Wong-Zakai approximation of solutions to reflecting stochastic differential equations on domains in Euclidean spaces*

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Abstract

In this paper, we study the Wong-Zakai approximation of the solution to the stochastic differential equation on a domain D in a Euclidean space with normal reflection at the boundary. We prove the L^p convergence of the approximation in $C([0,T] \to \bar{D})$ under some general conditions on D.

1 Introduction

Stochastic differential equations(SDEs) are defined as stochastic integral equations. The definition of the stochastic integrals is based on martingale theory although there are pathwise approaches to this problems via rough path theory recently. A simple relation between SDE and usual ordinal differential equation(=ODE) were found by Wong and Zakai [16]. That is, they consider Stratonovich SDE and corresponding ODE which is obtained by replacing the Brownian motion by the piecewise linear approximation and prove that the solution of the ODE converges to the solution of the Stratonovich SDE almost surely in the topology of uniform convergence when the approximation becomes finer. More general approximations of paths are found, *e.g.*, in [6]. When we consider SDE on a domain D in $\mathbb{R}^d$, we need to consider boundary conditions. In this paper, we study Wong-Zakai approximations of solutions to SDE with reflecting boundary conditions on $\bar{D}$ and prove the L^p convergence of them to the solution in $C([0,T] \to \bar{D})$. This is not a first study of Wong-Zakai approximation of reflecting SDE. Doss and Priouret [2] proved the uniform convergence of the Wong-Zakai approximations in probability in the case where ∂D is sufficiently smooth. Also Pettersson [8] proved the almost sure convergence in the case where D is a convex domain with the property (B) in Tanaka [15] and the diffusion coefficient is a constant matrix. This result was improved by Ren and Xu [10, 11]. They studied Stroock-Varadhan's type support theorem for stochastic variational inequalities and showed the convergence in probability in $C([0,T] \to \bar{D})$. This result corresponds to the case of reflecting SDEs on convex domains. Actually the existence and uniqueness of the solutions were proved by Lions and Sznitman [7] and Saisho [12] for more general domains. In such cases, Evans and Stroock [4] proved the weak convergence of the law of the Wong-Zakai approximations. Our

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results improve their weak convergence to L^p convergence in $C([0,T]\to\bar{D})$. We note that there are studies of Euler and Euler-Peano approximations of reflecting SDE. We refer them to the papers [13, 14] by Słomiński.

The paper is organized as follows. In Section 2, we state our main theorem (Theorem 2.9). First, we recall the basic results on the Skorohod problems and the existence and uniqueness of the strong solutions of reflecting SDE based on [7] and [12]. In particular, we explain the conditions on domains under which we will work. In Section 3, we prove L^p convergence of Euler-Peano approximations. In Section 4, we prove our main theorem by estimating the difference between the Euler-Peano and Wong-Zakai approximations.

2 Preliminary and main theorem

Let D be a non-empty open connected set in $\mathbb{R}^d$. In this paper, we do not assume the boundedness of the boundary of D or D itself. We define the set $\mathcal{N}_x$ of inward unit normal vectors at the boundary point $x\in\partial D$ by

$$\mathcal{N}_x=\cup_{r>0}\mathcal{N}_{x,r} \tag{2.1}$$

$$\mathcal{N}_{x,r}=\left\{\mathbf{n}\in\mathbb{R}^d \mid |\mathbf{n}|=1, B(x-r\mathbf{n},r)\cap D=\emptyset\right\}, \tag{2.2}$$

where $B(z,r)=\{y\in\mathbb{R}^d\mid |y-z|<r\}$, $z\in\mathbb{R}^d$, $r>0$. In this paper, the function space C_b^k denotes a set of k-times continuously differentiable functions such that all their derivatives and themselves are bounded. Let us recall conditions (A), (B), (C) following [12].

Definition 2.1. (1) *Condition* (A) (*uniform exterior sphere condition*). *There exists a constant* $r_0>0$ *such that*

$$\mathcal{N}_x=\mathcal{N}_{x,r_0}\neq\emptyset \quad \text{for any } x\in\partial D. \tag{2.3}$$

(2) *Condition* (B). *There exist constants* $\delta>0$ *and* $\beta\geq 1$ *satisfying: for any* $x\in\partial D$ *there exists a unit vector* l_x *such that*

$$(l_x,\mathbf{n})\geq\frac{1}{\beta} \qquad \text{for any } \mathbf{n}\in\cup_{y\in B(x,\delta)\cap\partial D}\mathcal{N}_y. \tag{2.4}$$

(3) *Condition* (C). *There exists a* C_b^2 *function* f *on* $\mathbb{R}^d$ *and a positive constant* γ *such that for any* $x\in\partial D$, $y\in\bar{D}$, $\mathbf{n}\in\mathcal{N}_x$ *it holds that*

$$(y-x,\mathbf{n})+\frac{1}{\gamma}\left((Df)(x),\mathbf{n}\right)|y-x|^2\geq 0. \tag{2.5}$$

Note that if D is a convex domain, the condition (A) holds for any r_0 and the condition (C) holds for $f\equiv 0$. The admissibility condition on D in [7] is the property that D can be approximated by domains with smooth boundary in a certain sense. In this paper, we do not use such a property and we refer it to [7]. Here we explain what Skorohod problem is. Let $w=w(t)$ $(0\leq t\leq T)$ be a continuous path on $\mathbb{R}^d$ with $w(0)\in\bar{D}$. The pair of paths (ξ,ϕ) on $\mathbb{R}^d$ is a solution of a Skorohod problem associated with w if the following properties hold.

(i) $\xi=\xi(t)$ $(0\leq t\leq T)$ is a continuous path in $\bar{D}$ with $\xi(0)=w(0)$.

(ii) It holds that $\xi(t) = w(t) + \phi(t)$ for all $0 \le t \le T$.

(iii) $\phi = \phi(t)$ $(0 \le t \le T)$ is a continuous bounded variation path on $\mathbb{R}^d$ such that $\phi(0) = 0$ and

$$\phi(t) = \int_0^t \mathbf{n}(s) d\|\phi\|_{[0,s]} \tag{2.6}$$

$$\|\phi\|_{[0,t]} = \int_0^t 1_{\partial D}(\xi(s)) d\|\phi\|_{[0,s]}. \tag{2.7}$$

where $\mathbf{n}(t) \in \mathcal{N}_{\xi(t)}$ if $\xi(t) \in \partial D$.

In the above, $\|\phi\|_{[0,t]}$ stands for the total variation norm of ϕ. See (2.10).

The existence and uniqueness of solutions were proved by Tanaka [15] for the convex domain with additional assumptions. Lions and Sznitman proved the existence and uniqueness under conditions (A), (B) and the admissibility of D. This was proved without the addmissibility condition by Saisho[12] as follows.

Theorem 2.2. *Assume conditions* (A) *and* (B). *Then there exists a unique solution to the Skorohod problem for any continuous path w. Moreover the mapping $\Gamma : w \mapsto \xi$ is continuous in the uniform convergence topology.*

Doss and Priouret [2] proved the convergence of Wong-Zakai approximation. They used the Lipschitz continuity of the Skorohod map $\Gamma : w \mapsto \xi$ in the half space case. Under conditions (A) and (B), it is proved that Γ is 1/2-Hölder continuous map in the uniform convergence topology. See [7, 12]. If Γ is Lipschitz continuous, Doss and Priouret's approach may be applicable. We use the notation $L(w) = \Gamma(w) - w$ which corresponds to the local time at the boundary ∂D.

The bounded variation norm of ϕ can be controlled by the supremum norm of w and the modulus of continuity. Such an estimate is proved by Tanaka [15] in the case of convex domains. Similar estimates are obtained by Saisho [12] without assumptions of the convexity. For our purpose, we need quantitative version of Saisho's estimate. To this end, we introduce the following quantities of the continuous path w. Let $0 < \theta \le 1$ and define

$$\|w\|_{\mathcal{H},[s,t],\theta} = \sup_{s \le u < v \le t} \frac{|w(v) - w(u)|}{|u - v|^\theta}. \tag{2.8}$$

Also we use the oscillation and the total variation of the path:

$$\|w\|_{\infty,[s,t]} = \max_{s \le u \le v \le t} |w(u) - w(v)|, \tag{2.9}$$

$$\|w\|_{[s,t]} = \sup_{\Delta} \sum_{k=1}^{N} |w(t_k) - w(t_{k-1})|, \tag{2.10}$$

where $\Delta = \{s = t_0 < \cdots < t_N = t\}$ is a partition of the interval $[s, t]$.

Lemma 2.3. *Assume* (A) *and* (B). *Let $0 < \theta \le 1$. Then there exist positive constants C_1, C_2, C_3 which depend only on θ, δ, β and r_0 in the Assumptions* (A) *and* (B) *such that*

$$\|\phi\|_{[s,t]} \le C_1 \left(1 + \|w\|_{\mathcal{H},[s,t],\theta}^{C_2} (t - s)\right) e^{C_3 \|w\|_{\infty,[s,t]}} \|w\|_{\infty,[s,t]} \qquad \textit{for all } 0 \le s < t \le T. \tag{2.11}$$

Proof. Let $0 \le s < t \le T$. The proof of this lemma is essentially the same as that of Proposition 3.1 in [12]. However, since the estimate in the above lemma is quantitative version of Proposition 3.1, we give the proof for the sake of completeness and reader's convenience. First we define a sequence of times inductively by

$$\begin{aligned}
T_0 &= \inf\{u \mid \xi(u) \in \partial D, s \le u \le t\}, \\
t_n &= \inf\{u \mid |\xi(u) - \xi(T_{n-1})| \ge \delta,\ T_{n-1} < u \le t\}, \qquad (n \ge 1) \\
T_n &= \inf\{u \mid \xi(u) \in \partial D,\ t_n \le u \le t\}. \qquad (n \ge 1)
\end{aligned}$$

We use the convention that the times are t if the sets on the RHS are empty. If $\xi(s) \notin \partial D$ and $T_0 = t$, ξ does not hit ∂D in the time interval $[s, t)$ and $\|\phi\|_{[s,t]} = 0$. Hence it is sufficient to consider other cases. In those cases, since ξ is a continuous path, there exists a minimum natural number N such that $t = T_N$. Let $T_{n-1} \le u < v \le T_n$ $(1 \le n \le N)$. We prove

$$\|\phi\|_{[u,v]} \le \beta \left(\|\xi\|_{\infty,[u,v]} + \|w\|_{\infty,[u,v]}\right). \tag{2.12}$$

Suppose $u, v \le t_n$. Let $l = l_{\xi(T_{n-1})}$. Then using the condition (B), we have

$$\begin{aligned}
(l, \xi(v) - \xi(u)) &= (l, w(v) - w(u)) + (l, \phi(v) - \phi(u)) \\
&= (l, w(v) - w(u)) + \int_u^v (l, \mathbf{n}(r)) d\|\phi\|_{[u,r]} \\
&\ge (l, w(v) - w(u)) + \frac{1}{\beta}\|\phi\|_{[u,v]},
\end{aligned}$$

which implies (2.12). Let us consider the case where $T_n > t_n$ and $v > t_n$. Since $\|\phi\|_{[t_n,v]} = 0$, we obtain

$$\|\phi\|_{[u,v]} = \|\phi\|_{[u,t_n]} \le \beta \left(\|\xi\|_{\infty,[u,t_n]} + \|w\|_{\infty,[u,t_n]}\right)$$

which implies (2.12). By Lemma 2.3 (ii) in [12], for any $0 \le s \le t \le T$,

$$|\xi(t) - \xi(s)|^2 \le |w(t) - w(s)|^2 + \frac{1}{r_0}\int_s^t |\xi(u) - \xi(s)|^2 d\|\phi\|_{[0,u]} + 2\int_s^t (w(t) - w(u), d\phi(u)). \tag{2.13}$$

