Remark on nondegeneracy of simple abelian varieties with many endomorphisms

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Abstract

We investigate a relationship between nondegeneracy of a simple abelian variety A over an algebraic closure of \mathbb{Q} and of its reduction A_0 . We prove that under some assumptions, nondegeneracy of A implies nondegeneracy of A_0 .

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Introduction

Let A be an abelian variety over an algebraically closure \mathbb{Q}^{alg} of \mathbb{Q} in \mathbb{C} . In this paper, we say that A is an abelian variety with many endomorphisms if the reduced degree of the \mathbb{Q} -algebra $\text{End}^0(A) := \text{End}(A) \otimes \mathbb{Q}$ is equal to $2 \dim A^{-1}$. This condition is equivalent to that A is of CM-type. An abelian variety A over \mathbb{Q}^{alg} is said to be nondegenerate if all the Hodge classes (see §1) on A are generated by divisor classes in the Hodge ring of A. If A is nondegenerate, then the Hodge conjecture holds for A. We know that products of elliptic curves over \mathbb{C} are nondegenerate (Tate [19], Murasaki [11], Imai [2], Murty [12]). However, there are examples which is degenerate but the Hodge conjecture holds (cf. [1] [16]). For other known results on the Hodge conjecture for abelian varieties, we refer to Gordon's article in Lewis's book [7, Appendix B].

¹The reduced degree of $\operatorname{End}^{0}(A)$ is always $\leq 2 \dim A$.

Let p be a prime number. Let \mathbb{F} be an algebraic closure of a finite field \mathbb{F}_p with p-elements. Let ℓ be a prime number different from p. An abelian variety A_0 over \mathbb{F} is said to be *nondegenerate* if all the ℓ -adic Tate classes (see §1) on A_0 are generated by divisor classes in the ℓ -adic étale cohomology ring of A_0 . If A_0 is nondegenerate, then the Tate conjecture holds for A_0 . Spiess [17] proved that products of elliptic curves over finite fields are nodegenerate. For certain abelian varieties over finite fields, nondegeneracy is known by Lenstra-Zarhin [6], Zarhin [24], Kowaloski [4]. However, there are examples which is not nondegenerate but the Tate conjecture holds ([10, Example 1.8]). For other known results on the Tate conjecture, we refer to [22].

Milne [9, Theorem] proved that if the Hodge conjecture holds for all CM abelian varieties over \mathbb{C} , then the Tate conjecture holds for all abelian varieties over the algebraic closure of a finite field. He furthermore studied a relationship between the Hodge conjecture for an abelian variety A with many endomorphisms over \mathbb{Q}^{alg} and the Tate conjecture for the reduction A_0/\mathbb{F} of A at a prime w of \mathbb{Q}^{alg} dividing p (see Theorem 1.2). Here, we note that by a result of Serre–Tate [13, Theorem 6], one can consider the reduction of A. However a relationship between nondegeneracy of $A/\mathbb{Q}^{\text{alg}}$ and of A_0/\mathbb{F} is not clear. In this paper, we investigate a relationship between nondegeneracy of certain simple abelian variety with many endomorphisms over \mathbb{Q}^{alg} and of its reduction. The following theorem is our main result.

Theorem 0.1. Let A be a simple abelian variety with many endomorphisms over \mathbb{Q}^{alg} .

(1) Assume that the CM-field $\operatorname{End}^{0}(A)$ is a abelian extension of \mathbb{Q} . If all powers of A are nondegenerate, then for any prime w of $\mathbb{Q}^{\operatorname{alg}}$, all powers of a simple factor of the reduction of A at w are nondegenerate.

(2) Let w be a prime of \mathbb{Q}^{alg} . Let A_0 be the reduction of A at w. Assume that the restriction of w to the Galois closure of the CM-field $\text{End}^0(A)$ is unramified over \mathbb{Q} and its absolute degree is one.

- (a) If the Hodge conjecture holds for all powers of A, then the Tate conjecture holds for all powers of A_0 .
- (b) All powers of A are nondegenerate if and only if all powers of A_0 are nondegenerate.

