FINITENESS CONDITIONS IN COVERS OF POINCARÉ DUALITY SPACES

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Abstract. A closed 4-manifold (or, more generally, a finite PD_4-space) has a finitely dominated infinite regular covering space if and only if either its universal covering space is finitely dominated or it is finitely covered by the mapping torus of a self homotopy equivalence of a PD_3-complex.

A space X is a *Poincaré duality space* if it has the homotopy type of a cell complex which satisfies Poincaré duality with local coefficients (with respect to some orientation character $w : \pi = \pi_1(X) \to \{\pm 1\}$). It is *finite* if the singular chain complex of the universal cover $\widetilde{X}$ is chain homotopy equivalent to a finite free $\mathbb{Z}[\pi]$-complex. (The PD-space X is homotopy equivalent to a Poincaré duality complex $\Leftrightarrow$ it is finitely dominated $\Leftrightarrow$ π is finitely presentable. See [2].) Closed manifolds are finite PD-complexes. The more general notion arises naturally in connection with Poincaré duality groups [4], and in considering covering spaces of manifolds [11].

In this note we show that finiteness hypotheses in two theorems about covering spaces of PD-complexes may be relaxed. Theorem 5 extends a criterion of Stark to all Poincaré duality groups. The main result is Theorem 6, which characterizes finite PD_4-spaces with finitely dominated infinite regular covering spaces.

1. some lemmas

Let X be a PD_n-space with fundamental group π. Let $\beta_i(X;\mathbb{Q}) = \dim_{\mathbb{Q}} H_i(X;\mathbb{Q})$ and $\beta_i^{(2)}(X) = \dim_{\mathcal{N}(\pi)} H_i(X;\mathcal{N}(\pi))$ be the ith rational Betti number and ith L^2 Betti number of X, respectively.

Lemma 1. *Let X be a PD_n-space with fundamental group π. Then $\Sigma\beta_i(X;\mathbb{Q}) < \infty$ and $\Sigma\beta_i^{(2)}(X) < \infty$. If X is finite then $\chi(X) = \Sigma(-1)^i\beta_i(X;\mathbb{Q}) = \Sigma(-1)^i\beta_i^{(2)}(X)$.*

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Proof. Since X is a PD_n-space and homology commutes with direct limits of coefficient modules so does cohomology. Therefore the singular chain complex of $\widetilde{X}$ is chain homotopy equivalent over $\mathbb{Z}[\pi]$ to a finite projective complex P_*, by the Brown-Strebel finiteness criterion [3]. Hence $H_i(X;\mathbb{Q}) = H_i(\mathbb{Q}\otimes_{\mathbb{Z}[\pi]} P_*)$ and $H_i(X;\mathcal{N}(\pi)) = H_i(\mathcal{N}(\pi)\otimes_{\mathbb{Z}[\pi]} P_*)$. The first assertion follows immediately. The proof of the L^2-Euler characteristic formula for finite complexes given in [12] is entirely homological, and requires only that C_* be chain homotopy equivalent to a finite free complex. □

If the strong Bass conjecture holds for π then the $L^{(2)}$-Euler characteristic formula holds even if X is not finite [6].

The following lemma is essentially from [7]. We shall use it in conjunction with universal coefficient spectral sequences.

Lemma 2. *Let G be a group and k be $\mathbb{Z}$ or a field, and let A be a $k[G]$-module which is free of finite rank m as a k-module. Then $Ext^q_{k[G]}(A,k[G]) \cong (H^q(G;k[G]))^m$ for all q.*

Proof. Let $(g\phi)(a) = g.\phi(g^{-1}a)$ for all $g\in G$ and $\phi\in Hom_k(A,k[G])$. Let $\{\alpha_i\}_{1\le i\le m}$ be a basis for A as a free k-module, and define a map $f : Hom_k(A,k[G]) \to k[G]^m$ by $f(\phi) = (\phi(\alpha_1),\dots,\phi(\alpha_m))$ for all $\phi\in Hom_k(A,k[G])$. Then f is an isomorphism of left $k[G]$-modules. The lemma now follows, since $Ext^q_{k[G]}(A,k[G]) \cong H^q(G;Hom_k(A,k[G]))$. (See Proposition III.2.2 of [4].) □

Lemma 3. *If $H^q(G;\mathbb{Z}[G])$ is 0 (respectively, finitely generated as an abelian group) for all $q\le q_0$ and B is a $\mathbb{Z}[G]$-module which is finitely generated as an abelian group then $Ext^q_{\mathbb{Z}[G]}(B,\mathbb{Z}[G])$ is 0 (respectively, finitely generated as an abelian group) for all $q\le q_0$.*

