \rangle [p_2 q] \langle q q_1 \rangle [q_1 q_2], \\ B &= -[4 p] \langle p 1 \rangle \langle p_2 q_2 \rangle [q p] \langle q q_1 \rangle [q_1 q_2], \end{aligned} \quad (6.14)$$

and identify,

$$\begin{aligned} \alpha &= \langle 1 p_2 \rangle [p_2 q], \\ \beta &= -[q p] \langle p 1 \rangle, \\ F &= [p 4] \langle q_2 p_2 \rangle \langle q q_1 \rangle [q_1 q_2]. \end{aligned} \quad (6.15)$$

Momentum conservation implies that $\alpha + \beta = -\langle 1 4 \rangle [4 q]$, and thus,

$$\frac{\alpha}{\alpha + \beta} = -\frac{\langle 1 p_1 \rangle [p_1 q]}{\langle 1 4 \rangle [4 q]} = -\frac{\langle 1 p_1 \rangle [p_1 3]}{\langle 1 4 \rangle [4 3]}, \quad (6.16)$$

where the second equality uses the proportionality of antiholomorphic spinors, $\tilde{\lambda}_q \propto \tilde{\lambda}_3$. (This proportionality holds only for some of the other six solutions \mathcal{S}_i in addition to \mathcal{S}_2 .)

The ratio thus simplifies to,

$$R = 1 + (4 - \mathcal{N}) \left(\frac{\langle 1 p_1 \rangle [p_1 3]}{\langle 1 4 \rangle [4 3]}\right) + (4 - \mathcal{N}) \left(\frac{\langle 1 p_1 \rangle [p_1 3]}{\langle 1 4 \rangle [4 3]}\right)^2. \quad (6.17)$$

We can solve for the explicit values of the cut momenta using the parametrization (3.4) with the external momenta cyclicly permuted (for the t -channel configuration), and using the on-shell values defining \mathcal{S}_2 given in eqs. (3.7) and (3.10). We find,

$$p_1^\mu \equiv p^\mu - k_4^\mu = \frac{s_{14}z}{2\langle 43\rangle[31]} \langle 4^- | \gamma^\mu | 1^- \rangle, \quad (6.18)$$

so that,

$$\begin{aligned} \langle 1 p_1 \rangle [p_1 3] &= \langle 1^- | \gamma^\mu | 3^- \rangle p_{1\mu} = \frac{s_{14}z}{2\langle 43\rangle[31]} \langle 1^- | \gamma^\mu | 3^- \rangle \langle 4^- | \gamma_\mu | 1^- \rangle \\ &= \frac{\langle 41 \rangle}{\langle 43 \rangle} s_{14}z, \end{aligned} \quad (6.19)$$

and thus,

$$\frac{\langle 1 p_1 \rangle [p_1 3]}{\langle 1 4 \rangle [4 3]} = \chi z. \quad (6.20)$$

This gives us our final form for the ratio,

$$R = 1 + (4 - \mathcal{N})\chi z + (4 - \mathcal{N})\chi^2 z^2, \quad (6.21)$$

and for the product of tree amplitudes,

$$\sum_{\text{SUSY multiplet}} \prod_{j=1}^6 A_j^{\text{tree}} \Big|_{\mathcal{S}_2} = -i s_{12} s_{23}^2 A_{---++}^{\text{tree}} \left(1 + (4 - \mathcal{N})\chi z + (4 - \mathcal{N})\chi^2 z^2 \right). \quad (6.22)$$

In this solution to the heptacut equations, the supersymmetric multiplet runs only in one of the loops. In other solutions (in particular, \mathcal{S}_6), the multiplet can run in both loops. The treatment of this case is similar but more elaborate. It turns out [77] that the sum over the multiplet can again be evaluated purely from the gluonic contributions. The main difference is that in this case there are three gluonic contributions A^4, B^4, C^4 (compared to the two in eq. (6.7)). One can again fix the relative sign of B and C by insisting that the $\mathcal{N} = 4$ supersymmetric result $-\frac{(A+B+C)^4}{\Delta}$ be equal to eq. (6.9), and from the obvious analog of eq. (6.11) one then finds the results for the supermultiplet sums for $\mathcal{N} = 4, 2, 1, 0$. These expressions can again be simplified as above.