Statement (2) of the theorem is almost a corollary of a result of Milne. We prove this theorem, using a result of Milne (Theorems 1.1 and 1.2) and a necessary and sufficient condition for nondegeneracy (Theorem 1.6, Theorem 1.8). The key (Proposition 2.1) is to compare the conditions of nondegeneracy over \mathbb{C} and \mathbb{F} by a result of Shimura–Taniyama on the prime ideal decomposition of Frobenius endomorphism.

This paper is organized as follows: In section 1, we recall Milne's results [9, 10] on the Hodge conjecture and the Tate conjecture. We also recall a necessary and sufficient condition for nondegeneracy of certain simple abelian varieties (Theorem 1.6, Theorem 1.8). In section 2, we prove a key proposition (Proposition 2.1) for our main result. Using the key proposition and results mentioned in section 1, we give a proof of Theorem 0.1. In the last section, using a result of Aoki [1] we give an example of a degenerate simple abelian variety over \mathbb{F} for which the Tate conjecture holds.

Notation.

For an abelian variety A with many endomorphisms over an algebraically closed field k, $\operatorname{End}^{0}(A)$ denotes $\operatorname{End}_{k}(A) \otimes \mathbb{Q}$, and C(A) denotes the center of $\operatorname{End}_{k}^{0}(A)$.

For a finite étale \mathbb{Q} -algebra $E, \Sigma_E := \text{Hom}(E, \mathbb{Q}^{\text{alg}})$. If E is a field Galois over \mathbb{Q} , we identify Σ_E with the Galois group $\text{Gal}(E/\mathbb{Q})$.

For a finite set S, \mathbb{Z}^S denotes the set of functions $f: S \to \mathbb{Z}$.

An affine algebraic group is of multiplicative type if it is commutative and its identity component is a torus. For such a group W over \mathbb{Q} , $\chi(W) :=$ Hom $(W_{\mathbb{Q}^{\text{alg}}}, \mathbb{G}_m)$ denotes the group of characters of W.

For a finite étale Q-algebra E, $(\mathbb{G}_m)_{E/\mathbb{Q}}$ denotes the Weil restriction $\operatorname{Res}_{E/\mathbb{Q}}(\mathbb{G}_m)$ which is characterized by $\chi((\mathbb{G}_m)_{E/\mathbb{Q}}) = \mathbb{Z}^{\Sigma_E}$.

1 The Hodge conjecture and the Tate conjecture for abelian varieties

We first recall a statement of conjectures.

The Hodge conjecture. Let A be an abelian variety of dimension g over \mathbb{C} . By $H_B^*(A, \mathbb{Q})$ we denote the Betti cohomology of A. For each integer i with $0 \leq i \leq g$, we define the space of the Hodge classes of degree i on A as follows:

$$H^{2i}_B(A, \mathbb{Q}) \cap H^i(A, \Omega^i).$$

We know that the image of a cycle map is contained in the space of the Hodge classes. A Hodge class is said to be *algebraic* if it belongs to the image of cycle map.

Conjecture. All Hodge classes on A are algebraic.

By the Lefschetz–Hodge theorem, all the Hodge classes of degree one are generated by divisor classes. Therefore A is nondegenerate if and only if all the Hodge classes on A are generated by the Hodge classes of degree one.

The Tate conjecture. Let \mathbb{F}_p be a finite field with *p*-elements and let \mathbb{F} be the algebraic closure of \mathbb{F}_p . Let A_1 be an abelian variety of dimension *g* over a finite subfield \mathbb{F}_q of \mathbb{F} . Let A_0 be the abelian variety $A_1 \otimes_{\mathbb{F}_q} \mathbb{F}$ over \mathbb{F} . By $H^{2i}(A_0, \mathbb{Q}_{\ell}(i))$ we denote the ℓ -adic étale cohomology group of A_0 . For each integer *i* with $0 \leq i \leq g$, we define the space of the ℓ -adic Tate classes of degree *i* on *A* as follows:

$$\lim_{L/\mathbb{F}_q} H^{2i}(A_0, \mathbb{Q}_\ell(i))^{\operatorname{Gal}(\mathbb{F}/L)}.$$

Here L/\mathbb{F}_q runs over all finite extensions of \mathbb{F}_q . We know that the image of the ℓ -adic étale cycle map is contained in the space of the Tate classes. A Tate class is said to be *algebraic* if it belongs to the image of the ℓ -adic étale cycle map.