Proof. Let T be the $\mathbb{Z}$-torsion submodule of B, and let H be the kernel of the action of G on T. Then T is a finite $\mathbb{Z}[G/H]$-module, and so is a quotient of a finitely generated free $\mathbb{Z}[G/H]$-module A. Let A_1 be the kernel of the projection from A to T. Clearly A and A_1 are $\mathbb{Z}[G]$-modules which are free of (the same) finite rank as abelian groups. We now apply the long exact sequence of $Ext^*_{\mathbb{Z}[G]}(-,\mathbb{Z}[G])$ together with Lemma 2 to the short exact sequences

$$0\to A_1\to A\to T\to 0$$

and

$$0\to T\to B\to B/T\to 0.$$

□

2. VIRTUAL POINCARÉ DUALITY GROUPS

Stark has shown that a finitely presentable group G of finite virtual cohomological dimension is a virtual Poincaré duality group if and only if it is the fundamental group of a closed PL manifold M whose universal cover $\widetilde{M}$ is homotopy finite [13]. The main step in showing the sufficiency of the latter condition involves showing first that G is of type vFP, and is established in [14]. If G_1 is an FP subgroup of finite index in G then $B = K(G_1, 1)$ is finitely dominated. Hence on applying the Gottlieb-Quinn Theorem to the fibration $\widetilde{M} \to M_1 \to B$ of the associated covering space M_1 it follows that $\widetilde{M}$ and B are Poincaré duality complexes. In particular, G_1 is a Poincaré duality group.

There are however Poincaré duality groups in every dimension $n \geq 4$ which are not finitely presentable. We shall give an analogue of Stark's sufficiency result for such groups, using an algebraic criterion instead of the Gottlieb-Quinn Theorem. In the next two results we shall assume that M is a PD_n-space with fundamental group π, M_ν is the covering space associated to a normal subgroup ν of π, $G = \pi/\nu$ and k is $\mathbb{Z}$ or a field.

Lemma 4. *Suppose that $H_p(M_\nu; k)$ is finitely generated for all $p \leq [n/2]$. Then $H_p(M_\nu; k)$ is finitely generated for all p if and only if $H^q(G; k[G])$ is finitely generated as a k-module for $q \leq [(n-1)/2]$, and then $H^q(G; k[G])$ is finitely generated as a k-module for all q. If $H^s(G; k[G]) = 0$ for $s < q$ then $H_{n-s}(M_\nu; k) = 0$ for $s < q$ and $H_{n-q}(M_\nu; k) \cong H^q(G; k[G])$.*

Proof. Let $E_2^{pq} = Ext^q_{k[G]}(H_p(M; k[G]), k[G]) \Rightarrow H^{p+q}(M; k[G])$ be the Universal Coefficient spectral sequence for the equivariant cohomology of M. Then $E_2^{pq} = Ext^q_{k[G]}(H_p(M_\nu; k), k[G])$, while $H^{p+q}(M; k[G]) \cong H_{n-p-q}(M_\nu; k)$, by Poincaré duality for M.

If $H^q(G; k[G])$ is finitely generated for $q \leq [(n-1)/2]$ then E_2^{pq} is finitely generated for all $p+q \leq [(n-1)/2]$, by Lemmas 2 and 3. Hence $H_p(M_\nu; k)$ is finitely generated for all $p \geq n - [(n-1)/2]$, and hence for all p. Conversely, if this holds and $H^s(G; k[G])$ is finitely generated for $s < q$ then E_r^{ps} is finitely generated for all $p \geq 0$, $r \geq 2$ and $s < q$. Since $H^q(M; k[G]) \cong H_{n-q}(M_\nu; k)$ is finitely generated as a k-module it follows that $H^q(G; k[G])$ is finitely generated as a k-module. Hence $H^q(G; k[G])$ is finitely generated for all q.