Hence

$$|\xi(t) - \xi(s)|^2 \le |w(t) - w(s)|^2 + 2\|w\|_{\infty,[s,t]}\|\phi\|_{[s,t]} + \frac{1}{r_0}\int_s^t |\xi(u) - \xi(s)|^2 d\|\phi\|_{[s,u]}. \tag{2.14}$$

By the Gronwall inequality (see Lemma 2.2 in [12]), we obtain

$$\begin{aligned}
|\xi(t) - \xi(s)|^2 &\le \left(\|w\|^2_{\infty,[s,t]} + 2\|w\|_{\infty,[s,t]}\|\phi\|_{[s,t]}\right) \exp\left(\|\phi\|_{[s,t]}/r_0\right) \\
&\le \left\{\left(1 + 1/\varepsilon^2\right)\|w\|^2_{\infty,[s,t]} + \varepsilon^2\|\phi\|^2_{[s,t]}\right\} \exp\left(\|\phi\|_{[s,t]}/r_0\right)
\end{aligned}$$

and

$$|\xi(t) - \xi(s)| \le \left\{(1 + 1/\varepsilon)\|w\|_{\infty,[s,t]} + \varepsilon\|\phi\|_{[s,t]}\right\} \exp\left(\|\phi\|_{[s,t]}/2r_0\right). \tag{2.15}$$

Here ε is any positive number. Now we prove that for any $T_{n-1} \le u \le v \le T_n$,

$$\|\phi\|_{[u,v]} \le \beta \left(G(\|w\|_{\infty,[u,v]}) + 2\right) \|w\|_{\infty,[u,v]}, \tag{2.16}$$

where

$$\begin{aligned} G(x) &= 4\left\{1 + \beta H(x)\right\} H(x), \\ H(x) &= \exp\left\{\beta\left(2\delta + x\right)/(2r_0)\right\}. \end{aligned}$$

We consider three cases (i) $T_{n-1} \le u < v \le t_n$, (ii) $t_n \le u < v \le T_n$, (iii) $u \le t_n < v \le T_n$. Let us consider the case (i). In this case $\|\xi\|_{\infty,[u,v]} \le 2\delta$. By combining this, (2.12) and (2.15), we have

$$\|\xi\|_{\infty,[u,v]} \le \left\{(1 + 1/\varepsilon)\|w\|_{\infty,[u,v]} + \varepsilon\beta\left(\|\xi\|_{\infty,[u,v]} + \|w\|_{\infty,[u,v]}\right)\right\} H(\|w\|_{\infty,[u,v]}).$$

Setting $\varepsilon = 1/\left(2\beta H(\|w\|_{\infty,[u,v]})\right)$, we obtain

$$\|\xi\|_{\infty,[u,v]} \le 4\left(1 + \beta H(\|w\|_{\infty,[u,v]})\right) H(\|w\|_{\infty,[u,v]})\|w\|_{\infty,[u,v]}. \tag{2.17}$$

We consider the case (ii). In this case, $\phi(r) = \phi(u)$ for all $u \le r \le v$ and $\|\xi\|_{\infty,[u,v]} = \|w\|_{\infty,[u,v]}$. Hence in the case of (iii),

$$\begin{aligned} \|\xi\|_{\infty,[u,v]} &\le \|\xi\|_{\infty,[u,t_n]} + \|\xi\|_{\infty,[t_n,v]} \\ &\le 4\left(1 + \beta H(\|w\|_{\infty,[u,v]})\right) H(\|w\|_{\infty,[u,v]})\|w\|_{\infty,[u,v]} + \|w\|_{\infty,[u,v]}. \end{aligned} \tag{2.18}$$

Consequently, by (2.12), the proof of (2.16) is finished. Using (2.16),

$$\begin{aligned} \|\phi\|_{[s,t]} &\le \beta \sum_{n=1}^{N} \left(G(\|w\|_{\infty,[T_{n-1},T_n]}) + 2\right) \|w\|_{\infty,[T_{n-1},T_n]} \\ &\le N\beta(G(\|w\|_{\infty,[s,t]}) + 2)\|w\|_{\infty,[s,t]}. \end{aligned} \tag{2.19}$$

We estimate N. Suppose $N \ge 2$. Since for any $1 \le n \le N-1$,

$$\begin{aligned} \delta &= |\xi(T_{n-1}) - \xi(t_n))| \\ &\le \|\xi\|_{\infty,[T_{n-1},T_n]} \\ &\le \left\{4\left(1 + \beta H(\|w\|_{\infty,[T_{n-1},T_n]})\right) H(\|w\|_{\infty,[T_{n-1},T_n]}) + 1\right\} \|w\|_{\infty,[T_{n-1},T_n]} \\ &\le \left\{4\left(1 + \beta H(\|w\|_{\infty,[T_{n-1},T_n]})\right) H(\|w\|_{\infty,[T_{n-1},T_n]}) + 1\right\} \|w\|_{\mathcal{H},[s,t],\theta}(T_n - T_{n-1})^{\theta}. \end{aligned} \tag{2.20}$$

Thus we have

$$T_n - T_{n-1} \ge \delta^{1/\theta}\left[\left\{4\left(1 + \beta H(\|w\|_{\infty,[T_{n-1},T_n]})\right) H\left(\|w\|_{\infty,[T_{n-1},T_n]}\right) + 1\right\} \|w\|_{\mathcal{H},[s,t],\theta}\right]^{-1/\theta}.$$

Summing the numbers on both sides from $n = 1$ to $n = N-1$, we obtain

$$t - s \ge (N-1)\delta^{1/\theta}\left[\left\{4\left(1 + \beta H(\|w\|_{\infty,[T_{n-1},T_n]})\right) H\left(\|w\|_{\infty,[T_{n-1},T_n]}\right) + 1\right\} \|w\|_{\mathcal{H},[s,t],\theta}\right]^{-1/\theta}$$

and

$$N - 1 \le \left[\delta^{-1}\left\{4\left(1 + \beta H(\|w\|_{\infty,[s,t]})\right) H(\|w\|_{\infty,[s,t]}) + 1\right\} \|w\|_{\mathcal{H},[s,t],\theta}\right]^{1/\theta}(t - s). \tag{2.21}$$

Clearly, this estimate is true when $N = 1$. The estimates (2.19) and (2.21) complete the proof of the lemma. ∎

It is easy to see that the term $\|w\|_{\mathcal{H},[s,t],\theta}$ in the above estimate can be replaced by a quantity defined by a modulus of continuity of w. We emphasize that we just need the continuity of w to estimate the bounded variation norm of ϕ. Also we note that this estimate is not sharp in the sense that the quantity on the RHS does not depend on the starting point x although $\|\phi\|_{[s,t]}$ does. If w is a continuous bounded variation path, we can prove the following estimate. This estimate is used to prove the exponential integrability of Y^N in the proof of Lemma 4.5.

Lemma 2.4. *Assume condition* (A) *and the existence of the solution ξ to the Skorohod problem for a continuous bounded variation path w. Then the total variation of the solution ξ has the estimate:*

$$\|\xi\|_{[s,t]} \le 2(\sqrt{2}+1)\|w\|_{[s,t]} \tag{2.22}$$

Proof. We write

$$\omega(s,t)=\|w\|_{[s,t]},\quad \eta_0(s,t)=|\xi(t)-\xi(s)|,\quad \eta(s,t)=\|\xi\|_{[s,t]}.$$

We use the estimate (2.13). Noting

$$\begin{aligned}\frac{1}{r_0}\int_s^t \eta_0(s,u)^2 d|\phi|_u &\le \frac{1}{r_0}\left(\int_s^t \eta(s,u)^2 d_u\eta(s,u)+\int_s^t \eta(s,u)^2 d_u\omega(s,u)\right)\\ &\le \frac{1}{r_0}\left(\frac{1}{3}\eta(s,t)^3+\eta(s,t)^2\omega(s,t)\right)=:k(s,t)\end{aligned} \tag{2.23}$$

and

$$\begin{aligned}\left|2\int_s^t (w(t)-w(u),d\phi(u))\right| &\le 2\left(\int_s^t \omega(u,t)d_u\omega(s,u)+\int_s^t \omega(u,t)d_u\eta(s,u)\right)\\ &\le 2\left(\int_s^t (\omega(s,t)-\omega(s,u))d_u\omega(s,u)+\int_s^t \omega(s,t)d_u\eta(s,u)\right)\\ &=\omega(s,t)^2+2\omega(s,t)\eta(s,t),\end{aligned}$$

we obtain

$$\eta_0(s,t)^2 \le 2\omega(s,t)^2+2\omega(s,t)\eta(s,t)+k(s,t) \tag{2.24}$$

and

$$2\eta_0(s,t)^2 \le \eta_0(s,t)^2+\eta(s,t)^2 \le \omega(s,t)^2+(\omega(s,t)+\eta(s,t))^2+k(s,t).$$

Therefore we have

$$\sqrt{2}\eta_0(s,t)\le 2\omega(s,t)+\eta(s,t)+\sqrt{k(s,t)}. \tag{2.25}$$

Note that

$$\eta(s,t)=\lim_{|\Delta|\to 0}\sum_{i=1}^n \eta_0(t_{i-1},t_i),$$

where Δ is a partition $s=t_0<\cdots<t_n=t$ and $|\Delta|=\sup_i(t_i-t_{i-1})$. We consider the term $\sqrt{k(s,t)}$. Using

$$\sqrt{k(s,t)}\le \sqrt{\eta(s,t)/(3r_0)}\,\eta(s,t)+\sqrt{\omega(s,t)/(r_0)}\,\eta(s,t)$$

and the additivity, $\eta(s,t)=\sum_{i=1}^n \eta(t_{i-1},t_i)$, we obtain

$$\sum_{i=1}^{n}\sqrt{k(t_{i-1},t_i)} \le \sup_i\left(\sqrt{\eta(t_{i-1},t_i)/(3r_0)}+\sqrt{\omega(t_{i-1},t_i)/(r_0)}\right)\eta(s,t)\to 0 \qquad \text{as } |\Delta|\to 0.$$

Thus, we get $\sqrt{2}\eta(s,t)\le 2\omega(s,t)+\eta(s,t)$ which proves the desired inequality. $\square$

Remark 2.5. Under the admissibility of the domain, Lions and Sznitman proved that $\|\phi\|_{[s,t]}\le \|w\|_{[s,t]}$ which implies $\|\xi\|_{[s,t]}\le 2\|w\|_{[s,t]}$. They use regularity property of the distance function from the boundary ∂D. So we may need some regularity condition on the boundary to prove such a stronger estimate. We note that there is a study of the regularity of the distance function, *e.g.*, [9]. However, the estimate (2.22) is enough for our purposes.

Let us recall the existence of strong solution and the uniqueness which is due to [15, 7, 12]. Let $(\Omega,\mathcal{F},P)$ be a complete probability space and $\mathcal{F}_t$ be the right-continuous filtration with the property that $\mathcal{F}_t$ contains all null sets of $(\Omega,\mathcal{F},P)$. Let $B=B(t)$ be an $\mathcal{F}_t$-Brownian motion on $\mathbb{R}^n$. Let $\sigma\in C(\mathbb{R}^d\to\mathbb{R}^n\otimes\mathbb{R}^d)$, $b\in C(\mathbb{R}^d\to\mathbb{R}^d)$ be continuous mappings. We consider an SDE with reflecting boundary condition on $\bar{D}$:

$$X(t)=x+\int_0^t\sigma(X(s))dB(s)+\int_0^t b(X(s))ds+\Phi(t), \tag{2.26}$$

where $x\in\bar{D}$. We denote this SDE by SDE(σ,b) simply. A pair of $\mathcal{F}_t$-adapted continuous processes $(X(t),\Phi(t))$ is called a solution to (2.26) if the following holds. Let

$$Y(t)=x+\int_0^t\sigma(X(s))dB(s)+\int_0^t b(X(s))ds \tag{2.27}$$

Then $(X(\cdot,\omega),\Phi(\cdot,\omega))$ is a solution of the Skorohod problem associated with $Y(\cdot,\omega)$ for almost all $\omega\in\Omega$. The following result is due to [12].