Conjecture. All Tate classes on A_0 are algebraic.

This is conjectured by Tate [19, Conjecture 1]. By a result of Tate [20], we know that for any abelian variety A_0 , all the Tate classes of degree one are generated by divisor classes on A_0 . Therefore A_0 is nondegenerate if and only if all the Tate classes on A_0 are generated by the Tate classes of degree one.

1.1 Necessary and sufficient condition

Let A be an abelian variety over an algebraically closed field k such that the reduced degree of $\operatorname{End}^{0}(A)$ is $2 \dim A$. In this case, A is said to have many endomorphisms. There are important algebraic groups of multiplicative type L(A), M(A), MT(A) and P(A) over \mathbb{Q} attached to A. Using these groups, Milne gave a necessary and sufficient condition for the Hodge conjecture and the Tate conjecture for abelian varieties with many endomorphisms.

Theorem 1.1 (Milne [10, p. 14, Theorem]). (1) Let A be an abelian variety with many endomorphisms over an algebraically closed field k of characteristic zero. Then $MT(A) \subset M(A) \subset L(A)$, and

- (i) the Hodge conjecture holds for all powers of A if and only if MT(A) = M(A);
- (ii) all powers of A are nondegenerate if and only if MT(A) = L(A).

(2) Let A_0 be an abelian variety over \mathbb{F} . Then $P(A_0) \subset M(A_0) \subset L(A_0)$, and

(i) the Tate conjecture holds for all powers of A_0 if and only if $P(A_0) = M(A_0)$;

(ii) all powers of A_0 are nondegenerate if and only if $P(A_0) = L(A_0)$.

For the relationship between the Hodge conjecture and the Tate conjecture, Milne proved the following:

Theorem 1.2 (Milne [10]). Let A be an abelian variety with many endomorphisms over \mathbb{Q}^{alg} and let A_0 be the reduction of A at a prime of \mathbb{Q}^{alg} . If the Hodge conjecture holds for all powers of A and

 $P(A_0) = L(A_0) \cap MT(A)$ (intersection inside L(A)),

then the Tate conjecture holds for all powers of A_0 .

In the rest of this subsection, we briefly recall the definitions of the groups L, M, MT and P associated to an abelian variety A over k (For more detail, see [8], [9] and [10]), and we recall a necessary and sufficient condition for nondegeneracy for certain simple abelian varieties (Theorem 1.6, Theorem 1.8).

Let A be an abelian variety with many endomorphisms over an algebraically closed field k. Put $E := \text{End}^0(A)$. Let C(A) be the center of E. We write A^{\vee} for the dual abelian variety of A. A polarization $\lambda : A \to A^{\vee}$ of A determines an involution of E which stabilizes C(A). The restriction of the involution to C(A) is independent of the choice of λ . By \dagger , we denote this restriction to C(A).

Definition 1.3 ([8, 4.3, 4.4], [9, p. 52–53], [10, A.3]). The Lefschetz group L(A) of A is the algebraic group over \mathbb{Q} such that

$$L(A)(R) = \{ \alpha \in (C(A) \otimes R)^{\times} \mid \alpha \alpha^{\dagger} \in R^{\times} \}$$

for all \mathbb{Q} -algebras R.

