

THE CONSISTENCY AND CONVERGENCE OF K-ENERGY MINIMIZING MOVEMENTS

JEFFREY STREETS

ABSTRACT. We show that K -energy minimizing movements agree with smooth solutions to Calabi flow as long as the latter exist. As corollaries we conclude that in a general Kähler class long time solutions of Calabi flow minimize both K -energy and Calabi energy. Lastly, by applying convergence results from the theory of minimizing movements, these results imply that long time solutions to Calabi flow converge in the weak distance topology to minimizers of the K -energy functional on the metric completion of the space of Kähler metrics, assuming one exists.

1. INTRODUCTION

Let (M^{2n}, ω, J) be a compact Kähler manifold. Let $\mathcal{H} = \{\phi \in C^\infty(M) \mid \omega + \sqrt{-1}\partial\bar{\partial}\phi > 0\}$ denote the space of Kähler potentials, and given $\phi \in \mathcal{H}$ let $\omega_\phi := \omega + \sqrt{-1}\partial\bar{\partial}\phi > 0$, and let s_ϕ denote the scalar curvature of the metric ω_ϕ . Furthermore, let $V = \text{Vol}(\omega_\phi)$, which is fixed for all ϕ , and set $\bar{s} = \frac{1}{V} \int_M s_\phi \omega_\phi^n$, which is also fixed for any ϕ . A one-parameter family of Kähler potentials ϕ_t is a solution of *Calabi flow* if

$$(1.1) \quad \frac{\partial}{\partial t} \phi = s_\phi - \bar{s}.$$

This flow was introduced by Calabi in his seminal paper [5] on extremal Kähler metrics. In [17] the author proved the general long time existence of certain weak solutions to Calabi flow, which we refer to herein as *K -energy minimizing movements*, (*KEMM* for short). In this paper we derive some further properties of these weak solutions focusing on their relationship to smooth Calabi flows, and we discuss the implications for Calabi flow.

The first main result is a consistency theorem relating KEMM and smooth solutions to Calabi flow. In particular we show that if the Calabi flow with some initial condition ϕ_0 exists smoothly on some time interval, then the KEMM with initial condition ϕ_0 agrees with the solution to Calabi flow on that interval.

Theorem 1.1. *Let (M^{2n}, ω, J) be a compact Kähler manifold. Given $\phi_0 \in \mathcal{H}$, suppose the Calabi flow with initial condition ϕ_0 , call it ϕ_t , exists smoothly on $[0, T)$. Let $\tilde{\phi}_t$ denote the K -energy minimizing movement with initial condition ϕ_0 . Then $\phi_t = \tilde{\phi}_t$ for all $t \in [0, T)$.*

This theorem has several direct consequences for smooth solutions to Calabi flow coming from the theory of minimizing movements. The first example is a bound on the growth of distance along a solution to Calabi flow.

Corollary 1.2. *Let (M^{2n}, ω, J) be a compact Kähler manifold. Given $\phi_0 \in \mathcal{H}$, if ϕ_t denotes a smooth Calabi flow with initial condition ϕ_0 on $[0, T]$, then there is a constant*

$C = C(\phi_0, \omega, T)$ such that for all $0 \leq s \leq t < T$ one has

$$d(\phi_s, \phi_t) \leq C(t-s)^{\frac{1}{2}}.$$

As the constant in the above corollary depends on T , it is difficult to apply this effectively for solutions existing on an infinite time interval. We provide one further corollary in this direction.

Corollary 1.3. *Let (M^{2n}, ω, J) be a compact Kähler manifold, and suppose the K -energy in $[\omega]$ is bounded below. Given $\phi_0 \in \mathcal{H}$, if ϕ_t denotes a smooth Calabi flow with initial condition ϕ_0 on $[0, T]$, then there is a constant $C = C(\nu(\phi_0), [\omega])$ such that for all $0 \leq s \leq t < T$ one has*

$$d(\phi_s, \phi_t) \leq C(t-s)^{\frac{1}{2}}.$$

Remark 1.4. A direct estimate of $d(\phi_0, \phi_t)$ along a solution to Calabi flow using the variation of length and the a priori bound on Calabi energy yields a linear growth rate for large times. Thus Corollaries 1.2 and 1.3 yield a nontrivial improvement of this estimate.

Another consequence of Theorem 1.1 shows that long time solutions of Calabi flow always realize the infimum of K -energy, denoted ν , and Calabi energy, denoted \mathcal{C} . A key component of the proof is a type of “evolutionary variational inequality” for the K -energy along a solution to Calabi flow. As this is of some independent interest we include the statement here.

Corollary 1.5. *Let (M^{2n}, ω, J) be a compact Kähler manifold. Given ϕ_t a solution to Calabi flow on $[0, T]$ and $\psi \in \mathcal{H}$, then one has*

$$d^2(\phi_{t+s}, \psi) \leq d^2(\phi_t, \psi) - 2s(\nu(\phi_{t+s}) - \nu(\psi))$$

for all $t, s \geq 0, t+s \leq T$.

Corollary 1.6. *Let (M^{2n}, ω, J) be a compact Kähler manifold. Given $\bar{\phi}_0 \in \bar{\mathcal{H}}$ the KEMM with initial condition $\bar{\phi}_0$ satisfies*

$$\lim_{t \rightarrow \infty} \bar{\nu}(\bar{\phi}_t) = \inf_{\phi \in \mathcal{H}} \nu(\phi).$$

Furthermore, given $\phi_t \in \mathcal{H}$ a solution of Calabi flow on $[0, \infty)$, one has

$$\lim_{t \rightarrow \infty} \nu(\phi_t) = \inf_{\phi \in \mathcal{H}} \nu(\phi).$$

Corollary 1.7. *Let (M^{2n}, ω, J) be a compact Kähler manifold. Given ϕ_t a solution of Calabi flow on $[0, \infty)$, one has*

$$\lim_{t \rightarrow \infty} \mathcal{C}(\phi_t) = \inf_{\phi \in \mathcal{H}} \mathcal{C}(\phi).$$

Next we establish convergence of K -energy minimizing movements to K -energy minimizers in the weak distance topology, assuming such a minimizer exists. This is a direct consequence of a general theorem on weak convergence of minimizing movements to minimizers proved by Bačák ([1] Theorem 1.5). As the proof of this theorem is ultimately spread through many papers, we include a self-contained exposition in §4 for convenience.

