# CLUSTER ALGEBRAS AND SINGULAR SUPPORTS OF PERVERSE SHEAVES

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ABSTRACT. We propose an approach to Geiss-Leclerc-Schroer's conjecture on the cluster algebra structure on the coordinate ring of a unipotent subgroup and the dual canonical base. It is based on singular supports of perverse sheaves on the space of representations of a quiver, which give the canonical base.

# INTRODUCTION

In [27], the author found an approach to the theory of cluster algebras, based on perverse sheaves on graded quiver varieties. This approach gave a link between two categorical frameworks for cluster algebras, the additive one via the cluster category by Buan et al. [5] and the multiplicative one via the category of representations of a quantum affine algebra by Hernandez-Leclerc [12]. See also the survey article [20].

In [27, §1.5], the author asked four problems in the to-do list, to which he thought that the same approach can be applied. Except the problem (3), they have been subsequently solved in works by Qin [28, 29] and Kimura-Qin [18].

Let us explain other related problems, for which the approach does not work. We need a new idea to attack these.

A problem, discussed in this paper, is a natural generalization of the problem (4) in the to-do list. The author asked to find a relation between the work of Geiss-Leclerc-Schröer [8] and [27] there. But the work [8] dealt with more general cases than those corresponding to [27]. A main conjecture says that every cluster monomial in the coordinate ring of a unipotent subgroup is a Lusztig's dual canonical base element. Later the theory is generalized to the q-analog, where the canonical base naturally lives [9]. Therefore it is desirable to find a relation between the cluster algebra structure and perverse sheaves on the space

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of quiver representations, which give the canonical base of the quantum enveloping algebra.

Another problem is not discussed here, but possibly related to the current one via [13]. In [12], Hernandez-Leclerc conjectured that the Grothendieck ring of representations of the quantum affine algebra has a structure of a cluster algebra so that every cluster monomial is a class of an irreducible representation. Those irreducible representations are simple perverse sheaves on graded quiver varieties. But what was proved in [27] is the special case of the conjecture only for a certain subalgebra of the Grothendieck ring. The first part of the conjecture has been subsequently proved in [13]. But the latter part, every cluster monomial is an irreducible representation, is still open.

In this paper, we propose an approach to the first problem. It is not fully developed yet. We will give a few results, which indicate that we are going in the right direction. The new idea is to use the singular support of a perverse sheaf, which is a lagrangian subvariety in the cotangent bundle. The latter is related to the representation theory of the preprojective algebra, which underlies the work [8].

The paper is organized as follows. In the first section, we briefly recall the canonical and semicanonical bases. In the second section, we review works of Geiss-Leclerc-Schröer [8, 9], where a quantum cluster algebra structure on a quantum unipotent subgroup is introduced. A subcategory, denoted by  $C_w$ , of the category of nilpotent representations of the preprojective algebra plays a crucial role. In the third section, we study how the singular support behaves under the restriction functor for perverse sheaves. The restriction functor gives a multiplication in the dual of the quantum enveloping algebra. Our main result is the estimate in Theorem 3.2. In the final section, we give two conjectures, which give links between the theory of [8, 9] and perverse sheaves via singular support.

Acknowledgments. The main conjecture (Conj. 4.2) was found in the spring of 2011, and has been mentioned to various people since then. The author thanks Masaki Kashiwara and Yoshihisa Saito for discussion on the conjecture. He also thanks the referee who points out a relation between the main conjecture and a conjecture in [7, §1.5].

# 1. Preliminaries

1(i). Quantum enveloping algebra. Let  $\mathfrak{g}$  be a symmetrizable Kac-Moody Lie algebra. We assume  $\mathfrak{g}$  is symmetric, as we use an approach to  $\mathfrak{g}$  via the Ringel-Hall algebra for a quiver. Let I be the index set of simple roots, P be the weight lattice, and  $P^*$  be its dual. Let  $\alpha_i$  denote the  $i^{\text{th}}$  simple root.

Let  $\mathbf{U}_q$  be the corresponding quantum enveloping algebra, that is a  $\mathbb{Q}(q)$ -algebra generated by  $e_i$ ,  $f_i$   $(i \in I)$ ,  $q^h$   $(h \in P^*)$  with certain relations. Let  $\mathbf{U}_q^-$  be the subalgebra generated by  $f_i$ . We set  $\mathrm{wt}(e_i) = \alpha_i$ ,  $\mathrm{wt}(f_i) = -\alpha_i$ ,  $\mathrm{wt}(q^h) = 0$ . Then  $\mathbf{U}_q$  is graded by P.

