Systematic study of the SO(10) symmetry breaking vacua in the matrix model for type IIB superstrings

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ABSTRACT: We study the properties of the space-time that emerges *dynamically* from the matrix model for type IIB superstrings in ten dimensions. We calculate the free energy and the extent of space-time using the Gaussian expansion method up to the third order. Unlike previous works, we study the $SO(d)$ symmetric vacua with all possible values of d within the range $2 \leq d \leq 7$, and observe clear indication of plateaus in the parameter space of the Gaussian action, which is crucial for the results to be reliable. The obtained results indeed exhibit systematic dependence on d, which turns out to be surprisingly similar to what was observed recently in an analogous work on the six-dimensional version of the model. In particular, we find the following properties: i) the extent in the shrunken directions is given by a constant, which does not depend on d; ii) the ten-dimensional volume of the Euclidean space-time is given by a constant, which does not depend on d except for $d = 2$; iii) The free energy takes the minimum value at $d = 3$. Intuitive understanding of these results is given by using the low-energy effective theory and some Monte Carlo results.

KEYWORDS: Matrix Models, Superstring Vacua.

Contents

1. Introduction

In recent years a lot of efforts have been devoted to constructing a stable (or long-lived meta-stable) vacuum in string theory, which is appropriate to describe our real world. In this kind of approach, however, there is no objective criterion to pick up one of the tremendously many possible vacua, which is commonly referred to as the landscape problem in the literature. It would be of course scientifically more desirable if a unique vacuum is chosen nonperturbatively and the chosen vacuum actually describes our real world. The aim of the present paper is to show that this possibility should not be totally forgotten as theorists' dream. Our explicit calculations rather suggest that we might be quite close to it although a big twist may still be needed to achieve the final goal.

In order to address such an issue, we certainly need a nonperturbative formulation of superstring theory. There are actually quite a few proposals made in the late 90s [1, 2, 3] after the discovery of D-branes. In this paper we study the type IIB matrix model [2], which is proposed as a nonperturbative formulation of type IIB superstring theory in ten dimensions. This model can be obtained formally by taking the zero-volume limit of $\mathrm{SU}(N)$ super Yang-Mills theory in ten dimensions.[1] The integer N can be viewed as a sort of regularization parameter, which should be taken to infinity eventually. In this formulation of superstring theory, the ten-dimensional target space is represented by the ten bosonic Hermitian matrices [4], which originate from the ten-dimensional gauge field

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[1]This relationship to $\mathrm{SU}(N)$ super Yang-Mills theory is considered at the level of classical action. Therefore the well-known gauge anomaly in the ten-dimensional super Yang-Mills theory is not an issue.

in the super Yang-Mills theory. The model has manifest $SO(10)$ invariance,[2] which is inherited from the super Yang-Mills theory. The question we would like to ask is whether this $SO(10)$ symmetry is spontaneously broken in the large-N limit down to $SO(4)$, which is the symmetry corresponding to our space-time.[3]

In ref. [6] two of the authors applied the Gaussian expansion method to this issue. The free energy of the $SO(d)$ symmetric vacua for $d = 2, 4, 6, 7$ was calculated up to the third order of the expansion, and $d = 4$ was found to give the smallest value. Moreover, the extent of space-time in the d directions was found to be larger than those in the remaining $(10 - d)$ directions. This result motivated higher order calculations up to the eighth order [7, 8, 9, 10, 11] for the $d = 4$ and $d = 7$ cases. While the results revealed interesting qualitative differences between the two cases, the situation turned out to be obscure.

In the Gaussian expansion method, the result depends on free parameters introduced in the Gaussian action. The crucial point is that one can still make a reliable prediction by identifying the "plateau region" in the parameter space, in which the result becomes almost constant. A practical approach is to obtain the points in the parameter space at which the result becomes stationary. As one goes to higher orders, one typically obtains more and more stationary points giving totally different results. However, if it turns out that there are quite a few points that give approximately the same results, one may regard it as indication of the plateau region. Such behaviors were indeed observed in various simple models and the obtained results confirmed the validity of the method. (For example, see refs. [7, 12, 13].) However, in the case of type IIB matrix model, the plateau region has not yet been identified unambiguously.

Recently it was suggested [14] that the above situation might be due to the extra symmetry Σ_d, which was imposed on the shrunken $(10 - d)$ directions in order to make the calculations feasible. By imposing a symmetry $SO(d) \times \Sigma_d \subset SO(10)$, which is stronger than just $SO(d)$, one can reduce the number of free parameters considerably. While this makes it much easier to obtain stationary points, it also makes the available stationary points strongly restricted by the chosen extra symmetry Σ_d. As a result, one might miss the opportunity to observe clear indication of plateaus for the $SO(d)$ symmetric vacua that might otherwise be there.

This possibility was noticed in a similar study for the six-dimensional version of the type IIB matrix model [14]. The model can be obtained formally by the zero-volume limit of $SU(N)$ super Yang-Mills theory in six dimensions, and it is expected to have properties analogous to the type IIB matrix model such as the spontaneous breakdown of the $SO(6)$ symmetry. On the technical side, the analysis becomes much easier than in the type IIB matrix model, and it was possible to perform calculations for all the values of d within the range $2 \leq d \leq 5$. Moreover, calculations were done imposing *only the SO(d) symmetry* up to order 3 for $d = 3, 4, 5$, and up to order 5 for $d = 4, 5$. Quite a few stationary

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[2]This should be considered as a virtue of this model, which is important for the issue we address in this paper. In any other proposal for a nonperturbative formulation of superstring/M theory, the space-time symmetry is not manifestly preserved.

[3]See ref. [5] for discussions on the idea of the emergent gravity, which is deeply related to the above scenario.

points were found for each d, and clear indication of plateaus was observed. This enabled reliable predictions for both the free energy and the extent of space-time, which exhibited interesting qualitative features summarized as follows.

i) The extent of space-time in the shrunken directions is given by a constant (r), which is independent of d. (universal "compactification" scale)

ii) The six-dimensional volume of the Euclidean space-time is given by a constant $(v \equiv \ell^6)$, which is independent of d except for $d = 2$. (constant volume property)

iii) The free energy takes the minimum value at $d = 3$.

Intuitive understanding for these properties is also given in ref. [14]. The properties i) and ii) can be understood by considering the low-energy effective theory, which is given in terms of a system similar to the branched polymer [4, 15]. The two dynamical scales r and ℓ, which characterize the properties i) and ii), respectively, are reproduced numerically by Monte Carlo simulation [16]. The property iii) as well as the anomaly for $d = 2$ in the property ii) can be understood from the properties of the fermion determinant [17].

With all these new insights, we redo the calculations for the type IIB matrix model and study the SO(d) symmetric vacua for all values of d within the range $2 \leq d \leq 7$. Unlike in the six-dimensional case, it is difficult to perform calculations imposing *only the SO(d) symmetry*. However, the calculations in the six-dimensional case [14] suggest that the stationary points obtained in the plateau region have all the variety of symmetries in the shrunken directions. Therefore, we perform calculations in the ten-dimensional case imposing SO(d) $\times \Sigma_d \subset$ SO(10) with various possible Σ_d for each d. In fact we exhaust all the possible extra symmetries that leave not more than five free parameters in the Gaussian action. By combining all the solutions obtained in this way, we were able to observe clear indication of plateaus already at the 3rd order calculations as we did in the six-dimensional case. This should be contrasted to the situation with the previous calculations for the type IIB matrix model, where one particular symmetry Σ_d was chosen for each d, and hence the number of solutions was not enough to clearly identify the plateau.

The free energy and the extent of space-time obtained for each d in the plateau region indeed exhibit systematic d dependence analogous to i)-iii) observed for the six-dimensional case. This is understandable theoretically given that the low-energy effective theory is described by a similar branched-polymer-like system [4, 15] and that the fermion determinant has similar properties [17]. Let us emphasize, however, that the Gaussian expansion method knows neither of these facts, and yet it revealed the same qualitative behaviors of the two models. Moreover, the free energy obtained for the SO(d) symmetric vacua is actually quite close to the value obtained from the formula for the partition function conjectured by Krauth, Nicolai and Staudacher [18], *for both the 6d and 10d models*. We consider these as strong evidence for the validity of the present calculations.

The rest of this article is organized as follows. In section 2 we define the model and the observable which serves as an order parameter for the spontaneous breaking of rotational SO(10) symmetry. In section 3 we describe the method we use to study the model. In

section 4 we present our results for the free energy and the extent of space-time in the SO(d) symmetric vacua ($2 \leq d \leq 7$). Section 5 is devoted to a summary and discussions. In appendix A we present the details of the ansatz we use to study each of the SO(d) symmetric vacua. In appendix B we derive the value of free energy from the formula for the partition function conjectured by Krauth, Nicolai and Staudacher.

2. The model and the order parameter

The type IIB matrix model can be obtained formally by the zero-volume limit of $D = 10$ SU(N) pure super Yang-Mills theory, and its partition function is given by

$$Z = \int dA \, d\Psi \, e^{-S_{\mathrm{b}}-S_{\mathrm{f}}} , \tag{2.1}$$

$$S_{\mathrm{b}} = -\frac{1}{4g^2} \operatorname{Tr}[A_\mu, A_\nu]^2 , \tag{2.2}$$

$$S_{\mathrm{f}} = -\frac{1}{2g^2} \operatorname{Tr}\left(\Psi_\alpha (C\Gamma^\mu)_{\alpha\beta}[A_\mu, \Psi_\beta]\right) . \tag{2.3}$$

Here A_μ ($\mu = 1, \cdots, 10$) are traceless $N \times N$ Hermitian matrices, whereas Ψ_α ($\alpha = 1, \cdots, 16$) are traceless $N \times N$ matrices with Grassmannian entries. The parameter g can be scaled out by appropriate redefinition of the matrices, and hence it is just a scale parameter rather than a coupling constant. We therefore set $g^2 N = 1$ from now on without loss of generality. The integration measure for A_μ and Ψ_α is given by

$$dA = \prod_{a=1}^{N^2-1} \prod_{\mu=1}^{10} \frac{dA_\mu^a}{\sqrt{2\pi}} , \qquad d\Psi = \prod_{a=1}^{N^2-1} \prod_{\alpha=1}^{16} d\Psi_\alpha^a , \tag{2.4}$$

where A_μ^a and Ψ_α^a are the coefficients in the expansion $A_\mu = \sum_{a=1}^{N^2-1} A_\mu^a T^a$ *etc.* with respect to the SU(N) generators T^a normalized as $\operatorname{Tr}(T^a T^b) = \frac{1}{2}\delta^{ab}$.

The model has an SO(10) symmetry, under which A_μ and Ψ_α transform as a vector and a Majorana-Weyl spinor, respectively. The 16×16 matrices Γ_μ are the gamma matrices after the Weyl projection, and C is the charge conjugation matrix, which satisfies $(\Gamma_\mu)^T = C\Gamma_\mu C^\dagger$ and $C^T = C$.

In order to discuss the spontaneous symmetry breaking (SSB) of SO(10) in the large-N limit, we consider the "moment of inertia" tensor [4, 19]

$$T_{\mu\nu} = \frac{1}{N} \operatorname{Tr}(A_\mu A_\nu) , \tag{2.5}$$

which is a 10×10 real symmetric tensor. We denote its eigenvalues as λ_j ($j = 1, \cdots, 10$) with the specific order

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{10} . \tag{2.6}$$

If the SO(10) is not spontaneously broken, the expectation values $\langle \lambda_j \rangle$ ($j = 1, \cdots, 10$) should be all equal in the large-N limit. Therefore, if we find that they are not equal, it implies that the SO(10) symmetry is spontaneously broken. Thus the expectation values

$\langle \lambda_j \rangle$ serve as an order parameter of the SSB. In ref. [17] it was found that the phase of the fermion determinant (or Pfaffian, strictly speaking) favors $d(\geq 3)$-dimensional configurations, which have λ_j $(j = d + 1, \cdots, 10)$ much smaller than the others. This suggests the possibility that the SO(10) symmetry is broken down to SO(d) with $d \geq 3$. Since the eigenvalue distribution of A_μ represents the extent of space-time in the type IIB matrix model [4], the above situation realizes the dynamical compactification to d-dimensional space-time.

In general one can obtain supersymmetric matrix models by taking the zero-volume limit of pure super Yang-Mills theories in $D = 3$, 4, 6 and 10 dimensions, where the $D = 10$ case corresponds to the type IIB matrix model. The convergence of the partition function for general D was investigated both numerically [18] and analytically [20]. The $D = 3$ model is ill-defined since the partition function is divergent. The $D = 4$ model has a real positive fermion determinant, and Monte Carlo simulation suggested the absence of the SSB of rotational symmetry [21]. (See also refs. [22, 23].) The $D = 6$ model and the $D = 10$ model both have a complex fermion determinant, whose phase is expected to play a crucial role [17, 24, 25, 26, 27] in the SSB of SO(D).

3. The Gaussian expansion method

Since there are no quadratic terms in the actions (2.2) and (2.3), we cannot perform a perturbative expansion in the ordinary sense. Finding the vacuum of this model is therefore a problem of solving a strongly coupled system. It is known that a certain class of matrix models can be solved exactly by using various large-N techniques, but the present model does not belong to such a category.[4] The use of the Gaussian expansion method in studying large-N matrix quantum mechanics has been advocated by Kabat and Lifschytz [28], and various black hole physics of the dual geometry has been discussed [29]. Applications to simplified versions of the type IIB matrix model were pioneered by refs. [30].

The starting point of the Gaussian expansion method is to introduce a Gaussian term S_0 and to rewrite the action $S = S_{\mathrm{b}} + S_{\mathrm{f}}$ as

$$S = (S_0 + S) - S_0 \; . \tag{3.1}$$

Then we can perform a perturbative expansion regarding the first term $(S_0 + S)$ as the "classical action" and the second term $(-S_0)$ as the "one-loop counter term". The results at finite order depend, of course, on the choice of the Gaussian term S_0, which contains many free parameters in general. However, it is known in various examples that there exists a region of parameters, in which the results obtained at finite order are almost constant. Therefore, if we can identify this "plateau region", we can make concrete predictions. It should be emphasized that the method enables us to obtain genuinely nonperturbative results, although most of the tasks involved are nothing more than perturbative calculations as emphasized in ref. [31].

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[4]Note, however, that the large-N limit simplifies the calculation in the Gaussian expansion method considerably since it allows us to consider only the planar diagrams.

There are some cases in which one finds more than one plateau regions in the parameter space. In that case, each of them is considered to correspond to a local minimum of the effective action, and the plateau which gives the smallest free energy corresponds to the true vacuum. These statements have been confirmed explicitly in exactly solvable matrix models [13].