Summing over all six solutions, and plugging the result into our master formulæ (5.2) and (5.3), taken with $u = \frac{1}{2}$ and $v = 1$, we find

$$\begin{aligned} c_1 &= -s_{12} s_{23}^2 A_{---++}^{\text{tree}} \left(1 + \frac{1}{4} (1 - \delta_{\mathcal{N},4}) (4 - \mathcal{N})! \chi (\chi + 1)^{\delta_{\mathcal{N},1}} \right), \\ c_2 &= \frac{3}{2} s_{23}^2 A_{---++}^{\text{tree}} (1 - \delta_{\mathcal{N},4}) (4 - \mathcal{N})! \chi (\chi + 1)^{\delta_{\mathcal{N},1}}, \end{aligned} \quad (6.23)$$

valid for $\mathcal{N} = 4, 2, 1$.

C. The s -channel contribution to $A_4^{2\text{-loop}}(1^-, 2^+, 3^-, 4^+)$

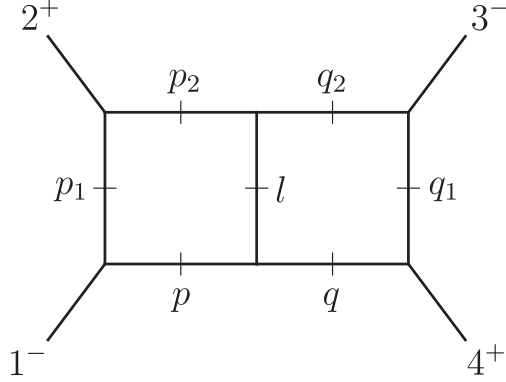


FIG. 10: The heptacut for the s -channel contribution to $A_4^{2\text{-loop}}(1^-, 2^+, 3^-, 4^+)$.

The heptacut for the s -channel contribution to $A_4^{2\text{-loop}}(1^-, 2^+, 3^-, 4^+)$ is shown in fig. 10. We will evaluate this contribution in two different ways, illustrating both the result's independence of the precise choice of contour, and also illustrating the potential advantages of a judicious choice of contour in a given calculation.

Rather than using our master formulæ (5.2) and (5.3), let us evaluate the augmented heptacut integral for a general contour, before imposing the constraint equations. Adding up the contributions from all six solutions, we find

$$\sum_{i=1}^6 \oint_{\Gamma_i} \frac{dz}{z(z+\chi)} \prod_{j=1}^6 A_j^{\text{tree}}(z) = -i s_{12}^2 s_{23} A_{-+-+}^{\text{tree}} \left[\sum_{j=1}^6 \frac{a_{1,j} - a_{2,j}}{\chi} - (4 - \mathcal{N}) \frac{a_{1,6} - a_{3,6} - a_{2,5}}{(\chi+1)^2} + \left(\left(1 - \frac{1}{2} \delta_{\mathcal{N},4}\right) \frac{\chi^{4-\mathcal{N}} + 1}{(\chi+1)^{4-\mathcal{N}}} - 1 \right) \left(\frac{a_{1,3} - a_{2,3}}{\chi} + \frac{a_{1,4} - a_{2,4}}{\chi} \right) \right]. \quad (6.24)$$

In this expression, we need to impose the constraint equations in order to restrict the evaluation to a valid contour; and then we would seek to project onto each basis integral in turn. Now, suppose we can find a pair of solutions to the constraint equations which projects onto the first or second basis integral, respectively, *and* in addition, satisfies $a_{1,3} - a_{2,3} + a_{1,4} - a_{2,4} = 0$. Using such a contour would set the second line of eq. (6.24) equal to zero and therefore produce a particularly simple algebraic expression for c_1 and c_2 directly,

without need for additional simplification. Choosing $u = \frac{1}{3}$ and $v = 1$ in P_1 and P_2 gives such a contour.