Theorem 1.8. *Let (M^{2n}, ω, J) be compact Kähler manifold. Suppose $\bar{\phi} \in \bar{\mathcal{H}}$ is a minimizer for $\bar{\nu}$. Given $\bar{\phi}_0 \in \bar{\mathcal{H}}$, the KEMM with initial condition $\bar{\phi}_0$ exists for all time and converges weakly to a minimizer for $\bar{\nu}$ in the distance topology.*

Combining Theorems 1.1 and 1.8, and using the result of Chen-Tian [9] that constant scalar curvature metrics are minimizers of K -energy yields the obvious corollary:

Corollary 1.9. *Let (M^{2n}, ω, J) be a compact Kähler manifold and suppose $\phi_\infty \in \mathcal{H}$ satisfies $s_{\phi_\infty} \equiv c$. Then any solution to Calabi flow which exists smoothly on $[0, \infty)$ converges weakly to a minimizer for $\bar{\nu}$ in the distance topology.*

Remark 1.10. Corollary 1.9 represents an affirmative qualitative answer to a question implicit in [10]. In particular, in [10] Donaldson proposes four possibilities for the convergence behavior of long time solutions of Calabi flow. The simplest case is that of convergence to a constant scalar curvature metric, assuming one exists. Corollary 1.9 schematically takes this form, but the statement hides a subtlety that prevents it from being a complete answer to this conjecture. In particular, while it follows from the work of Chen-Tian [9] that constant scalar curvature metrics represent the only minima of K -energy, it does not immediately follow that the minimizers for $\bar{\nu}$ are all in \mathcal{H} , and so are minimizers for ν . Given the convexity properties of ν though this seems likely to be true. What is missing is a kind of “effective uniqueness” statement which says that any sequence of points in \mathcal{H} realizing the infimum of K -energy converges in the weak distance topology to a constant scalar curvature metric. A result of this kind together with the corresponding convergence statement for Calabi flow is established for the (anti)canonical Kähler class on a Kähler-Einstein manifold by Berman [3].

Here is an outline of the rest of the paper. In §2 we review the construction of K -energy minimizing movements and discuss some of their properties. Then in §3 we give the proof of Theorem 1.1 and the related corollaries. We end in §4 with the proof of Theorem 1.8 and Corollary 1.9.

Acknowledgments: The author would like to thank Miroslav Bačák for informing the author of his work, and also Robert Berman and Weiyong He for interesting discussions on Calabi flow.

2. REVIEW OF K-ENERGY MINIMIZING MOVEMENTS

The method of minimizing movements consists of employing an implicit Euler scheme to generate gradient lines of functionals on metric spaces. A foundational work in this direction is the paper of Mayer [15], proving a general existence result in the case of convex functionals on metric spaces with nonpositive curvature in the sense of Alexandrov. In [17] we showed that one can use this theorem to construct minimizing movement solutions to Calabi flow which exist for all time. In this section we give a very brief review of this construction, and then record some further aspects of the theory of minimizing movement established in [15].

2.1. The space of Kähler metrics. Let (M^{2n}, ω, J) be a compact Kähler manifold. As in the introduction, we denote the space of Kähler metrics in $[\omega]$, thought of as a space of Kähler potentials, by \mathcal{H} :

$$\mathcal{H} = \{\phi \in C^\infty(M) \mid \omega + \sqrt{-1}\partial\bar{\partial}\phi > 0\}.$$

This space is an infinite dimensional manifold modeled locally on $C^\infty(M)$. In particular, formally one observes that for every $\phi \in \mathcal{H}$, $T_\phi\mathcal{H} \cong C^\infty(M)$, and we can define a Riemannian

metric, called the Mabuchi-Semmes-Donaldson metric ([14], [16], [11]) by

$$\langle \alpha, \beta \rangle_\phi := \int_M \alpha \beta \omega_\phi^n.$$

As shown in ([14], [16], [11]), this metric has formally nonpositive curvature. One can adapt the usual definition of the length of a curve to this situation, and it was shown in the work of Chen [7] that the space \mathcal{H} is convex by $C^{1,1}$ geodesics, and moreover the distance function induced by the length functional does indeed induce a metric space structure on \mathcal{H} , which we denote by d . Given this setup, we let $(\overline{\mathcal{H}}, d)$ denote the metric space completion of (\mathcal{H}, d) .

2.2. Long time existence of K-energy minimizing movements. Recall that the K -energy functional can be defined on \mathcal{H} by

$$\nu(\phi) = - \int_0^1 \int_M (s(\omega_\phi) - \bar{s}) \dot{\phi} \omega_\phi^n dt.$$

Where $\phi_t : [0, 1] \rightarrow \mathcal{H}$ is any smooth map such that $\phi_0 = 0$ and $\phi_1 = \phi$. We aim to construct minimizing movements of ν in $\overline{\mathcal{H}}$, so first we must extend the domain of ν to $\overline{\mathcal{H}}$.

Definition 2.1. Let (X, d) be a metric space and $f : X \rightarrow \mathbb{R}$ a lower semicontinuous function. If (\overline{X}, d) denotes the completion of (X, d) , we define the *lower semicontinuous extension of f* by

$$(2.1) \quad \overline{f}(x) := \begin{cases} f(x) & x \in X \\ \liminf_{x_n \rightarrow x} f(x_n) & x \in \overline{X} \setminus X. \end{cases}$$

One can easily show that \overline{f} is indeed lower semicontinuous (cf. [17] Lemma 5.10).

Definition 2.2. Let (M^{2n}, ω, J) be a compact Kähler manifold. Observe that by Theorem 3.1 the function ν is lower semicontinuous on \mathcal{H} . Thus we set

$$\overline{\nu} : \overline{\mathcal{H}} \rightarrow \mathbb{R}$$

to be the lower semicontinuous extension of $\nu : \mathcal{H} \rightarrow \mathbb{R}$ in the sense of Definition 2.1.

Definition 2.3. Let (M^{2n}, ω, J) be a compact Kähler manifold. Fix $\phi \in \overline{\mathcal{H}}$ and $\tau > 0$. Let

$$(2.2) \quad \mathcal{F}_{\phi, \tau}(\psi) = \frac{d^2(\phi, \psi)}{2\tau} + \overline{\nu}(\psi).$$

Furthermore, set

$$(2.3) \quad \mu_{\phi, \tau} := \inf_{\psi \in \mathcal{H}} \mathcal{F}_{\phi, \tau}(\psi).$$

The quantity μ is sometimes referred to as a *Moreau-Yosida approximation* of the given functional, in this case ν . Finally, we define the *resolvent operator*

$$W_\tau : \overline{\mathcal{H}} \rightarrow \overline{\mathcal{H}}$$

by the property

$$\mathcal{F}_{\phi, \tau}(W_\tau(\phi)) = \mu_{\phi, \tau}.$$

The fact that there exists a unique minimizer for $\mathcal{F}_{\phi, \tau}$ and so the map W_τ is shown as part of the proof of ([15] Theorem 1.13).

One interprets the resolvent operator formally as $W_\tau = (I + \tau \nabla \bar{\nu})^{-1}$, and as such one expects that the formal gradient line for $\bar{\nu}$ with initial condition ϕ can be constructed by $\phi_t = \lim_{n \rightarrow \infty} W_{\frac{t}{n}}^n(\phi_0)$. This formal picture was justified in the case of convex functions on metric spaces of nonpositive curvature in the work of Mayer ([15] Theorem 1.13). In [17] we applied this theorem to assert the long time existence of formal gradient lines of $\bar{\nu}$ with arbitrary initial data.