As the Gaussian action for the present model, we consider the most general one that preserves the SU(N) symmetry. Note, in particular, that we have to allow the Gaussian action to break the SO(10) symmetry so that we can study the SSB of SO(10). In practice we are going to restrict the parameter space by imposing the SO(d) symmetry with $2 \leq d \leq 7$. The plateau region identified for each d corresponds to a local minimum which breaks the SO(10) symmetry spontaneously. By comparing the free energy, we can determine which local minimum is actually the true vacuum.

Making use of the SO(10) symmetry of the model, we can always bring the Gaussian action into the form

$$S_0 = S_{0b} + S_{0f} \ , \tag{3.2}$$

$$S_{0b} = \frac{N}{2} \sum_{\mu=1}^{10} M_\mu \operatorname{Tr}(A_\mu)^2 \ , \tag{3.3}$$

$$S_{0f} = \frac{N}{2} \sum_{\alpha,\beta=1}^{16} \mathcal{A}_{\alpha\beta} \operatorname{Tr}(\Psi_\alpha \Psi_\beta) \ , \tag{3.4}$$

where M_μ and $\mathcal{A}_{\alpha\beta}$ are arbitrary parameters. The 16×16 complex matrix $\mathcal{A}_{\alpha\beta}$ can be expanded in term of the gamma matrices as

$$\mathcal{A}_{\alpha\beta} = \sum_{\mu,\nu,\rho=1}^{10} \frac{i}{3!} m_{\mu\nu\rho} (C\Gamma_\mu \Gamma_\nu^\dagger \Gamma_\rho)_{\alpha\beta} \ , \tag{3.5}$$

using a 3-form $m_{\mu\nu\rho}$. We can then rewrite the partition function (2.1) as

$$Z = Z_0 \langle e^{-(S-S_0)} \rangle_0 \ , \tag{3.6}$$

$$Z_0 = \int \mathrm{d}A \, \mathrm{d}\Psi \, e^{-S_0} \ , \tag{3.7}$$

where $\langle \cdot \rangle_0$ is a vacuum expectation value with respect to the partition function Z_0. From this we find that the free energy $F = -\log Z$ can be expand as

$$F = \sum_{k=0}^{\infty} f_k \ ,$$

$$f_0 = -\log Z_0 \ ,$$

$$f_k = -\sum_{l=0}^{k} \frac{(-1)^{k-l}}{(k+l)!} \, _{k+l}C_{k-l} \Big\langle (S_b - S_0)^{k-l} (S_f)^{2l} \Big\rangle_{C,0} \qquad \text{for } k \geq 1 \ , \tag{3.8}$$

where the subscript 'C' in $\langle \cdot \rangle_{C,0}$ implies that the connected part is taken. The expansion is organized in such a way that it corresponds to the loop expansion regarding the insertion

of the 2-point vertex $(-S_0)$ as a contribution from the one-loop counterterm. Similarly the expectation value of an observable $\mathcal{O}$ can be evaluated as

$$\langle \mathcal{O} \rangle = \langle \mathcal{O} \rangle_0 + \sum_{k=1}^{\infty} O_k \ ,$$

$$O_k = \sum_{l=0}^{k} \frac{(-1)^{k-l}}{(k+l)!} \, {}_{k+l}\mathrm{C}_{k-l} \langle \mathcal{O} \, (S_{\mathrm{b}} - S_0)^{k-l} \, (S_{\mathrm{f}})^{2l} \rangle_{\mathrm{C},0} \ . \tag{3.9}$$

In practice we truncate the series expansion at some finite order. Then the free energy (3.8) and the observable (3.9) depend on the free parameters M_μ and $\mathcal{A}_{\alpha\beta}$ in the Gaussian action. We search for the values of parameters, at which the free energy becomes stationary by solving the "self-consistency equations"

$$\frac{\partial}{\partial M_\mu} F = 0 \ , \qquad \frac{\partial}{\partial m_{\mu\nu\rho}} F = 0 \ , \tag{3.10}$$

and estimate F and $\langle \mathcal{O} \rangle$ at the solutions. As we increase the order of the expansion, the number of solutions increases. If we find that there are many solutions close to each other in the parameter space which give similar results for the free energy and the observables, we may identify the region as a plateau.

In actual calculation it is convenient to derive the series expansion (3.8) in the following way. First we consider the action

$$\tilde{S} = S_0 + \epsilon S_{\mathrm{b}} + \sqrt{\epsilon} S_{\mathrm{f}} \ , \tag{3.11}$$

and the partition function

$$\tilde{Z} = \int \mathrm{d}A \, \mathrm{d}\Psi \, e^{-\tilde{S}} \ , \tag{3.12}$$

where ϵ is a fictitious expansion parameter. Next we calculate the free energy in the ϵ-expansion as

$$\tilde{F} = -\log \tilde{Z} = \sum_{k=0}^{\infty} \epsilon^k \tilde{f}_k \ . \tag{3.13}$$

Each term $\tilde{f}_k$ depends on the parameters M_μ and $m_{\mu\nu\lambda}$ in the Gaussian action S_0. Then we substitute these parameters as

$$M_\mu \to (1 - \epsilon) \, M_\mu \ , \qquad m_{\mu\nu\rho} \to (1 - \epsilon) \, m_{\mu\nu\rho} \ , \tag{3.14}$$

reorganize the series with respect to ϵ, and set ϵ to 1. In this way we reproduce the expression (3.8). The action (3.11) is introduced to obtain the ordinary perturbation theory for the first term in (3.1), and the final step (3.14) corresponds to taking account of the second term in (3.1) as the one-loop counter term. The main task is to obtain the series (3.13), which is nothing more than what is required for ordinary perturbation theory.[5] We use a similar procedure to obtain the expansion (3.9) for the observables.

[5]Further simplification is possible by exploiting the fact that the free energy F is related to the two-particle irreducible (2PI) free energy through the Legendre transformation [7]. The number of Feynman diagrams decreases considerably by the restriction to 2PI diagrams. While we do not use this technique for the 3rd order calculations in the present work, it would be crucial in performing higher order calculations.

ansatz	discrete symmetry for shrunken directions	symbol
SO(7)		○
SO(6)	$\mathbb{Z}_{4(7,8,9,10|1)}$	○
	$\mathbb{Z}_{3(8,9,10)}$	●
	$\mathbb{Z}_{2(7,8|1)}$	△
SO(5)	$\mathbb{Z}_{5(6,7,8,9,10)}$	○
	$\mathbb{Z}_{4(6,7,8,9|1)}$	●
	$\mathbb{Z}_{4(6,7,8,9|10)}$	△
	$\mathbb{Z}_{2(6,7|1)} \times \mathbb{Z}_{3(8,9,10)}$	▲
	$\mathbb{Z}_{2(6,7|8,9,10)} \times \mathbb{Z}_{3(8,9,10)}$	▽
	$\mathbb{Z}_{2(6,7|1)} \times \mathbb{Z}_{2(8,9|1)}$	▼
	$\mathbb{Z}_{2(6,7|10)} \times \mathbb{Z}_{2(8,9|10)} \times \mathbb{Z}_{2(\{6,7\},\{8,9\}|1,10)}$	◇
SO(4)	$\mathbb{Z}_{2(5,6|1)} \times \mathbb{Z}_{2(7,8|1)} \times \mathbb{Z}_{2(9,10|1)}$	○
	$\mathbb{Z}_{3(5,6,7)} \times \mathbb{Z}_{3(8,9,10)} \times \mathbb{Z}_{2(\{5,6,7\},\{8,9,10\}|1)}$	●
	$\mathbb{Z}_{4(5,6,7,8|1)} \times \mathbb{Z}_{2(9,10|1)}$	△
	$\mathbb{Z}_{5(6,7,8,9,10)} \times \mathbb{Z}_{2(-|1,6,7,8,9,10)}$	▲
	$\mathbb{Z}_{3(5,6,7)} \times \mathbb{Z}_{3(8,9,10)} \times \mathbb{Z}_{2(-|1,8,9,10)}$	▽
SO(3)	$\mathbb{Z}_{2(4,5|1)} \times \mathbb{Z}_{2(6,7|1)} \times \mathbb{Z}_{2(8,9|1)} \times \mathbb{Z}_{3(\{4,5\},\{6,7\},\{8,9\})}$	○
	$\mathbb{Z}_{2(4,5|10)} \times \mathbb{Z}_{2(6,7|10)} \times \mathbb{Z}_{2(8,9|10)}$ $\times\mathbb{Z}_{3(\{4,5\},\{6,7\},\{8,9\})} \times \mathbb{Z}_{2(\{4,5\},\{6,7\})}$	●
	$\mathbb{Z}_{2(4,5|10)} \times \mathbb{Z}_{2(6,7|10)} \times \mathbb{Z}_{2(8,9|10)}$ $\times\mathbb{Z}_{3(\{4,5\},\{6,7\},\{8,9\})} \times \mathbb{Z}_{2(-|1,10)}$	△
	$\mathbb{Z}_{3(4,5,6)} \times \mathbb{Z}_{3(7,8,9)} \times \mathbb{Z}_{2(\{4,5,6\},\{7,8,9\}|1)} \times \mathbb{Z}_{2(-|1,10)}$	▲
	$\mathbb{Z}_{3(4,5,6)} \times \mathbb{Z}_{2(7,8|1)} \times \mathbb{Z}_{2(9,10|1)} \times \mathbb{Z}_{2(\{7,8\},\{9,10\})}$	▽
	$\mathbb{Z}_{3(4,5,6)} \times \mathbb{Z}_{4(7,8,9,10|1)} \times \mathbb{Z}_{2(-|7,8,9,10)}$	▼
	$\mathbb{Z}_{2(4,5|1)} \times \mathbb{Z}_{5(6,7,8,9,10)} \times \mathbb{Z}_{2(-|1,6,7,8,9,10)}$	◇
	$\mathbb{Z}_{6(5,6,7,8,9,10|1)} \times \mathbb{Z}_{2(-|5,6,7,8,9,10)}$	◆
	$\mathbb{Z}_{2(4,5|1)} \times \mathbb{Z}_{5(6,7,8,9,10|4,5)}$	
SO(2)	$\mathbb{Z}_{2(3,4|1)} \times \mathbb{Z}_{2(5,6|1)} \times \mathbb{Z}_{2(7,8|1)} \times \mathbb{Z}_{2(9,10|1)}$ $\times\mathbb{Z}_{4(\{3,4\},\{5,6\},\{7,8\},\{9,10\})}$	○
	$\mathbb{Z}_{4(3,4,5,6|1)} \times \mathbb{Z}_{4(7,8,9,10|1)} \times \mathbb{Z}_{2(\{3,4,5,6\},\{7,8,9,10\})}$	●
	$\mathbb{Z}_{3(3,4,5)} \times \mathbb{Z}_{3(6,7,8)} \times \mathbb{Z}_{2(\{3,4,5\},\{6,7,8\}|1)}$ $\times\mathbb{Z}_{2(9,10|1)} \times \mathbb{Z}_{2(-|3,4,5,6,7,8)}$	△
	$\mathbb{Z}_{3(3,4,5)} \times \mathbb{Z}_{3(6,7,8)} \times \mathbb{Z}_{2(\{3,4,5\},\{6,7,8\}|1)}$ $\times\mathbb{Z}_{2(9,10|1)} \times \mathbb{Z}_{2(-|9,10)}$	▲

Table 1: The list of the SO(d) ansatz ($d = 2,\cdots,7$), which leave us with not more than 5 parameters in the Gaussian action. The symbols shown in the right-most column are used in figs. 1, 2, 4 and 5 to represent the solutions to the self-consistency equations. For the SO(3) $\times \mathbb{Z}_{2(4,5|1)} \times \mathbb{Z}_{5(6,7,8,9,10|4,5)}$ ansatz, we find no solutions in the range $-2 < f < 7$ displayed in fig. 1, and hence we do not assign any symbol.

Since we are interested in the large-N limit, we list up Feynman diagrams in the double-line notation, and keep only the planar diagrams when evaluating the free energy (3.8) and the observable (3.9).

There are many free parameters in the Gaussian action; *i.e.*, we get 10 from M_μ and 120 from $m_{\mu\nu\rho}$. There are as many self-consistency equations as these parameters.

Unfortunately it seems impossible to solve them in full generality. However, it is reasonable to expect that some subgroup of SO(10) symmetry such as SO(d) with $2 \leq d \leq 7$ remains unbroken. This allows us to impose the corresponding symmetry on the Gaussian action, and hence to reduce the number of parameters.

In order to study the SO(d) symmetric vacuum, we impose the SO(d) symmetry on the Gaussian action. For $d \geq 4$, this leads to $M_1 = \cdots = M_d$ and $m_{\mu\nu\rho} = 0$ unless μ, ν and ρ are three different numbers in $\{d+1, \cdots, 10\}$. Therefore we obtain $(11-d)$ parameters from M_μ and $_{10-d}C_3$ parameters from $m_{\mu\nu\rho}$. For $d = 3$, m_{123} can also be non-zero, and for $d = 2$, $m_{12\mu}$ with $\mu = 3, 4, \cdots, 10$ can also be non-zero. Thus the total number of parameters is 5, 9, 16, 27, 44, 73 for $d = 7, 6, 5, 4, 3, 2$, respectively. For $d \geq 8$, the SO(d) symmetry enforces $m_{\mu\nu\rho} = 0$, which makes the Gaussian expansion ill-defined. However, the SO(d) symmetric vacua with $d \geq 8$ are expected to exist, and they should be realized as a plateau with the SO(d) ansatz ($d \leq 7$). See ref. [11] for an explicit example of this claim for the SO(10) symmetric vacuum.

It turned out that solving the self-consistency equations (3.10) is possible for the number of parameters not more than 5. Except for $d = 7$, we therefore impose extra discrete symmetries (Σ_d) in the shrunken directions $x_{d+1}, \cdots, x_{10}$, where we require that SO(d)$\times\Sigma_d$ is a subgroup of SO(10). Let us introduce the following notations. First, $\mathbb{Z}_{p(i_1, \cdots, i_p)}$ represents the group of cyclic permutations acting on $x_{i_1}, \cdots, x_{i_p}$. Similarly, $\mathbb{Z}_{p(\{i_1, j_1\}, \cdots, \{i_p, j_p\})}$ represents the group of cyclic permutations acting on the sets $\{x_{i_1}, x_{j_1}\}, \cdots, \{x_{i_p}, x_{j_p}\}$. We also combine them with a reflection. For instance, the symbol $\mathbb{Z}_{p(i_1, \cdots, i_p | j_1, \cdots, j_q)}$ implies that we make a reflection $(x_{j_1}, \cdots, x_{j_q}) \mapsto (-x_{j_1}, \cdots, -x_{j_q})$ together with a cyclic permutation of $x_{i_1} \cdots, x_{i_p}$. The symbol $\mathbb{Z}_{2(-|j_1, \cdots, j_q)}$ implies that we make a reflection without any cyclic permutations. Note that the permutation is odd if the integer p is even and the number of elements in the set to be permuted is odd. In that case, we have to combine it with a reflection in an odd number of directions in order to make the imposed symmetry an element of SO(10). Table 1 shows the list of the ansatz that leave us with not more than 5 parameters. In the previous works [6, 7, 8, 9, 10, 11] one particular extra symmetry was chosen for each d so that the number of parameters is reduced to 3. All the extra symmetries used in the literature are included in the list of ansatz we study. See appendix A for more detail.