The quantum enveloping algebra  $\mathbf{U}_q$  is a Hopf algebra. We have a coproduct  $\Delta: \mathbf{U}_q \to \mathbf{U}_q \otimes \mathbf{U}_q$ . It does not preserve  $\mathbf{U}_q^-$ , but Lusztig introduced its modification  $r: \mathbf{U}_q^- \to \mathbf{U}_q^- \otimes \mathbf{U}_q^-$  such that  $r(f_i) = f_i \otimes 1 + 1 \otimes f_i$  and r is an algebra homomorphism with respect to the multiplication on  $\mathbf{U}_q^- \otimes \mathbf{U}_q^-$  given by

(1.1) 
$$(x_1 \otimes y_1) \cdot (x_2 \otimes y_2) = q^{-(\operatorname{wt} x_2, \operatorname{wt} y_1)} x_1 x_2 \otimes y_1 y_2,$$

where  $x_i, y_i$  are homogeneous elements. We call r the twisted coproduct.

Let  $\mathbf{A} = \mathbb{Z}[q, q^{-1}]$ . Then  $\mathbf{U}_q^-$  has an  $\mathbf{A}$ -subalgebra  $_{\mathbf{A}}\mathbf{U}_q^-$  generated by q-divided powers  $f_i^{(n)} = f_i^n / [n]!$ , where  $[n] = (q^n - q^{-n}) / (q - q^{-1})$ and  $[n]! = [n][n-1] \cdots [1]$ . Then r induces  $_{\mathbf{A}}\mathbf{U}_q^- \to _{\mathbf{A}}\mathbf{U}_q^- \otimes _{\mathbf{A}}\mathbf{U}_q^-$ , which is denoted also by r.

1(ii). Perverse sheaves on the space of quiver representations and the canonical base. Consider the Dynkin diagram  $\mathcal{G} = (I, E)$ for the Kac-Moody Lie algebra  $\mathfrak{g}$ , where I is the set of vertices, and Ethe set of edges. Note that  $\mathcal{G}$  does not have an edge loop, i.e., an edge connecting a vertex to itself.

Let H be the set of pairs consisting of an edge together with its orientation. So we have #H = 2#E. For  $h \in H$ , we denote by i(h)(resp. o(h)) the incoming (resp. outgoing) vertex of h. For  $h \in H$  we denote by  $\overline{h}$  the same edge as h with the reverse orientation. Choose and fix an orientation  $\Omega$  of the graph, i.e., a subset  $\Omega \subset H$  such that  $\overline{\Omega} \cup \Omega = H, \ \Omega \cap \overline{\Omega} = \emptyset$ . The pair  $(I, \Omega)$  is called a *quiver*.

Let  $V = (V_i)_{i \in I}$  be a finite dimensional *I*-graded vector space over  $\mathbb{C}$ . The dimension of V is a vector

$$\dim V = (\dim V_i)_{i \in I} \in \mathbb{Z}_{\geq 0}^I.$$

We define a vector space by

$$\mathbf{E}_{V} \stackrel{\text{def.}}{=} \bigoplus_{h \in \Omega} \operatorname{Hom}(V_{\mathrm{o}(h)}, V_{\mathrm{i}(h)}).$$

Let  $G_V$  be an algebraic group defined by

$$G_V \stackrel{\text{def.}}{=} \prod_i \operatorname{GL}(V_i).$$

Its Lie algebra is the direct sum  $\bigoplus_i \mathfrak{gl}(V_i)$ . The group  $G_V$  acts on  $\mathbf{E}_V$  by

$$B = (B_h)_{h \in \Omega} \mapsto g \cdot B = (g_{\mathbf{i}(h)} B_h g_{\mathbf{o}(h)}^{-1})_{h \in \Omega}.$$

The space  $\mathbf{E}_V$  parametrizes isomorphism classes of representations of the quiver with the dimension vector dim V together with a linear base of the underlying vector space compatible with the *I*-grading. The action of the group  $G_V$  is induced by the change of bases.