Theorem 2.6. *Assume D satisfies conditions* (A) *and* (B) *and σ and b are bounded and global Lipschitz maps. Then there exists a unique strong solution to* (2.26).

Here we note the following. This follows from Garsia-Rodemich-Rumsey's estimate.

Lemma 2.7. *Let $F=F(t,\omega)$ be a $\mathbb{R}^d$-valued continuous process with the property that for all $p\ge 1$*

$$E[|F(t)-F(s)|^{2p}]\le C_p|t-s|^p \qquad 0\le s\le t\le T. \tag{2.28}$$

Then for all $0<\theta<1$ and $p\ge 1$ there exist constants $C'_{p,\theta}$ which depends only on C_p and θ such that

$$E[\|F\|^p_{\mathcal{H},[0,T],\theta/2}]\le C'_{p,\theta}. \tag{2.29}$$

If σ is bounded, then the quadratic variation of $M(t)=\int_0^t\sigma(X(s))dB(s)$ is bounded and we see the exponential integrability of $\max_{0\le t\le T}|M(t)|$. Therefore, using Lemma 2.3 and Lemma 2.7 and Burkholder-Davis-Gundy's inequality, we immediately obtain the following estimate.

Lemma 2.8. *Assume the same assumptions as in Theorem* 2.6*. Let* $p \geq 1$*. There exists a positive constant* C_p *such that*

$$E[\|X\|_{\infty,[s,t]}^{2p}] \leq C_p|t-s|^p, \tag{2.30}$$

$$E[\|\Phi\|_{[s,t]}^{2p}] \leq C_p|t-s|^p. \tag{2.31}$$

From now on, we always assume that σ belongs to C_b^2 and b belongs to C_b^1. Now, we are going to explain our main theorem. Let $N \in \mathbb{N}$. Let $X^N(t)$ be the solution to the reflecting ODE:

$$X^N(t) = x + \int_0^t \sigma(X^N(s))dB^N(s) + \int_0^t b(X^N(s))ds + \Phi^N(t), \tag{2.32}$$

where

$$B^N(t) = B(t_{k-1}^N) + \frac{\Delta_N B_k}{\Delta_N}(t - t_{k-1}^N) \qquad t_{k-1}^N \leq t \leq t_k^N, \tag{2.33}$$

$$\Delta_N B_k = B(t_k^N) - B(t_{k-1}^N), \qquad \Delta_N = T/N, \qquad t_k^N = \frac{kT}{N}. \tag{2.34}$$

We already explained the existence of the strong solution to a reflecting SDE driven by a Brownian motion. The definition of the solution to the above equation is similar to reflecting SDE. The existence and uniqueness of the solutions follows from Theorem 2.6. We prove an existence and uniqueness theorem when the driving path is a continuous bounded variation path in Section 4. The following is our main theorem. In this paper, we do not intend to obtain the best order. The order given below is probably far from best.

Theorem 2.9. *Assume* (A)*,* (B) *and* (C)*. Let* X *be the solution to* $\mathrm{SDE}(\sigma, \tilde{b})$*, where* $\tilde{b} = b + \frac{1}{2}\mathrm{tr}(D\sigma)(\sigma)$*. Let* $0 < \theta < 1$*. For any* $p \geq 1$*, there exists a positive constant* $C_{p,T,\theta}$ *such that for all* $N \in \mathbb{N}$*,*

$$E\left[\max_{0\leq t\leq T} |X^N(t) - X(t)|^{2p}\right] \leq C_{p,T,\theta}\Delta_N^{\theta/6}. \tag{2.35}$$

As we noted, although this estimate may not be good, by this result and Borel-Cantelli lemma, we can conclude

$$\lim_{N\to\infty} \max_{0\leq t\leq T} |X^{2^N}(t) - X(t)| = 0 \qquad \text{almost surely.} \tag{2.36}$$

In order to prove this theorem, we need the Euler-Peano approximation of the solution. We explain the Euler-Peano approximation in the next Section.

3 Euler-Peano approximation

In this section, we consider the Euler-Peano approximation X_E^N of X. For $0 \leq k \leq N$, set $t_k^N = kT/N$. Let us define $X_E^N(t)$ $(0 \leq t \leq T)$ as the solution to the Skorohod problem inductively which is given by $X_E^N(0) = x \in \mathbb{R}^d$ and

$$\begin{aligned} X_E^N(t) &= X_E^N(t_{k-1}^N) + \sigma(X_E^N(t_{k-1}^N))(B(t) - B(t_{k-1}^N)) + b(X_E^N(t))(t - t_k^N) \\ &\quad + \Phi_E^N(t) - \Phi_E^N(t_{k-1}^N) \qquad t_{k-1}^N \leq t \leq t_k^N. \end{aligned} \tag{3.1}$$

In other words, X_E^N satisfies

$$X_E^N(t) = x + \int_0^t \sigma(X_E^N(\pi_N(s)))dB(s) + \int_0^t b(X_E^N(\pi_N(s)))ds + \Phi_E^N(t), \tag{3.2}$$

where $\pi_N(t) = \max\{t_k^N \mid t_k^N \le t\}$. Define

$$Y_E^N(t) = x + \int_0^t \sigma(X_E^N(\pi_N(s)))dB(s) + \int_0^t b(X_E^N(\pi_N(s)))ds. \tag{3.3}$$

Then by the definition of the solution of the SDE, it holds that

$$X_E^N(t) = \Gamma\left(Y_E^N\right)(t). \tag{3.4}$$

We prove

Theorem 3.1. *Assume* (A), (B) *and* (C)*. Then for any $p \ge 1$, there exists $C_p > 0$ such that*

$$E\left[\max_{0\le t\le T} |X_E^N(t) - X(t)|^{2p}\right] \le C_p \Delta_N^p. \tag{3.5}$$

This estimate was already proved in [13] for general convex domains under the conditions that σ and b are bounded and global Lipschitz continuous. Also the readers may find a result of local version of Euler-Peano and Euler approximation under the conditions (A) and (B) only in that paper.

To prove this theorem, we need the following lemma.

Lemma 3.2. *Assume* (A) *and* (B)*. Let $p \ge 1$. There exists a positive constant C_p which is independent of N such that*

$$E[\|X_E^N\|_{\infty,[s,t]}^{2p}] \le C_p|t-s|^p, \tag{3.6}$$

$$E\left[\|\Phi_E^N\|_{[s,t]}^{2p}\right] \le C_p|t-s|^p. \tag{3.7}$$

Proof. It suffices to prove (3.7). Since $M_E^N(t) = \int_0^t \sigma(X_E^N(\pi_N(s)))dB(s)$ is a martingale whose quadratic variation is uniformly bounded for N, we see that

$$\sup_N E[\exp(a \max_{0\le t\le T} |M_E^N(t)|)] < \infty$$

for all $a > 0$. Thus by Lemma 2.3, Lemma 2.7 and Burkholder-Davis-Gundy's inequality, we complete the proof. □

Proof of Theorem 3.1. The following proof is a modification of that of Lemma 3.1 in [7]. Note that we need just Lipschitz continuity of σ and b and their boundedness in the proof below. It suffices to prove the case where $p \ge 2$. Define

$$\begin{aligned} Z^N(t) &= X_E^N(t) - X(t), \\ \mu_N(t) &= e^{-\frac{2}{\gamma}(f(X_E^N(t)+f(X(t))))}, \\ k_N(t) &= \mu_N(t)|Z^N(t)|^2. \end{aligned}$$

Then we have

$$
\begin{aligned}
&dk_N(t)\\
&= \mu_N(t)\Bigg\{ 2\left(Z^N(t), (\sigma(X_E^N(\pi_N(t))) - \sigma(X(t)))dB(t)\right)\\
&\quad + 2\left(Z^N(t), b(X_E^N(\pi_N(t))) - b(X(t))\right)dt\\
&\quad + \mathrm{tr}\left(({}^t\sigma\sigma)(X_E^N(\pi_N(t)))\right)dt + \mathrm{tr}\left(({}^t\sigma\sigma)(X(t))\right)dt\\
&\quad - \mathrm{tr}\left(({}^t\sigma(X(t))\sigma(X_E^N(\pi_N(t))))\right) - \mathrm{tr}\left(({}^t\sigma(X_E^N(\pi_N(t)))\sigma(X(t)))\right)dt\Bigg\}\\
&\quad + 2\mu_N(t)\left(Z^N(t), d\Phi_E^N(t) - d\Phi(t)\right)\\
&\quad - \frac{2\mu_N(t)}{\gamma}\left|Z^N(t)\right|^2\left\{\left((Df)(X_E^N(t)), d\Phi_E^N(t)\right) + ((Df)(X(t)), d\Phi(t))\right\}\\
&\quad - \frac{2\mu_N(t)}{\gamma}\left|Z^N(t)\right|^2\left\{\left((Df)(X_E^N(t)), \sigma(X_E^N(\pi_N(t)))dB(t)\right) + ((Df)(X(t)), \sigma(X(t))dB(t))\right\}\\
&\quad + R_N(t)dt,
\end{aligned}
\tag{3.8}
$$

where

$$
\begin{aligned}
R(t) = {}& \frac{4\mu_N(t)}{\gamma}\left((Df)(X_E^N(t)), \sigma(X_E^N(\pi_N(t)))^t\left(\sigma(X(t)) - \sigma(X_E^N(\pi_N(t)))\right)\left(Z^N(t)\right)\right)dt\\
&+ \frac{4\mu_N(t)}{\gamma}\left((Df)(X(t)), \sigma(X(t))^t\left(\sigma(X_E^N(\pi_N(t))) - \sigma(X(t))\right)\left(Z^N(t)\right)\right)dt\\
&- \frac{2\mu_N(t)}{\gamma}|Z^N(t)|^2\left(\left((Df)(X_E^N(t)), b(X_E^N(\pi_N(t)))\right)dt + ((Df)(X(t)), b(X(t)))\,dt\right)\\
&- \frac{\mu_N(t)}{\gamma}|Z^N(t)|^2\Big\{\mathrm{tr}(D^2f)(X_E^N(t))(\sigma(X_E^N(\pi_N(t))\cdot, \sigma(X_E^N(\pi_N(t)))))\\
&\qquad\qquad + \mathrm{tr}(D^2f)(X(t))(\sigma(X(t)\cdot, \sigma(X(t))\cdot))\Big\}dt\\
&+ \frac{2\mu_N(t)}{\gamma^2}|(Df)(X_E^N(t))(\sigma(X_E^N(\pi_N(t)))) + (Df)(X(t))(\sigma(X(t)))|^2|Z^N(t)|^2dt.
\end{aligned}
\tag{3.9}
$$

Note that by condition (C),

$$
\begin{aligned}
&\left(X_E^N(t) - X(t), d\Phi_E^N(t) - d\Phi(t)\right)\\
&\qquad - \frac{1}{\gamma}\left|X_E^N(t) - X(t)\right|^2\left\{\left((Df)(X_E^N(t)), d\Phi_E^N(t)\right) + ((Df)(X(t)), d\Phi(t))\right\} \le 0
\end{aligned}
\tag{3.10}
$$

and $\sup_{0\le t\le T} E[|X_E^N(t) - X_E^N(\pi_N(t))|^p] \le C\Delta_N^{p/2}$. As for the first term on the RHS of (3.8), using Bukholder-Davis-Gundy's inequality, we get for any $0 \le T' \le T$,

$$\begin{aligned}
&E\left[\sup_{0\le t\le T'}\left|\int_0^t \mu_N(s)\left(X_E^N(s)-X(s),\sigma(X_E^N(\pi_N(s)))-\sigma(X(s))dB(s)\right)\right|^p\right]\\
&\le CE\left[\left(\int_0^{T'}|X_E^N(t)-X(t)|^4dt\right)^{p/2}\right]+CE\left[\left(\int_0^{T'}|X_E^N(\pi_N(t))-X_E^N(t)|^4dt\right)^{p/2}\right]\\
&\le C_TE\left[\int_0^{T'}k_N(t)^pdt\right]+C_TE\left[\int_0^{T'}|X_E^N(\pi_N(t))-X_E^N(t)|^{2p}dt\right]\\
&\le C_T\int_0^{T'}E[k_N(t)^p]dt+C_T\Delta_N^p. \qquad (3.11)
\end{aligned}$$

We can estimate the other terms similarly and we obtain

$$E\left[\sup_{0\le t\le T'}k_N(t)^p\right]\le C_T\Delta_N^p+C_T\int_0^{T'}E\left[\sup_{0\le s\le t}k_N(s)^p\right]dt. \qquad (3.12)$$

By the Gronwall inequality, this implies the desired estimate. □

4 Proof of main theorem

First, we prove the existence and uniqueness of the solution to reflecting ODE driven by a continuous bounded variation path.