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In case that $k = \mathbb{C}$, we can describe L(A) as a subgroup of $(\mathbb{G}_m)_{E/\mathbb{Q}}$ in terms of characters as follows ([10, A.7]): L(A) is a subgroup of $(\mathbb{G}_m)_{E/\mathbb{Q}}$ whose character group is

$$\frac{\mathbb{Z}^{\Sigma_E}}{g \in \mathbb{Z}^{\Sigma_E} \mid g = \iota g \text{ and } \sum g(\sigma) = 0\}}.$$
(1.1)

Here ιg is a function sending an element σ of Σ_E to $g(\iota \sigma)$, and $\sum g(\sigma)$ denotes $\sum_{\sigma \in \Sigma_E} g(\sigma).$

In case that $k = \mathbb{F}$, L(A) is a subgroup of $(\mathbb{G}_m)_{C(A)/\mathbb{Q}}$ whose character group is

$$\frac{\mathbb{Z}^{\Sigma_{C(A)}}}{\{g \in \mathbb{Z}^{\Sigma_{C(A)}} \mid g = \iota g \text{ and } \sum g(\sigma) = 0\}}.$$
(1.2)

Definition 1.4. Jannsen [3] proved that the category of motives generated by abelian varieties over \mathbb{F} with the algebraic cycles modulo numerical equivalence as the correspondences is Tannakian. The group M(A) is defined as the fundamental group of the Tannakian subcategory of this category generated by A and the Tate object.

Definition 1.5. When the characteristic of k is zero, the Mumford-Tate group MT(A) is defined to be the largest algebraic subgroup of L(A) fixing the Hodge classes on all powers of A.

When A is simple and the characteristic of k is zero, we describe a condition for which a character of L(A) is trivial on MT(A). To give the condition, we introduced notion of CM-type.

Let *E* be a CM-algebra. A subset Φ of Σ_E is called *CM-type* of *E* if $\Sigma_E = \Phi \cup \iota \Phi$ and $\Phi \cap \iota \Phi = \phi$. Here ι is complex conjugation on \mathbb{C} .

When $E = \text{End}^{0}(A)$, the action of E on $\Gamma(A, \Omega^{1})$ defines a CM-type of E.

Now assume that A is simple. Let Φ be the CM-type of the CM-field $\operatorname{End}^{0}(A)$. A character g of L(A) is trivial on MT(A) if and only if

$$\sum_{\sigma \in \Phi} g(\tau \circ \sigma) = 0 \tag{1.3}$$

for all $\tau \in \operatorname{Gal}(\mathbb{Q}^{\operatorname{alg}}/\mathbb{Q})$.

For nondegeneracy of certain abelian varieties, the following result is known:

Theorem 1.6. Let A be a simple abelian variety with many endomorphisms over \mathbb{Q}^{alg} . Let Φ be the CM-type of the CM-field $E := \text{End}^{0}(A)$ defined by the action of E on $\Gamma(A, \Omega^{1})$. Assume that E is a abelian extension over \mathbb{Q} with its Galois group G. Then all powers of A are nondegenerate if and only if

$$\sum_{\sigma \in \Phi} \chi(\sigma) \neq 0$$

for any character χ of G such that $\chi(\iota) = -1$.

For a proof of the theorem, see [5].

Definition 1.7 ([9, §4], [10, A.7]). Let k be the algebraic closure \mathbb{F} of a finite filed \mathbb{F}_p . Let A_1 be a model of A and let π_1 be the Frobenius endomorphism of A_1 . Then the group P(A) is the smallest algebraic subgroup of L(A) containing some power of π_1 . It is independent of the choice of A_1 .

When A/\mathbb{F} is simple, we describe a condition for which a character of L(A) is trivial on P(A). To give the condition, we introduce some notion about Weil numbers.

A Weil q-number of weight *i* is an algebraic number α such that $q^N \alpha$ is an algebraic integer for some *N* and the complex absolute value $|\sigma(\alpha)|$ is $q^{i/2}$, for all embeddings $\sigma : \mathbb{Q}[\alpha] \to \mathbb{C}$. We know that π_1 is a Weil q-number of weight one. Then π_1 is a unit at all primes of $\mathbb{Q}[\pi_1]$ not dividing *p*. We define the *slope function* s_{π_1} of π_1 as follows: for any prime **p** dividing *p* of a field containing π_1 ,

$$s_{\pi}(\mathfrak{p}) = \frac{\operatorname{ord}_{\mathfrak{p}}(\pi_1)}{\operatorname{ord}_{\mathfrak{p}}(q)}.$$
(1.4)

The slope function determines a Weil q-number up to a root of unity. From the definition of Weil numbers, $s_{\pi_1}(\mathfrak{p}) + s_{\pi_1}(\iota \mathfrak{p}) = 1$.