In [22, 24] Lusztig introduced a full subcategory  $\mathcal{P}_V$  of the abelian category of perverse sheaves on  $\mathbf{E}_V$ . Its definition is not recalled here. See [22, §2] or [24, Chap. 9]. Its objects are  $G_V$ -equivariant.

Let  $\mathscr{D}(\mathbf{E}_V)$  be the bounded derived category of complexes of sheaves of  $\mathbb{C}$ -vector spaces over  $\mathbf{E}_V$ . Let  $\mathcal{Q}_V$  be the full subcategory of  $\mathscr{D}(\mathbf{E}_V)$ consisting of complexes that are isomorphic to finite direct sums of complexes of the form L[d] for  $L \in \mathcal{P}_V$ ,  $d \in \mathbb{Z}$ .

Let  $\mathscr{K}(\mathcal{Q}_V)$  be the Grothendieck group of  $\mathcal{Q}_V$ , that is the abelian group with generators (L) for isomorphism classes of objects L of  $\mathcal{Q}_V$ with relations (L) + (L') = (L'') whenever L'' is isomorphic to  $L \oplus L'$ . It is a module over  $\mathbf{A} = \mathbb{Z}[q, q^{-1}]$ , where q corresponds to the shift of complexes in  $\mathcal{Q}_V$ . Then  $\mathscr{K}(\mathcal{Q}_V)$  is a free **A**-module with a basis (L)where L runs over  $\mathcal{P}_V$ .

Let us consider the direct sum  $\bigoplus_{V} \mathscr{K}(\mathcal{Q}_{V})$  over all isomorphism classes of finite dimensional *I*-graded vector spaces. Let  $S_{i}$  be the *I*graded vector space with dim  $S_{i} = 1$ , dim  $S_{j} = 0$  for  $j \neq i$ . Then the corresponding space  $\mathbf{E}_{S_{i}}$  is a single point. Let  $1_{i}$  be the constant sheaf on  $\mathbf{E}_{S_{i}}$ , viewed as an element in  $\mathscr{K}(\mathcal{Q}_{S_{i}})$ . Then Lusztig defined a multiplication and a twisted coproduct on  $\bigoplus_{V} \mathscr{K}(\mathcal{Q}_{V})$  such that the **A**-algebra homomorphism

$$\Phi\colon {}_{\mathbf{A}}\mathbf{U}_q^- \to \bigoplus_V \mathscr{K}(\mathcal{Q}_V)$$

with  $\Phi(f_i) = 1_i$  is an isomorphism respecting twisted coproducts [22, 24]. The construction was motivated by an earlier work by Ringel [30].

We here recall the definition of the twisted coproduct on  $\bigoplus_V \mathscr{K}(\mathcal{Q}_V)$ . For the definition of the multiplication, see the original papers.

Let W be an I-graded subspace of V. Let T = V/W. Let E(W) be the subspace of  $\mathbf{E}_V$  consisting of  $B \in \mathbf{E}_V$  which preserves W. We consider the diagram

(1.2) 
$$\mathbf{E}_T \times \mathbf{E}_W \xleftarrow{\kappa} E(W) \xrightarrow{\iota} \mathbf{E}_V,$$

where  $\iota$  is the inclusion and  $\kappa$  is the map given by assigning to  $B \in E(W)$ , its restriction to W and the induced map on T.

Consider the functor

$$\operatorname{Res} \stackrel{\text{def.}}{=} \kappa_! \iota^*(\bullet)[d] \colon \mathscr{D}(\mathbf{E}_W) \to \mathscr{D}(\mathbf{E}_T \times \mathbf{E}_W),$$

where d is a certain explicit integer, whose definition is omitted here as it is not relevant for the discussion in this paper.

It is known that Res sends  $\mathcal{Q}_W$  to the subcategory  $\mathcal{Q}_{T,W}$  of  $\mathscr{D}(\mathbf{E}_T \times \mathbf{E}_W)$  consisting of complexes that are isomorphic to finite direct sums of complexes of the form  $(L \boxtimes L')[d]$  for  $L \in \mathcal{P}_T$ ,  $L' \in \mathcal{P}_W$ ,  $d \in \mathbb{Z}$ . Therefore we have an induced **A**-linear homomorphism

Res: 
$$\mathscr{K}(\mathcal{Q}_V) \to \mathscr{K}(\mathcal{Q}_{T,W}) \cong \mathscr{K}(\mathcal{Q}_T) \otimes_{\mathbf{A}} \mathscr{K}(\mathcal{Q}_W).$$

We take direct sum over V, T, W to get a homomorphism of an algebra with respect to the twisted multiplication (1.1). It corresponds to r on  ${}_{\mathbf{A}}\mathbf{U}_{q}^{-}$  under  $\Phi$ .