Proposition 4.1. *Assume the conditions* (A) *and* (B) *hold. Let $w = w(t)$ $(0 \le t \le T)$ be a continuous bounded variation path on $\mathbb{R}^n$. Then there exists a unique continuous bounded variation path $x(t)$ on $\mathbb{R}^d$ satisfying the reflecting ODE:*

$$x(t)=x+\int_0^t\sigma(x(s))dw(s)+\int_0^t b(x(s))ds+\Phi(t), \qquad 0\le t\le T. \qquad (4.1)$$

Proof. The following proof is a modification of the proof of Theorem 5.1 in [12]. Note that the boundedness and the continuity of σ and b are sufficient for the existence of the solutions. Let us consider the partition of $[0,T]$ by $t_k^N = kT/N$. Let x^N be the Euler-Peano approximation of the solution, that is, let us define x^N as the solution of the Skorohod problem with $x^N(0) = x$:

$$\begin{aligned}
x^N(t)&=x^N(t_{k-1}^N)+\sigma(x^N(t_{k-1}^N))(w(t)-w(t_{k-1}^N))\\
&\quad+b(x^N(t_{k-1}^N))(t-t_{k-1}^N)+\Phi^N(t)-\Phi^N(t_{k-1}^N) \qquad t_{k-1}^N\le t\le t_k^N. \qquad (4.2)
\end{aligned}$$

Let

$$y^N(t)=x+\int_0^t\sigma(x^N(\pi_N(s))dw(s)+\int_0^t b(x^N(\pi_N(s)))ds \qquad (4.3)$$

Then $\{y^N\}$ is a family of uniformly bounded equicontinuous paths defined on $[0,T]$ with values in $\mathbb{R}^d$. Therefore by the Arzela-Ascoli theorem, there exists a subsequence $\{y^{N_k}\}$ which converges in the uniform convergence topology. We denote the limit by y^∞. Then by the continuity of the

Skorohod map in Theorem 2.2, $x^{N_k}(=\Gamma(y^{N_k}))$, $\Phi^{N_k}(=L(y^{N_k})$ also converges to a continuous paths, say, x^∞, Φ^∞, in uniform convergence topology. Clearly, the pair (x^∞, Φ^∞) is a solution of a Skorohod problem associated with y^∞. Taking the limit $N_k \to \infty$ in (4.3), we have

$$y^\infty(t) = x + \int_0^t \sigma(x^\infty(s))dw(s) + \int_0^t b(x^\infty(s)))ds. \tag{4.4}$$

This shows that (x^∞, Φ^∞) is a solution of the reflecting ODE. We can check the uniqueness in a similar manner to Theorem 5.1 in [12]. Note that the boundedness of σ and b and their Lipschitz continuity are sufficient for the proof. □

Remark 4.2. We may prove the existence of the solution of reflecting ODE when the driving path is just p-variation path, where $1 \le p < 2$ using Davie's argument [1]. We will study this problem hopefully together with more general rough differential equation corresponding to the case of $p \ge 2$ in future's paper.

From now on, for simplicity, we may denote $\Delta_N B_k$, Δ_N, t_k^N by ΔB_k, Δ, t_k. By the definition, it holds that

$$X^N(t) = X^N(t_{k-1}) + \int_{t_{k-1}}^t \sigma(X^N(s))\frac{\Delta B_k}{\Delta}ds + \int_{t_{k-1}}^t b(X^N(s))ds \tag{4.5}$$

$$+ \Phi^N(t) - \Phi^N(t_{k-1}) \qquad t_{k-1} \le t \le t_k. \tag{4.6}$$

Clearly, $X^N(t_{k-1})$ is $\mathcal{F}_{t_{k-1}}$-measurable. Let

$$Y^N(t) = x + \int_0^t \sigma(X^N(s))dB^N(s) + \int_0^t b(X^N(s))ds. \tag{4.7}$$

Then $X^N = \Gamma(Y^N)$ and $\Phi^N = L(Y^N)$.

Lemma 4.3. *Assume* (A) *and* (B). *Fix $N \in \mathbb{N}$. Let $t_{k-1} \le s \le t \le t_k$. The constant C below is independent of t, s, k, N.*
(1) *The following relations hold.*

$$Y^N(t) - Y^N(t_{k-1}) = \int_{t_{k-1}}^t \sigma(X^N(s))\frac{\Delta B_k}{\Delta}ds + \int_{t_{k-1}}^t b(X^N(s))ds \tag{4.8}$$

and

$$|Y^N(t) - Y^N(s)| \le C\left(|\Delta B_k|\frac{t-s}{\Delta} + t - s\right) \tag{4.9}$$

$$\|\Phi^N\|_{[s,t]} \le C\left(|\Delta B_k|\frac{t-s}{\Delta} + t - s\right). \tag{4.10}$$

(2) *We have*

$$\int_{t_{k-1}}^{t} \sigma(X^N(s))\frac{\Delta B_k}{\Delta}ds = \sigma(X^N(t_{k-1}))\frac{\Delta B_k}{\Delta}(t-t_{k-1})$$
$$+\int_{t_{k-1}}^{t}\left(\int_{t_{k-1}}^{s}(D\sigma)(X^N(r))\sigma(X^N(r))\frac{\Delta B_k}{\Delta}dr\right)\frac{\Delta B_k}{\Delta}ds$$
$$+\int_{t_{k-1}}^{t}\left(\int_{t_{k-1}}^{s}(D\sigma)(X^N(r))(b(X^N(r)))dr\right)\frac{\Delta B_k}{\Delta}ds$$
$$+\int_{t_{k-1}}^{t}\left(\int_{t_{k-1}}^{s}(D\sigma)(X^N(r))d\Phi^N(r)\right)\frac{\Delta B_k}{\Delta}ds$$
$$= I_0^k(t)+I_1^k(t)+I_2^k(t)+I_3^k(t). \tag{4.11}$$

Let $I_4^k(t)=\int_{t_{k-1}}^t b(X^N(s))ds$. *Then*

$$|I_1^k(t)| \le C|\Delta B_k|^2\frac{(t-t_{k-1})^2}{\Delta^2}, \tag{4.12}$$
$$|I_2^k(t)| \le C|\Delta B_k|\frac{(t-t_{k-1})^2}{\Delta}, \tag{4.13}$$
$$|I_3^k(t)| \le C\left(|\Delta B_k|^2\left(\frac{t-t_{k-1}}{\Delta}\right)^2+\frac{(t-t_{k-1})^2}{\Delta}|\Delta B_k|\right), \tag{4.14}$$
$$|I_4^k(t)| \le C(t-t_{k-1}). \tag{4.15}$$

Proof. The proof of the equation (4.8) and (4.11) is a simple calculation. The estimate in (4.9) follows from (4.8). Hence the estimate (4.10) follows from this estimate and Lemma 2.4. By the boundedness of $\sigma, D\sigma, b$, we get (4.12), (4.13), (4.15). Using (4.10),

$$|I_3^k(t)| \le C\|\Phi^N\|_{[t_{k-1},t]}\frac{(t-t_{k-1})|\Delta B_k|}{\Delta}$$
$$\le C\left(|\Delta B_k|^2\left(\frac{t-t_{k-1}}{\Delta}\right)^2+\frac{(t-t_{k-1})^2}{\Delta}|\Delta B_k|\right). \tag{4.16}$$

This complete the proof. □

Lemma 4.4. *Assume* (A) *and* (B). *Let* $p \ge 1$. *There exists a positive constant* C_p *which is independent of* N *such that for all* $0 \le s \le t \le T$,

$$E[\|Y^N\|_{\infty,[s,t]}^{2p}] \le C_p|t-s|^p. \tag{4.17}$$

Proof. Pick two points $0 \le s \le t \le T$. First consider the case where there exists $1 \le k \le N$ such that $t_{k-1} \le s \le t \le t_k$. Then by (4.9),

$$\max_{s\le u\le v\le t}|Y^N(u)-Y^N(v)| \le C(|\Delta B_k|\frac{t-s}{\Delta}+t-s). \tag{4.18}$$

Hence $E[\|Y^N\|_{\infty,[s,t]}\|^{2p}] \le C_p(t-s)^p$. If $t_{k-1} \le s \le t_k < t \le t_{k+1}$ for some k, noting

$$\|Y^N\|_{\infty,[s,t]} \le \|Y^N\|_{\infty,[s,t_k]} + \|Y^N\|_{\infty,[t_k,t]}, \tag{4.19}$$

we can use the estimate in the first case. We consider the other cases. Let us choose $1 \le l < m-1 \le N$ such that $t_{l-1} \le s \le t_l < t_{m-1} \le t \le t_m$. Then

$$\begin{aligned} &Y^N(t) - Y^N(s) \\ &= \sum_{n=0}^{4} \left\{ I_n^l(t_l) - I_n^l(s) + \sum_{k=l+1}^{m-1} (I_n^k(t_k) - I_n^k(t_{k-1})) + I_n^m(t) - I_n^m(t_{m-1}) \right\} \\ &= \sum_{n=0}^{4} (J_n^N(t) - J_n^N(s)). \end{aligned} \tag{4.20}$$

Note that $\{J_n^N(t) \mid 0 \le t \le T\}$ are continuous processes and it suffices to estimate $E[\|J_n^N\|_{\infty,[s,t]}^{2p}]$. First let us consider the term J_0^N. Let $M^N(t)$ be a continuous $\mathcal{F}_t$-martingale such that

$$M^N(t) = \int_0^t \sigma(X^N(\pi_N(s)))dB(s). \tag{4.21}$$

Then J_0^N is the piecewise linear approximation of M^N at the times $\{t_k\}_{k=1}^N$. Therefore,

$$\begin{aligned} \|J_0^N\|_{\infty,[s,t]} &\le \max_{l-1\le k,k'\le m} |M^N(t_k) - M^N(t_{k'})| \\ &\le 2 \max_{l-1\le k\le m} |M^N(t_k) - M^N(t_l)| \\ &\le 2 \max_{t_{l-1}\le r\le t_m} |M^N(r) - M^N(t_l)|. \end{aligned} \tag{4.22}$$

Using Doob's inequality, we get

$$E[\|J_0^N\|_{\infty,[s,t]}^{2p}] \le C_p(t_m - t_{l-1})^p \le 3^p C_p (t-s)^p. \tag{4.23}$$

Next we consider the term J_3^N. By the estimate (4.14), we have

$$\|J_3^N\|_{\infty,[s,t]} \le C \sum_{k=l}^{m} \left(|\Delta B_k|^2 + \Delta \cdot |\Delta B_k|\right) \le C \left(\sum_{k=1}^{m} |\Delta B_k|^2 \right) + C\Delta(t-s). \tag{4.24}$$

Note that

$$\{\Delta B_k\}_{k=l}^m = \sqrt{\Delta}\{\xi_k\}_{k=l}^m \quad \text{in law}, \tag{4.25}$$

where $\{\xi_k\}_{k=l}^m$ are i.i.d. random vectors whose common distribution is the normal distribution on $\mathbb{R}^n$ with 0 mean and identity covariance matrix. Hence

$$E\left[\|J_3^N\|_{\infty,[s,t]}^{2p}\right] \le C_p \Delta^{2p} E\left[\left(\sum_{k=l}^{m} |\xi_k|^2 \right)^{2p} \right] + C_p (t-s)^{4p}. \tag{4.26}$$