The final assertion is an immediate consequence of duality and the universal coefficient spectral sequence. □

Theorem 5. *If $H_p(M_\nu;k)$ is finitely generated for all p then G is FP_∞ over k and $H^s(G;k[G]) \neq 0$ for some $s \leq n$. If moreover $k = \mathbb{Z}$ and $v.c.d.G < \infty$ then G is virtually a PD_r-group, for some $r \leq n$.*

Proof. Let $C_*(\widetilde{M})$ be the equivariant chain complex of the universal covering space $\widetilde{M}$. Since M is a PD_n-space $C_*(\widetilde{M})$ is chain homotopy equivalent to a finite projective $\mathbb{Z}[\pi]$-complex. Hence $C_*(M_\nu;k) = k[G] \otimes_{\mathbb{Z}[\pi]} C_*(\widetilde{M})$ is chain homotopy equivalent to a finite projective $k[G]$-complex. The arguments of [14] apply equally well with coefficients k a field (instead of $\mathbb{Z}$), and thus the hypotheses of Lemma 4 imply that G is FP_∞ over k.

If $v.c.d.G < \infty$ we may assume without loss of generality that $c.d.G < \infty$, and so G is FP. Since $H_q(M_\nu;\mathbb{Z}))$ is finitely generated for all q the groups $H^s(G;\mathbb{Z}[G])$ are all finitely generated, and since $H_0(M_\nu;\mathbb{Z}) = \mathbb{Z}$ we must have $H^s(G;\mathbb{Z}[G]) \neq 0$ for some $s \leq n$, by Lemma 4. Then G is a PD_s-group, by Theorem 3 of [7]. □

A finitely generated group G is a *weak PD_r-group* if $H^r(G;\mathbb{Z}[G]) \cong \mathbb{Z}$ and $H^q(G;\mathbb{Z}[G]) = 0$ for $q \neq r$. Theorem 5 complements the main result of [11], in which it is shown that if the $\mathbb{Z}[\nu]$-chain complex $C_*(\widetilde{M_\nu}) = C_*(\widetilde{M})|_\nu$ has finite $[n/2]$-skeleton and G is a weak PD_r-group then M_ν is a PD_{n-r}-space.

For each $n \geq 2$ and $k \geq \binom{n+1}{2}$ there are weak PD_k-groups which act freely and cocompactly on $S^{2n-1} \times \mathbb{R}^k$, but which are not virtually torsion-free [8]. Thus if $r \geq 6$ weak PD_r-groups need not be virtual PD_r-groups, and so the other conditions in Theorem 5 do not imply that $v.c.d.G < \infty$, in general. Weak PD_1-groups have two ends, and so are virtually $\mathbb{Z}$, while FP_2 weak PD_2-groups are virtual PD_2-groups [1]. Little is known about the intermediate cases $r = 3, 4$ or 5. In particular, it is not known whether a group G of type FP_∞ such that $H^3(G;\mathbb{Z}[G]) \cong \mathbb{Z}$ must be a virtual PD_3-group. (The fact that local homology manifolds which are homology 2-spheres are standard may be some slight evidence for this being true.)

Stark's argument for realization in the finitely presentable case can be adapted to show that any virtual PD_n-group acts freely on a 1-connected homotopy finite complex, with quotient a PD_m-space for some $m \geq n$. However finite presentability is needed in order to obtain a free *cocompact* action on a 1-connected complex. A natural converse to Theorem 5 (analogous to Stark's realization result) might be that every virtual PD group G acts freely and cocompactly on some connected manifold X with $H_q(X;\mathbb{Z})$ finitely generated for all q. It would suffice to show that $G \cong \pi/\nu$ where π is a finitely presentable

vPD-group and ν is a normal subgroup such that $H_*(\nu;\mathbb{Z})$ is finitely generated. For there is a closed PL manifold M with $\pi_1(M)\cong\pi$ and $\widetilde{M}$ homotopy finite, by Stark's result. The quotient group G acts freely and cocompactly on M_ν, and a spectral sequence argument shows that $H_*(M_\nu;\mathbb{Z})$ is finitely generated.

3. FINITELY DOMINATED COVERING SPACES OF PD_4-SPACES

Let M be a PD_4-space with fundamental group π, and suppose that M has a finitely dominated infinite regular covering space M_ν. Then $\nu=\pi_1(M_\nu)$ is finitely presentable and π/ν has one or two ends. In [9] we showed that if π/ν has two ends then M is the mapping torus of a self homotopy equivalence of a PD_3-complex, while if π/ν has one end and ν is FP_3 then either the universal covering space $\widetilde{M}$ is contractible or homotopy equivalent to S^2. We shall show here that the hypothesis that ν be FP_3 is redundant if M is a closed 4-manifold, or more generally if M is a finite PD_4-space.

The results from [9] used in the next theorem were originally formulated in terms of PD_4-complexes. The arguments given in [9] apply equally well to PD_4-spaces, since they need only the $L^{(2)}$-Euler characteristic formula of Lemma 1 above.