4. Results

For each ansatz, we first obtain the free energy up to the third order as a function of the free parameters in the Gaussian action. By differentiating the free energy with respect to the free parameters, we obtain the self-consistency equations, which we solve numerically by Mathematica. The free energy evaluated at each solution is plotted in fig. 1 for the SO(d) ansatz ($2 \leq d \leq 7$) described in the previous section. More precisely, we actually plot "the free energy density" defined as

$$f = \lim_{N \to \infty} \left\{ \frac{F}{N^2 - 1} - (-3 \log N) \right\} , \quad \text{where } F = -\log Z . \tag{4.1}$$
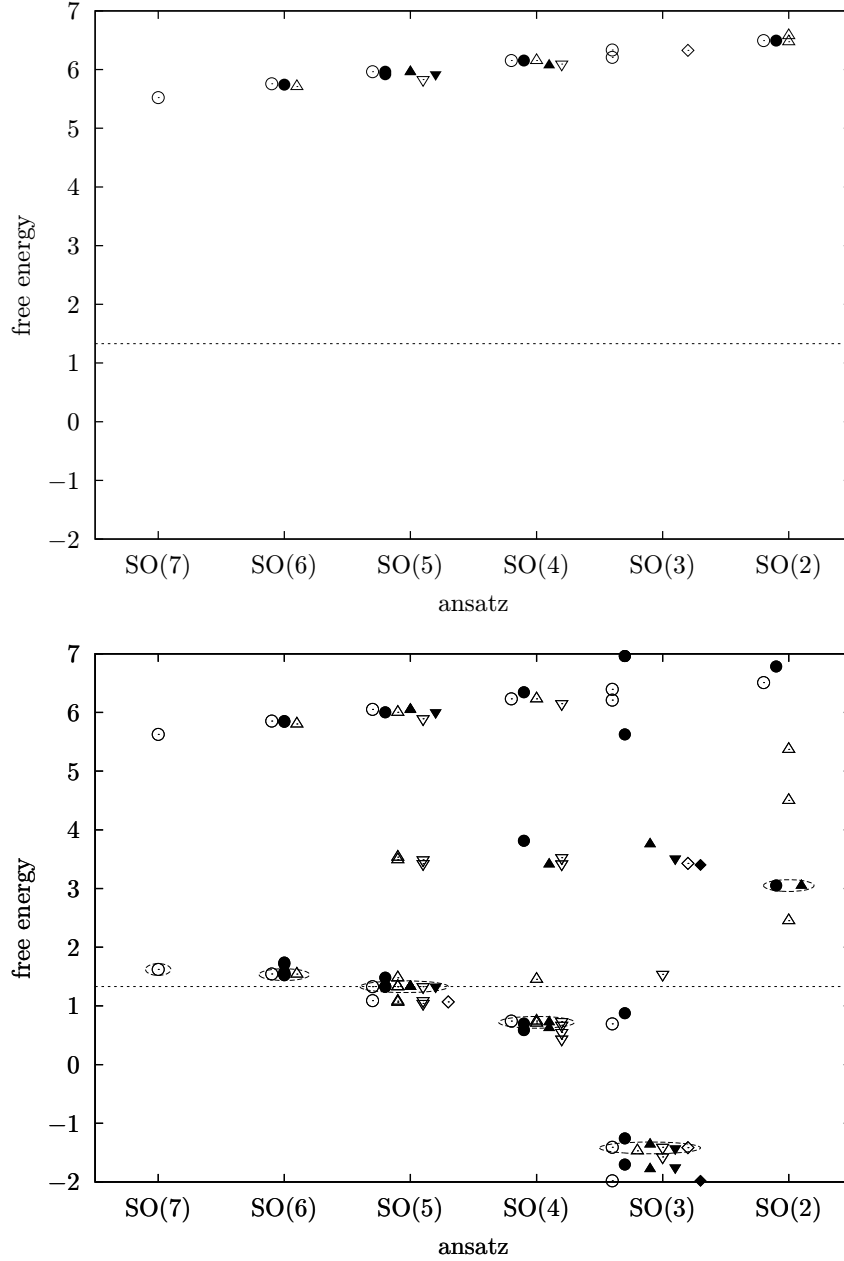
Figure 1: The free energy density (4.1) evaluated at the solutions to the self-consistency equations at the orders 1 (Top) and 3 (Bottom). Each symbol represents the largest symmetry in Table 1 that the solution has. The horizontal line represents the value $\log 8 - \frac{3}{4} = 1.32944...$ obtained from the KNS conjecture. The data points surrounded by the dashed lines correspond to the "physical solutions", which indicate the plateau region.
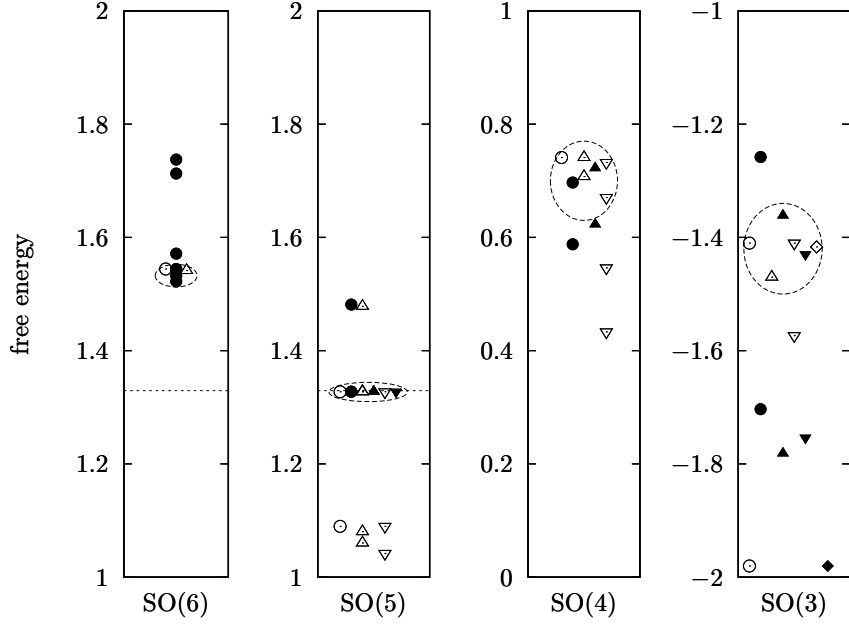
Figure 2: The zoom-up of fig. 1 (Bottom) for $d = 6, 5, 4, 3$ near the "physical solutions", which are surrounded by the dashed line.
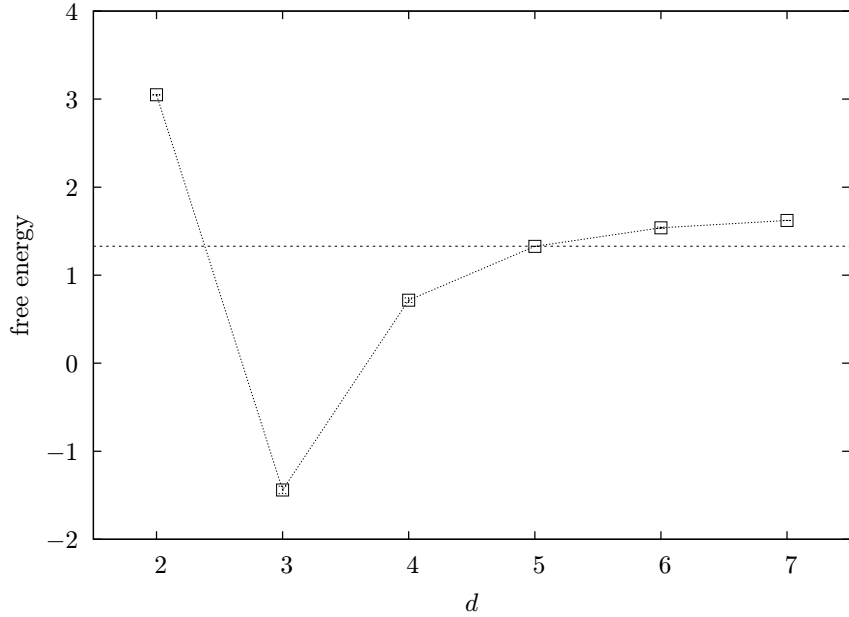

Figure 3: The free energy density averaged over the "physical solutions" for each d is plotted against d. We put error bars representing the mean square error when there are more than one physical solutions. The horizontal line represents the KNS value $f = \log 8 - \frac{3}{4} = 1.32944\ldots$, and the dotted line connecting the data points is drawn to guide the eye.

The horizontal line $\log 8 - \frac{3}{4} = 1.32944...$ represents the result obtained in appendix B from the analytic formula of the partition function conjectured by Krauth, Nicolai and Staudacher (KNS) [18]. At order 3, the "physical solutions" identified for each d as described below are surrounded by a dashed line. In fig. 2 we zoom up the region containing these solutions for $d = 6, 5, 4, 3$, which gives the free energy density $f \sim 1.5$, 1.3, 0.7, -1.4, respectively.

In fig. 3 we show the free energy density obtained by averaging over the physical solutions for each d at order 3. We put error bars representing the mean square error when there are more than one physical solutions. The result decreases monotonically as d decreases from 7 to 3, and it becomes much larger for $d = 2$. Thus, the SO(3) symmetric vacuum gives the smallest free energy density. The d-dependence of the free energy density is quite analogous to the one observed in the six-dimensional case [14]. There the value of the free energy tends to decrease slightly as one goes from order 3 to order 5. Considering such artifacts due to truncation, we speculate that the KNS conjecture actually refers to the partition function for the SO(10) symmetric vacuum.

Let us then explain how we identify the "physical solutions". For that we also refer to the results for the extent of space-time. In appendix A we give the explicit form of $\langle T_{\mu\nu} \rangle$ for each ansatz. Let us note that it is not diagonal in general, and therefore we need to diagonalize it in order to obtain the eigenvalues $\langle \lambda_j \rangle$. Then we find that some solutions give smaller values in the directions involved in the preserved SO(d) symmetry than in the remaining directions. Such solutions are not shown in the figures, and will not be considered in what follows. For the SO(d) ansatz, the d large eigenvalues $\langle \lambda_j \rangle$ $(1 \leq j \leq d)$ are equal due to the imposed SO(d) symmetry, and we denote the corresponding value as R^2. The remaining $(10 - d)$ eigenvalues for each solution turn out to be quite close to each other and we denote the mean value as r^2. In figs. 4 and 5, we plot the values of R^2 and r^2 evaluated at the solutions for each ansatz.

At order 3, we find a set of solutions for $3 \leq d \leq 6$ giving similar values for the free energy density f and the extent of space-time R^2 and r^2. We consider this as the concentration of solutions [7], which indicates a plateau region in the space of parameters explained in section 3. Thus we can pick up the "physical solutions" for $3 \leq d \leq 6$ without much ambiguity. For the SO(2) ansatz, we find two solutions close to each other with $f \sim 3.0$, $R^2 \sim 3.6$ and $r^2 \sim 0.11$, which we identify as the physical solutions. For the SO(7) ansatz, we obtain only two solutions at order 3, which are not close to each other. From fig. 1 (Bottom), however, it looks reasonable to identify the one with smaller free energy density as the physical solution.