Since $S_{m,l} = \sum_{k=l}^{m}(|\xi_k|^2 - n)$ belongs to the Wiener chaos of order 2, there exists a constant C_q ($q \geq 1$) which is independent of m, l such that

$$\|S_{m,l}\|_{L^q} \leq C_q \|S_{m,l}\|_{L^2}. \tag{4.27}$$

This follows from the hypercontractivity of the Ornstein-Uhlenbeck operator. See [5]. Therefore

$$\begin{aligned} E\left[\left(\sum_{k=l}^{m}|\xi_k|^2\right)^{2p}\right] &= E\left[\{S_{m,l} + n(m-l+1)\}^{2p}\right] \\ &\leq C_p\|S_{m,l}\|_{L^2}^{2p} + C_p\{n(m-l+1)\}^{2p} \\ &\leq C_p\{n(m-l+1)\}^{p} + C_p\{n(m-l+1)\}^{2p}. \end{aligned} \tag{4.28}$$

Thus

$$\begin{aligned} E\left[\|J_3^N\|_{\infty,[s,t]}^{2p}\right] &\leq C_p n^p \Delta^p (t-s)^p + C_p n^{2p}(t-s)^{2p} + C_p (t-s)^{4p} \\ &\leq C_p (t-s)^{2p}. \end{aligned} \tag{4.29}$$

We can estimate other terms in a similar way and we complete the proof. □

Lemma 4.5. *Assume* (A) *and* (B). *Let* $p \geq 1$. *There exists a positive number* C_p *which is independent of* N *such that for all* $0 \leq s \leq t \leq T$,

$$E[\|X^N\|_{\infty,[s,t]}^{2p}] \leq C_p|t-s|^p, \tag{4.30}$$

$$E[\|\Phi^N\|_{[s,t]}^{2p}] \leq C_p|t-s|^p. \tag{4.31}$$

Proof. It suffices to prove (4.31). By checking the exponential integrability of $\|Y^N\|_{\infty,[0,T]}$, we can prove this by using the fact $\Phi^N = L(Y^N)$, Lemma 2.3, Lemma 4.4 and Lemma 2.7. We prove that for any $a > 0$, there exists N_0 such that

$$\sup_{N \geq N_0} E[e^{a\|Y^N\|_{\infty,[0,T]}}] < \infty. \tag{4.32}$$

By the estimate (4.9),

$$\max_{0\leq t\leq T}|Y^N(t)| \leq \max_{0\leq k\leq N}|Y^N(t_k)| + C\max_{1\leq k\leq N}|\Delta B_k| + C/N. \tag{4.33}$$

Because $\sup_N E[e^{a\max_{1\leq k\leq N}|\Delta B_k|}] < \infty$ for all $a > 0$, it is sufficient to prove

$$\sup_{N\geq N_0} E[e^{a\max_{0\leq k\leq N}|Y^N(t_k)|}] \tag{4.34}$$

By the decomposition and the estimates of Y^N in Lemma 4.3, we have

$$\max_{0\leq k\leq N}|Y^N(t_k)| \leq C + \max_{0\leq k\leq N}|M^N(t_k)| + C\sum_{k=1}^{N}|\Delta B_k|^2, \tag{4.35}$$

where $\{M^N(t)\}$ is the continuous martingale which is defined in (4.21). Since the quadratic variation is bounded, we have $\sup_N E[e^{a\max_{0\le t\le T}|M^N(t)|}] < \infty$ for any a. Also

$$
\begin{aligned}
E\left[\exp\left(\sum_{k=1}^N Ca|\Delta B_k|^2\right)\right] &= \prod_{k=1}^N E[e^{Ca|\Delta B_k|^2}] \\
&= \prod_{k=1}^N \int_{\mathbb{R}^n} \exp\left(\frac{CaT}{N}|x|^2 - \frac{1}{2}|x|^2\right)\frac{1}{(2\pi)^{n/2}}dx \\
&= \left(1 - \frac{2CaT}{N}\right)^{-nN/2} \to e^{CaTn} \quad \text{as } N\to\infty. \qquad (4.36)
\end{aligned}
$$

These imply (4.32) and the proof is finished. □

The following is a key lemma for the proof of L^p convergence of Wong-Zakai approximation.

Lemma 4.6. *Assume* (A), (B) *and* (C). *Let X_E^N be the Euler-Peano approximation to SDE$(\sigma,\tilde{b})$, where $\tilde{b} = b + \frac{1}{2}\mathrm{tr}(D\sigma)(\sigma)$. Then for any $0<\theta<1$, there exists a positive constant C_θ such that for all N,*

$$
\sup_{0\le k\le N} E\left[|X^N(t_k^N) - X_E^N(t_k^N)|^2\right] \le C_\theta \cdot \Delta_N^{\theta/2}. \qquad (4.37)
$$

Remark 4.7. The order of convergence in (4.37) is, roughly speaking, half of that of the Wong-Zakai approximation to the SDE without reflection term. This convergence order can be expected by the 1/2-Hölder continuity of the Skorohod map. Consider two Skorohod equations $\xi = w + \phi$, $\xi' = w' + \phi'$. Then it was proved in [12] (see also [7]) that under the assumptions (A) and (B),

$$
\begin{aligned}
|\xi(t) - \xi'(t)|^2 \le & \left\{|w(t)-w'(t)|^2 + 4\left(\|\phi\|_{[0,t]} + \|\phi'\|_{[0,t]}\right)\max_{0\le s\le t}|w(s)-w'(s)|\right\} \\
& \exp\left\{\left(\|\phi\|_{[0,t]} + \|\phi'\|_{[0,t]}\right)/r_0\right\}, \qquad 0\le t\le T. \qquad (4.38)
\end{aligned}
$$

By this 1/2-Hölder continuity of the Skorohod map Γ, we obtain

$$
E[|\Gamma(B)_t - \Gamma(B^N)_t|^2] \le C\Delta_N^{\theta/2}, \qquad (4.39)
$$

where $0<\theta<1$. By examining the proof in [12], one can replace the term $\|\phi\|_{[0,t]} + \|\phi'\|_{[0,t]}$ in (4.38) by $\|\phi-\phi'\|_{[0,t]}$. We are not sure whether or not this change gives better estimates than the above. Of course the estimate in (4.38) is a pathwise estimate and there are no reason that the pathwise estimate gives good estimate for the expectation also. Of course, if D is a half space (or convex polyhedron, see [3]) in a Euclidean space, then Γ is Lipschitz continuous and the upper bound in (4.39) is $O(\Delta_N^\theta)$. Also, it seems that the calculation in [2] also gives the convergence speed $O(\Delta_N^\theta)$ for Wong-Zakai approximations of general reflecting SDEs in the half space case. However, We do not know examples of reflecting SDE for which the slow convergence speed $\Delta_N^{\theta/2}$ really appear.

In the proof of this lemma, the integrals which contains $\mathcal{F}_t$-semimartingales and non-adapted bounded variation processes, *e.g.* Wong-Zakai approximation $X_N(t)$ appear. Hence we need the following definition of the integrals.

Lemma 4.8. *Let $X(t), Y(t)$ be $\mathcal{F}_t$-continuous semimartingales and $A(t)$ be bounded variation continuous process. Suppose that $\sup_{0\le t\le T}\{|X(t)|+|Y(t)|\}+|A(\cdot)|_{[0,T]}\in L^p$ for all p. Define*

$$\int_0^t X(s)A(s)dY(s)=\lim_{N\to\infty}\sum_{k=1}^N X(t_{k-1}^N)A(t_{k-1}^N)(Y(t_k^N)-Y(t_{k-1}^N)), \tag{4.40}$$

$$\langle XA,Y\rangle_t=\lim_{N\to\infty}\sum_{k=1}^N\left((X(t_k^N)A(t_k^N))-(X(t_{k-1}^N)A(t_{k-1}^N))\right)(Y(t_k^N)-Y(t_{k-1}^N)), \tag{4.41}$$

where $t_k^N=tk/N$. These converge in probability and it holds that

$$\int_0^t X(s)A(s)dY(s)=\int_0^t A(s)dZ(s)=A(t)Z(t)-\int_0^t Z(s)dA(s), \tag{4.42}$$

$$\langle XA,Y\rangle_t=\int_0^t A(s)d\langle X,Y\rangle_s, \tag{4.43}$$

where $Z(s)=\int_0^t X(s)dY(s)$ is usual Ito integral and the RHS of (4.42) is Riemann-Stieltjes integral.

Let us consider a set of stochastic processes $\mathbb{S}$ which consists of a finite sum of product process $Y(t)A(t)$. Here $Y(t)$ is a $\mathcal{F}_t$- continuous semimartingale and $A(t)$ is a continuous bounded variation process which is not necessarily $\mathcal{F}_t$-adapted and $\sup_{0\le t\le T}|Y(t)|+\|A\|_{[0,T]}\in\cap_{p\ge1}L^p$. Then this class is stable under the stochastic integral in the sense of the above lemma. In the calculation below, we use the integrals of stochastic processes in this sense. Moreover the following chain rule holds.

Lemma 4.9. *Let $Y,Z\in\mathbb{S}$. Then*

$$Y(t)Z(t)=Y(0)Z(0)+\int_0^t Y(s)dZ(s)+\int_0^t Z(s)dY(s)+\langle Y,Z\rangle_t, \tag{4.44}$$

where $\langle Y,Z\rangle_t$ is defined similarly to Lemma 4.8.

The above two lemmas are proved by a standard argument (integration by parts) and we omit the proof. In the proof of Lemma 4.6, we use estimates on the expectations of the integrals in the above sense. We introduce a family of iterated integrals. Let $\mathcal{S}$ be a set of stochastic processes which consists of the processes $g(Y(t))$ where g is a C^1 function with values in $\mathbb{R}$ with bounded derivative and

$$Y=X^N, X_E^N, B, B^N, \Phi^N(t), \Phi_E^N(t). \tag{4.45}$$

We define a set $\mathcal{S}_i$ of two parameter processes $f=f(s,t)$ $(0\le s\le t\le T)$ inductively. Let $\mathcal{S}_0=\{1\}$. The set S_i $(i\ge1)$ consists of finite sums of

$$\prod_{k=1}^j f_k(s,t), \qquad \int_s^t g(s,u)df_0(u), \tag{4.46}$$

where $f_k\in S_{i_k}$ $\sum_{k=1}^j i_k=i$ $i_k\ge1$ and $f_0\in\mathcal{S}, g\in\mathcal{S}_{i-1}$. Inductively, we see that $f=f(s,t)\in\mathcal{S}_i$ is equal to a finite sum of $g(s)h(t)$, where $g,h\in\mathbb{S}$. Therefore, the integral in (4.46) is meaningful. For these random variables, we have the following estimate.

Lemma 4.10. *Let $t_{k-1}^N \le s \le t \le t_k^N$. Let $p \ge 1$. For any $f \in \mathcal{S}_i$ ($i \in \mathbb{N}$), there exists $C_p > 0$ which is independent of N, k such that*

$$\| \max_{s \le u \le v \le t} f(u,v) \|_{L^p} \le C(p)(t-s)^{i/2}. \tag{4.47}$$

Proof. In this proof, we say that $f \in \cup_{i \ge 0} \mathcal{S}_i$ is adapted when the following holds. The definition is given inductively by

(i) $1 \in \mathcal{S}_0$ is adapted,

(ii) Let f be finite linear sums of processes in (4.46). Then f is adapted if all f_k ($1 \le k \le j$) and g are adapted and $f_0 = g(Y(t))$, where $Y = X_E^N, B, \Phi_E^N$ and g is a C^1 function with bounded derivative.