Recall that  $\mathscr{K}(\mathcal{Q}_V)$  has an **A**-basis (*L*), where *L* runs over  $\mathcal{P}_V$ . Taking direct sum over *V*, and pulling back by the isomorphism  $\Phi$ , we get an **A**-basis of  ${}_{\mathbf{A}}\mathbf{U}_q^-$ . This is Lusztig's canonical basis. Let us denote it by  $\mathscr{B}(\infty)$ .

Kashiwara gave an algebraic approach to  $\mathscr{B}(\infty)$ . See [15] and references therein for detail. He first introduced an  $\mathbf{A}_0$ -form  $\mathscr{L}(\infty)$  of  $\mathbf{U}_q^-$ , where  $\mathbf{A}_0 = \{f \in \mathbb{Q}(q) \mid f \text{ is regular at } q = 0\}$ . Then he also defined a basis, called the *crystal base* of  $\mathscr{L}(\infty)/q\mathscr{L}(\infty)$ . Then he introduced the global crystal base of  $\mathbf{U}_q^-$ , which descends to the crystal base of  $\mathscr{L}(\infty)/q\mathscr{L}(\infty)$ . It turns out that the global crystal base and the canonical base are the same. See [11].

In this paper, we do not distinguish the crystal base and the global base, that is the canonical base. We denote both by  $\mathscr{B}(\infty)$ .

When we want to emphasize that a canonical base element  $b \in \mathscr{B}(\infty)$  is a perverse sheaf, we denote it by  $L_b$ .

1(iii). Dual canonical base. There exists a unique symmetric bilinear form ( , ) on  $\mathbf{U}_q^-$  satisfying

$$(1,1) = 1, \qquad (f_i, f_j) = \delta_{ij},$$
$$(r(x), y \otimes z) = (x, yz) \quad \text{for } x, y, z \in \mathbf{U}_q^-.$$

Our normalization is different from [24, Ch. 1] and follows Kashiwara's as in [19, 9].

Under (, ), we can identify the graded dual algebra of  $\mathbf{U}_q^-$ , an algebra with the multiplication given by r, with  $\mathbf{U}_q^-$  itself.

Let  $\mathscr{B}^{\mathrm{up}}(\infty)$  denote the dual base of  $\mathscr{B}(\infty)$  with respect to (, ). It is called the *dual canonical base* of  $\mathbf{U}_q^-$ .

For  $b_1, b_2, b_3 \in \mathscr{B}(\infty)$ , let us define  $r_{b_3}^{b_1, b_2} \in \mathbf{A}$  by

$$r(b_3) = \sum_{b_1, b_2 \in \mathscr{B}(\infty)} r_{b_3}^{b_1, b_2} b_1 \otimes b_2.$$

Let  $b_1^{\text{up}}, b_2^{\text{up}}, b_3^{\text{up}} \in \mathscr{B}^{\text{up}}(\infty)$  be the dual elements corresponding to  $b_1$ ,  $b_2$ ,  $b_3$  respectively. Then we have

$$b_1^{\rm up}b_2^{\rm up} = \sum_{b_3^{\rm up}\in \mathscr{B}^{\rm up}(\infty)} r_{b_3}^{b_1,b_2}b_3^{\rm up}.$$

Thus the structure constant is given by  $r_{b_3}^{b_1,b_2}$ .

1(iv). Lusztig's lagrangian subvarieties and crystal. Let us introduce Lusztig's lagrangian subvariety in the cotangent space of the space of quiver representations.

The dual space to  $\mathbf{E}_V$  is

$$\mathbf{E}_{V}^{*} = \bigoplus_{h \in \overline{\Omega}} \operatorname{Hom}(V_{\mathrm{o}(h)}, V_{\mathrm{i}(h)}).$$

The group  $G_V$  acts on  $\mathbf{E}_V^*$  in the same way as on  $\mathbf{E}_V$ .