Theorem 2.4. ([17] Theorem 1.3) *Given $\phi_0 \in \overline{\mathcal{H}}$, there exists a continuous map*

$$\phi_t : [0, \infty) \rightarrow \overline{\mathcal{H}}$$

such that for all $t > 0$ one has

$$\phi_t = \lim_{n \rightarrow \infty} W_{\frac{t}{n}}^n(\phi_0),$$

and moreover

$$\lim_{t \rightarrow 0} \phi_t = \phi_0.$$

2.3. Further aspects of Mayer's Theorem. In this subsection we record a number of further properties of the minimizing movements constructed in Mayer's Theorem. These generally speaking are generalizations of certain properties obviously satisfied for smooth gradient flows. While the statements we reference in [15] apply to the general setup of that paper, we have specialized the statements to our situation for convenience. To begin we record a definition of the lower slope of $\bar{\nu}$ at a point.

Definition 2.5. Given $\phi_0 \in \overline{\mathcal{H}}$ satisfying $\bar{\nu}(\phi_0) < \infty$, let

$$|\nabla_- \bar{\nu}|(\phi_0) = \max \left\{ \limsup_{\phi \rightarrow \phi_0} \frac{\bar{\nu}(\phi_0) - \bar{\nu}(\phi)}{d(\phi_0, \phi)}, 0 \right\}$$

One should think of this quantity roughly speaking as the norm of the gradient of $\bar{\nu}$, which corresponds to the speed of a minimizing movement. In particular, we have

Theorem 2.6. ([15] Theorem 2.17) *Let ϕ_t be a KEMM. Then for all $t \geq 0$ one has*

$$\lim_{s \rightarrow 0^+} \frac{d(\phi_{t+s}, \phi_t)}{s} = |\nabla_- \bar{\nu}|(\phi_t).$$

Part of the proof of this theorem is a lemma we require asserting the intuitively clear statement that if the infinitesimal variation of distance along a minimizing movement is zero at some point, then the flow is constant from that point on.

Lemma 2.7. ([15] Lemma 2.15) *Let ϕ_t be a KEMM. If*

$$\lim_{s \rightarrow 0^+} \frac{d(\phi_{t_0+s}, \phi_{t_0})}{s} = 0,$$

then $\phi_t = \phi_{t_0}$ for all $t \geq t_0$.

We also later require the again intuitively clear point that the lower gradient is itself nonincreasing upon taking resolvents due to convexity of ν .

Lemma 2.8. ([15] Lemma 2.23) *Given $\phi \in \overline{\mathcal{H}}$ and $\tau > 0$,*

$$|\nabla_- \bar{\nu}|(W_\tau(\phi)) \leq |\nabla_- \bar{\nu}|(\phi).$$

Moreover, if ϕ_t denotes a KEMM and $t \geq s \geq 0$, then

$$|\nabla_{-\bar{\nu}}|(\phi_t) \leq |\nabla_{-\bar{\nu}}|(\phi_s).$$

It is more difficult to define the *direction* of the minimizing movement. The next theorem asserts roughly speaking that there is a unique steepest direction associated to minimizing movement solutions.

Theorem 2.9. ([15] Theorem 2.16) *Let ϕ_t be a KEMM. Assume $0 < |\nabla_{-\bar{\nu}}|(\phi_{t_0})$ and let $\phi^i \rightarrow \phi_{t_0}$ be any sequence of points satisfying*

$$(2.4) \quad \lim_{i \rightarrow \infty} \frac{\bar{\nu}(\phi_{t_0}) - \bar{\nu}(\phi^i)}{d(\phi_{t_0}, \phi^i)} = |\nabla_{-\bar{\nu}}|(\phi_{t_0}).$$

Then there exists a sequence $s_i \rightarrow 0^+$ such that

$$\lim_{i \rightarrow \infty} \frac{d(\phi^i, \phi_{t_0+s_i})}{d(\phi^i, \phi_{t_0})} = 0.$$

Remark 2.10. In the sequel we will use the fact that the values s_i above are chosen so that

$$d(\phi_{t_0}, \phi_{t_0+s_i}) = d(\phi_{t_0}, \phi^i).$$

Furthermore, it was shown by Calabi-Chen [6] that the distance between pairs of points is nonincreasing under Calabi flow. This fact can be generalized to K -energy minimizing movements. Note that we have cited [17], although in some sense as a corollary to Theorem 2.4 it is implicit in [15].

Theorem 2.11. ([17] Theorem 1.4) *Let (M^{2n}, ω, J) be a compact Kähler manifold. If ϕ_t, ψ_t are K -energy minimizing movements, then for all $t \geq 0$ one has*

$$d(\phi_t, \psi_t) \leq d(\phi_0, \psi_0).$$

Lastly, for the proof of Corollary 1.7 we require a technical lemma of Mayer whose proof we reproduce in our simpler case here.

Lemma 2.12. ([15] Lemma 2.8) *Given $\phi_t \in \overline{\mathcal{H}}$ a KEMM and $\psi \in \overline{\mathcal{H}}$, then for any $s, t \geq 0$ one has*

$$d^2(\phi_{t+s}, \psi) \leq d^2(\phi_t, \psi) - 2s(\bar{\nu}(\phi_{t+s}) - \bar{\nu}(\psi)).$$

Proof. This is the statement of [15] Lemma 2.8 in the special case $S = 0$. By the semigroup properties it suffices to show the statement for $t = 0$. Recall that W_τ denotes the resolvent operator. Let $\eta_\lambda : [0, 1] \rightarrow \overline{\mathcal{H}}$ denote the unique geodesic connecting $W_\tau(\phi_0)$ to ψ .

$$\begin{aligned} \mathcal{F}_{\phi_0, \tau}(W_\tau(\phi_0)) &= \frac{d^2(W_\tau(\phi_0), \phi_0)}{2\tau} + \bar{\nu}(W_\tau(\phi_0)) \\ &\leq \frac{d^2(\eta_\lambda, \phi_0)}{2\tau} + \bar{\nu}(\eta_\lambda) \\ &\leq (1 - \lambda)\bar{\nu}(W_\tau(\phi_0)) + \lambda\bar{\nu}(\psi) \\ &\quad + \frac{1}{2\tau} \left((1 - \lambda)d^2(W_\tau(\phi_0), \phi_0) + \lambda d^2(\phi_0, \psi) - \lambda(1 - \lambda)d^2(W_\tau(\phi_0), \psi) \right) \\ &= \mathcal{F}_{\phi_0, \tau}(W_\tau(\phi_0)) + \lambda \left(\bar{\nu}(\psi) - \bar{\nu}(W_\tau(\phi_0)) + \frac{d^2(\phi_0, \psi)}{2\tau} - \frac{d^2(W_\tau(\phi_0), \phi_0)}{2\tau} \right) \\ &\quad - \lambda(1 - \lambda) \frac{d^2(W_\tau(\phi_0), \psi)}{2\tau}. \end{aligned}$$

By subtracting $\mathcal{F}_{\phi_0, \tau}(W_\tau(\phi_0))$ from both sides and dividing by λ we obtain

$$d^2(W_\tau(\phi_0), \psi) \leq d^2(\phi_0, \psi) - 2\tau (\bar{\nu}(W_\tau(\phi_0)) - \bar{\nu}(\psi)).$$

Now iterate this inequality n times with $\tau = \frac{s}{n}$. Since $\bar{\nu}(W_{\frac{s}{n}}^k(\phi_0)) \geq \bar{\nu}(W_{\frac{s}{n}}^n(\phi_0))$ this implies

$$d^2(W_{\frac{s}{n}}^n(\phi_0), \psi) \leq d^2(\phi_0, \psi) - 2s \left(\bar{\nu}(W_{\frac{s}{n}}^n(\phi_0)) - \bar{\nu}(\psi) \right)$$

Sending $n \rightarrow \infty$ and using the lower semicontinuity of $\bar{\nu}$ yields the lemma. \square

3. THE CONSISTENCY THEOREM

In this section we prove Theorem 1.1. To begin we record an inequality of Chen relating the K -energy and distance in \mathcal{H} .