By an induction on i, it is easy to check that the set $\mathcal{S}_i$ is equal to the set of finite sums of two parameter processes

$$\left(\prod_{k=1}^{l} \int_s^t g_k(s,u) dA^k(u) \right) \cdot h(s,t), \tag{4.48}$$

where

(a) A^k is a bounded variation process in $\mathcal{S}$ and $g_k \in \mathcal{S}_{i_k}$. When $l = 0$, we set this product term as 1.

(b) $h \in \mathcal{S}_j$ is adapted.

(c) the indices i_k, j satisfy $\sum_{k=1}^{l} (i_k + 1) + j = i$.

Using this fact, Lemma 3.2, Lemma 4.5, Lemma 4.3 (4.9) and Lemma 2.4, we can complete the proof of the desired result by an induction on i. □

Proof of Lemma 4.6. We write

$$\begin{aligned} Z^N(t) &= X_E^N(t) - X^N(t), \\ \rho_N(t) &= e^{-\frac{2}{\gamma}(f(X_E^N(t)+f(X^N(t))))}, \\ m_N(t) &= \rho_N(t)|Z^N(t)|^2. \end{aligned}$$

Let $t_{k-1} \le t \le t_k$. By Lemma 4.9,

$$
\begin{aligned}
&dm_N(t) \\
&= \rho_N(t)\Bigg\{2\left(Z^N(t), \sigma(X_E^N(t_{k-1}))dB(t) - \sigma(X^N(t))\frac{\Delta B_k}{\Delta}dt\right) \\
&\quad + 2\left(Z^N(t), \tilde{b}(X_E^N(t_{k-1})) - b(X^N(t))\right)dt + \operatorname{tr}\left[({}^t\sigma\sigma)(X_E^N(t_{k-1}))\right]dt\Bigg\} \\
&\quad + 2\rho_N(t)\left(Z^N(t), d\Phi_E^N(t) - d\Phi^N(t)\right) \\
&\quad - \frac{2\rho_N(t)}{\gamma}\left|Z^N(t)\right|^2\left\{\left((Df)(X_E^N(t)), d\Phi_E^N(t)\right) + \left((Df)(X^N(t)), d\Phi^N(t)\right)\right\} \\
&\quad - \frac{2\rho_N(t)}{\gamma}\left|Z^N(t)\right|^2\Bigg\{\left((Df)(X_E^N(t)), \sigma(X_E^N(t_{k-1}))dB(t)\right) \\
&\qquad\qquad + \left((Df)(X^N(t)), \sigma(X^N(t))\frac{\Delta B_k}{\Delta}dt\right)\Bigg\} \\
&\quad - \frac{4\rho_N(t)}{\gamma}\sum_i \left((Df)(X_E^N(t)), \sigma(X_E^N(t_{k-1}))e_i\right)\left(Z^N(t), \sigma(X_E^N(t_{k-1}))e_i\right)dt \\
&\quad - \frac{2\rho_N(t)}{\gamma}|Z^N(t)|^2\left\{\left((Df)(X_E^N(t)), \tilde{b}(X_E^N(t_{k-1}))\right)dt + \left((Df)(X^N(t)), b(X^N(t))\right)dt\right\} \\
&\quad - \frac{\rho_N(t)}{\gamma}|Z^N(t)|^2 \operatorname{tr}(D^2 f)(X_E^N(t))\left[\sigma(X_E^N(t_{k-1})\cdot, \sigma(X_E^N(t_{k-1}))\cdot\right]dt \\
&\quad + \frac{2\rho_N(t)}{\gamma^2}|Z^N(t)|^2|(Df)(X_E^N(t))(\sigma(X_E^N(t_{k-1})))|^2 dt, \qquad (4.49)
\end{aligned}
$$

where $\{e_i\}$ is a c.o.n.s of $\mathbb{R}^n$. After integrating both sides from t_{k-1} to t_k, we see that the sum of the integral of the second term and the third term on the RHS is non-positive by the condition (C), Therefore

$$
m_N(t_k) \le m_N(t_{k-1}) + \sum_{k=1}^{6} I_k, \qquad (4.50)
$$

where

$$
\begin{aligned}
I_1 = \int_{t_{k-1}}^{t_k} \rho_N(t)\Bigg\{&2\left(Z^N(t), \sigma(X_E^N(t_{k-1}))dB(t) - \sigma(X^N(t))\frac{\Delta B_k}{\Delta}dt\right) \\
&+ 2\left(Z^N(t), \tilde{b}(X_E^N(t_{k-1})) - b(X^N(t))\right)dt + \operatorname{tr}\left(({}^t\sigma\sigma)(X_E^N(t_{k-1}))\right)dt\Bigg\}
\end{aligned}
$$

$$
I_2 = -\int_{t_{k-1}}^{t_k} \frac{4\rho_N(t)}{\gamma}\sum_i \left((Df)(X_E^N(t)), \sigma(X_E^N(t_{k-1}))e_i\right)\left(Z^N(t), \sigma(X_E^N(t_{k-1}))e_i\right)dt
$$

$$
\begin{aligned}
I_3 = -\int_{t_{k-1}}^{t_k} \frac{2}{\gamma} m_N(t)\Bigg\{&\left((Df)(X_E^N(t)), \sigma(X_E^N(t_{k-1}))dB(t)\right) \\
&+ \left((Df)(X^N(t)), \sigma(X^N(t))\frac{\Delta B_k}{\Delta}dt\right)\Bigg\}
\end{aligned}
$$

$$I_4 = -\int_{t_{k-1}}^{t_k} \frac{2}{\gamma} m_N(t) \left\{ \left((Df)(X_E^N(t)), \tilde{b}(X_E^N(t_{k-1})) \right) dt + \left((Df)(X^N(t)), b(X^N(t)) \right) dt \right\}$$

$$I_5 = -\int_{t_{k-1}}^{t_k} \frac{m_N(t)}{\gamma} \mathrm{tr}(D^2 f)(X_E^N(t)) \left[\sigma(X_E^N(t_{k-1})\cdot, \sigma(X_E^N(t_{k-1}))\cdot \right] dt$$

$$I_6 = \int_{t_{k-1}}^{t_k} \frac{2m_N(t)}{\gamma^2} |(Df)(X_E^N(t))(\sigma(X_E^N(t_{k-1})))|^2 dt.$$

Let $a_k = E[m_N(t_k)]$. We prove that there exists a positive constant C and $0 < \theta < 1$ which is independent of N and a non-negative sequence $\{b_k\}$ such that

$$a_k \le \left(1 + \frac{CT}{N}\right) a_{k-1} + b_k \qquad 1 \le k \le N$$
$$\sum_{k=1}^N b_k \le C \left(\frac{T}{N}\right)^{\theta/2}.$$

Then we get

$$\begin{aligned} a_k &\le \left(1 + \frac{CT}{N}\right)^2 a_{k-2} + \left(1 + \frac{CT}{N}\right) b_{k-1} + b_k \\ &\le \left(1 + \frac{CT}{N}\right)^k a_0 + \sum_{i=0}^{k-1} \left(1 + \frac{CT}{N}\right)^i b_{k-i} \\ &\le e^{CT} \sum_{i=1}^k b_i \le C_T \left(\frac{T}{N}\right)^{\theta/2} \end{aligned}$$

which is the desired estimate. We consider I_k $(k = 4, 5, 6)$. By Lemma 4.10, we have

$$\|m_N(t) - m_N(t_{k-1})\|_{L^p} \le C\Delta^{1/2} \qquad t_{k-1} \le t \le t_k.$$

Thus

$$|E[I_k]| \le C \left(a_{k-1}\Delta + \Delta^{3/2}\right).$$

So our task is to estimate I_1, I_2, I_3. We consider I_1. We rewrite

$$I_1 = J_1 + J_2 + J_3 + J_4,$$

where

$$\begin{aligned} J_1 = 2\int_{t_{k-1}}^{t_k} \rho_N(t_{k-1}) \Bigg\{ &\left(Z^N(t), \sigma(X_E^N(t_{k-1}))dB(t) - \sigma(X^N(t_{k-1}))\frac{\Delta B_k}{\Delta} dt \right) \\ &+ \frac{1}{2}\mathrm{tr}\left({}^t\sigma\sigma\right)(X_E^N(t_{k-1}))dt \Bigg\} \end{aligned}$$

$$J_2 = 2\int_{t_{k-1}}^{t_k} \rho_N(t)\Big(Z^N(t), \tilde{b}(X_E^N(t_{k-1})) - b(X^N(t)) - \big(\sigma(X^N(t)) - \sigma(X^N(t_{k-1}))\big)\,\frac{\Delta B_k}{\Delta}\Big)dt,$$

$$J_3 = 2\int_{t_{k-1}}^{t_k} (\rho_N(t) - \rho_N(t_{k-1}))\left(Z^N(t), \sigma(X_E^N(t_{k-1}))dB(t) - \sigma(X^N(t_{k-1}))\frac{\Delta B_k}{\Delta}dt\right),$$

$$J_4 = \int_{t_{k-1}}^{t_k} (\rho_N(t) - \rho_N(t_{k-1}))\,\mathrm{tr}\left({}^t\sigma\sigma\right)(X_E^N(t_{k-1}))dt.$$

First we estimate J_1. Let

$$\tilde{J}_1 = 2\int_{t_{k-1}}^{t_k} \rho_N(t_{k-1})\Bigg\{\left(Z^N(t) - Z^N(t_{k-1}), \sigma(X_E^N(t_{k-1}))dB(t) - \sigma(X^N(t_{k-1}))\frac{\Delta B_k}{\Delta}dt\right) + \frac{1}{2}\mathrm{tr}\left({}^t\sigma\sigma\right)(X_E^N(t_{k-1}))dt\Bigg\}.$$

Then $E[J_1 - \tilde{J}_1] = 0$. So it suffices to estimate the expectation of $\tilde{J}_1$. We rewrite

$$\tilde{J}_1 = \sum_{k=1}^{4} \tilde{J}_{1,k},$$

where

$$\tilde{J}_{1,1} = 2\int_{t_{k-1}}^{t_k} \rho_N(t_{k-1})\Bigg\{\Big(\sigma(X_E^N(t_{k-1}))(B(t) - B(t_{k-1})) - \sigma(X^N(t_{k-1}))\frac{\Delta B_k}{\Delta}(t - t_{k-1}), \sigma(X_E^N(t_{k-1}))dB(t) - \sigma(X^N(t_{k-1}))\frac{\Delta B_k}{\Delta}dt\Big) + \frac{1}{2}\mathrm{tr}\left({}^t\sigma\sigma\right)(X_E^N(t_{k-1}))dt\Bigg\}.$$

$$\tilde{J}_{1,2} = -2\int_{t_{k-1}}^{t_k} \rho_N(t_{k-1})\Bigg\{\Big(\int_{t_{k-1}}^{t} \left(\sigma(X^N(s)) - \sigma\left(X^N(t_{k-1})\right)\right)\frac{\Delta B_k}{\Delta}ds, \sigma(X_E^N(t_{k-1}))dB(t) - \sigma(X^N(t_{k-1}))\frac{\Delta B_k}{\Delta}dt\Big)\Bigg\}$$

$$\tilde{J}_{1,3} = 2\int_{t_{k-1}}^{t_k} \rho_N(t_{k-1})\Bigg\{\Big(\tilde{b}(X_E^N(t_{k-1})))(t - t_{k-1}) - \int_{t_{k-1}}^{t} b(X^N(s))ds, \sigma(X_E^N(t_{k-1}))dB(t) - \sigma(X^N(t_{k-1}))\frac{\Delta B_k}{\Delta}dt\Big)\Bigg\}$$

$$\tilde{J}_{1,4} = 2\int_{t_{k-1}}^{t_k} \rho_N(t_{k-1})\Bigg\{\Big((\Phi_E^N(t) - \Phi_E^N(t_{k-1})) - (\Phi^N(t) - \Phi^N(t_{k-1})),$$
$$\sigma(X_E^N(t_{k-1}))dB(t) - \sigma(X^N(t_{k-1}))\frac{\Delta B_k}{\Delta}dt\Big)\Bigg\}$$