We define a *Weil germ* to be an equivalent class ² of Weil numbers. For a Weil germ π , the slope function of π are the slope function (see (1.4)) of any representative of π .

Now assume that A/\mathbb{F} is simple. Let π_A denote the germ represented by π_1 . Milne's result on the character of P(A) is the following ([10, A.7]): let g be a character of L(A). Then g is trivial on P(A) if and only if

$$\sum_{\sigma \in \Sigma_{C(A)}} g(\sigma) s_{\sigma \pi_A}(\mathfrak{p}) = 0.$$
(1.5)

for all primes \mathfrak{p} dividing p of a field containing all conjugates $\sigma(\pi_1)$.

Using Theorem 1.1 and (1.2) (1.5), we obtain the following:

Theorem 1.8 ([18]). Let A_0 be a simple abelian variety over \mathbb{F} . Assume that $C(A_0)$ is abelian extension of \mathbb{Q} with its Galois group G_0 . Let \mathfrak{p} be a prime of $C(A_0)$ dividing p. Then any power of A_0 is nondegenerate if and only if

$$\sum_{\sigma \in G_0} s_{\pi}(\sigma \mathfrak{p}) \chi(\sigma) \neq 0$$

for any character χ of G_0 such that $\chi(\iota) = -1$.

This is an analogous result to Theorem 1.6 for simple abelian varieties over \mathbb{F} .

²Let π be a Weil p^f -number and let π' be a Weil $p^{f'}$ -number. We say π and π' are equivalent if $\pi^{f'} = {\pi'}^f \cdot \zeta$ for some root of unity ζ .

2 Proof of the main theorem

We prove Theorem 0.1 using the theorems mention in the previous section. We first fix the notation:

A : a simple abelian variety with many endomorphisms over \mathbb{Q}^{alg}

w: a prime of \mathbb{Q}^{alg} dividing p

 A_0 : a simple factor of the reduction of A at w

- E: the CM-field End⁰(A)
- Φ : the CM-type of E,

 $\varphi\,$: the characteristic function of $\Phi\,$

- E_0 : the center of End⁰(A_0) (E_0 is a subfield of E)
 - π : the Weil germ attached to A_0

 K/\mathbb{Q} : a finite Galois extension which include all conjugate of E $G := \operatorname{Gal}(K/\mathbb{Q})$

- \mathfrak{p} : the restriction of w to K
- $G_{\mathfrak{p}}$: the decomposition group of \mathfrak{p} in K

The following proposition is a key in a proof of our main result.

Proposition 2.1. Let the notation as above. Then for any $\sigma \in \Sigma_{E_0}$ and any $\tau \in G$,

$$s_{\sigma\pi}(\tau\mathfrak{p}) = \frac{1}{|G_{\mathfrak{p}}|} \sum_{h \in G_{\mathfrak{p}}} \varphi(h\tau^{-1} \circ \sigma).$$

Proof. We identify Σ_E with $\operatorname{Hom}_{\mathbb{Q}}(E, K)$. Let $\sigma \in \Sigma_{E_0}$ and $\tau \in G$. Let $\tilde{\sigma} \in G$ be a lift of σ . Since E_0 is equal to the smallest subfield of $\mathbb{Q}^{\operatorname{alg}}$ containing a representative of π , we have $s_{\sigma\pi}(\tau \mathfrak{p}) = s_{\tilde{\sigma}\pi}(\tau \mathfrak{p})$. Therefore we may fix the lift $\tilde{\sigma} \in G$ for each $\sigma \in \Sigma_{E_0}$. Then we have $s_{\tilde{\sigma}\pi}(\tau \mathfrak{p}) = s_{\pi}(\tilde{\sigma}^{-1}\tau \mathfrak{p})$. By a theorem of Shimura–Taniyama (see Tate [21, Lemma 5]), $s_{\pi}(\tilde{\sigma}^{-1}\tau \mathfrak{p})$ is given as follows