Theorem 3.1. ([8] Theorem 1.2) *Let $\phi_0, \phi_1 \in \mathcal{H}$. Then $\nu(\phi_1) \geq \nu(\phi_0) - d(\phi_0, \phi_1)\sqrt{\mathcal{C}(\phi_0)}$.*

Lemma 3.2. *If $\phi \in \mathcal{H}$, then $|\nabla_- \bar{\nu}|(\phi) = \sqrt{\mathcal{C}(\phi)}$.*

Proof. Let $\{\phi_i\} \in \overline{\mathcal{H}}$ be a sequence in $\overline{\mathcal{H}}$ converging to ϕ in the distance topology. For each ϕ_i choose a sequence $\{\phi_i^j\} \in \mathcal{H}$ converging to ϕ_i such that

$$\lim_{j \rightarrow \infty} \nu(\phi_i^j) = \bar{\nu}(\phi_i).$$

We then compute using Theorem 3.1,

$$\begin{aligned} \frac{\bar{\nu}(\phi) - \bar{\nu}(\phi_i)}{d(\phi, \phi_i)} &= \lim_{j \rightarrow \infty} \frac{\bar{\nu}(\phi) - \bar{\nu}(\phi_i^j)}{d(\phi, \phi_i)} \\ &\leq \lim_{j \rightarrow \infty} \frac{\bar{\nu}(\phi) + \left(d(\phi, \phi_i^j)\sqrt{\mathcal{C}(\phi)} - \bar{\nu}(\phi) \right)}{d(\phi, \phi_i)} \\ &= \sqrt{\mathcal{C}(\phi)} \lim_{j \rightarrow \infty} \frac{d(\phi, \phi_i^j)}{d(\phi, \phi_i)} \\ &= \sqrt{\mathcal{C}(\phi)}. \end{aligned}$$

Since ϕ_i was arbitrary, we conclude that $|\nabla_- \bar{\nu}| \leq \sqrt{\mathcal{C}(\phi)}$.

To show the reverse inequality, we must find an appropriate test curve. Unsurprisingly, the right thing to pick is the solution to Calabi flow with initial condition $\phi = \phi_0$. As observed in [5] the Calabi flow equation is strictly parabolic, and so we have a short-time solution to Calabi flow ϕ_t on $[0, \epsilon)$. First, observe that if $\mathcal{C}(\phi) = 0$ then the desired inequality holds automatically. Thus assume $\mathcal{C}(\phi) \neq 0$ and choose ϵ' sufficiently small that $\mathcal{C}(\phi) \neq 0$ for all $t \in [0, \epsilon')$. Observe that for $T \in [0, \epsilon')$ we have by direct calculation

$$\nu(\phi_0) - \nu(\phi_T) = \int_0^T \mathcal{C}(\phi_t) dt.$$

Moreover, using ϕ_t itself as a test curve in the definition of distance we obtain

$$\begin{aligned}
d(\phi_0, \phi_T) &\leq \int_0^T \left(\int_M \left(\frac{\partial \phi}{\partial t} \right)^2 \omega_{\phi_t}^n \right)^{\frac{1}{2}} dt \\
&= \int_0^T \sqrt{\mathcal{C}(\phi_t)} dt \\
&\leq \left(\int_0^T dt \right)^{\frac{1}{2}} \left(\int_0^T \mathcal{C}(\phi_t) dt \right)^{\frac{1}{2}} \\
&= \sqrt{T} \left(\int_0^T \mathcal{C}(\phi_t) dt \right)^{\frac{1}{2}}.
\end{aligned}$$

Combining these two statements yields

$$\begin{aligned}
\lim_{T \rightarrow 0} \frac{\bar{\nu}(\phi) - \bar{\nu}(\phi_T)}{d(\phi, \phi_T)} &= \lim_{T \rightarrow 0} \frac{\nu(\phi) - \nu(\phi_T)}{d(\phi, \phi_T)} \\
&\geq \lim_{T \rightarrow 0} \frac{\left(\int_0^T \mathcal{C}(\phi_t) dt \right)^{\frac{1}{2}}}{\sqrt{T}} \\
&\geq \lim_{T \rightarrow 0} \frac{(T\mathcal{C}(\phi_0) - CT^2)^{\frac{1}{2}}}{\sqrt{T}} \\
&= \sqrt{\mathcal{C}(\phi_0)} = \sqrt{\mathcal{C}(\phi)}.
\end{aligned}$$

This finishes the proof of the lemma. Observe that as part the proof we have shown that if ϕ_t denotes a smooth solution to Calabi flow on $[0, T)$, then for all $t_0 \in [0, T)$ one has

$$(3.1) \quad \lim_{h \rightarrow 0} \frac{\bar{\nu}(\phi_{t_0+h}) - \bar{\nu}(\phi_{t_0})}{d(\phi_{t_0}, \phi_{t_0+h})} = \sqrt{\mathcal{C}(\phi_{t_0})} = |\nabla_- \bar{\nu}|(\phi_{t_0}).$$

Also, this has the further implication that, with the same setup,

$$(3.2) \quad \lim_{h \rightarrow 0} \frac{d(\phi_{t_0}, \phi_{t_0+h})}{h} = \sqrt{\mathcal{C}(\phi_{t_0})}.$$

□

Proof of Theorem 1.1. Since both ϕ_t and $\tilde{\phi}_t$ are continuous paths in the distance topology, the function $f(t) = d(\phi_t, \tilde{\phi}_t)$ is continuous, and $f(0) = 0$ since $\phi_0 = \tilde{\phi}_0$. By standard measure-theoretic lemmas it suffices to show that

$$D^+ f(t) := \limsup_{h \rightarrow 0^+} \frac{f(t+h) - f(t)}{h} \leq 0$$

for all $t \in [0, T)$. Suppose this were false, and there existed $\tau \in [0, T)$ and a sequence $t_k \rightarrow 0$ such that

$$(3.3) \quad \lim_{k \rightarrow \infty} \frac{d(\phi_{\tau+t_k}, \tilde{\phi}_{\tau+t_k}) - d(\phi_\tau, \tilde{\phi}_\tau)}{t_k} > 0.$$