In fig. 6 we plot the result for R^2 and r^2 averaged over all the physical solutions for each d. We put error bars representing the mean square error when there are more than one physical solutions. We find that r^2 stays almost constant at $r^2 = 0.1 \sim 0.15$, which seems to be universal for all the SO(d) symmetric vacua with $2 \leq d \leq 7$. On the other hand, the results for R^2 are found to be larger for smaller d. Let us test whether this behavior is consistent with the constant-volume property [14], which is given by $R^d \tilde{r}^{10-d} \approx \ell^{10}$, where $\tilde{r}^2$ represents the universal "compactification" scale, which we leave as a free parameter in the present analysis.
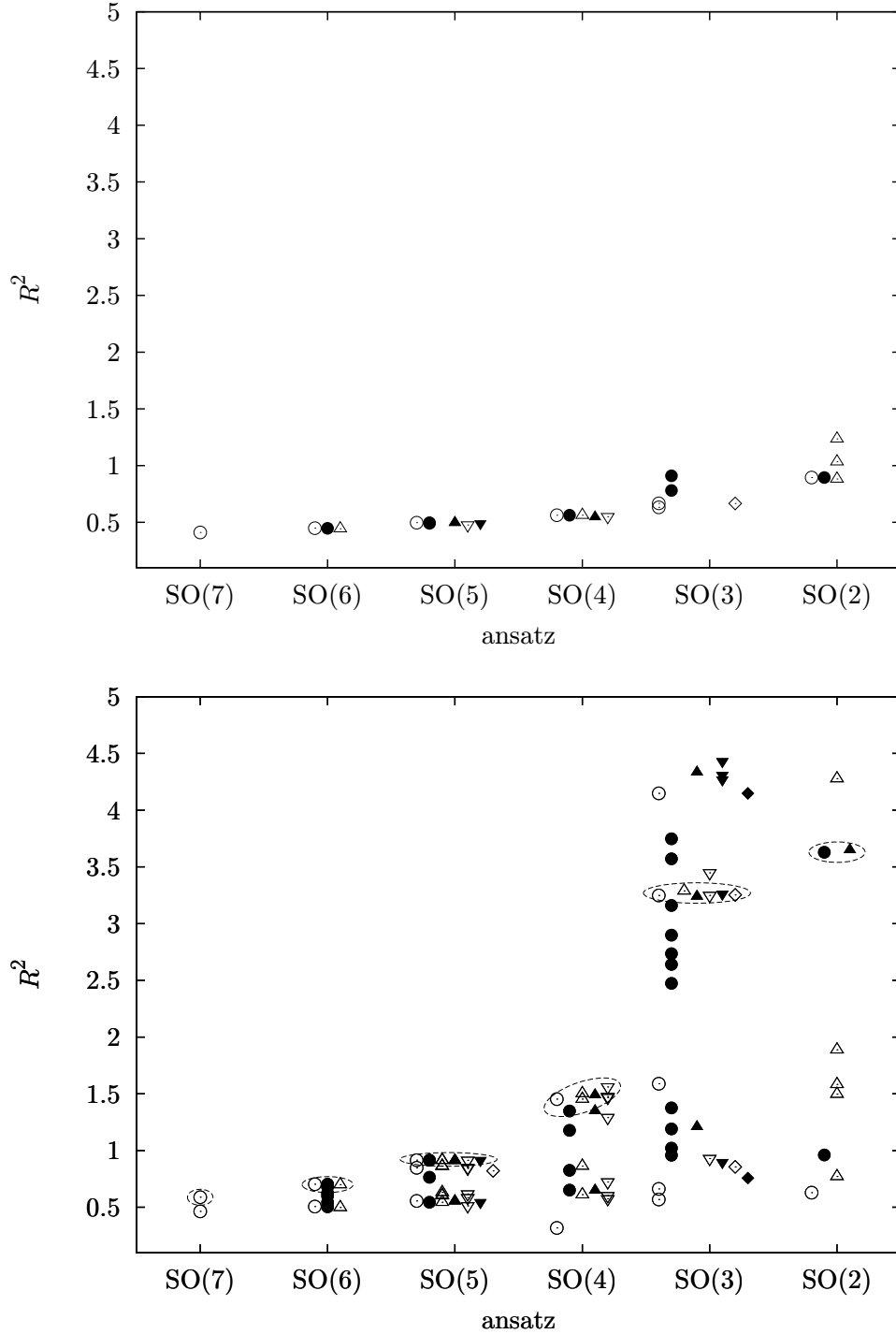
Figure 4: The extent of space-time R^2 in the extended directions evaluated at the solutions to the self-consistency equations at the first order (Top) and the third order (Bottom). Each symbol represents the largest symmetry in Table 1 that the solution has. The data points surrounded by the dashed lines correspond to the "physical solutions", which indicate the plateau region.
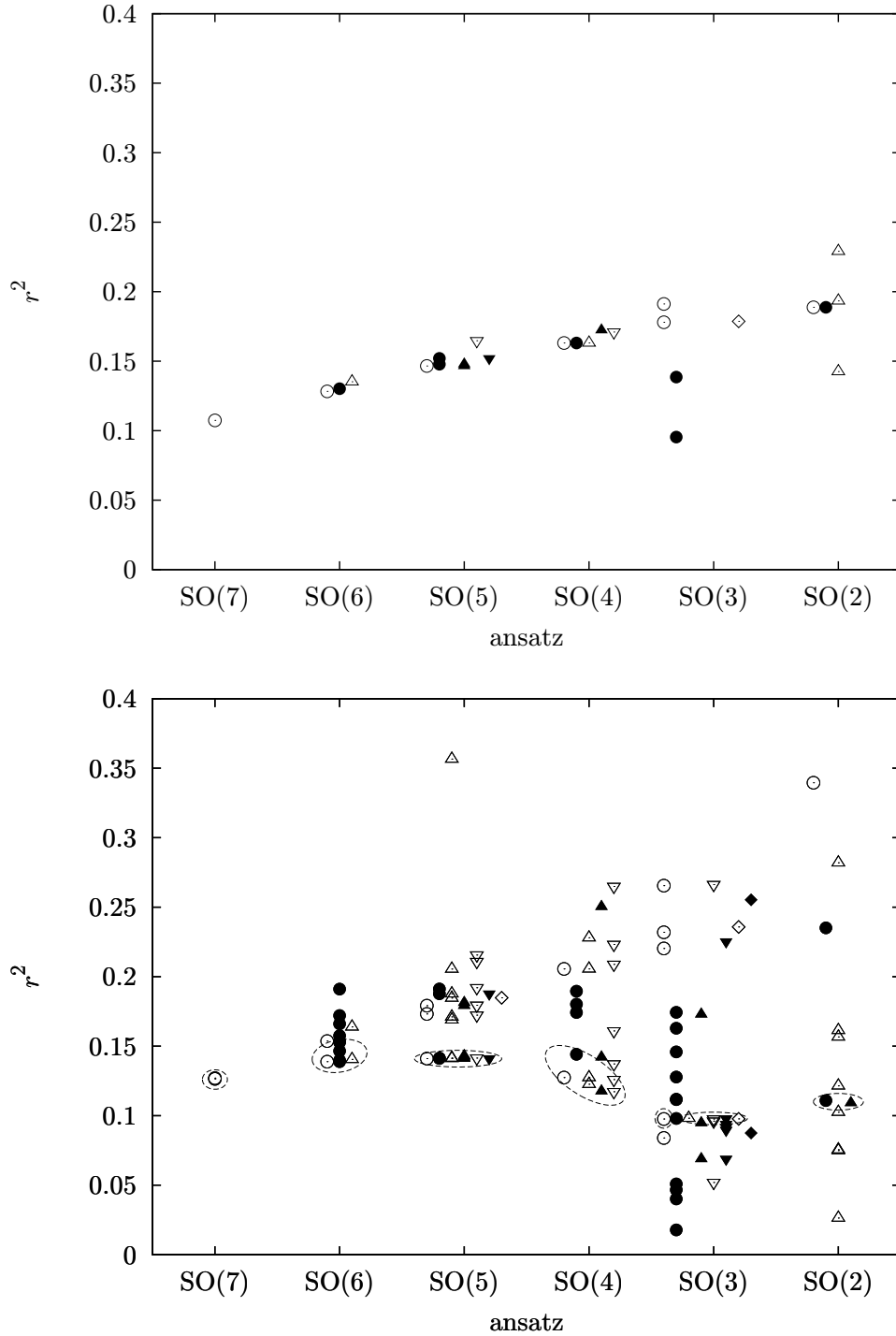
Figure 5: The extent of space-time r^2 in the shrunken directions evaluated at the solutions to the self-consistency equations at the first order (Top) and the third order (Bottom). Each symbol represents the largest symmetry in Table 1 that the solution has. The data points surrounded by the dashed lines correspond to the "physical solutions", which indicate the plateau region.
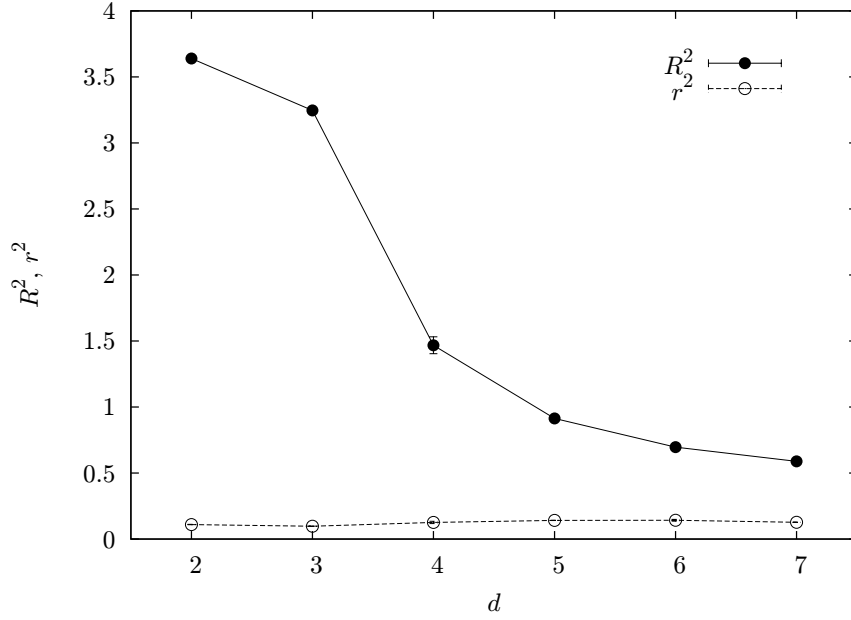
Figure 6: The extent of space-time R^2 and r^2 in the extended and shrunken directions, respectively, are plotted against d after taking the average over the "physical solutions" for each d. We put error bars representing the mean square error when there are more than one physical solutions. The solid and dashed lines connecting the data points are drawn to guide the eye.
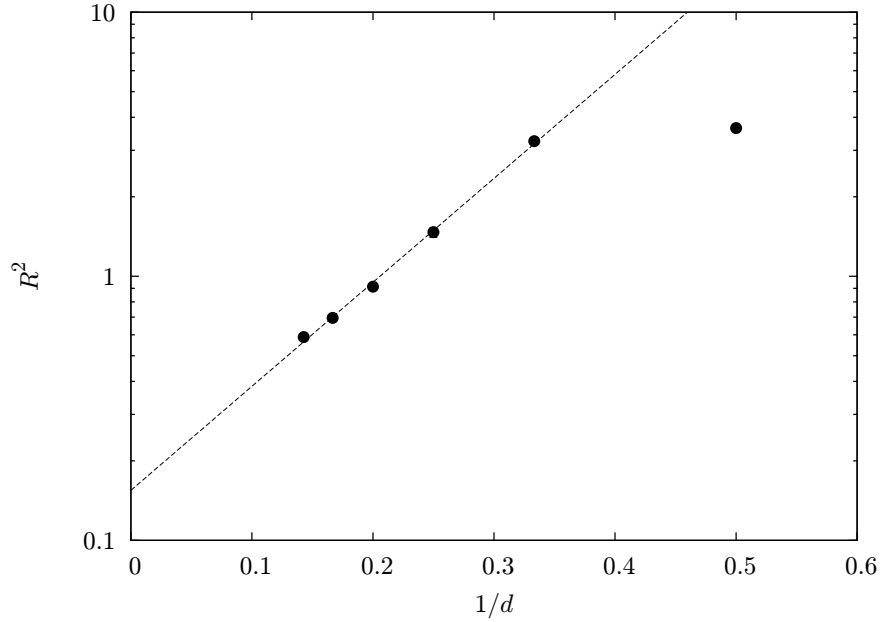

Figure 7: The extent of space-time R^2 in the extended directions shown in fig. 6 is plotted in the log scale against $1/d$. The straight dashed line represents a fit to the behavior (4.2) excluding $d = 2$, from which we obtain $\tilde{r}^2 = 0.155$ and $\ell^2 = 0.383$.

From the constant-volume property, we obtain

$$\log R^2 \approx \frac{10}{d} \log \left(\frac{\ell^2}{\tilde{r}^2} \right) + \log \tilde{r}^2 \ . \tag{4.2}$$

In fig. 7 we therefore plot R^2 in the log scale against $1/d$. Indeed we find that the results can be fitted to a straight line except for $d = 2$. From the fit, we obtain $\ell^2 = 0.383$ and $\tilde{r}^2 = 0.155$. Note that the obtained value for $\tilde{r}^2$ is consistent with r^2 obtained directly by the Gaussian expansion method as the extent of space-time in the shrunken directions. The obtained value for ℓ^2 is reproduced by Monte Carlo simulation [16].

5. Summary and discussions

In this paper we have applied the Gaussian expansion method to the type IIB matrix model, which was conjectured to be a nonperturbative formulation of type IIB superstring theory in ten dimensions. In particular, we have investigated the dynamical properties of the model associated with the SSB of SO(10), which was speculated to realize the dynamical compactification. Unlike previous works, we studied the SO(d) symmetric vacua for $2 \le d \le 7$ systematically. Moreover, we were able to observe clear indication of plateaus, which is crucial for the method to be reliable. This was made possible by allowing various extra symmetries in the shrunken directions as suggested by the success of a similar work on the six-dimensional version of the model.

Indeed our results bear surprising similarity to the results for the 6d case. The free energy was found to decrease as d is lowered until one reaches $d = 3$, and then it becomes much larger for $d = 2$. This implies that the SO(10) symmetry is spontaneously broken down to SO(3). The "compactification" scale is universal for all d. The extent of space-time obeys the constant-volume property for $d \ge 3$, while the volume for $d = 2$ is a bit smaller. All these properties are common to the 6d and 10d cases, and they can be understood intuitively as follows. (See ref. [14] for more detailed discussions.)

First of all, the long-distance effective theory of these models can be obtained by integrating out the off-diagonal elements of the matrices perturbatively [4, 15]. One then finds that the effective theory for the diagonal elements is described by a branched-polymer-like system, where the diagonal elements of the bosonic matrices A_μ are identified with the coordinates of the vertices in the polymer. This naturally explains the constant-volume property since the branched polymer tends to occupy a fixed volume for entropic reasons. The anomaly for $d = 2$ can be understood as a consequence of the fact that the fermion determinant disfavors two-dimensional configurations [17]. The volume that appears in the description of the constant-volume property can be calculated by Monte Carlo simulation, and it agrees quite well with the value suggested by the Gaussian expansion method for both 6d and 10d cases [16].

The driving force for the SSB comes from the phase of the fermion determinant [17] as has been clearly demonstrated in a simplified non-supersymmetric model [25], which was studied by both the Gaussian expansion method [26] and a Monte Carlo method [27], giving consistent results. There the free energy was determined by subtle competition between the

effect of the phase, which favors smaller d, and the dynamics of the phase-quenched model, which favors larger d. In the present supersymmetric matrix models, the latter effect is suppressed by $1/N$ [16, 24], and the free energy is determined essentially by the effect of the phase. This explains the monotonic decrease of the free energy for smaller d. The anomaly for $d = 2$ can be understood again from the property of the fermion determinant.

The extent in the shrunken directions is determined by the fluctuation of the off-diagonal elements since the diagonal elements cannot fluctuate in the shrunken directions due to the strong effect of the phase. At the leading order of the long-distance approximation, the dynamics of the off-diagonal elements is given by the Gaussian action, which couples weakly to the dynamics of the diagonal elements. This explains the universal compactification scale.

Since our results are obtained at the 3rd order of the Gaussian expansion, it remains to be seen whether the main conclusions change qualitatively at higher orders. In fact the 6d case has been studied both at the 3rd and 5th orders. The results did change slightly, but the qualitative features seemed to be robust. It is nonetheless desirable to perform the 5th order calculations in the 10d case as well. It would be also important to perform Monte Carlo studies analogous to [27] and to see whether the properties suggested by our work can be reproduced.

Our calculations clearly demonstrate the SSB of SO(10), which is itself an interesting dynamical property of the type IIB matrix model. However, in view of the scenario for dynamical compactification, there are two problems. One is that it seems more likely that the minimum of free energy occurs at $d = 3$ instead of $d = 4$, at least within the approximation of the present work. The other is that the ratio of the extent of space-time in the extended directions and that in the shrunken directions is finite for all d, as suggested by the constant volume property and the universal compactification scale. We consider that the nontrivial dynamics of the type IIB matrix model suggested by the present work confirms the validity of the basic idea to use matrices as the microscopic degrees of freedom in formulating superstring theory nonperturbatively. The detailed properties of the space-time emerging from the model, however, suggest that it still lacks some important ingredient in order to be capable of describing our real world.