$$s_{\pi}(\tilde{\sigma}^{-1}\tau\mathfrak{p}) = \frac{|\Phi(\tilde{\sigma}^{-1}\tau\mathfrak{p})|}{|\Sigma_{E}(\tilde{\sigma}^{-1}\tau\mathfrak{p})|}$$

where

$$\Sigma_E(\tilde{\sigma}^{-1}\tau \mathfrak{p}) := \{ f \in \Sigma_E \mid \tilde{\sigma}^{-1}\tau \mathfrak{p} = f^{-1}\mathfrak{p} \cdots (*) \}$$
$$\Phi(\tilde{\sigma}^{-1}\tau \mathfrak{p}) := \Phi \cap \Sigma_E(\tilde{\sigma}^{-1}\tau \mathfrak{p}).$$

Here (*) means that for any $x \in E$,

$$v_{\tau \mathfrak{p}}(\tilde{\sigma}(x)) = v_{\mathfrak{p}}(f(x)).$$

Now we consider condition (*). Let $\tilde{f} \in G$ be a lift of f. Then we have the following equivalences

condition (*)
$$\iff \tilde{\sigma}^{-1}\tau \mathfrak{p}$$
 and $\tilde{f}^{-1}\mathfrak{p}$ lie over the same prime of E
 $\iff \tilde{\sigma}^{-1}\tau \mathfrak{p} = \eta \tilde{f}^{-1}\mathfrak{p}$ for some $\eta \in \operatorname{Gal}(K/E)$
 $\iff \tilde{f}\eta^{-1}\tilde{\sigma}^{-1}\tau \in G_{\mathfrak{p}}$ for some $\eta \in \operatorname{Gal}(K/E)$
 $\iff \tilde{f} \in G_{\mathfrak{p}}\tau^{-1}\tilde{\sigma}\operatorname{Gal}(K/E)$

If $\tilde{f} \in G_{\mathfrak{p}}\tau^{-1}\tilde{\sigma}\operatorname{Gal}(K/E)$, then any lifts of f are also in $G_{\mathfrak{p}}\tau^{-1}\tilde{\sigma}\operatorname{Gal}(K/E)$. Hence the property that \tilde{f} belongs to $G_{\mathfrak{p}}\tau^{-1}\tilde{\sigma}\operatorname{Gal}(K/E)$ is independent of the choice of the lift of f.

For $h_1, h_2 \in G_{\mathfrak{p}}$ and $\eta_1, \eta_2 \in \operatorname{Gal}(K/E)$, we have

$$(h_1\tau^{-1}\tilde{\sigma}\eta_1)^{-1}(h_2\tau^{-1}\tilde{\sigma}\eta_2) \in \operatorname{Gal}(K/E) \iff \tilde{\sigma}^{-1}\tau h_1^{-1}h_2\tau^{-1}\tilde{\sigma} \in \operatorname{Gal}(K/E) \iff h_1^{-1}h_2 \in \operatorname{Gal}(K/\tau^{-1}\tilde{\sigma}(E)).$$

From the above argument, we have

$$\begin{aligned} |\Sigma_E(\tilde{\sigma}^{-1}\tau \mathfrak{p})| &= |G_{\mathfrak{p}}/G_{\mathfrak{p}} \cap \operatorname{Gal}(K/\tau^{-1}\tilde{\sigma}(E))| \\ &= \frac{|G_{\mathfrak{p}}|}{|G_{\mathfrak{p}} \cap \operatorname{Gal}(K/\tau^{-1}\tilde{\sigma}(E))|} \end{aligned}$$