Let ψ_t denote the K -energy minimizing movement with initial condition ϕ_τ . For notational convenience we will shift the time variable so that ψ is thought to exist on $[\tau, \infty)$, i.e. $\psi_\tau = \phi_\tau$. Let us first rule out a particular case. Suppose $|\nabla_- \bar{\nu}|(\phi_\tau) = 0$. It follows from Theorem 2.6

and Lemma 2.7 that $\psi_t = \psi_\tau$ for all $t \geq \tau$. Moreover, from Lemma 3.2 we conclude that $\mathcal{C}(\phi_\tau) = 0$, and so ϕ_τ is a stationary solution of Calabi flow, i.e. $\phi_t = \phi_\tau = \psi_\tau = \psi_t$ for all $t \geq \tau$. It thus follows from Theorem 2.11 that

$$d(\phi_{\tau+t_k}, \tilde{\phi}_{\tau+t_k}) = d(\psi_\tau, \tilde{\phi}_{\tau+t_k}) = d(\psi_{\tau+t_k}, \tilde{\phi}_{\tau+t_k}) \leq d(\psi_\tau, \tilde{\phi}_\tau) = d(\phi_\tau, \tilde{\phi}_\tau),$$

contradicting (3.3).

Now we assume $|\nabla_{-v}|(\phi_\tau) \neq 0$. The strategy of what follows is summarized in Figure 1. On the one hand the K -energy minimizing movement with initial condition ϕ_τ , denoted ψ_τ , ought to stay within the dotted line by the distance nonincreasing property of Theorem 2.11. But on the other hand we can show that ψ agrees to first order with ϕ at τ , contradicting this fact. To begin, observe that by (3.1) the sequence of points $\{\phi_{\tau+t_k}\}$ satisfies (2.4), and

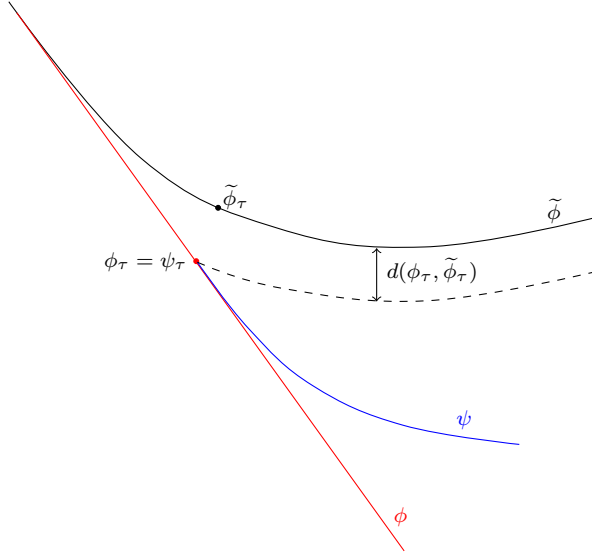


FIGURE 1. A Hypothetical Bifurcation

therefore by Theorem 2.9 we conclude that there exists a sequence of times $s_k \rightarrow 0$ such that

$$(3.4) \quad \lim_{k \rightarrow \infty} \frac{d(\phi_{\tau+t_k}, \psi_{\tau+s_k})}{d(\phi_{\tau+t_k}, \psi_\tau)} = 0.$$

Observe then that by the triangle inequality and Theorem 2.11 we have

$$\begin{aligned} \lim_{k \rightarrow \infty} \frac{d(\tilde{\phi}_{\tau+t_k}, \phi_{\tau+t_k}) - d(\tilde{\phi}_\tau, \phi_\tau)}{t_k} &\leq \lim_{k \rightarrow \infty} \frac{d(\tilde{\phi}_{\tau+t_k}, \psi_{\tau+t_k}) + d(\psi_{\tau+t_k}, \phi_{\tau+t_k}) - d(\tilde{\phi}_\tau, \psi_\tau)}{t_k} \\ &\leq \lim_{k \rightarrow \infty} \frac{d(\psi_{\tau+t_k}, \phi_{\tau+t_k})}{t_k} \\ &\leq \lim_{k \rightarrow \infty} \frac{d(\psi_{\tau+t_k}, \psi_{\tau+s_k}) + d(\psi_{\tau+s_k}, \phi_{\tau+t_k})}{t_k} \\ &= I + II. \end{aligned}$$

We claim that both I and II are zero, contradicting the hypothesis (3.3) and finishing the proof. For the term I we recall from Remark 2.10 that the times s_k are chosen so that

$$d(\phi_\tau, \psi_{\tau+s_k}) = d(\phi_\tau, \phi_{\tau+t_k}) =: d_k.$$

Note also that by (3.2) we have that

$$d_k = d(\phi_\tau, \phi_{\tau+t_k}) = t_k |\nabla_{-\bar{\nu}}|(\phi_\tau) + t_k o(t_k).$$

Also, by Theorem 2.6 we conclude that

$$d_k = d(\phi_\tau, \psi_{\tau+s_k}) = d(\psi_\tau, \psi_{\tau+s_k}) = s_k |\nabla_{-\bar{\nu}}|(\psi_\tau) + s_k o(s_k).$$

By combining the two equations above and using that $|\nabla_{-\bar{\nu}}|(\phi_\tau) = |\nabla_{-\bar{\nu}}|(\psi_\tau) \neq 0$, a number of consequences follow. In particular we conclude that there exist constants $C, k_0 > 0$ such that for all $k \geq k_0$ one has

$$s_k \leq Ct_k.$$

We assume that all values of k below satisfy $k \geq k_0$. In particular, using the three equations displayed above we conclude that

$$|t_k - s_k| = t_k o(t_k) + s_k o(s_k) = t_k o(t_k).$$

Now assume that for some given k we have $t_k > s_k$. By Theorem 2.6 we conclude that there is a constant $C > 0$ so that

$$d(\psi_\tau, \psi_{\tau+(t_k-s_k)}) \leq C(t_k - s_k) \leq Ct_k o(t_k).$$

By Theorem 2.11 we obtain

$$d(\psi_{\tau+s_k}, \psi_{\tau+t_k}) \leq d(\psi_\tau, \psi_{\tau+(t_k-s_k)}) \leq Ct_k o(t_k).$$

The case $s_k > t_k$ yields a similar inequality. Thus finally we conclude that

$$I = \lim_{k \rightarrow \infty} \frac{d(\psi_{\tau+s_k}, \psi_{\tau+t_k})}{t_k} \leq \lim_{k \rightarrow \infty} \frac{t_k o(t_k)}{t_k} = 0.$$

For the term II we note using (3.2) and (3.4) that

$$\begin{aligned} \lim_{k \rightarrow \infty} \frac{d(\psi_{\tau+s_k}, \phi_{\tau+t_k})}{t_k} &= \lim_{k \rightarrow \infty} \frac{d(\phi_{\tau+t_k}, \phi_\tau)}{t_k} \cdot \frac{d(\psi_{\tau+s_k}, \phi_{\tau+t_k})}{d(\phi_{\tau+t_k}, \phi_\tau)} \\ &= \left(\lim_{k \rightarrow \infty} \frac{d(\phi_{\tau+t_k}, \phi_\tau)}{t_k} \right) \cdot \left(\lim_{k \rightarrow \infty} \frac{d(\phi_{\tau+t_k}, \psi_{\tau+s_k})}{d(\phi_{\tau+t_k}, \psi_\tau)} \right) \\ &= |\nabla_{-\bar{\nu}}|(\phi_\tau) \cdot 0 \\ &= 0. \end{aligned}$$

□

Next we give the proof of Corollaries 1.2, 1.3, 1.6, and 1.7.