By a simple calculation,

$$E[\tilde{J}_{1,1}] = E\left[\rho_N(t_{k-1})\|\sigma(X_E^N(t_{k-1})) - \sigma(X^N(t_{k-1}))\|_{H.S.}^2\right](t_k - t_{k-1}) \le Ca_k\,\Delta.$$

By Lemma 4.10, we have

$$E[|\tilde{J}_{1,2}|] \le C\Delta^{3/2}.$$

It is easy to see that $E[|\tilde{J}_{1,3}|] \le C\Delta^{3/2}$. By integrating by parts

$$|E[\tilde{J}_{1,4}]| \le CE\left[\left(\|\Phi_E^N\|_{[t_{k-1},t_k]} + \|\Phi^N\|_{[t_{k-1},t_k]}\right)\|B\|_{\infty,[t_{k-1},t_k]}\right] =: c_k.$$

We have

$$\sum_{k=1}^N c_k \le C\|\left(\|\Phi_E^N\|_{[0,T]} + \|\Phi^N\|_{[0,T]}\right)\|_{L^2}\|\max_k \|B\|_{\infty,[t_{k-1},t_k]}\|_{L^2} \le C\Delta^{\theta/2}.$$

We estimate J_2.

$$\begin{aligned}
J_2 &= 2\int_{t_{k-1}}^{t_k}\rho_N(t)\Big(Z^N(t), \tilde{b}(X_E^N(t_{k-1})) - \tilde{b}(X^N(t_{k-1})) + b(X^N(t_{k-1})) - b(X^N(t))\Big)dt\\
&\quad + 2\int_{t_{k-1}}^{t_k}\rho_N(t)\Big(Z^N(t), \tilde{b}(X^N(t_{k-1})) - b(X^N(t_{k-1}))\\
&\qquad\qquad - \left(\sigma(X^N(t)) - \sigma(X^N(t_{k-1}))\right)\frac{\Delta B_k}{\Delta}\Big)dt\\
&= J_{2,1} + J_{2,2}.
\end{aligned}$$

By rewriting $Z^N(t) = Z^N(t_{k-1}) + Z^N(t) - Z^N(t_{k-1})$ and using Lemma 4.10 and the Schwarz inequality, we get

$$E[|J_{2,1}|] \le C(a_k\Delta + \Delta^{3/2}).$$

We consider $J_{2,2}$. Let

$$\begin{aligned}
\tilde{J}_{2,2} = 2\int_{t_{k-1}}^{t_k}\rho_N(t_{k-1})\Big(&Z^N(t_{k-1}), \tilde{b}(X^N(t_{k-1})) - b(X^N(t_{k-1}))\\
&- \left(\sigma(X^N(t)) - \sigma(X^N(t_{k-1}))\right)\frac{\Delta B_k}{\Delta}\Big)dt.
\end{aligned}$$

By Lemma 4.10, we have

$$|E[J_{2,2} - \tilde{J}_{2,2}]| \le C\Delta^{3/2}.$$

Noting

$$
\begin{aligned}
&\sigma(X^N(t)) - \sigma(X^N(t_{k-1})) \\
&= \int_{t_{k-1}}^{t} (D\sigma)(X^N(s)) \left(\sigma(X^N(s))\frac{\Delta B_k}{\Delta}\right) ds + \int_{t_{k-1}}^{t} (D\sigma)(X^N(s))(b(X^N(s)))ds \\
&\quad + \int_{t_{k-1}}^{t} (D\sigma)(X^N(s))(d\Phi^N(s))
\end{aligned}
$$

and for any ξ,

$$
E\left[\int_{t_{k-1}}^{t_k} (t-t_{k-1})(D\sigma)(\xi)\left(\sigma(\xi)\frac{\Delta B_k}{\Delta}\right)\left(\frac{\Delta B_k}{\Delta}\right)dt\right] = \frac{1}{2}\mathrm{tr}(D\sigma)(\xi)(\sigma(\xi)) = (\tilde{b}-b)(\xi),
$$

we obtain

$$
|E[\tilde{J}_{2,2}]| \le CE\left[\|\Phi^N\|_{[t_{k-1},t_k]}\|B\|_{\infty,[t_{k-1},t_k]}\right] + C\Delta^{3/2}.
$$

Therefore, as before, we get $|E[J_{2,2}]| \le C\Delta^{3/2} + c_k$, where c_k is a non-negative number such that $\sum_{k=1}^N c_k \le C\Delta^{\theta/2}$. By Lemma 4.10, we have $E[|J_4|] \le C\Delta^{3/2}$. We estimate J_3 together with I_2 and a term $I_{3,2}$ which is defined below. We estimate I_3. We rewrite

$$
I_3 = I_{3,1} + I_{3,2} + I_{3,3} + I_{3,4} + I_{3,5} + I_{3,6} + I_{3,7},
$$

where

$$
\begin{aligned}
&I_{3,1} \\
&= -\frac{4}{\gamma}\int_{t_{k-1}}^{t_k}\rho_N(t)\int_{t_{k-1}}^{t}\left(Z^N(s)-Z^N(t_{k-1}), \sigma(X_E^N(t_{k-1}))dB(s) - \sigma(X^N(s))\frac{\Delta B_k}{\Delta}ds\right) \\
&\times\left\{\left((Df)(X_E^N(t)), \sigma(X_E^N(t_{k-1}))dB(t)\right) + \left((Df)(X^N(t)), \sigma(X^N(t))\frac{\Delta B_k}{\Delta}dt\right)\right\},
\end{aligned}
$$

$$
\begin{aligned}
I_{3,2} = &-\frac{4}{\gamma}\int_{t_{k-1}}^{t_k}\rho_N(t)\int_{t_{k-1}}^{t}\left(Z^N(t_{k-1}), \sigma(X_E^N(t_{k-1}))dB(s) - \sigma(X^N(s))\frac{\Delta B_k}{\Delta}ds\right) \\
&\times\left\{\left((Df)(X_E^N(t)), \sigma(X_E^N(t_{k-1}))dB(t)\right) + \left((Df)(X^N(t)), \sigma(X^N(t))\frac{\Delta B_k}{\Delta}dt\right)\right\},
\end{aligned}
$$

$$
\begin{aligned}
I_{3,3} = &-\frac{4}{\gamma}\int_{t_{k-1}}^{t_k}\rho_N(t)\int_{t_{k-1}}^{t}\left(Z^N(s), \tilde{b}(X_E^N(t_{k-1})) - b(X^N(s))\right)ds \\
&\times\left\{\left((Df)(X_E^N(t)), \sigma(X_E^N(t_{k-1}))dB(t)\right) + \left((Df)(X^N(t)), \sigma(X^N(t))\frac{\Delta B_k}{\Delta}dt\right)\right\},
\end{aligned}
$$

$$
\begin{aligned}
I_{3,4} = &-\frac{4}{\gamma}\int_{t_{k-1}}^{t_k}\rho_N(t)\int_{t_{k-1}}^{t}\left(Z^N(s)-Z^N(t_{k-1}), d\Phi_E^N(s) - d\Phi^N(s)\right) \\
&\times\left\{\left((Df)(X_E^N(t)), \sigma(X_E^N(t_{k-1}))dB(t)\right) + \left((Df)(X^N(t)), \sigma(X^N(t))\frac{\Delta B_k}{\Delta}dt\right)\right\},
\end{aligned}
$$

$$I_{3,5} = -\frac{4}{\gamma}\int_{t_{k-1}}^{t_k} \rho_N(t)\left(Z^N(t_{k-1}), (\Phi_E^N(t)-\Phi_E^N(t_{k-1})) - (\Phi^N(t)-\Phi^N(t_{k-1}))\right)$$
$$\times\left\{\left((Df)(X_E^N(t)), \sigma(X_E^N(t_{k-1}))dB(t)\right) + \left((Df)(X^N(t)), \sigma(X^N(t))\frac{\Delta B_k}{\Delta}dt\right)\right\},$$

$$I_{3,6} = -\frac{2}{\gamma}|Z^N(t_{k-1})|^2\int_{t_{k-1}}^{t_k} \rho_N(t)\left\{\left((Df)(X_E^N(t)), \sigma(X_E^N(t_{k-1}))dB(t)\right)\right.$$
$$\left. + \left((Df)(X^N(t)), \sigma(X^N(t))\frac{\Delta B_k}{\Delta}dt\right)\right\},$$
$$I_{3,7} = -\frac{2}{\gamma}\int_{t_{k-1}}^{t_k} \rho_N(t)(t-t_{k-1})\mathrm{tr}\left(({}^t\sigma\sigma)(X_E^N(t_{k-1}))\right)\left\{\left((Df)(X_E^N(t)), \sigma(X_E^N(t_{k-1}))dB(t)\right)\right.$$
$$\left. + \left((Df)(X^N(t)), \sigma(X^N(t))\frac{\Delta B_k}{\Delta}dt\right)\right\}.$$

As for $I_{3,1}, I_{3,3}, I_{3,4}, I_{3,7}$, by Lemma 4.10, it is easy to see

$$|E[I_{3,k}]| \le C\Delta^{3/2}.$$

We consider $I_{3,6}$. By (4.9) and Lemma 2.4, we have

$$\|X^N\|_{[t_{k-1},t_k]} \le C\left(|\Delta B_k| + \Delta\right).$$

Using this estimate and (3.6), we have

$$\left|E\left[\int_{t_{k-1}}^{t_k} \rho_N(t)\left\{\left((Df)(X_E^N(t)), \sigma(X_E^N(t_{k-1}))dB(t)\right)\right.\right.\right.$$
$$\left.\left.\left. + \left((Df)(X^N(t)), \sigma(X^N(t))\frac{\Delta B_k}{\Delta}dt\right)\right\}\right|\mathcal{F}_{t_{k-1}}\right]\right| \le C\Delta.$$

Hence, $|E[I_{3,6}]| \le Ca_{k-1}\Delta$. Using Lemma 4.10, we have there exists non-negative random variable $I'_{3,5}$ such that $E[I'_{3,5}] \le C\Delta^{3/2}$ and

$$|I_{3,5}| \le C|Z^N(t_{k-1})|G_k + I'_{3,5} \le \frac{C}{2}\left(|Z^N(t_{k-1})|^2G_k + G_k\right) + I'_{3,5},$$

where

$$G_k = \left(\max_{t_{k-1}\le t\le t_k}\left|\int_{t_{k-1}}^{t}\left((Df)(X_E^N(s)), \sigma(X_E^N(t_{k-1}))dB(s)\right)\right| + \|B\|_{\infty,[t_{k-1},t_k]}\right)$$
$$\times\left(\|\Phi_E^N\|_{[t_{k-1},t_k]} + \|\Phi^N\|_{[t_{k-1},t_k]}\right).$$

Using $E[G_k|\mathcal{F}_{t_{k-1}}] \le C\Delta$, we obtain

$$E[|I_{3,5}|] \le C(a_{k-1}\Delta + E\left[G_k\right]) + C\Delta^{3/2}.$$

Since

$$\sum_{k=1}^{N} E[G_k] \le E\Bigg[\left(\|\Phi_E^N\|_{[0,T]} + \|\Phi^N\|_{[0,T]}\right) \times \max_k \left(\max_{t_{k-1}\le t\le t_k} \left| \int_{t_{k-1}}^{t} \left((Df)(X_E^N(s)), \sigma(X_E^N(t_{k-1}))dB(s)\right) \right| + \|B\|_{\infty,[t_{k-1},t_k]} \right) \Bigg] \le C\Delta^{\theta/2},$$

we obtain the desired estimate for $I_{3,5}$. Finally we estimate $I_2 + J_3 + I_{3,2}$. First we rewrite J_3. Note that

$$\begin{aligned}
\rho_N(t) - \rho_N(t_{k-1}) = &-\frac{2}{\gamma}\rho_N(t_{k-1})\Bigg\{ \Big((Df)(X_E^N(t_{k-1})), \sigma(X_E^N(t_{k-1}))(B(t) - B(t_{k-1}))\Big) \\
&+ \left((Df)(X^N(t_{k-1})), \sigma(X^N(t_{k-1}))\frac{\Delta B_k}{\Delta}(t - t_{k-1})\right) \Bigg\} \\
&- \frac{2}{\gamma}\rho_N(t_{k-1})\Bigg\{ \Big((Df)(X_E^N(t_{k-1})), \tilde{b}(X_E^N(t_{k-1}))(t - t_{k-1})\Big) \\
&+ \big((Df)(X^N(t_{k-1})), b(X^N(t_{k-1}))(t - t_{k-1})\big) \Bigg\} \\
&- \frac{2}{\gamma}\rho_N(t_{k-1})\Bigg\{ \Big((Df)(X_E^N(t_{k-1})), \Phi_E^N(t) - \Phi_E^N(t_{k-1})\Big) \\
&+ \big((Df)(X^N(t_{k-1})), \Phi^N(t) - \Phi^N(t_{k-1})\big) \Bigg\} + \tilde{\rho}(t_{k-1}, t).
\end{aligned}$$