Theorem 6. *Let M be a finite PD_4-space with fundamental group π, and let ν be an infinite normal subgroup of π such that $G=\pi/\nu$ has one end and the associated covering space M_ν is finitely dominated. Then G is of type FP_∞ and M is aspherical.*

Proof. Let k be $\mathbb{Z}$ or a field. Then G is of type FP_∞ and $H^q(G;k[G])$ is finitely generated as a k-module for all q, by Lemma 4 and Theorem 5. Moreover $Ext^q_{k[\pi]}(H_p(M_\nu;k),k[\pi])=0$ for $q\le 1$ and all p, since G has one end, and so $H_q(M_\nu;k)=0$ for $q\ge 3$. In particular, $H^2(G;\mathbb{Z}[G])\cong H_2(M_\nu;\mathbb{Z})$ is torsion-free, and so is a free abelian group of finite rank.

We may assume that M_ν is not acyclic and G is not virtually a PD_2-group, by Theorem 3.9 of [9]. Therefore $H^2(G;k[G])=0$ for all k, by the main result of [1]. Hence $H_2(M_\nu;\mathbb{F}_p)=0$ for all primes p, so $H_1(M_\nu;\mathbb{Z})$ is torsion-free and nonzero. Therefore $H^s(G;\mathbb{Z}[G])=H_{4-s}(M_\nu;\mathbb{Z})=0$ for $s<3$ and $H^3(G;\mathbb{Z}[G])\cong H_1(M_\nu;\mathbb{Z})=\nu/\nu'$ is a nontrivial finitely generated abelian group. Therefore $\nu/\nu'\cong H^3(G;\mathbb{Z}[G])\cong\mathbb{Z}$ [7].

Thus we may assume that M_ν is an homology circle. Let $\tilde{G}=\pi/\nu'$ and let $t\in\tilde{G}$ represent a generator of the infinite cyclic group ν/ν'. Let M'_ν be the covering space associated to the subgroup ν'. Since M_ν is finitely dominated a Wang sequence argument shows that $H_q(M'_\nu;k)$

is a finitely generated $k[t,t^{-1}]$-module on which $t-1$ acts invertibly, for all $q>0$. Then $H_q(M'_\nu;\mathbb{F}_p)$ is finitely generated for all primes p and all $q>0$. Now $H^s(\tilde{G};k[\tilde{G}])=0$ for all k and all $s<4$, by a Lyndon-Hochschild-Serre spectral sequence argument. Therefore $H_q(M'_\nu;\mathbb{F}_p)=0$ for all primes p and all $q>0$, by Lemma 4. Nontrivial finitely generated $\mathbb{Z}[t,t^{-1}]$-modules have nontrivial finite quotients, and so we may conclude that M'_ν is acyclic.

Since M is a PD_4-space $C_*(\widetilde{M})$ is chain homotopy equivalent to a finite projective $\mathbb{Z}[\pi]$-complex C_*. Thus $D_*=\mathbb{Z}\otimes_{\mathbb{Z}[\nu']}C_*$ is a finite projective $\mathbb{Z}[\tilde{G}]$-complex, and is a resolution of $\mathbb{Z}$. Therefore $\tilde{G}$ is a PD_4-group. (In particular, we see again that $G=\tilde{G}/(\nu/\nu')$ is FP_∞.)

Since ν/ν' is a torsion-free abelian normal subgroup of $\tilde{G}$ the group ring $\mathbb{Z}[\tilde{G}]$ has a flat extension R, obtained by localising with respect to the nonzero elements of $\mathbb{Z}[t,t^{-1}]$, such that $R\otimes_{\mathbb{Z}[\tilde{G}]}\mathbb{Z}=0$. (See page 23 of [9] and the references there.) Hence $R\otimes_{\mathbb{Z}[\tilde{G}]}D_*$ is a contractible complex of finitely generated projective R-modules.

We may in fact assume that C_* is a finite *free* $\mathbb{Z}[\pi]$-complex, since M is a *finite* PD_4-space. It follows that $\chi(M)=\chi(R\otimes_{\mathbb{Z}[\tilde{G}]}D_*)=0$. Since ν is an infinite FP_2 normal subgroup of π and π/ν has one end $\beta_1^{(2)}(\pi)=0$ and $H^s(\pi;\mathbb{Z}[\pi])=0$ for $s\le 2$. Therefore M is aspherical, by Corollary 3.5.2 of [9]. □

With this result we may now reformulate Theorem 3.9 of [9] as follows.