Proof of Corollary 1.2. This follows directly from [15] Theorem 2.2. A precise estimate of the constant B as referenced in that statement is given in [17] Lemma 5.8, and has the claimed dependencies. □

Proof of Corollary 1.3. Again we want to apply [15] Theorem 2.2. One observes that the proof is in a sense a formal application of the convexity property and the estimate of [15] Lemma 1.11. This lemma claims that given $\phi_0 \in \overline{\mathcal{H}}$, $T > 0$, there is a constant B with complicated dependencies such that for all $\tau > 0$ and $j \in \mathbb{N}$ such that $j\tau \leq T$, one has

$$d^2(\phi_0, W_\tau^j(\phi_0)) \leq Bj\tau.$$

Therefore it suffices to prove this estimate with weaker dependencies in this setting. So, by the definition of the resolvent operator, we have

$$\frac{d^2(W_\tau^{j+1}(\phi_0), W_\tau^j(\phi_0))}{2\tau} \leq \bar{\nu}(W_\tau^j(\phi_0)) - \bar{\nu}(W_\tau^{j+1}(\phi_0)).$$

Therefore by the triangle inequality and the Cauchy-Schwarz inequality we obtain

$$\begin{aligned} d^2(\phi_0, W_\tau^j(\phi_0)) &\leq \left(\sum_{i=0}^{j-1} d(W_\tau^i(\phi_0), W_\tau^{i+1}(\phi_0)) \right)^2 \\ &\leq j \sum_{i=0}^{j-1} d^2(W_\tau^i(\phi_0), W_\tau^{i+1}(\phi_0)). \end{aligned}$$

Combining these inequalities and using that $\bar{\nu}$ is bounded below yields

$$d^2(\phi_0, W_\tau^j(\phi_0)) \leq 2j\tau (\bar{\nu}(\phi_0) - \bar{\nu}(W_\tau^j(\phi_0))) \leq C(\bar{\nu}(\phi_0), [\omega])j\tau.$$

Thus we have obtained the required bound depending only on $\bar{\nu}(\phi_0)$ and the underlying Kähler class $[\omega]$. The corollary follows. \square

Proof of Corollary 1.6. The first statement follows directly from [15] Theorem 2.39 and the fact that

$$\inf_{\bar{\phi} \in \bar{\mathcal{H}}} \bar{\nu}(\bar{\phi}) = \inf_{\phi \in \mathcal{H}} \nu(\phi).$$

Thus the second statement follows directly from the first and Theorem 1.1. \square

Proof of Corollary 1.7. Suppose the claim is false, i.e. for the given Calabi flow ϕ_t ,

$$\beta := \lim_{t \rightarrow \infty} \mathcal{C}(\phi_t) > \inf_{\phi \in \mathcal{H}} \mathcal{C}(\phi).$$

Choose $\psi \in \mathcal{H}$ such that $\mathcal{C}(\psi) = \alpha < \beta$, and let ψ_t denote the KEMM with initial condition ψ guaranteed by Theorem 2.4. Note that by Lemmas 2.8 and 3.2 it follows that for all $t \geq 0$ one has

$$|\nabla_{-\bar{\nu}}|(\psi_t) \leq |\nabla_{-\bar{\nu}}|(\psi_0) = \mathcal{C}(\psi) = \alpha.$$

Also, it follows from [15] Corollary 2.18 that for a general KEMM ψ_t one has

$$(3.5) \quad \bar{\nu}(\psi_0) - \bar{\nu}(\psi_t) = \int_0^t |\nabla_{-\bar{\nu}}|^2(\psi_s) ds \leq t\alpha.$$

Also, applying the gradient flow property to the Calabi flow solution ϕ_t yields

$$\bar{\nu}(\phi_0) - \bar{\nu}(\phi_t) \geq \beta t.$$

Combining these inequalities yields that, for a constant C depending on i and all $t \geq 0$, one has

$$(3.6) \quad \bar{\nu}(\psi_t) \geq \bar{\nu}(\phi_t) + (\beta - \alpha)t - C.$$

Applying Lemma 2.12 and Theorem 2.11 we obtain

$$\begin{aligned} (3.7) \quad 0 &\leq d^2(\psi_{t+s}, \phi_t) \\ &\leq d^2(\psi_t, \phi_t) - 2s (\bar{\nu}(\psi_{t+s}) - \bar{\nu}(\phi_t)) \\ &\leq d^2(\psi_0, \phi_0) - 2s (\bar{\nu}(\psi_t) - \bar{\nu}(\phi_t)) + 2s (\bar{\nu}(\psi_t) - \bar{\nu}(\psi_{t+s})). \end{aligned}$$

Now choose $s = 1$ and observe by (3.5) that

$$(3.8) \quad \bar{\nu}(\psi_t) - \bar{\nu}(\psi_{t+1}) = \int_t^{t+1} |\nabla_{-\bar{\nu}}|^2(\psi_s) ds \leq C.$$

Thus plugging in (3.6) and (3.8) into (3.7) yields

$$0 \leq C - 2(\beta - \alpha)t,$$

which is a contradiction for sufficiently large t . The corollary follows. \square

4. CONVERGENCE RESULTS

In this section we prove Theorem 1.8. To begin we define a notion of weak convergence in NPC spaces first introduced by Jost [13].

Definition 4.1. Let (X, d) be a complete NPC space. Given $\{x_n\} \subset X$ a bounded sequence and a point $x \in X$, the *asymptotic radius of $\{x_n\}$ around x* is

$$r(\{x_n\}, x) = \limsup_{n \rightarrow \infty} d(x_n, x).$$

Moreover, the *asymptotic radius of $\{x_n\}$* is

$$r(\{x_n\}) = \inf_{x \in X} r(\{x_n\}, x).$$

Also, we declare that $x \in X$ is the *asymptotic center of $\{x_n\}$* if $r(\{x_n\}, x) = r(\{x_n\})$. We will show in Lemma 4.3 that the asymptotic center always exists and is unique. Finally, we say that $\{x_n\}$ *weakly converges to x* if x is the asymptotic center of every subsequence of $\{x_n\}$. We say that $z \in X$ is a *weak cluster point* of $\{x_n\}$ if there is a subsequence which converges weakly to z .

Remark 4.2. The definition of weak convergence above is a generalization of the definition of weak convergence in Hilbert spaces.