Next we calculate $|\Phi(\tilde{\sigma}^{-1}\tau \mathfrak{p})|$. Since $h\tau^{-1}\tilde{\sigma}(x) = \tau^{-1}\tilde{\sigma}(x)$ for any $h \in \operatorname{Gal}(K/\tau\tilde{\sigma}(E))$ and for any $x \in E$, we have

$$\varphi(h\tau^{-1}\circ\sigma)=\varphi(\tau^{-1}\circ\sigma).$$

Therefore we have

$$\Phi(\tilde{\sigma}^{-1}\tau\mathfrak{p})| = \frac{1}{|G_{\mathfrak{p}} \cap \operatorname{Gal}(K/\tau^{-1}\tilde{\sigma}(E))|} \sum_{h \in G_{\mathfrak{p}}} \varphi(h\tau^{-1} \circ \sigma).$$

Hence we have

$$s_{\sigma\pi}(\tau\mathfrak{p}) = \frac{1}{|G_{\mathfrak{p}}|} \sum_{h \in G_{\mathfrak{p}}} \varphi(h\tau^{-1} \circ \sigma).$$

Before starting the proof of (1) of Theorem 0.1, we prepare some notation. By the assumption that E is abelian over \mathbb{Q} , we may take K = E. We write G for the Galois group of E/\mathbb{Q} and G_0 for the Galois group of E_0/\mathbb{Q} . We identify Σ_E (resp. Σ_{E_0}) with G (resp. G_0). Then there is an exact sequence of finite abelian groups

$$1 \longrightarrow G_1 \longrightarrow G \longrightarrow G_0 \longrightarrow 1,$$

where $G_1 = \operatorname{Gal}(E/E_0)$.

We define the subgroup \hat{G}^- of the character group of G as follows:

$$\hat{G^-} := \{ \chi : G \longrightarrow \mathbb{C}^{\times} \mid \chi(\iota) = -1 \}.$$

Here $\iota \in G$ is the complex conjugation. Similarly to \hat{G}^- , we define the subgroup \hat{G}_0^- of the character group of G_0 . Since G_0 is a quotient group of G, we consider \hat{G}_0^- as the subgroup of \hat{G}^- :

$$\hat{G}_0^- = \{ \chi \in \hat{G}^- \mid \chi(G_1) = 1 \}.$$

Remark 2.2. Since p is completely decomposed in E_0 (cf. [18, Proposition 3.5]), the decomposition group $G_{\mathfrak{p}}$ of \mathfrak{p} is contained in G_1 . If \mathfrak{p} is unramified and its absolute degree is one, then $E = E_0$ and hence $G = G_0$.

Proof of (1) of Theorem 0.1. Let \mathfrak{p}_0 be the prime $\mathfrak{p} \cap E_0$ of E_0 . By Proposition 2.1, for any $\chi \in \hat{G}_0^-$ we have

$$\sum_{\sigma \in G_0} s_{\pi}(\sigma \mathfrak{p}_0) \chi(\sigma) = \frac{1}{|G_1|} \sum_{\sigma \in G} s_{\pi}(\sigma \mathfrak{p}) \chi(\sigma)$$
$$= \frac{1}{|G_1|} \sum_{\sigma \in G} \frac{1}{|G_{\mathfrak{p}}|} \sum_{h \in G_{\mathfrak{p}}} \varphi(h\sigma^{-1}) \chi(\sigma)$$
$$= \frac{1}{|G_1|} \sum_{i \in G_{\mathfrak{p}}} \sum_{\sigma \in G} \varphi(h\sigma^{-1}) \chi(\sigma)$$
$$= \frac{1}{|G_1|} \sum_{\sigma \in G} \varphi(\sigma^{-1}) \chi(\sigma)$$
$$= \frac{1}{|G_1|} \sum_{\sigma \in \Phi} \bar{\chi}(\sigma).$$

From this, we obtain that for any $\chi \in \hat{G_0^-} \subset \hat{G^-}$,

$$\sum_{\sigma \in G_0} s_{\pi}(\sigma \mathfrak{p}_0) \chi(\sigma) \neq 0 \quad \text{if and only if} \quad \sum_{\sigma \in \Phi} \bar{\chi}(\sigma) \neq 0.$$

Therefore the assertion follows from Theorem 1.6 and Theorem 1.8. $\hfill \Box$

Proof of (2) of Theorem 0.1. To prove the assertion, by Theorem 1.1 and Theorem 1.2, it suffices to show that $L(A) = L(A_0)$ and $MT(A) = P(A_0)$.