Corollary. *A finite PD_4-space M has a finitely dominated infinite regular covering space if and only if either M is aspherical, or $\widetilde{M}\simeq S^2$, or M has a 2-fold cover which is homotopy equivalent to the mapping torus of a self-homotopy equivalence of a PD_3-complex. If M has a finitely dominated regular covering space and is not aspherical it is a PD_4-complex.*

Proof. Only the final sentence needs any comment. If $\widetilde{M}\simeq S^2$ then $\pi_1(M)$ is virtually a PD_2-group and so is finitely presentable. This is also clear if M has a 2-fold cover which is the mapping torus of a self-homotopy equivalence of a PD_3-complex. Thus in each case M is a PD_4-complex. □

There are PD_n groups of type FF which are not finitely presentable, for each $n\ge 4$ [5]. The corresponding $K(G,1)$ spaces are aspherical finite PD_n-spaces which are not PD_n-complexes.

The hypothesis that M be finite is used only in the final paragraph of the proof of Theorem 6, in the appeal to Corollary 3.5.2 of [9] and in the calculation of $\chi(M)$. (If we assumed instead that $v.c.d.G < \infty$ then we could use multiplicativity of the Euler characteristic to show that $\chi(M) = 0$.)

A more substantial issue is that the argument for Theorem 6 does not appear to extend to the case when ν is an ascendant subgroup of π, as considered in [10] (where the FP_3 condition is also used). Is there an argument along the following lines? Let C_* be a finite projective $\mathbb{Z}[\pi]$-complex with $H_0(C_*) \cong \mathbb{Z}$ and $H_1(C_*) = 0$. Show that $Hom_{\mathbb{Z}[\pi]}(H_2(C_*), \mathbb{Z}[\pi]) = 0$ if $[\pi : \nu] = \infty$ and $C_*|_\nu$ is chain homotopy equivalent to a finite projective $\mathbb{Z}[\nu]$-complex. If so, the proofs of Theorem 3.9 of [9] and Theorem 6 of [10] would apply, without needing to assume that ν is FP_3 or that M is finite.

4. PD_4-complexes with π_3 finitely generated

We conclude with an alternative characterization of PD_4-complexes as in the Corollary to Theorem 6. Recall that there is a natural exact sequence of left $\mathbb{Z}[\pi]$-modules

$$H_4(\widetilde{X}; \mathbb{Z}) \to \Gamma_W(\Pi) \to \pi_3(X) \to H_3(\widetilde{X}; \mathbb{Z}) \to 0,$$

where Γ_W is the quadratic functor of Whitehead and the third homomorphism is the Hurewicz homomorphism.

Theorem 7. *Let M be a PD_4-complex with infinite fundamental group π. Then the following are equivalent:*

1. *either M is aspherical, or $\widetilde{M} \sim S^2$ or S^3;*
2. *$\widetilde{M}$ is homotopy finite;*
3. *$\pi_3(M)$ is finitely generated as an abelian group;*
4. *π has finitely many ends and $\pi_2(M)$ is finitely generated as an abelian group.*

Proof. Clearly (1) $\Rightarrow$ (2) $\Rightarrow$ (3) and (4). Since π is finitely presentable, $E^2\mathbb{Z}$ is torsion free, and so $\Pi = \pi_2(M)$ is torsion free also. If $\pi_3(M)$ is finitely generated as an abelian group then $H_3(\widetilde{M}; \mathbb{Z})$ and $\Gamma_W(\Pi)$ are finitely generated. Hence π has finitely many ends and Π is finitely generated. Thus (3) $\Rightarrow$ (4).

If (4) holds then π has one or two ends and $Hom(\Pi; \mathbb{Z}[\pi]) = 0$. Hence $E^2\mathbb{Z} \cong \Pi$, by the evaluation exact sequence. If π has one end then either $E^2\mathbb{Z} = \Pi = 0$, in which case M is aspherical, or both are infinite cyclic, in which case $\widetilde{M} \simeq S^2$. If π has two ends we may assume without loss of generality that $\pi \cong \mathbb{Z}$. But then $\Pi = 0$ and $\widetilde{M} \simeq S^3$. Thus (4) $\Rightarrow$ (1). ∎

In particular, if π is infinite and either $\pi_3(M) = 0$ or π has one end and $\pi_2(M) = 0$ then M is aspherical. However, if $M = \#^r S^1 \times S^3$ for some $r > 1$ then $\pi_2(M) = 0$ but $\pi_3(M)$ is not finitely generated.

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