Lemma 4.3. *Given (X, d) a complete NPC space and $\{x_n\}$ a bounded sequence in X , there exists a unique asymptotic center for $\{x_n\}$.*

Proof. First we show uniqueness. Suppose x and y are both asymptotic centers of $\{x_n\}$. By hypothesis we have

$$\limsup_{n \rightarrow \infty} d(x_n, x) = r(\{x_n\}, x) = r(\{x_n\}) = r(\{x_n\}, y) = \limsup_{n \rightarrow \infty} d(x_n, y).$$

Consider the geodesic $\gamma : [0, 1] \rightarrow X$ connecting x to y . Fix any $t \in (0, 1)$. By applying the triangle inequality we conclude for any n ,

$$\begin{aligned} d^2(x_n, \gamma(t)) &\leq (1-t)d^2(x_n, x) + td^2(x_n, y) - t(1-t)d^2(x, y) \\ &\leq (1-t)r^2(\{x_n\}, x) + tr^2(\{x_n\}, y) - t(1-t)d^2(x, y) \\ &= r^2(\{x_n\}) - t(1-t)d^2(x, y). \end{aligned}$$

Taking the limsup as n goes to infinity we obtain

$$\begin{aligned} r(\{x_n\}) &= \inf_{x \in X} r(\{x_n\}, x) \\ &\leq r(\{x_n\}, \gamma(t)) \\ &\leq \sqrt{r^2(\{x_n\}) - t(1-t)d^2(x, y)}. \end{aligned}$$

It follows that $d(x, y) = 0$, and so $x = y$ as required.

Now we show existence. As the sequence is bounded, certainly there exists $x \in X$ such that $r(\{x_n\}, x) < \infty$. Thus choose a sequence $\{y_n\}$ realizing $\inf_{x \in X} r(\{x_n\}, x)$. By repeating the argument estimate above for uniqueness shows directly that $\{y_n\}$ is a Cauchy sequence. Since X is complete there exists a limit point y_∞ which is an asymptotic center. \square

Lemma 4.4. ([4] Proposition 2.4 pg. 176) *Let (X, d) be a complete NPC space. Let C denote a complete convex subset of X . Then*

- (1) *For every $x \in X$, there exists a unique point $\pi_C(x) \in C$ such that $d(x, \pi_C(x)) = d(x, C)$.*
- (2) *If γ denotes the geodesic connecting x to $\pi_C(x)$ and $y \in \gamma$ then $\pi_C(x) = \pi_C(y)$*
- (3) *If $x \in X \setminus C$ and $y \in C$ satisfies $\pi_C(x) \neq y$, then $\alpha(x, \pi_C(x), y) \geq \frac{\pi}{2}$.*

Proof. (1) Consider a sequence of points $\{y_n\} \in C$ realizing $d(x, C)$. We will show that $\{y_n\}$ is a Cauchy sequence, which simultaneously establishes existence and uniqueness.

(2) This follows directly from the triangle inequality.

(3) First note that if $\alpha(x, \pi_C(x), y) < \frac{\pi}{2}$, then by choosing x' on the geodesic connecting x to $\pi_C(x)$ very close to $\pi_C(x)$ and likewise choosing y' on the geodesic connecting y to $\pi_C(x)$ sufficiently close to $\pi_C(x)$, we can guarantee that the corresponding angle in the comparison triangle $\overline{\Delta}(x', \pi_C(x), y') \subset \mathbb{R}^2$ is also less than $\frac{\pi}{2}$. But using the NPC condition this implies that there is a point $p \in C$ on the geodesic connecting $\pi_C(x)$ to y satisfying $d(x', p) < d(x', \pi_C(x))$. But by (2) we have $d(x', \pi_C(x)) = d(x', C)$, a contradiction. \square

Lemma 4.5. (cf. [12] Proposition 5.2) *If a bounded sequence $\{x_n\} \subset X$ converges weakly to a point $x \in X$ then for any geodesic γ passing through x , one has*

$$\lim_{n \rightarrow \infty} d(x, \pi_\gamma(x_n)) = 0.$$

Proof. If the claim were false then let γ denote a geodesic segment containing x such that

$$\lim_{n \rightarrow \infty} d(x, \pi_\gamma(x_n)) \neq 0$$

Observe that since the segment γ is compact there exists a subsequence $\{x_{n_i}\}$ and a point $y \in \gamma, y \neq x$ such that $\{\pi_\gamma x_{n_i}\} \rightarrow y$. By the definition of the projection operator, for all n_i one has $d(x_{n_i}, \pi_\gamma(x_{n_i})) \leq d(x_{n_i}, x)$. Taking the limit as $i \rightarrow \infty$ yields

$$\lim_{i \rightarrow \infty} d(x_{n_i}, y) \leq \lim_{i \rightarrow \infty} d(x_{n_i}, x).$$

Since x is the asymptotic center of every subsequence of $\{x_n\}$, it follows from the above inequality that y is the asymptotic center of the sequence $\{x_{n_i}\}$. However, by Lemma 4.3 asymptotic centers are unique and so $y = x$, a contradiction. \square

Lemma 4.6. ([2] Lemma 3.1) *Let C denote a closed convex subset of a complete NPC space (X, d) . If $\{x_n\} \subset C$ and $\{x_n\}$ converges weakly to x , then $x \in C$.*

Proof. Suppose $x \notin C$. Let $\gamma : [0, 1] \rightarrow X$ denote the geodesic connecting x to $\pi_C(x)$. We aim to show that $\pi_\gamma(x_n) = \pi_C(x)$ for all n .

We argue by contradiction and assume $\pi_\gamma(x_n) \neq \pi_C(x)$. Observe that if $\pi_\gamma(x_n) \in C$, then by Lemma 4.4 (2) we would have $\pi_\gamma(x_n) = \pi_C \pi_\gamma(x_n) = \pi_C x$, thus $\pi_\gamma(x_n) \notin C$. Applying Lemma 4.4 (3) we conclude

$$\alpha(\pi_\gamma(x_n), \pi_C(x), x_n) = \alpha(\pi_\gamma(x_n), \pi_C \pi_\gamma(x_n), x_n) \geq \frac{\pi}{2}.$$

On the other hand γ is itself a closed convex set. Moreover note that $x_n \notin \gamma$ for otherwise $\pi_\gamma(x_n) = x_n \in C$, contradicting the argument above that $\pi_\gamma(x_n) \notin C$. Thus applying Lemma 4.4 (3) to the set γ we conclude

$$\alpha(x_n, \pi_\gamma(x_n), \pi_C(x)) \geq \frac{\pi}{2}.$$

Since $\alpha(\pi_\gamma(x_n), x_n, \pi_C(x)) > 0$, these three inequalities contradict the triangle inequality for NPC space, finishing the proof of the claim that $\pi_\gamma(x_n) = \pi_C(x)$. It follows that

$$\lim_{n \rightarrow \infty} d(\pi_\gamma(x_n), x) = \lim_{n \rightarrow \infty} d(\pi_C(x), x) > 0.$$

But since $\{x_n\}$ converges weakly to x , Lemma 4.5 guarantees that for the geodesic γ ,

$$\lim_{n \rightarrow \infty} d(\pi_\gamma(x_n), x) = 0.$$

This is a contradiction, and so $x \in C$. □

Lemma 4.7. ([1] Lemma 3.1) *Let (X, d) be a complete NPC space. If $f : X \rightarrow (-\infty, \infty]$ is a lower semicontinuous convex function, then it is weakly lower semicontinuous.*

Proof. If the claim were false, we could find $x \in X$ and $\{x_n\} \in X$ converging weakly to x such that

$$\liminf_{n \rightarrow \infty} f(x_n) < f(x).$$

In particular, there is a subsequence x_{n_k} and $\epsilon > 0$ such that $f(x_{n_k}) < f(x) - \epsilon$ for all k . Let C denote the closure of the convex hull of $\{x_{n_k}\}$. From convexity and lower semicontinuity of f we conclude that for all $y \in C$ one has

$$f(y) < f(x) - \epsilon.$$

However, by Lemma 4.6 we conclude that $x \in C$, and thus we obtain

$$f(x) < f(x) - \epsilon,$$

a contradiction. □

Next we record the convergence theorem for minimizing movements proved by Bačák mentioned in the introduction.