This gives us the results,

$$\begin{aligned} c_1 &= -s_{12}^2 s_{23} A_{-+--+}^{\text{tree}} \left(1 - \frac{3}{4}(4 - \mathcal{N}) \frac{\chi}{(\chi + 1)^2} \right), \\ c_2 &= -\frac{3}{2} s_{12} s_{23} A_{-+--+}^{\text{tree}} \frac{4 - \mathcal{N}}{(\chi + 1)^2}, \end{aligned} \quad (6.25)$$

valid for $\mathcal{N} = 4, 2, 1$. The t -channel contribution can be obtained by exchanging $s_{12} \leftrightarrow s_{23}$ and $\chi \rightarrow \chi^{-1}$.

If we compare the expressions obtained above for the coefficients c_i to those obtained using the choice suggested in Section V, $u = \frac{1}{2}$ and $v = 1$, we find that the expressions are equal by virtue of the identity

$$\frac{1}{\chi} \left(\left(1 - \frac{1}{2} \delta_{\mathcal{N},4} \right) \frac{\chi^{4-\mathcal{N}} + 1}{(\chi + 1)^{4-\mathcal{N}}} - 1 \right) = -\frac{4 - \mathcal{N}}{(\chi + 1)^2}, \quad (6.26)$$

valid for $\mathcal{N} = 4, 2, 1$. This identity can of course easily be proven without reference to the current discussion, but the point we wish to emphasize is that the flexibility in choosing contours suggests certain algebraic simplifications which are not immediately obvious.

The double box coefficients given in eqs. (6.4, 6.5, 6.23, 6.25) agree with the $\mathcal{O}(\epsilon^0)$ terms of the corresponding coefficients, supplied to us by Lance Dixon [78], in the amplitudes computed by Bern, De Freitas, and Dixon [54].

VII. CONCLUSIONS

In this paper, we have taken the first step to extending the maximal generalized unitarity method to two loops. Cutting propagators can be viewed as deforming the original real loop-momentum contours of integration to contours encircling the global poles of the integrand. At two loops, there is a variety of such poles. We can evaluate the integral along many different linear combinations of these contours. However, our choices are restricted by the requirement that the evaluation along any contour respect the vanishing of certain insertions of Levi-Civita symbols, as well as of total derivatives arising from integration-by-parts identities. We derived the corresponding constraint equations for the massless double box, and showed how to use their solutions to obtain simple formulæ, eqs. (5.2) and (5.3), for the

coefficients of the two double box basis integrals to leading order in the dimensional regulator ϵ . To derive these equations, we adopted a parametrization of the loop momenta and solved explicitly for the maximal cuts, a heptacut in our case, and identified the additional poles present in the remaining degree of freedom.

We expect that the approach given in this paper — parametrize the basis integrals; solve the on-shell equations; identify the poles in the remaining degrees of freedom; impose all constraint equations — will apply to the full set of integrals required for two-loop amplitudes, both to the four-dimensional cuts considered here, and more generally to the D -dimensional cuts required for a complete calculation of the amplitude.

Acknowledgments

We are grateful to Henrik Johansson for many useful discussions and comments on the manuscript. We also wish to thank Lance Dixon for sharing unpublished results on the integral coefficients of the amplitudes computed in ref. [54]. We have also benefited from discussions with Simon Badger, Till Bargheer, Emil Bjerrum-Bohr, Ruth Britto, Darren Forde, Lisa Freyhult, Janusz Gluza, Gregory Korchemsky, Donal O’Connell, Mikael Passare and David Skinner. KJL thanks the Niels Bohr Institute for hospitality where part of this work was carried out. This work is supported by the European Research Council under Advanced Investigator Grant ERC-AdG-228301. KJL gratefully acknowledges financial support from Kungliga Vetenskapsakademien under Project no. FOA10V-133. This research also used resources of Academic Technology Services at UCLA.

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