Here $\tilde{\rho} \in \mathcal{S}_2$. Hence we can neglect the term $\tilde{\rho}$ to estimate J_3 by Lemma 4.10. Also we can estimate the terms containing Φ_E^N, Φ^N in a similar way to $\tilde{J}_{1,4}, \tilde{J}_{2,2}, I_{3,5}$. We can estimate the term containing $b, \tilde{b}$ by Lemma 4.10. Consequently, we can replace the term J_3 by $\tilde{J}_3$:

$$\begin{aligned}
\tilde{J}_3 = &-\frac{4}{\gamma}\int_{t_{k-1}}^{t_k} \rho_N(t_{k-1})\Bigg\{ \Big((Df)(X_E^N(t_{k-1})), \sigma(X_E^N(t_{k-1}))(B(t) - B(t_{k-1}))\Big) \\
&+ \left((Df)(X^N(t_{k-1})), \sigma(X^N(t_{k-1}))\frac{\Delta B_k}{\Delta}(t - t_{k-1})\right) \Bigg\} \\
&\times \left(Z^N(t_{k-1}), \sigma(X_E^N(t_{k-1}))dB(t) - \sigma(X^N(t_{k-1}))\frac{\Delta B_k}{\Delta}dt\right).
\end{aligned}$$

Also, similarly, we can replace $I_{3,2}$ by $\tilde{I}_{3,2}$:

$$\tilde{I}_{3,2} = -\frac{4}{\gamma}\int_{t_{k-1}}^{t_k} \rho_N(t_{k-1}) \times \left(Z^N(t_{k-1}), \sigma(X_E^N(t_{k-1}))(B(t) - B(t_{k-1})) - \sigma(X^N(t_{k-1}))\frac{\Delta B_k}{\Delta}(t - t_{k-1}) \right) \times \left\{ \left((Df)(X_E^N(t_{k-1})), \sigma(X_E^N(t_{k-1}))dB(t)\right) + \left((Df)(X^N(t_{k-1})), \sigma(X^N(t_{k-1}))\frac{\Delta B_k}{\Delta}dt \right) \right\}.$$

By a simple calculation, we have

$$\begin{aligned} & E\left[\tilde{J}_3 \mid \mathcal{F}_{t_{k-1}}\right] \\ &= \frac{2\Delta}{\gamma}\rho_N(t_{k-1})\sum_i \left((Df)(X_E^N(t_{k-1})), \sigma(X_E^N(t_{k-1}))e_i\right)\left(Z^N(t_{k-1}), \sigma(X^N(t_{k-1}))e_i\right) \\ &\quad - \frac{2\Delta}{\gamma}\rho_N(t_{k-1})\sum_i \left((Df)(X^N(t_{k-1})), \sigma(X^N(t_{k-1}))e_i\right)\left(Z^N(t_{k-1}), \sigma(X_E^N(t_{k-1}))e_i\right) \\ &\quad + \frac{2\Delta}{\gamma}\rho_N(t_{k-1})\sum_i \left((Df)(X^N(t_{k-1})), \sigma(X^N(t_{k-1}))e_i\right)\left(Z^N(t_{k-1}), \sigma(X^N(t_{k-1}))e_i\right), \end{aligned}$$

$$\begin{aligned} & E\left[\tilde{I}_{3,2} \mid \mathcal{F}_{t_{k-1}}\right] \\ &= -\frac{2\Delta}{\gamma}\rho_N(t_{k-1})\sum_i \left(Z^N(t_{k-1}), \sigma(X_E^N(t_{k-1}))e_i\right)\left((Df)(X^N(t_{k-1})), \sigma(X^N(t_{k-1}))e_i\right) \\ &\quad + \frac{2\Delta}{\gamma}\rho_N(t_{k-1})\sum_i \left(Z^N(t_{k-1}), \sigma(X^N(t_{k-1}))e_i\right)\left((Df)(X_E^N(t_{k-1})), \sigma(X_E^N(t_{k-1}))e_i\right) \\ &\quad + \frac{2\Delta}{\gamma}\rho_N(t_{k-1})\sum_i \left(Z^N(t_{k-1}), \sigma(X^N(t_{k-1}))e_i\right)\left((Df)(X^N(t_{k-1})), \sigma(X^N(t_{k-1}))e_i\right). \end{aligned}$$

Hence

$$\begin{aligned} & E\left[\tilde{J}_3 + \tilde{I}_{3,2}\right] \\ &= Ca_{k-1}\Delta \\ &\quad + \frac{4\Delta}{\gamma}E\left[\rho_N(t_{k-1})\sum_i \left(Z^N(t_{k-1}), \sigma(X^N(t_{k-1}))e_i\right)\left((Df)(X^N(t_{k-1})), \sigma(X^N(t_{k-1}))e_i\right)\right]. \end{aligned}$$

Consequently,

$$\left| E\left[I_2 + \tilde{J}_3 + \tilde{I}_{3,2}\right]\right| \le C\left(a_{k-1}\Delta + \Delta^{3/2}\right).$$

This completes the proof. □

Remark 4.11. Note that some parts in the above estimate for $a_k = E[m_n(t_k)]$ are rough. In the case where $\partial D = \emptyset$, that is, $D = \mathbb{R}^d$, the local time terms vanish. In this case, the estimate

$$a_k \le (1 + C\Delta)a_{k-1} + C\Delta^2 \tag{4.51}$$

might be true. The bad term $\Delta^{\theta/2}$ essentially comes from the estimates on local time terms if $\partial D \neq \emptyset$.

Lemma 4.12. *Assume the same assumptions in Lemma* 4.6 *and consider the same SDE. Let* $0 < \theta < 1$. *Then there exists a positive constant* $C_{p,T,\theta}$ *such that*

$$E\left[\max_{0\le t\le T}|X^N(t)-X_E^N(t)|^{2p}\right] \le C_{p,T,\theta}\Delta_N^{\theta/6}. \tag{4.52}$$

Proof. Let N_0 be a natural number and choose a sufficiently large natural number N. Pick partition points $\{s_k\}_{k=1}^{N_0} \subset \{t_k^N\}_{k=1}^N$ such that

$$|s_k - t_k^{N_0}| \le \frac{T}{N}.$$

Let $t_{k-1}^{N_0} \le t \le t_k^{N_0}$. Then

$$\begin{aligned}|X^N(t)-X_E^N(t)| &\le |X^N(t)-X^N(t_k^{N_0})| + |X^N(t_k^{N_0})-X^N(s_k)| + |X^N(s_k)-X_E^N(s_k)| \\ &\quad + |X_E^N(s_k)-X_E^N(t_k^{N_0})| + |X_E^N(t_k^{N_0})-X_E^N(t)|\end{aligned}$$

and

$$\begin{aligned}\max_{0\le t\le T}|X^N(t)-X_E^N(t)| &\le 2\max_{0\le s\le t\le T, |t-s|\le T/N_0}|X^N(t)-X^N(s)| \\ &\quad + 2\max_{0\le s\le t\le T, |t-s|\le T/N_0}|X_E^N(t)-X_E^N(s)| \\ &\quad + \sum_{k=1}^{N_0}|X^N(s_k)-X_E^N(s_k)|.\end{aligned}$$

Therefore

$$\begin{aligned}E\left[\max_{0\le t\le T}|X^N(t)-X_E^N(t)|^{2p}\right] &\le C_{p,\theta}3^{2p-1}2^{2p}\left(\frac{T}{N_0}\right)^{p\theta} + N_0^{2p-1}\sum_{k=1}^{N_0}E\left[|X^N(s_k)-X_E^N(s_k)|^{2p}\right] \\ &\le C_{p,\theta}\left(3^{2p-1}2^{2p+1}\left(\frac{T}{N_0}\right)^{p\theta} + N_0^{2p}\left(\frac{T}{N}\right)^{\theta/2}\right).\end{aligned}$$

Here we use the uniform moment estimate for X_E^N, X^N and Lemma 4.6 and Lemma 2.7. Hence setting N_0 as the integer part of $N^{1/6p}$, we obtain the desired estimate. ☐

Proof of main theorem. The proof follows from Theorem 3.1 and Lemma 4.12. ☐

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References

[1] A.M. Davie, Differential equations driven by rough paths: an approach via discrete approximations, Appl. Math. Res. Express. AMRX 2007, no. 2, Art. ID abm009, 40 pp.

[2] H. Doss and P. Priouret, Support d'un processus de réflexion, Z. Wahrsch. Verw. Gebiete 61 (1982), no. 3, 327345.

[3] P. Dupuis and H. Ishii, On Lipschitz continuity of the solution mapping to the Skorokhod problem, with applications. Stochastics Stochastics Rep. 35 (1991), no. 1, 3162.

[4] L.C. Evans and D.W. Stroock, An approximation scheme for reflected stochastic differential equations, Stochastic Process. Appl. 121 (2011), no. 7, 14641491.

[5] V.I. Bogachev, Gaussian measures, Mathematical Surveys and Monographs, 62, American Mathematical Society, Providence, RI, 1998.

[6] N. Ikeda and S. Watanabe, Stochastic differential equations and diffusion processes. North-Holland Mathematical Library, 24. North-Holland Publishing Co., Amsterdam-New York; Kodansha, Ltd., Tokyo, 1981.

[7] P.L. Lions and A.S. Sznitman, Stochastic differential equations with reflecting boundary conditions, Comm. Pure Appl. Math. 37 (1984), no. 4, 511537.

[8] R. Pettersson, Wong-Zakai approximations for reflecting stochastic differential equations. Stochastic Anal.Appl. 17 (1999), no. 4, 609617.

[9] R. Poliquin, R. Rockafellar, R. T.; L. Thibault, Local differentiability of distance functions. Trans. Amer. Math. Soc. 352 (2000), no. 11, 52315249.

[10] J. Ren and S. Xu, A transfer principle for multivalued stochastic differential equations. J. Funct. Anal. 256 (2009), no. 9, 27802814.

[11] J.Ren and S. Xu, Support theorem for stochastic variational inequalities. (English summary) Bull. Sci. Math. 134 (2010), no. 8, 826856.

[12] Y. Saisho, Stochastic differential equations for multi-dimensional domain with reflecting boundary, Probab. Theory Related Fields 74 (1987), no. 3, 455477.

[13] L. Słomiński, On approximation of solutions of multidimensional SDEs with reflecting boundary conditions. Stochastic Process. Appl. 50 (1994), no. 2, 197219.

[14] L. Słomiński, Euler's approximations of solutions of SDEs with reflecting boundary. Stochastic Process. Appl. 94 (2001), no. 2, 317337.

[15] H. Tanaka, Stochastic differential equations with reflecting boundary condition in convex regions, Hiroshima Math. J. 9 (1979), no. 1, 163177.

[16] E. Wong and M. Zakai, On the relation between ordinary and stochastic differential equations. Internat. J. Engrg. Sci. 3 (1965) 213229.