Theorem 4.8. ([1] Theorem 1.5) *Given (X, d) a complete NPC space and $f : X \rightarrow (-\infty, \infty]$ a lower semicontinuous convex function. Assume that f attains its minimum on X . Then for all $x \in X$, the f -minimizing movement with initial condition x converges weakly to a minimizer of f as $\lambda \rightarrow \infty$.*

Proof. Fix a sequence $\{t_n\} \rightarrow \infty$, and for notational convenience let $x_n = F_{t_n}x$. Let

$$C = \{y \in X \mid f(y) = \inf_{x \in X} f(x)\}.$$

By assumption $C \neq \emptyset$. Note that it is clear that given any $y \in C$, the minimizing movement with initial condition y is stationary, i.e. $F_t y = y$ for all $t \geq 0$. It then follows from the distance nonincreasing property of f -minimizing movements that for any $x \in X$, $y \in C$ and $n > m$ one has $d(x_n, y) \leq d(x_m, y)$.

Now we claim that if all weak cluster points of $\{x_n\}$ lie in C then there is a unique weak cluster point of $\{x_n\}$ in C . Suppose $c_1, c_2 \in C$ are weak cluster points of $\{x_n\}$. In particular, there exists a subsequence $\{x_{n_k}\}$ converging weakly to c_1 and a subsequence $\{x_{m_l}\}$ converging

weakly to c_2 . Without loss of generality let us assume that $r(\{x_{n_k}\}) \leq r(\{x_{m_l}\})$. Fix $\epsilon > 0$, and then fix $K \in \mathbb{N}$ such that $d(x_{n_k}, c_1) < r(\{x_{n_k}\}) + \epsilon$ for all $k \geq K$. By the distance nonincreasing property we immediately conclude that

$$d(x_{m_k}, c_1) < r(\{x_{n_k}\}) + \epsilon \leq r(\{x_{n_l}\}) + \epsilon$$

for all l large enough to ensure $n_l \geq n_K$. It follows that c_1 is an asymptotic center for $\{x_{m_k}\}$, but these are unique by Lemma 4.3, and so $c_1 = c_2$.

We now show that all weak cluster points of $\{x_n\}$ do indeed lie in C . With the claim of uniqueness above, the proof will be finished. By ([15] Theorem 2.39) the sequence $\{x_n\}$ is minimizing for f , i.e.

$$\lim_{n \rightarrow \infty} f(x_n) = \inf_{x \in X} f(x).$$

Since f is weakly lower semicontinuous by Lemma 4.7, we conclude that all weak cluster points of $\{x_n\}$ are in C . \square

Proof of Theorem 1.8. In [17] Lemma 5.9 it was established that $(\overline{\mathcal{H}}, d)$ is an NPC space, and in [17] §5 we established that $\overline{\nu}$ is a lower semicontinuous convex function. The theorem follows directly from Theorem 4.8. \square

Proof of Corollary 1.9. It follows from [9] Theorem 1.1.2 that ϕ_∞ is a minimizer for ν , as defined on \mathcal{H} . From the definition of $\overline{\nu}$ it follows immediately that ϕ_∞ is a minimizer for $\overline{\nu}$. The corollary follows from Theorem 1.8. \square

REFERENCES

- [1] M. Bačák, *The proximal point algorithm in metric spaces*, Israel Journal of Mathematics (2012), 1-13.
- [2] M. Bačák, I. Searston, B. Sims *Alternating projections in CAT(0) spaces*, J. Math. Anal. and App., Vol. 385, Issue 2, 599-607.
- [3] R. Berman, *A thermodynamical formalism for Monge-Ampere equations, Moser-Trudinger inequalities and Kähler-Einstein metrics*, arXiv:1011.3976.
- [4] M. Bridson, A. Haefliger, *Metric spaces of non-positive curvature*, Springer, Berlin, 1999.
- [5] E. Calabi, *Extremal Kähler metrics*, Seminar on Differential Geometry, Princeton University Press, 1982.
- [6] E. Calabi, X.X. Chen, *The space of Kähler metrics II*, J. Diff. Geom. 61 (2002), 173-193.
- [7] X.X. Chen, *The space of Kähler metrics*, J. Diff. Geom. 56 (2000), 189-234.
- [8] X.X. Chen, *The space of Kähler metrics III - On the lower bound of the Calabi energy and geodesic distance*, Invent. Math. 175 (2009) 453-503.
- [9] X.X. Chen, G. Tian, *Geometry of Kähler metrics and foliations by holomorphic discs* Publ. Math. de L'IHES, Vol. 107, No. 1, 1-107.
- [10] S.K. Donaldson, *Conjectures in Kähler geometry*, Strings and geometry, 71-78, Clay Math. Proc., 3, Amer. Math. Soc., Providence, RI 2004.
- [11] S.K. Donaldson, *Symmetric spaces, Kähler geometry and Hamiltonian dynamics*, in Northern California Symplectic Geometry Seminar, Amer. Math. Soc. Transl. Ser. 2, 196, Amer. Math. Soc., Providence, 1999, 13-33.
- [12] R. Espínola, A. Fernández-Léon, *CAT(k)-spaces, weak convergence and fixed points*, J. Math. Anal. Appl. 353 (2009), no. 1, 410-427.
- [13] J. Jost, *Equilibrium maps between metric spaces*, Calc. Var. PDE 2 (1994), 173-204.
- [14] T. Mabuchi, *Some symplectic geometry on compact Kähler manifolds (1)*, Osaka J. Math 24 (1987), 227-252.
- [15] U. Mayer, *Gradient flows on nonpositively curved metric spaces and harmonic maps*, Comm. Anal. Geom., Vol. 6, No. 2, 199-253, 1998.

- [16] S. Semmes, *Complex Monge-Ampere equations and symplectic manifolds*, Amer. J. Math. 114 (1992), 495-550.
- [17] _____, *Long time existence of minimizing movement solutions to Calabi flow*, arXiv:1208.2718.

ROWLAND HALL, UNIVERSITY OF CALIFORNIA, IRVINE, IRVINE, CA 92617
E-mail address: `jstreets@uci.edu`