Acknowledgments

A. Details of the ansatz

In this section we describe the details of the ansatz, which are listed in Table 1. We exhaust all the symmetries $SO(d) \times \Sigma_d$ with $2 \leq d \leq 7$, which reduce the number of parameters

in the Gaussian action to 5 or less. We also describe the general form of the moment of inertia tensor (2.5) for each ansatz.

SO(7) ansatz

- SO(7)

  5 parameters $(M, M_8, M_9, M_{10}, \tilde{m})$

$$M_\mu = (\underbrace{M, \cdots, M}_{7}, M_8, M_9, M_{10}) , \quad m_{8,9,10} = \tilde{m} , \quad \text{and zero otherwise.} \quad \text{(A.1)}$$

  The moment of inertia tensor takes the form

$$\langle T_{\mu\nu} \rangle = \begin{pmatrix} \overline{\mathrm{SO}(7)\,\mathrm{part}} & \\ & \begin{array}{ccc} C & \times & \times \\ \times & C' & \times \\ \times & \times & C'' \end{array} \end{pmatrix} . \quad \text{(A.2)}$$

  Here and henceforth "$\times$" represents a component which is found to vanish up to order 3 by explicit calculation, although there is no such symmetry that enforces it.

  By further imposing the symmetry $\mathbb{Z}_{3(8,9,10)}$, which enforces $M_8 = M_9 = M_{10}$, one obtains the "SO(7) ansatz" used in refs. [6, 7, 8, 9, 10, 11].

SO(6) ansatz

- SO(6) $\times \mathbb{Z}_{4(7,8,9,10|1)}$

  3 parameters $(M, \tilde{M}, \tilde{m})$

$$M_\mu = (\underbrace{M, \cdots, M}_{6}, \tilde{M}, \tilde{M}, \tilde{M}, \tilde{M}) ,$$

$$m_{7,8,9} = m_{8,9,10} = m_{7,9,10} = m_{7,8,10} = \tilde{m} , \quad \text{and zero otherwise.} \quad \text{(A.3)}$$

  The moment of inertia tensor takes the form

$$\langle T_{\mu\nu} \rangle = \begin{pmatrix} \overline{\mathrm{SO}(6)\,\mathrm{part}} & \\ & \begin{array}{cccc} C & \alpha & \alpha & \alpha \\ \alpha & C & \alpha & \alpha \\ \alpha & \alpha & C & \alpha \\ \alpha & \alpha & \alpha & C \end{array} \end{pmatrix} . \quad \text{(A.4)}$$

  This ansatz is equivalent to the "SO(6) ansatz" used in ref. [6].

- SO(6) $\times \mathbb{Z}_{3(8,9,10)}$

  5 parameters $(M, \tilde{M}, \tilde{M}', \tilde{m}, \tilde{m}')$

$$M_\mu = (\underbrace{M, \cdots, M}_{6}, \tilde{M}, \tilde{M}', \tilde{M}', \tilde{M}') ,$$

$$m_{7,8,9} = m_{7,9,10} = -m_{7,8,10} = \tilde{m} , \quad m_{8,9,10} = \tilde{m}' , \quad \text{and zero otherwise.} \quad \text{(A.5)}$$

The moment of inertia tensor takes the form

$$\langle T_{\mu\nu}\rangle = \begin{pmatrix} \begin{array}{c|cccc} \text{SO(6) part} & & & & \\ \hline & C & \beta & \beta & \beta \\ & \beta & C' & \alpha & \alpha \\ & \beta & \alpha & C' & \alpha \\ & \beta & \alpha & \alpha & C' \end{array} \end{pmatrix} . \tag{A.6}$$

- $\mathrm{SO}(6) \times \mathbb{Z}_{2(7,8|1)}$

  5 parameters $(M, \tilde{M}, \tilde{M}', \tilde{M}'', \tilde{m})$

  $$M_\mu = (\underbrace{M, \cdots, M}_{6}, \tilde{M}, \tilde{M}, \tilde{M}', \tilde{M}'') \ ,$$

  $$m_{7,9,10} = m_{8,9,10} = \tilde{m} \ , \quad \text{and zero otherwise.} \tag{A.7}$$

  The moment of inertia tensor takes the form

$$\langle T_{\mu\nu}\rangle = \begin{pmatrix} \begin{array}{c|cc|c|c} \text{SO(6) part} & & & & \\ \hline & C & \alpha & \times & \times \\ & \alpha & C & \times & \times \\ \hline & \times & \times & C' & \times \\ \hline & \times & \times & \times & C'' \end{array} \end{pmatrix} . \tag{A.8}$$

SO(5) ansatz

- $\mathrm{SO}(5) \times \mathbb{Z}_{5(6,7,8,9,10)}$

  4 parameters $(M, \tilde{M}, \tilde{m}, \tilde{m}')$

  $$M_\mu = (\underbrace{M, \cdots, M}_{5}, \underbrace{\tilde{M}, \cdots, \tilde{M}}_{5}) \ ,$$

  $$m_{6,7,8} = m_{7,8,9} = m_{8,9,10} = m_{6,9,10} = m_{6,7,10} = \tilde{m} \ ,$$

  $$m_{6,7,9} = m_{7,8,10} = m_{6,8,9} = m_{7,9,10} = m_{6,8,10} = \tilde{m}' \ , \quad \text{and zero otherwise.} \tag{A.9}$$

  The moment of inertia tensor takes the form

$$\langle T_{\mu\nu}\rangle = \begin{pmatrix} \begin{array}{c|ccccc} \text{SO(5) part} & & & & & \\ \hline & C & \alpha & \alpha & \alpha & \alpha \\ & \alpha & C & \alpha & \alpha & \alpha \\ & \alpha & \alpha & C & \alpha & \alpha \\ & \alpha & \alpha & \alpha & C & \alpha \\ & \alpha & \alpha & \alpha & \alpha & C \end{array} \end{pmatrix} . \tag{A.10}$$

- $\mathrm{SO}(5) \times \mathbb{Z}_{4(6,7,8,9|1)}$

5 parameters $(M, \tilde{M}, \tilde{M}', \tilde{m}, \tilde{m}')$

$$M_\mu = (\underbrace{M, \cdots, M}_{5}, \tilde{M}, \tilde{M}, \tilde{M}, \tilde{M}, \tilde{M}') \ ,$$

$$m_{6,7,8} = m_{7,8,9} = m_{6,8,9} = m_{6,7,9} = \tilde{m} \ ,$$

$$m_{6,7,10} = m_{7,8,10} = m_{8,9,10} = -m_{6,9,10} = \tilde{m}' \ , \quad \text{and zero otherwise.} \qquad (A.11)$$

The moment of inertia tensor takes the form

$$\langle T_{\mu\nu} \rangle = \begin{pmatrix} \begin{array}{c|cccc|c} \text{SO(5) part} & & & & & \\ \hline & C & \alpha & \alpha & \alpha & \beta \\ & \alpha & C & \alpha & \alpha & \beta \\ & \alpha & \alpha & C & \alpha & \beta \\ & \alpha & \alpha & \alpha & C & \beta \\ \hline & \beta & \beta & \beta & \beta & C' \end{array} \end{pmatrix} . \qquad (A.12)$$

- $SO(5) \times \mathbb{Z}_{4(6,7,8,9|10)}$

  5 parameters $(M, \tilde{M}, \tilde{M}', \tilde{m}, \tilde{m}')$

$$M_\mu = (\underbrace{M, \cdots, M}_{5}, \tilde{M}, \tilde{M}, \tilde{M}, \tilde{M}, \tilde{M}') \ ,$$

$$m_{6,7,8} = m_{7,8,9} = m_{6,8,9} = m_{6,7,9} = \tilde{m} \ ,$$

$$m_{6,7,10} = -m_{7,8,10} = m_{8,9,10} = m_{6,9,10} = \tilde{m}' \ , \quad \text{and zero otherwise.} \qquad (A.13)$$

The moment of inertia tensor takes the form

$$\langle T_{\mu\nu} \rangle = \begin{pmatrix} \begin{array}{c|cccc|c} \text{SO(5) part} & & & & & \\ \hline & C & \alpha & \alpha & \alpha & 0 \\ & \alpha & C & \alpha & \alpha & 0 \\ & \alpha & \alpha & C & \alpha & 0 \\ & \alpha & \alpha & \alpha & C & 0 \\ \hline & 0 & 0 & 0 & 0 & C' \end{array} \end{pmatrix} . \qquad (A.14)$$

- $SO(5) \times \mathbb{Z}_{2(6,7|1)} \times \mathbb{Z}_{3(8,9,10)}$

  5 parameters $(M, \tilde{M}, \tilde{M}', \tilde{m}, \tilde{m}')$

$$M_\mu = (\underbrace{M, \cdots, M}_{5}, \tilde{M}, \tilde{M}, \tilde{M}', \tilde{M}', \tilde{M}') \ ,$$

$$m_{6,8,9} = m_{6,9,10} = -m_{6,8,10} = m_{7,8,9} = m_{7,9,10} = -m_{7,8,10} = \tilde{m} \ ,$$

$$m_{8,9,10} = \tilde{m}' \ , \quad \text{and zero otherwise.} \qquad (A.15)$$

The moment of inertia tensor takes the form

$$\langle T_{\mu\nu}\rangle = \begin{pmatrix} \begin{array}{c|cc|ccc} \text{SO(5) part} & & & & & \\ \hline & C & \alpha & \beta & \beta & \beta \\ & \alpha & C & \beta & \beta & \beta \\ \hline & \beta & \beta & C' & \gamma & \gamma \\ & \beta & \beta & \gamma & C' & \gamma \\ & \beta & \beta & \gamma & \gamma & C' \end{array} \end{pmatrix} . \tag{A.16}$$

- $\text{SO}(5) \times \mathbb{Z}_{2(6,7|8,9,10)} \times \mathbb{Z}_{3(8,9,10)}$

  5 parameters $(M, \tilde{M}, \tilde{M}', \tilde{m}, \tilde{m}')$

$$M_\mu = (\underbrace{M, \cdots, M}_{5}, \tilde{M}, \tilde{M}, \tilde{M}', \tilde{M}', \tilde{M}') ,$$

$$m_{6,8,9} = m_{7,8,9} = m_{6,9,10} = m_{7,9,10} = -m_{6,8,10} = -m_{7,8,10} = \tilde{m} ,$$

$$m_{6,7,8} = m_{6,7,9} = m_{6,7,10} = \tilde{m}' , \quad \text{and zero otherwise.} \tag{A.17}$$

The moment of inertia tensor takes the form

$$\langle T_{\mu\nu}\rangle = \begin{pmatrix} \begin{array}{c|cc|ccc} \text{SO(5) part} & & & & & \\ \hline & C & \alpha & 0 & 0 & 0 \\ & \alpha & C & 0 & 0 & 0 \\ \hline & 0 & 0 & C' & \beta & \beta \\ & 0 & 0 & \beta & C' & \beta \\ & 0 & 0 & \beta & \beta & C' \end{array} \end{pmatrix} . \tag{A.18}$$

- $\text{SO}(5) \times \mathbb{Z}_{2(6,7|1)} \times \mathbb{Z}_{2(8,9|1)}$

  5 parameters $(M, \tilde{M}, \tilde{M}', \tilde{M}'', \tilde{m})$

$$M_\mu = (\underbrace{M, \cdots, M}_{5}, \tilde{M}, \tilde{M}, \tilde{M}', \tilde{M}', \tilde{M}'') ,$$

$$m_{6,8,10} = m_{7,8,10} = m_{6,9,10} = m_{7,9,10} = \tilde{m} , \quad \text{and zero otherwise.} \tag{A.19}$$

The moment of inertia tensor takes the form

$$\langle T_{\mu\nu}\rangle = \begin{pmatrix} \begin{array}{c|cc|cc|c} \text{SO(5) part} & & & & & \\ \hline & C & \alpha & \times & \times & \times \\ & \alpha & C & \times & \times & \times \\ \hline & \times & \times & C' & \beta & \times \\ & \times & \times & \beta & C' & \times \\ \hline & \times & \times & \times & \times & C'' \end{array} \end{pmatrix} . \tag{A.20}$$

- $\text{SO}(5) \times \mathbb{Z}_{2(6,7|10)} \times \mathbb{Z}_{2(8,9|10)} \times \mathbb{Z}_{2(\{6,7\},\{8,9\}|1,10)}$

5 parameters $(M, \tilde{M}, \tilde{M}', \tilde{m}, \tilde{m}')$

$$M_\mu = (\underbrace{M, \cdots, M}_{5}, \tilde{M}, \tilde{M}, \tilde{M}, \tilde{M}, \tilde{M}') \ ,$$

$$m_{6,8,10} = -m_{7,8,10} = -m_{6,9,10} = m_{7,9,10} = \tilde{m} \ ,$$

$$m_{6,7,10} = -m_{8,9,10} = \tilde{m}' \ , \quad \text{and zero otherwise.} \tag{A.21}$$

The moment of inertia tensor takes the form

$$\langle T_{\mu\nu} \rangle = \begin{pmatrix} \begin{array}{c|cc|cc|c} \text{SO(5) part} & & & & & \\ \hline & C & \alpha & \times & \times & 0 \\ & \alpha & C & \times & \times & 0 \\ \hline & \times & \times & C & \alpha & 0 \\ & \times & \times & \alpha & C & 0 \\ \hline & 0 & 0 & 0 & 0 & C' \end{array} \end{pmatrix} . \tag{A.22}$$

SO(4) ansatz

- SO(4) $\times \ \mathbb{Z}_{2(5,6|1)} \times \mathbb{Z}_{2(7,8|1)} \times \mathbb{Z}_{2(9,10|1)}$

  5 parameters $(M, \tilde{M}, \tilde{M}', \tilde{M}'', \tilde{m})$

$$M_\mu = (M, M, M, M, \tilde{M}, \tilde{M}, \tilde{M}', \tilde{M}', \tilde{M}'', \tilde{M}'') \ ,$$

$$m_{5,7,9} = m_{6,7,9} = m_{5,8,9} = m_{6,8,9} = m_{5,7,10} = m_{6,7,10}$$

$$= m_{5,8,10} = m_{6,8,10} = \tilde{m} \ , \quad \text{and zero otherwise.} \tag{A.23}$$

The moment of inertia tensor takes the form

$$\langle T_{\mu\nu} \rangle = \begin{pmatrix} \begin{array}{c|cc|cc|cc} \text{SO(4) part} & & & & & & \\ \hline & C & \alpha & \times & \times & \times & \times \\ & \alpha & C & \times & \times & \times & \times \\ \hline & \times & \times & C' & \beta & \times & \times \\ & \times & \times & \beta & C' & \times & \times \\ \hline & \times & \times & \times & \times & C'' & \gamma \\ & \times & \times & \times & \times & \gamma & C'' \end{array} \end{pmatrix} . \tag{A.24}$$

By further imposing $\mathbb{Z}_{3(\{5,6\},\{7,8\},\{9,10\})}$, which enforces $\tilde{M} = \tilde{M}' = \tilde{M}''$, one obtains the "SO(4) ansatz" used in ref. [6].