The  $G_V$ -action preserves the natural pairing between  $\mathbf{E}_V$  and  $\mathbf{E}_V^*$ . Considering  $\mathbf{E}_V \oplus \mathbf{E}_V^*$  as a symplectic manifold, we have the moment map  $\mu = (\mu_i) : \mathbf{E}_V \oplus \mathbf{E}_V^* \to \bigoplus_i \mathfrak{gl}(V_i)$  given by

$$\mu_i(B) = \sum_{\mathbf{i}(h)=i} \varepsilon(h) B_h B_{\overline{h}},$$

where B has components  $B_h$  for both  $h \in \Omega$  and  $\overline{\Omega}$ , and  $\varepsilon(h) = 1$  if  $h \in \Omega$  and -1 otherwise.

Lusztig's lagrangian  $\Lambda_V$  [21, 22] is defined as

(1.3) 
$$\Lambda_V \stackrel{\text{def.}}{=} \{ B \in \mathbf{E}_V \oplus \mathbf{E}_V^* \, | \, \mu(B) = 0, B \text{ is nilpotent} \}.$$

This space parametrizes isomorphism classes of nilpotent representation of the preprojective algebra associated with the quiver  $(I, \Omega)$ , together with a linear base of the underlying vector space compatible with the *I*-grading. The action of the group  $G_V$  is induced by the change of bases. The preprojective algebra is denoted by  $\Lambda$  in this paper.

This is a lagrangian subvariety in  $\mathbf{E}_V \oplus \mathbf{E}_V^*$ . (It was proved that  $\Lambda_V$  is half-dimensional in  $\mathbf{E}_V \oplus \mathbf{E}_V^*$  in [22, 12.3]. And the same argument shows that it is also a lagrangian. Otherwise use [26, Th. 5.8] and take the limit  $W \to \infty$ .)

Let Irr  $\Lambda_V$  be the set of irreducible components of  $\Lambda_V$ . Lusztig defined a structure of an abstract crystal (see [16, §3] for the definition)

on Irr  $\Lambda_V$  in [21], and Kashiwara-Saito proved that it is isomorphic to the underlying crystal of the canonical base  $\mathscr{B}(\infty)$  of  $\mathbf{U}_a^-$  [16].

We denote by  $\Lambda_b$  the irreducible component of  $\Lambda_V$  corresponding to a canonical base element  $b \in \mathscr{B}(\infty)$ .

1(v). **Dual semicanonical base.** Let  $C(\Lambda_V)$  be the Q-vector space of Q-valued constructible functions over  $\Lambda_V$ , which is invariant under the  $G_V$ -action. Lusztig defined an operator  $C(\Lambda_T) \times C(\Lambda_W) \to C(\Lambda_V)$  for  $V = T \oplus W$ , under which the direct sum  $\bigoplus_V C(\Lambda_V)$  is an associative algebra (see [22, §12]).

If  $V = S_i$ , then  $\Lambda_{S_i}$  is a single point. Let  $1_i$  be the constant function on  $\Lambda_{S_i}$  with the value 1. Let  $C_0$  be the subalgebra of  $\bigoplus_V C(\Lambda_V)$ generated by the elements  $1_i$   $(i \in I)$ , and let  $C_0(\Lambda_V) = C_0 \cap C(\Lambda_V)$ . Then Lusztig (see [22, Th. 12.13]) proved that  $C_0$  is isomorphic to the universal enveloping algebra  $\mathbf{U}(\mathbf{n})$  of the lower triangular subalgebra  $\mathbf{n}$ of  $\mathfrak{g}$  by  $f_i \mapsto 1_i$ .

Note that we have an embedding  $\Lambda_T \times \Lambda_W \to \Lambda_V$  given by the direct sum, where  $V = T \oplus W$  as above. Then the restriction defines an operator  $C(\Lambda_V) \to C(\Lambda_T) \otimes C(\Lambda_W)$ . Geiss-Leclerc-Schröer proved that it sends  $C_0(\Lambda_V)$  to  $C_0(\Lambda_T) \otimes C_0(\Lambda_W)$ , and gives the natural cocommutative coproduct on  $\mathbf{U}(\mathfrak{n})$  under the isomorphism  $C_0 \cong \mathbf{U}(\mathfrak{n})$  (see [7, §4]).