We first show that $L(A) = L(A_0)$. By the assumption that \mathfrak{p} is unramified and its absolute degree is one, we obtain that $E = E_0$ from a result of Shimura–Taniyama [14, p. 100, Theorem 2]. By the description (1.1) (1.2) of the character group of L(A) and $L(A_0)$, we have $L(A) = L(A_0)$.

Next we show that $MT(A) = P(A_0)$. We easily see that the condition (1.3) of triviality on MT(A) of a character of L(A) is described in terms of the characteristic function φ of the CM-type Φ as follows: for all $\tau \in \text{Gal}(\mathbb{Q}^{\text{alg}}/\mathbb{Q})$,

$$\sum_{\sigma \in \Sigma_E} \varphi(\tau^{-1} \circ \sigma) g(\sigma) = 0.$$
(2.1)

On the other hand, the assumption on \mathfrak{p} implies that $G_{\mathfrak{p}} = 1$. Therefore, by Proposition 2.1, we obtain that for all $\tau \in \operatorname{Gal}(\mathbb{Q}^{\operatorname{alg}}/\mathbb{Q})$,

$$s_{\sigma\pi}(\tau\mathfrak{p}) = \varphi(\tau^{-1} \circ \sigma).$$

From this equation and the equality $E = E_0$, the condition (1.3) of triviality on $P(A_0)$ of a character of $L(A)(= L(A_0))$ is described as follows: for all $\tau \in \text{Gal}(\mathbb{Q}^{\text{alg}}/\mathbb{Q})$,

$$\sum_{\sigma \in \Sigma_{E_0}} g(\sigma) s_{\sigma\pi}(\tau \mathfrak{p}) = \sum_{\sigma \in \Sigma_E} \varphi(\tau^{-1} \circ \sigma) g(\sigma) = 0.$$
 (2.2)

Since $L(A) = L(A_0)$ and conditions (2.1) (2.2) are coincide, we obtain that $MT(A) = P(A_0)$. This completes the proof.

3 Example

From Theorem 0.1 and a result of Aoki [1] on CM abelian varieties of Fermat type, we obtain examples of a simple *degenerate* abelian variety A_0 over \mathbb{F} for which the Tate conjecture holds. Here we give a such example.

Let m = 27 and let $\alpha = (1, 9, 17)$. Here α is an element of the set \mathcal{A}_m^1 defined as follows:

$$\mathcal{A}_{m}^{1} := \{ \alpha = (a_{0}, a_{1}, a_{2}) \in (\mathbb{Z}/m\mathbb{Z})^{3} \mid a_{i} \not\equiv 0 \pmod{m}, a_{0} + a_{1} + a_{2} \equiv 0 \pmod{m} \}$$

We define a subset Φ_{α} of $\mathbb{Z}/m\mathbb{Z}$ as

$$\Phi_{\alpha} := \{ t \in \mathbb{Z}/m\mathbb{Z} \mid \langle ta_0 \rangle + \langle ta_1 \rangle + \langle ta_2 \rangle = m \}$$

where for any $c \in \mathbb{Z}/m\mathbb{Z}$ we denote by $\langle c \rangle$ the least natural number such that $\langle c \rangle \equiv c \mod m$. Then let $A = A_{\alpha}$ be a simple abelian variety with CM-type $(\mathbb{Q}(\mu_m), \Phi_{\alpha})$. Then by a result of Aoki [1, Theorem 2.1], A is degenerate and the Hodge conjecture holds for all powers of A.

On the other hand, let A_0 be a simple factor of the reduction of A at a prime w of \mathbb{Q}^{alg} dividing a prime p. By Theorem 0.1 (2), we see that if $p \equiv 1 \mod m$, then A_0 is degenerate and the Tate conjecture holds for all powers of A_0 . Furthermore, using Theorem 1.8, one can see that if $p^9 \equiv 1 \mod m$ then all powers of A_0 are nondegenerate.

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