- SO(4) $\times \ \mathbb{Z}_{3(5,6,7)} \times \mathbb{Z}_{3(8,9,10)} \times \mathbb{Z}_{2(\{5,6,7\},\{8,9,10\}|1)}$

4 parameters $(M, \tilde{M}, \tilde{m}, \tilde{m}')$

$$M_\mu = (M, M, M, M, \underbrace{\tilde{M}, \cdots, \tilde{M}}_{6}) \ ,$$

$$m_{5,6,7} = m_{8,9,10} = \tilde{m} \ ,$$

$$m_{5,6,8} = m_{5,6,9} = m_{5,6,10} = m_{6,7,8} = m_{6,7,9} = m_{6,7,10}$$

$$= -m_{5,7,8} = -m_{5,7,9} = -m_{5,7,10} = m_{5,8,9} = m_{6,8,9} = m_{7,8,9}$$

$$= m_{5,9,10} = m_{6,9,10} = m_{7,9,10} = -m_{5,8,10} = -m_{6,8,10} = -m_{7,8,10} = \tilde{m}' \ ,$$

$$\text{and zero otherwise.} \tag{A.25}$$

The moment of inertia tensor takes the form

$$\langle T_{\mu\nu} \rangle = \left(\begin{array}{c|c} \text{SO(4) part} & \\ \hline & \begin{array}{ccc|ccc} C & \alpha & \alpha & \beta & \beta & \beta \\ \alpha & C & \alpha & \beta & \beta & \beta \\ \alpha & \alpha & C & \beta & \beta & \beta \\ \hline \beta & \beta & \beta & C & \alpha & \alpha \\ \beta & \beta & \beta & \alpha & C & \alpha \\ \beta & \beta & \beta & \alpha & \alpha & C \end{array} \end{array} \right) \ . \tag{A.26}$$

By imposing $\text{SO}(3) \times \text{SO}(3)$ instead of $\mathbb{Z}_3 \times \mathbb{Z}_3$, which enforces $\tilde{m}' = 0$, one obtains the "SO(4) ansatz" used in refs. [7, 8, 9, 10, 11].

- $\text{SO}(4) \times \mathbb{Z}_{4(5,6,7,8|1)} \times \mathbb{Z}_{2(9,10|1)}$

  5 parameters $(M, \tilde{M}, \tilde{M}', \tilde{m}, \tilde{m}')$

$$M_\mu = (M, M, M, M, \tilde{M}, \tilde{M}, \tilde{M}, \tilde{M}, \tilde{M}', \tilde{M}') \ ,$$

$$m_{5,6,9} = m_{6,7,9} = m_{7,8,9} = -m_{5,8,9} = m_{5,6,10} = m_{6,7,10} = m_{7,8,10} = -m_{5,8,10} = \tilde{m} \ ,$$

$$m_{5,6,7} = m_{6,7,8} = m_{5,7,8} = m_{5,6,8} = \tilde{m}' \ , \quad \text{and zero otherwise.} \tag{A.27}$$

The moment of inertia tensor takes the form

$$\langle T_{\mu\nu} \rangle = \left(\begin{array}{c|c} \text{SO(4) part} & \\ \hline & \begin{array}{cccc|cc} C & \alpha & \alpha & \alpha & \beta & \beta \\ \alpha & C & \alpha & \alpha & \beta & \beta \\ \alpha & \alpha & C & \alpha & \beta & \beta \\ \alpha & \alpha & \alpha & C & \beta & \beta \\ \hline \beta & \beta & \beta & \beta & C' & \gamma \\ \beta & \beta & \beta & \beta & \gamma & C' \end{array} \end{array} \right) \ . \tag{A.28}$$

- $\text{SO}(4) \times \mathbb{Z}_{5(6,7,8,9,10)} \times \mathbb{Z}_{2(-|1,6,7,8,9,10)}$

  5 parameters $(M, \tilde{M}, \tilde{M}', \tilde{m}, \tilde{m}')$

$$M_\mu = (M, M, M, M, \tilde{M}, \underbrace{\tilde{M}', \cdots, \tilde{M}'}_{5}) \ ,$$

$$m_{5,6,7} = m_{5,7,8} = m_{5,8,9} = m_{5,9,10} = -m_{5,6,10} = \tilde{m} \ ,$$

$$m_{5,6,8} = m_{5,7,9} = m_{5,8,10} = -m_{5,6,9} = -m_{5,7,10} = \tilde{m}' \ , \quad \text{and zero otherwise.} \tag{A.29}$$

The moment of inertia tensor takes the form

$$\langle T_{\mu\nu}\rangle = \begin{pmatrix} \begin{array}{c|c} \text{SO(4) part} & \\ \hline & \begin{array}{c|ccccc} C & \times & \times & \times & \times & \times \\ \hline \times & C' & \alpha & \alpha & \alpha & \alpha \\ \times & \alpha & C' & \alpha & \alpha & \alpha \\ \times & \alpha & \alpha & C' & \alpha & \alpha \\ \times & \alpha & \alpha & \alpha & C' & \alpha \\ \times & \alpha & \alpha & \alpha & \alpha & C' \end{array} \end{array} \end{pmatrix}. \tag{A.30}$$

- $\text{SO}(4) \times \mathbb{Z}_{3(5,6,7)} \times \mathbb{Z}_{3(8,9,10)} \times \mathbb{Z}_{2(-|1,8,9,10)}$

  5 parameters $(M, \tilde{M}, \tilde{M}', \tilde{m}, \tilde{m}')$

$$M_\mu = (M, M, M, M, \tilde{M}, \tilde{M}, \tilde{M}, \tilde{M}', \tilde{M}', \tilde{M}') \ ,$$

$$m_{5,8,9} = m_{5,9,10} = -m_{5,8,10} = m_{6,8,9} = m_{6,9,10} = -m_{6,8,10}$$

$$= m_{7,8,9} = m_{7,9,10} = -m_{7,8,10} = \tilde{m} \ ,$$

$$m_{5,6,7} = \tilde{m}' \ , \quad \text{and zero otherwise.} \tag{A.31}$$

The moment of inertia tensor takes the form

$$\langle T_{\mu\nu}\rangle = \begin{pmatrix} \begin{array}{c|c|c} \text{SO(4) part} & & \\ \hline & \begin{array}{ccc} C & \alpha & \alpha \\ \alpha & C & \alpha \\ \alpha & \alpha & C \end{array} & \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \\ \hline & \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} & \begin{array}{ccc} C' & \beta & \beta \\ \beta & C' & \beta \\ \beta & \beta & C' \end{array} \end{array} \end{pmatrix}. \tag{A.32}$$

SO(3) ansatz

- $\text{SO}(3) \times \mathbb{Z}_{2(4,5|1)} \times \mathbb{Z}_{2(6,7|1)} \times \mathbb{Z}_{2(8,9|1)} \times \mathbb{Z}_{3(\{4,5\},\{6,7\},\{8,9\})}$

  5 parameters $(M, \tilde{M}, \tilde{M}', \tilde{m}, \tilde{m}')$

$$M_\mu = (M, M, M, \underbrace{\tilde{M}, \cdots, \tilde{M}}_{6}, \tilde{M}') \ ,$$

$$m_{4,6,8} = m_{5,6,8} = m_{4,7,8} = m_{5,7,8} = m_{4,6,9} = m_{5,6,9} = m_{4,7,9} = m_{5,7,9} = \tilde{m} \ ,$$

$$m_{4,6,10} = m_{5,6,10} = m_{4,7,10} = m_{5,7,10} = -m_{4,8,10} = -m_{5,8,10}$$

$$= -m_{4,9,10} = -m_{5,9,10} = m_{6,8,10} = m_{7,8,10} = m_{6,9,10} = m_{7,9,10} = \tilde{m}' \ ,$$

$$\text{and zero otherwise.} \tag{A.33}$$

The moment of inertia tensor takes the form

$$
\langle T_{\mu\nu}\rangle =
\left(
\begin{array}{c|cc|cc|cc|c}
\text{SO(3) part} & & & & & & & \\
\hline
& C & \alpha & \beta & \beta & \beta & \beta & \gamma \\
& \alpha & C & \beta & \beta & \beta & \beta & \gamma \\
\hline
& \beta & \beta & C & \alpha & \beta & \beta & \gamma \\
& \beta & \beta & \alpha & C & \beta & \beta & \gamma \\
\hline
& \beta & \beta & \beta & \beta & C & \alpha & \gamma \\
& \beta & \beta & \beta & \beta & \alpha & C & \gamma \\
\hline
& \gamma & \gamma & \gamma & \gamma & \gamma & \gamma & C'
\end{array}
\right). \tag{A.34}
$$

• $\mathrm{SO(3)} \times \mathbb{Z}_{2(4,5|10)} \times \mathbb{Z}_{2(6,7|10)} \times \mathbb{Z}_{2(8,9|10)} \times \mathbb{Z}_{3(\{4,5\},\{6,7\},\{8,9\})} \times \mathbb{Z}_{2(\{4,5\},\{6,7\})}$

  5 parameters $(M, \tilde{M}, \tilde{M}', m, \tilde{m})$

$$
M_\mu = (M, M, M, \underbrace{\tilde{M}, \cdots, \tilde{M}}_{6}, \tilde{M}') \ ,
$$

$$
m_{1,2,3} = m \ , \quad m_{4,5,10} = m_{6,7,10} = m_{8,9,10} = \tilde{m} \ , \quad \text{and zero otherwise.} \tag{A.35}
$$

The moment of inertia tensor takes the form

$$
\langle T_{\mu\nu}\rangle =
\left(
\begin{array}{c|cc|cc|cc|c}
\text{SO(3) part} & & & & & & & \\
\hline
& C & \times & \times & \times & \times & \times & 0 \\
& \times & C & \times & \times & \times & \times & 0 \\
\hline
& \times & \times & C & \times & \times & \times & 0 \\
& \times & \times & \times & C & \times & \times & 0 \\
\hline
& \times & \times & \times & \times & C & \times & 0 \\
& \times & \times & \times & \times & \times & C & 0 \\
\hline
& 0 & 0 & 0 & 0 & 0 & 0 & C'
\end{array}
\right). \tag{A.36}
$$

• $\mathrm{SO(3)} \times \mathbb{Z}_{2(4,5|10)} \times \mathbb{Z}_{2(6,7|10)} \times \mathbb{Z}_{2(8,9|10)} \times \mathbb{Z}_{3(\{4,5\},\{6,7\},\{8,9\})} \times \mathbb{Z}_{2(-|1,10)}$

  4 parameters $(M, \tilde{M}, \tilde{M}', \tilde{m})$

$$
M_\mu = (M, M, M, \underbrace{\tilde{M}, \cdots, \tilde{M}}_{6}, \tilde{M}') \ ,
$$

$$
m_{4,6,8} = m_{5,6,8} = m_{4,7,8} = m_{5,7,8} = m_{4,6,9} = m_{5,6,9} = m_{4,7,9} = m_{5,7,9} = \tilde{m} \ ,
$$
$$
\text{and zero otherwise.} \tag{A.37}
$$

The moment of inertia tensor takes the form

$$
\langle T_{\mu\nu}\rangle =
\left(
\begin{array}{c|cc|cc|cc|c}
\text{SO(3) part} & & & & & & & \\
\hline
& C & \alpha & \times & \times & \times & \times & 0 \\
& \alpha & C & \times & \times & \times & \times & 0 \\
\hline
& \times & \times & C & \alpha & \times & \times & 0 \\
& \times & \times & \alpha & C & \times & \times & 0 \\
\hline
& \times & \times & \times & \times & C & \alpha & 0 \\
& \times & \times & \times & \times & \alpha & C & 0 \\
\hline
& 0 & 0 & 0 & 0 & 0 & 0 & C'
\end{array}
\right). \tag{A.38}
$$

- $\mathrm{SO}(3) \times \mathbb{Z}_{3(4,5,6)} \times \mathbb{Z}_{3(7,8,9)} \times \mathbb{Z}_{2(\{4,5,6\},\{7,8,9\}|1)} \times \mathbb{Z}_{2(-|1,10)}$

  5 parameters $(M, \tilde{M}, \tilde{M}', \tilde{m}, \tilde{m}')$

$$M_\mu = (M, M, M, \underbrace{\tilde{M}, \cdots, \tilde{M}}_{6}, \tilde{M}') \ ,$$

$$m_{4,5,\nu} = m_{5,6,\nu} = -m_{4,6,\nu} = m_{7,8,\mu} = m_{8,9,\mu} = -m_{7,9,\mu} = \tilde{m} \ ,$$
$$\text{where } \mu = 4, 5, 6 \text{ and } \nu = 7, 8, 9 \ ,$$
$$m_{4,5,6} = m_{7,8,9} = \tilde{m}' \ , \quad \text{and zero otherwise.} \tag{A.39}$$

  The moment of inertia tensor takes the form

$$\langle T_{\mu\nu} \rangle = \left(
\begin{array}{c|ccc|ccc|c}
\text{SO(3) part} & & & & & & & \\
\hline
& C & \alpha & \alpha & \beta & \beta & \beta & 0 \\
& \alpha & C & \alpha & \beta & \beta & \beta & 0 \\
& \alpha & \alpha & C & \beta & \beta & \beta & 0 \\
\hline
& \beta & \beta & \beta & C & \alpha & \alpha & 0 \\
& \beta & \beta & \beta & \alpha & C & \alpha & 0 \\
& \beta & \beta & \beta & \alpha & \alpha & C & 0 \\
\hline
& 0 & 0 & 0 & 0 & 0 & 0 & C'
\end{array}
\right) . \tag{A.40}$$

- $\mathrm{SO}(3) \times \mathbb{Z}_{3(4,5,6)} \times \mathbb{Z}_{2(7,8|1)} \times \mathbb{Z}_{2(9,10|1)} \times \mathbb{Z}_{2(\{7,8\},\{9,10\})}$