Let Y be an irreducible component of  $\Lambda_V$ . Then consider the functional  $\rho_Y \colon C(\Lambda_V) \to \mathbb{Q}$  given by taking the value on a dense open subset of Y. Then  $\{\rho_Y \mid Y \in \operatorname{Irr} \Lambda_V\}$  gives a base of  $C_0(\Lambda_V)$ . This follows from [23, §3] together with the result of Kashiwara-Saito mentioned above. Under the isomorphism  $\mathbf{U}(\mathfrak{n}) \cong C_0$ , the base  $\rho_Y$  is called the *dual semicanonical base* of  $\mathbf{U}(\mathfrak{n})^*_{\operatorname{gr}}$ , where  $\mathbf{U}(\mathfrak{n})^*_{\operatorname{gr}}$  denote the graded dual of  $\mathbf{U}(\mathfrak{n})$ .

We have a natural bijection  $b^{\text{up}} \mapsto \rho_{\Lambda_b}$  between the dual canonical base and the dual semicanonical base. However  $\rho_{\Lambda_b}$  is different from the specialization of  $b^{\text{up}}$  at q = 1 in general. See [7, §1.5] for a counterexample.

#### 2. Cluster Algebras and Quantum Unipotent subgroups

We fix a Weyl group element w throughout this section. Let  $\Delta_w^+ = \Delta^+ \cap w(-\Delta^+)$ , where  $\Delta^+$  is the set of positive roots. Then  $\mathfrak{n}(w) = \bigoplus_{\alpha \in \Delta_w^+} \mathfrak{g}_{-\alpha}$  is a Lie subalgebra of  $\mathfrak{g}$ , where  $\mathfrak{g}_{-\alpha}$  is the root subspace corresponding to the root  $-\alpha$ .

2(i). Quantum unipotent subgroup. Let us briefly recall the qanalog of the universal enveloping algebra  $\mathbf{U}(\mathfrak{n}(w))$  of  $\mathfrak{n}(w)$ , denoted by  $\mathbf{U}_q^-(w)$ . See [24, Ch. 40], [19, §4] and [9] for more detail. (It is denoted by  $A_q(\mathfrak{n}(w))$  in [9].)

Let  $T_i$  be the braid group operator corresponding to  $i \in I$ , where  $T_i = T''_{i,1}$  in the notation in [24]. Choose a reduced expression  $w = s_{i_1}s_{i_2}\cdots s_{i_\ell}$ . Then it gives  $\beta_p = s_{i_1}s_{i_2}\cdots s_{i_{p-1}}(\alpha_{i_p})$ , and we have  $\Delta_w^+ = \{\beta_p\}_{1 \leq p \leq \ell}$ . We define a *root vector* 

$$T_{i_1}T_{i_2}\cdots T_{i_{p-1}}(f_{i_p}).$$

Let  $\mathbf{c} = (c_1, \ldots, c_\ell) \in \mathbb{Z}_{\geq 0}^{\ell}$ . We multiply *q*-divided powers of root vectors in the order given by  $\beta_1, \ldots, \beta_\ell$ :

$$L(\mathbf{c}) = f_{i_1}^{(c_1)} T_{i_1}(f_{i_2}^{(c_2)}) \cdots (T_{i_1} \cdots T_{i_{\ell-1}})(f_{i_{\ell}}^{(c_{\ell})}).$$

Then the  $\mathbb{Q}(q)$ -subspace spanned by  $L(\mathbf{c})$  ( $\mathbf{c} \in \mathbb{Z}_{\geq 0}^{\ell}$ ) is independent of the choice of a reduced expression of w. This subspace is  $\mathbf{U}_{q}^{-}(w)$ . Moreover  $L(\mathbf{c})$  gives a basis of  $\mathbf{U}_{q}^{-}(w)$ . It can be shown that  $\mathbf{U}_{q}^{-}(w)$  is a subalgebra of  $\mathbf{U}_{q}^{-}$ .

It is known that  $\mathbf{U}_q^-(w)$  is compatible with the dual canonical base, i.e.,  $\mathbf{U}_q^-(w) \cap \mathscr{B}^{\mathrm{up}}(\infty)$  is a base of  $\mathbf{U}_q^-(w)$ . This is an interpretation of the main result due to Lusztig [25, Th. 1.2], based on an earlier work by Saito [31]. (See [19, Th. 4.25] for the current statement.)