  5 parameters $(M, \tilde{M}, \tilde{M}', \tilde{m}, \tilde{m}')$

$$M_\mu = (M, M, M, \tilde{M}, \tilde{M}, \tilde{M}, \tilde{M}', \tilde{M}', \tilde{M}', \tilde{M}') \ ,$$

$$m_{4,5,7} = m_{4,5,8} = m_{5,6,7} = m_{5,6,8} = -m_{4,6,7} = -m_{4,6,8}$$
$$= m_{4,5,9} = m_{4,5,10} = m_{5,6,9} = m_{5,6,10} = -m_{4,6,9} = -m_{4,6,10} = \tilde{m} \ ,$$
$$m_{4,5,6} = \tilde{m}' \ , \quad \text{and zero otherwise.} \tag{A.41}$$

  The moment of inertia tensor takes the form

$$\langle T_{\mu\nu} \rangle = \left(
\begin{array}{c|ccc|cc|cc}
\text{SO(3) part} & & & & & & & \\
\hline
& C & \alpha & \alpha & \gamma & \gamma & \gamma & \gamma \\
& \alpha & C & \alpha & \gamma & \gamma & \gamma & \gamma \\
& \alpha & \alpha & C & \gamma & \gamma & \gamma & \gamma \\
\hline
& \gamma & \gamma & \gamma & C' & \beta & \beta & \beta \\
& \gamma & \gamma & \gamma & \beta & C' & \beta & \beta \\
\hline
& \gamma & \gamma & \gamma & \beta & \beta & C' & \beta \\
& \gamma & \gamma & \gamma & \beta & \beta & \beta & C'
\end{array}
\right) . \tag{A.42}$$

- $\mathrm{SO}(3) \times \mathbb{Z}_{3(4,5,6)} \times \mathbb{Z}_{4(7,8,9,10|1)} \times \mathbb{Z}_{2(-|7,8,9,10)}$

  5 parameters $(M, \tilde{M}, \tilde{M}', \tilde{m}, \tilde{m}')$

$$M_\mu = (M, M, M, \tilde{M}, \tilde{M}, \tilde{M}, \tilde{M}', \tilde{M}', \tilde{M}', \tilde{M}') \ ,$$

$$m_{4,7,8} = m_{4,8,9} = m_{4,9,10} = -m_{4,7,10} = m_{5,7,8} = m_{5,8,9}$$
$$= m_{5,9,10} = -m_{5,7,10} = m_{6,7,8} = m_{6,8,9} = m_{6,9,10} = -m_{6,7,10} = \tilde{m} \ ,$$
$$m_{4,5,6} = \tilde{m}' \ , \quad \text{and zero otherwise.} \tag{A.43}$$

The moment of inertia tensor takes the form

$$\langle T_{\mu\nu}\rangle =
\left(
\begin{array}{c|ccc|cccc}
\text{SO(3) part} & & & & & & & \\
\hline
 & C & \alpha & \alpha & 0 & 0 & 0 & 0 \\
 & \alpha & C & \alpha & 0 & 0 & 0 & 0 \\
 & \alpha & \alpha & C & 0 & 0 & 0 & 0 \\
\hline
 & 0 & 0 & 0 & C' & \times & \times & \times \\
 & 0 & 0 & 0 & \times & C' & \times & \times \\
 & 0 & 0 & 0 & \times & \times & C' & \times \\
 & 0 & 0 & 0 & \times & \times & \times & C'
\end{array}
\right) .
\tag{A.44}$$

- $\mathrm{SO}(3) \times \mathbb{Z}_{2(4,5|1)} \times \mathbb{Z}_{5(6,7,8,9,10)} \times \mathbb{Z}_{2(-|1,6,7,8,9,10)}$

  5 parameters $(M, \tilde{M}, \tilde{M}', \tilde{m}, \tilde{m}')$

$$M_\mu = (M, M, M, \tilde{M}, \tilde{M}, \underbrace{\tilde{M}', \cdots, \tilde{M}'}_{5}) ,$$

$$m_{4,6,7} = m_{4,7,8} = m_{4,8,9} = m_{4,9,10} = -m_{4,6,10}$$
$$= m_{5,6,7} = m_{5,7,8} = m_{5,8,9} = m_{5,9,10} = -m_{5,6,10} = \tilde{m} ,$$
$$m_{4,6,8} = m_{4,7,9} = m_{4,8,10} = -m_{4,6,9} = -m_{4,7,10}$$
$$= m_{5,6,8} = m_{5,7,9} = m_{5,8,10} = -m_{5,6,9} = -m_{5,7,10} = \tilde{m}' ,$$
$$\text{and zero otherwise.}
\tag{A.45}$$

The moment of inertia tensor takes the form

$$\langle T_{\mu\nu}\rangle =
\left(
\begin{array}{c|cc|ccccc}
\text{SO(3) part} & & & & & & & \\
\hline
 & C & \alpha & 0 & 0 & 0 & 0 & 0 \\
 & \alpha & C & 0 & 0 & 0 & 0 & 0 \\
\hline
 & 0 & 0 & C' & \beta & \beta & \beta & \beta \\
 & 0 & 0 & \beta & C' & \beta & \beta & \beta \\
 & 0 & 0 & \beta & \beta & C' & \beta & \beta \\
 & 0 & 0 & \beta & \beta & \beta & C' & \beta \\
 & 0 & 0 & \beta & \beta & \beta & \beta & C'
\end{array}
\right) .
\tag{A.46}$$

- $\mathrm{SO}(3) \times \mathbb{Z}_{6(5,6,7,8,9,10|1)} \times \mathbb{Z}_{2(-|5,6,7,8,9,10)}$

  5 parameters $(M, \tilde{M}, \tilde{M}', \tilde{m}, \tilde{m}')$

$$M_\mu = (M, M, M, \tilde{M}, \underbrace{\tilde{M}', \cdots, \tilde{M}'}_{6}) ,$$

$$m_{4,5,6} = m_{4,6,7} = m_{4,7,8} = m_{4,8,9} = m_{4,9,10} = -m_{4,5,10} = \tilde{m} ,$$
$$m_{4,5,7} = m_{4,6,8} = m_{4,7,9} = m_{4,8,10} = -m_{4,5,9} = -m_{4,6,10} = \tilde{m}' ,$$
$$\text{and zero otherwise.}
\tag{A.47}$$

The moment of inertia tensor takes the form

$$\langle T_{\mu\nu} \rangle = \begin{pmatrix} \begin{array}{c|c} \text{SO(3) part} & \\ \hline \end{array} & \\ & \begin{array}{c|cccccc} C & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & C' & \alpha & \alpha & \alpha & \alpha & \alpha \\ 0 & \alpha & C' & \alpha & \alpha & \alpha & \alpha \\ 0 & \alpha & \alpha & C' & \alpha & \alpha & \alpha \\ 0 & \alpha & \alpha & \alpha & C' & \alpha & \alpha \\ 0 & \alpha & \alpha & \alpha & \alpha & C' & \alpha \\ 0 & \alpha & \alpha & \alpha & \alpha & \alpha & C' \end{array} \end{pmatrix} . \tag{A.48}$$

- $\mathrm{SO}(3) \times \mathbb{Z}_{2(4,5|1)} \times \mathbb{Z}_{5(6,7,8,9,10|4,5)}$

  5 parameters $(M, \tilde{M}, \tilde{M}', \tilde{m}, \tilde{m}')$

$$M_\mu = (M, M, M, \tilde{M}, \tilde{M}, \underbrace{\tilde{M}', \cdots, \tilde{M}'}_{5}) \ ,$$

$$m_{6,7,8} = m_{7,8,9} = m_{8,9,10} = m_{6,9,10} = m_{6,7,10} = \tilde{m} \ ,$$

$$m_{6,7,9} = m_{7,8,10} = m_{6,8,9} = m_{7,9,10} = m_{6,8,10} = \tilde{m}' \ ,$$

$$\text{and zero otherwise.} \tag{A.49}$$

The moment of inertia tensor takes the form

$$\langle T_{\mu\nu} \rangle = \begin{pmatrix} \begin{array}{c|c} \text{SO(3) part} & \\ \hline \end{array} & \\ & \begin{array}{cc|ccccc} C & \times & 0 & 0 & 0 & 0 & 0 \\ \times & C & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & C' & \alpha & \alpha & \alpha & \alpha \\ 0 & 0 & \alpha & C' & \alpha & \alpha & \alpha \\ 0 & 0 & \alpha & \alpha & C' & \alpha & \alpha \\ 0 & 0 & \alpha & \alpha & \alpha & C' & \alpha \\ 0 & 0 & \alpha & \alpha & \alpha & \alpha & C' \end{array} \end{pmatrix} . \tag{A.50}$$

SO(2) ansatz

- $\mathrm{SO}(2) \times \mathbb{Z}_{2(3,4|1)} \times \mathbb{Z}_{2(5,6|1)} \times \mathbb{Z}_{2(7,8|1)} \times \mathbb{Z}_{2(9,10|1)} \times \mathbb{Z}_{4(\{3,4\},\{5,6\},\{7,8\},\{9,10\})}$

  3 parameters $(M, \tilde{M}, \tilde{m})$

$$M_\mu = (M, M, \underbrace{\tilde{M}, \cdots, \tilde{M}}_{8}) \ ,$$

$$m_{3,5,7} = m_{3,6,7} = m_{3,5,8} = m_{3,6,8} = m_{4,5,7} = m_{4,6,7} = m_{4,5,8} = m_{4,6,8}$$

$$= m_{5,7,9} = m_{5,8,9} = m_{5,7,10} = m_{5,8,10} = m_{6,7,9} = m_{6,8,9} = m_{6,7,10} = m_{6,8,10}$$

$$= m_{3,7,9} = m_{3,8,9} = m_{3,7,10} = m_{3,8,10} = m_{4,7,9} = m_{4,8,9} = m_{4,7,10} = m_{4,8,10}$$

$$= m_{3,5,9} = m_{3,6,9} = m_{3,5,10} = m_{3,6,10} = m_{4,5,9} = m_{4,6,9} = m_{4,5,10} = m_{4,6,10} = \tilde{m} \ ,$$

$$\text{and zero otherwise.} \tag{A.51}$$

The moment of inertia tensor takes the form

$$
\langle T_{\mu\nu}\rangle =
\left(
\begin{array}{c|cc|cc|cc|cc}
\text{SO(2) part} & & & & & & & & \\
\hline
& C & \alpha & \beta & \beta & \beta & \beta & \beta & \beta \\
& \alpha & C & \beta & \beta & \beta & \beta & \beta & \beta \\
\hline
& \beta & \beta & C & \alpha & \beta & \beta & \beta & \beta \\
& \beta & \beta & \alpha & C & \beta & \beta & \beta & \beta \\
\hline
& \beta & \beta & \beta & \beta & C & \alpha & \beta & \beta \\
& \beta & \beta & \beta & \beta & \alpha & C & \beta & \beta \\
\hline
& \beta & \beta & \beta & \beta & \beta & \beta & C & \alpha \\
& \beta & \beta & \beta & \beta & \beta & \beta & \alpha & C
\end{array}
\right) . \tag{A.52}
$$

This ansatz is equivalent to the "SO(2) ansatz" used in ref. [6].

- $SO(2) \times \mathbb{Z}_{4(3,4,5,6|1)} \times \mathbb{Z}_{4(7,8,9,10|1)} \times \mathbb{Z}_{2(\{3,4,5,6\},\{7,8,9,10\})}$

4 parameters $(M, \tilde{M}, \tilde{m}, \tilde{m}')$

$$
M_\mu = (M, M, \underbrace{\tilde{M}, \cdots, \tilde{M}}_{8}) \ ,
$$

$$
m_{3,4,5} = m_{4,5,6} = m_{3,5,6} = m_{3,4,6} = m_{7,8,9} = m_{8,9,10} = m_{7,9,10} = m_{7,8,10} = \tilde{m} \ ,
$$
$$
m_{3,4,7} = m_{4,5,7} = m_{5,6,7} = -m_{3,6,7} = m_{3,4,8} = m_{4,5,8} = m_{5,6,8} = -m_{3,6,8}
$$
$$
= m_{3,4,9} = m_{4,5,9} = m_{5,6,9} = -m_{3,6,9} = m_{3,4,10} = m_{4,5,10} = m_{5,6,10} = -m_{3,6,10}
$$
$$
= m_{3,7,8} = m_{3,8,9} = m_{3,9,10} = -m_{3,7,10} = m_{4,7,8} = m_{4,8,9} = m_{4,9,10} = -m_{4,7,10}
$$
$$
= m_{5,7,8} = m_{5,8,9} = m_{5,9,10} = -m_{5,7,10} = m_{6,7,8} = m_{6,8,9} = m_{6,9,10} = -m_{6,7,10} = \tilde{m}' \ ,
$$
$$
\text{and zero otherwise.} \tag{A.53}
$$

The moment of inertia tensor takes the form

$$
\langle T_{\mu\nu}\rangle =
\left(
\begin{array}{c|cccc|cccc}
\text{SO(2) part} & & & & & & & & \\
\hline
& C & \alpha & \alpha & \alpha & \beta & \beta & \beta & \beta \\
& \alpha & C & \alpha & \alpha & \beta & \beta & \beta & \beta \\
& \alpha & \alpha & C & \alpha & \beta & \beta & \beta & \beta \\
& \alpha & \alpha & \alpha & C & \beta & \beta & \beta & \beta \\
\hline
& \beta & \beta & \beta & \beta & C & \alpha & \alpha & \alpha \\
& \beta & \beta & \beta & \beta & \alpha & C & \alpha & \alpha \\
& \beta & \beta & \beta & \beta & \alpha & \alpha & C & \alpha \\
& \beta & \beta & \beta & \beta & \alpha & \alpha & \alpha & C
\end{array}
\right) . \tag{A.54}
$$

- $SO(2) \times \mathbb{Z}_{3(3,4,5)} \times \mathbb{Z}_{3(6,7,8)} \times \mathbb{Z}_{2(\{3,4,5\},\{6,7,8\}|1)} \times \mathbb{Z}_{2(9,10|1)} \times \mathbb{Z}_{3(-|3,4,5,6,7,8)}$

5 parameters $(M, \tilde{M}, \tilde{M}', \tilde{m}, \tilde{m}')$

$$M_\mu = (M, M, \underbrace{\tilde{M}, \cdots, \tilde{M}}_{6}, \tilde{M}', \tilde{M}') \ ,$$

$$m_{3,4,9} = m_{4,5,9} = -m_{3,5,9} = m_{6,7,9} = m_{7,8,9} = -m_{6,8,9}$$

$$= m_{3,4,10} = m_{4,5,10} = -m_{3,5,10} = m_{6,7,10} = m_{7,8,10} = -m_{6,8,10} = \tilde{m} \ ,$$

$$m_{3,6,9} = m_{4,6,9} = m_{5,6,9} = m_{3,7,9} = m_{4,7,9} = m_{5,7,9}$$

$$= m_{3,8,9} = m_{4,8,9} = m_{5,8,9} = m_{3,6,10} = m_{4,6,10} = m_{5,6,10}$$

$$= m_{3,7,10} = m_{4,7,10} = m_{5,7,10} = m_{3,8,10} = m_{4,8,10} = m_{5,8,10} = \tilde{m}' \ ,$$

$$\text{and zero otherwise.} \tag{A.55}$$

The moment of inertia tensor takes the form

$$\langle T_{\mu\nu} \rangle = \left(
\begin{array}{c|ccc|ccc|cc}
\text{SO(2) part} & & & & & & & & \\
\hline
& C & \alpha & \alpha & \times & \times & \times & 0 & 0 \\
& \alpha & C & \alpha & \times & \times & \times & 0 & 0 \\
& \alpha & \alpha & C & \times & \times & \times & 0 & 0 \\
\hline
& \times & \times & \times & C & \alpha & \alpha & 0 & 0 \\
& \times & \times & \times & \alpha & C & \alpha & 0 & 0 \\
& \times & \times & \times & \alpha & \alpha & C & 0 & 0 \\
\hline
& 0 & 0 & 0 & 0 & 0 & 0 & C' & \beta \\
& 0 & 0 & 0 & 0 & 0 & 0 & \beta & C'
\end{array}
\right) \ . \tag{A.56}$$

- $SO(2) \times \mathbb{Z}_{3(3,4,5)} \times \mathbb{Z}_{3(6,7,8)} \times \mathbb{Z}_{2(\{3,4,5\},\{6,7,8\}|1)} \times \mathbb{Z}_{2(9,10|1)} \times \mathbb{Z}_{2(-|9,10)}$

5 parameters $(M, \tilde{M}, \tilde{M}', \tilde{m}, \tilde{m}')$

$$M_\mu = (M, M, \underbrace{\tilde{M}, \cdots, \tilde{M}}_{6}, \tilde{M}', \tilde{M}') \ ,$$

$$m_{3,4,6} = m_{4,5,6} = -m_{3,5,6} = m_{3,4,7} = m_{4,5,7} = -m_{3,5,7}$$

$$= m_{3,4,8} = m_{4,5,8} = -m_{3,5,8} = m_{3,6,7} = m_{3,7,8} = -m_{3,6,8}$$

$$= m_{4,6,7} = m_{4,7,8} = -m_{4,6,8} = m_{5,6,7} = m_{5,7,8} = -m_{5,6,8} = \tilde{m} \ ,$$

$$m_{3,4,5} = m_{6,7,8} = \tilde{m}' \ , \quad \text{and zero otherwise.} \tag{A.57}$$

The moment of inertia tensor takes the form

$$\langle T_{\mu\nu} \rangle = \left(
\begin{array}{c|ccc|ccc|cc}
\text{SO(2) part} & & & & & & & & \\
\hline
& C & \alpha & \alpha & \beta & \beta & \beta & 0 & 0 \\
& \alpha & C & \alpha & \beta & \beta & \beta & 0 & 0 \\
& \alpha & \alpha & C & \beta & \beta & \beta & 0 & 0 \\
\hline
& \beta & \beta & \beta & C & \alpha & \alpha & 0 & 0 \\
& \beta & \beta & \beta & \alpha & C & \alpha & 0 & 0 \\
& \beta & \beta & \beta & \alpha & \alpha & C & 0 & 0 \\
\hline
& 0 & 0 & 0 & 0 & 0 & 0 & C' & \times \\
& 0 & 0 & 0 & 0 & 0 & 0 & \times & C'
\end{array}
\right) \ . \tag{A.58}$$

B. Free energy from the Krauth-Nicolai-Staudacher conjecture

In this section we derive the value of free energy from the analytic formula for the partition function conjectured by Krauth, Nicolai and Staudacher (KNS) [18]. This conjecture was obtained by combining earlier analytic works [32, 33] with Monte Carlo results at small N [18]. For the $D = 10$ model, the formula reads

$$
Z_{\mathrm{KNS}} = \int \mathrm{d}A \, \mathrm{d}\Psi \, e^{-S_{\mathrm{KNS}}} = \frac{2^{\frac{N(N+1)}{2}} \pi^{\frac{N-1}{2}}}{2\sqrt{N} \prod_{k=1}^{N-1} k!} \times \sum_{m|N} \frac{1}{m^2} \, ,
\tag{B.1}
$$

$$
S_{\mathrm{KNS}} = \frac{2}{N}(S_{\mathrm{b}} + S_{\mathrm{f}}) \, ,
\tag{B.2}
$$

where S_{b} and S_{f} are defined by (2.2) and (2.3) respectively, and the sum runs over all the divisors of N. The value of this sum is smaller than $\sum_{m=1}^{\infty} \frac{1}{m^2} = \frac{\pi^2}{6}$ for arbitrary N, and hence it does not contribute to the "free energy density" (4.1) in the large-N limit.

As one can see from (B.2), the definition of the action S_{KNS} differs from ours by the factor of $2/N$. In order to absorb this factor, we introduce the rescaled variables $A'_\mu = (2/N)^{1/4} A_\mu$ and $\Psi'_\alpha = (2/N)^{3/8} \Psi_\alpha$, whose integration measure is given by

$$
\mathrm{d}A' \, \mathrm{d}\Psi' = \left(\frac{N}{2}\right)^{\frac{7}{2}(N^2-1)} \mathrm{d}A \, \mathrm{d}\Psi \, .
\tag{B.3}
$$

As a result, the partition function (2.1) can be obtained as

$$
Z = \left(\frac{N}{2}\right)^{\frac{7}{2}(N^2-1)} Z_{\mathrm{KNS}} = 2^{-3N^2 + \frac{N}{2} + \frac{5}{2}} \pi^{\frac{N-1}{2}} N^{\frac{7}{2}N^2 - 4} \left(\prod_{k=1}^{N-1} k!\right)^{-1} \times \sum_{m|N} \frac{1}{m^2} \, .
\tag{B.4}
$$

From this, we obtain the large-N asymptotics for $F = -\log Z$ as

$$
\frac{F}{N^2 - 1} = -3\log N + \left(\log 8 - \frac{3}{4}\right) + O\left(\frac{\log N}{N^2}\right) \, .
\tag{B.5}
$$

The Gaussian expansion method reproduces the first term of (B.5) correctly for any ansatz. Substituting (B.5) into the definition (4.1) of the free energy density, we obtain $f = \log 8 - \frac{3}{4} = 1.32944...$ as a prediction from the KNS conjecture.

References

[1] T. Banks, W. Fischler, S. H. Shenker and L. Susskind, *M theory as a matrix model: A conjecture*, *Phys. Rev.* **D 55** (1997) 5112 [`hep-th/9610043`].

[2] N. Ishibashi, H. Kawai, Y. Kitazawa and A. Tsuchiya, *A large N reduced model as superstring*, *Nucl. Phys.* **B 498** (1997) 467 [`hep-th/9612115`].

[3] R. Dijkgraaf, E. P. Verlinde and H. L. Verlinde, *Matrix string theory*, *Nucl. Phys.* **B 500** (1997) 43 [`hep-th/9703030`].

[4] H. Aoki, S. Iso, H. Kawai, Y. Kitazawa and T. Tada, *Space-time structures from IIB matrix model*, *Prog. Theor. Phys.* **99** (1998) 713 [`hep-th/9802085`].

[5] H. Steinacker, *Emergent geometry and gravity from matrix models: An introduction*, *Class. Quant. Grav.* **27**, 133001 (2010);
J. Lee and H. S. Yang, *Quantum gravity from noncommutative spacetime*, `arXiv:1004.0745`.

[6] J. Nishimura and F. Sugino, *Dynamical generation of four-dimensional space-time in the IIB matrix model*, *JHEP* **05** (2002) 001 [`hep-th/0111102`].

[7] H. Kawai, S. Kawamoto, T. Kuroki, T. Matsuo and S. Shinohara, *Mean field approximation of IIB matrix model and emergence of four-dimensional space-time*, *Nucl. Phys.* **B 647** (2002) 153 [`hep-th/0204240`].

[8] H. Kawai, S. Kawamoto, T. Kuroki and S. Shinohara, *Improved perturbation theory and four-dimensional space-time in IIB matrix model*, *Prog. Theor. Phys.* **109** (2003) 115 [`hep-th/0211272`].

[9] T. Aoyama, H. Kawai and Y. Shibusa, *Stability of 4-dimensional space-time from IIB matrix model via improved mean field approximation*, *Prog. Theor. Phys.* **115** (2006) 1179 [`hep-th/0602244`].

[10] T. Aoyama and H. Kawai, *Higher order terms of improved mean field approximation for IIB matrix model and emergence of four-dimensional space-time*, *Prog. Theor. Phys.* **116** (2006) 405 [`hep-th/0603146`].

[11] T. Aoyama and Y. Shibusa, *Improved perturbation method and its application to the IIB matrix model*, *Nucl. Phys.* **B 754** (2006) 48 [`hep-th/0604211`].

[12] J. Nishimura, T. Okubo and F. Sugino, *Convergence of the Gaussian expansion method in dimensionally reduced Yang-Mills integrals*, *JHEP* **10** (2002) 043 [`hep-th/0205253`].

[13] J. Nishimura, T. Okubo and F. Sugino, *Testing the Gaussian expansion method in exactly solvable matrix models*, *JHEP* **10** (2003) 057 [`hep-th/0309262`].

[14] T. Aoyama, J. Nishimura and T. Okubo, *Spontaneous breaking of the rotational symmetry in dimensionally reduced super Yang-Mills models*, *Prog. Theor. Phys.* **125** (2011) 537 [`arXiv:1007.0883`].

[15] J. Ambjorn, K. N. Anagnostopoulos, W. Bietenholz, T. Hotta and J. Nishimura, *Monte Carlo studies of the IIB matrix model at large N*, *JHEP* **07** (2000) 011 [`hep-th/0005147`].

[16] K. N. Anagnostopoulos, T. Azuma, and J. Nishimura, work in progress.

[17] J. Nishimura and G. Vernizzi, *Spontaneous breakdown of Lorentz invariance in IIB matrix model*, *JHEP* **04** (2000) 015 [`hep-th/0003223`]; *Brane world generated dynamically from string type IIB matrices*, *Phys. Rev. Lett.* **85** (2000) 4664 [`hep-th/0007022`].

[18] W. Krauth, H. Nicolai and M. Staudacher, *Monte Carlo approach to M theory*, *Phys. Lett.* **B 431** (1998) 31 [`hep-th/9803117`].

[19] T. Hotta, J. Nishimura and A. Tsuchiya, *Dynamical aspects of large N reduced models*, *Nucl. Phys.* **B 545** (1999) 543 [`hep-th/9811220`].

[20] P. Austing and J. F. Wheater, *The convergence of Yang-Mills integrals*, *JHEP* **02** (2001) 028 [`hep-th/0101071`]; *JHEP* **04** (2001) 019 [`hep-th/0103159`].

[21] J. Ambjorn, K. N. Anagnostopoulos, W. Bietenholz, T. Hotta and J. Nishimura, *Large N dynamics of dimensionally reduced 4D SU(N) super Yang-Mills theory*, *JHEP* **07** (2000) 013 [`hep-th/0003208`].

[22] Z. Burda, B. Petersson and J. Tabaczek, *Geometry of reduced supersymmetric 4D Yang-Mills integrals*, *Nucl. Phys.* **B 602** (2001) 399 [`hep-lat/0012001`].

[23] J. Ambjorn, K. N. Anagnostopoulos, W. Bietenholz, F. Hofheinz and J. Nishimura, *On the spontaneous breakdown of Lorentz symmetry in matrix models of superstrings*, *Phys. Rev.* **D 65** (2002) 086001 [`hep-th/0104260`].

[24] K. N. Anagnostopoulos and J. Nishimura, *New approach to the complex-action problem and its application to a nonperturbative study of superstring theory*, *Phys. Rev.* **D 66** (2002) 106008 [`hep-th/0108041`].

[25] J. Nishimura, *Exactly solvable matrix models for the dynamical generation of space-time in superstring theory*, *Phys. Rev.* **D 65** (2002) 105012 [`hep-th/0108070`].

[26] J. Nishimura, T. Okubo and F. Sugino, *Gaussian expansion analysis of a matrix model with the spontaneous breakdown of rotational symmetry*, *Prog. Theor. Phys.* **114** (2005) 487 [`hep-th/0412194`].

[27] K. N. Anagnostopoulos, T. Azuma and J. Nishimura, *A general approach to the sign problem: The factorization method with multiple observables*, *Phys. Rev.* **D 83** (2011) 054504 [`arXiv:1009.4504`];
A practical solution to the sign problem in a matrix model for dynamical compactification, [`arXiv:1108.1534`].

[28] D. Kabat and G. Lifschytz, *Approximations for strongly-coupled supersymmetric quantum mechanics*, *Nucl. Phys.* **B 571** (2000) 419 [`hep-th/9910001`].

[29] D. Kabat, G. Lifschytz and D.A. Lowe, *Black hole thermodynamics from calculations in strongly-coupled gauge theory*, *Phys. Rev. Lett.* **86** (2001) 1426 [`hep-th/0007051`];
D. Kabat, G. Lifschytz and D.A. Lowe, *Black hole entropy from non-perturbative gauge theory*, *Phys. Rev.* **D 64** (2001) 124015 [`hep-th/0105171`];
N. Iizuka, D. Kabat, G. Lifschytz and D.A. Lowe, *Probing black holes in non-perturbative gauge theory*, *Phys. Rev.* **D 65** (2002) 024012 [`hep-th/0108006`].

[30] S. Oda and F. Sugino, *Gaussian and mean field approximations for reduced Yang-Mills integrals*, *JHEP* **03** (2001) 026 [`hep-th/0011175`];
F. Sugino, *Gaussian and mean field approximations for reduced 4D supersymmetric Yang-Mills integral*, *JHEP* **07** (2001) 014 [`hep-th/0105284`].

[31] P. M. Stevenson, *Optimized perturbation theory*, *Phys. Rev.* **D 23** (1981) 2916.

[32] M. B. Green and M. Gutperle, *D particle bound states and the D instanton measure*, *JHEP* **01** (1998) 005 [`hep-th/9711107`].

[33] G. W. Moore, N. Nekrasov and S. Shatashvili, *D particle bound states and generalized instantons*, *Commun. Math. Phys.* **209** (2000) 77 [`hep-th/9803265`].