Let  $\mathscr{B}^{\mathrm{up}}(w) \stackrel{\text{def.}}{=} \mathbf{U}_q^-(w) \cap \mathscr{B}^{\mathrm{up}}(\infty)$ . Let  $\mathscr{B}(w) \subset \mathscr{B}(\infty)$  be the corresponding subset in the canonical base. Then [25, Prop. 8.3] gives a parametrization of  $\mathscr{B}(w)$  as follows. Let  $\mathscr{L}(\infty)$  be the  $\mathbf{A}_0$ -form used in the definition of the crystal base. Then one shows  $L(\mathbf{c}) \in \mathscr{L}(\infty)$  and the set  $\{L(\mathbf{c}) \mod q\mathscr{L}(\infty)\}$  is equal to  $\mathscr{B}(w)$ , where  $\mathscr{B}(w)$  is considered as a subset of  $\mathscr{L}(\infty)/q\mathscr{L}(\infty)$ . Let us denote by  $b(\mathbf{c}) \in \mathscr{B}(w)$  the canonical base element corresponding to  $L(\mathbf{c}) \mod q\mathscr{L}(\infty)$ . Therefore  $b(\mathbf{c}) \equiv L(\mathbf{c}) \mod q\mathscr{L}(\infty)$ .

The argument in the proof of [2, Th. 3.13] shows that the transition matrix between the base  $\{L(\mathbf{c})\}$  and  $\{b(\mathbf{c})\}$  is upper triangular with respect to the lexicographic order on  $\{\mathbf{c}\}$ .

It is known that  $\{L(\mathbf{c})\}$  is orthogonal with respect to (, ) (see [24, Prop. 40.2.4]). Therefore we can deduce the corresponding relation between  $b^{\text{up}}(\mathbf{c})$  and  $L^{\text{up}}(\mathbf{c}) = L(\mathbf{c})/(L(\mathbf{c}), L(\mathbf{c}))$ , where  $b^{\text{up}}(\mathbf{c})$  is the dual canonical base element corresponding to  $b(\mathbf{c})$ . (See [19, Th. 4.29].)

2(ii).  $\mathscr{B}(w)$  and Kashiwara operators. Let us give a characterization of  $\mathscr{B}(w)$  in terms of Kashiwara operators on  $\mathscr{B}(\infty)$ . Recall that  $\mathscr{B}(\infty)$  is an abstract crystal, and hence has maps wt:  $\mathscr{B}(\infty) \to P$ ,  $\varepsilon_i: \mathscr{B}(\infty) \to \mathbb{Z}, \varphi_i: \mathscr{B}(\infty) \to \mathbb{Z} \ (i \in I)$  together with Kashiwara operators  $\tilde{e}_i: \mathscr{B}(\infty) \to \mathscr{B}(\infty) \sqcup \{0\}, \ \tilde{f}_i: \mathscr{B}(\infty) \to \mathscr{B}(\infty) \sqcup \{0\}$  satisfying certain axioms. We denote by  $u_{\infty}$  the element in  $\mathscr{B}(\infty)$  corresponding to 1 in  $\mathbf{U}_q^-$ . There is also an operator  $*: \mathscr{B}(\infty) \to \mathscr{B}(\infty)$ , which corresponds to the anti-involution  $*: \mathbf{U}_q^- \to \mathbf{U}_q^-$  given by  $f_i \mapsto f_i$ . Therefore we have another set of maps and operators  $\varepsilon_i^* = \varepsilon_i *, \ \varphi_i^* = \varphi_i *, \ \tilde{e}_i^* = *\tilde{e}_i *, \ \tilde{f}_i^* = *\tilde{f}_i *.$ 

Let  $w = s_{i_1}s_{i_2}\cdots s_{i_\ell}$  as above. For  $i = i_1$ , we have

$$\varepsilon_i(b) = c_1,$$
  

$$\tilde{e}_i b(\mathbf{c}) = \begin{cases} b(\mathbf{c}') & \text{if } c_1 \neq 0, \\ 0 & \text{if } c_1 = 0, \end{cases} \qquad \tilde{f}_i b(\mathbf{c}) = b(\mathbf{c}'')$$

where  $\mathbf{c}' = (c_1 - 1, c_2, \dots, c_\ell)$ ,  $\mathbf{c}'' = (c_1 + 1, c_2, \dots, c_\ell)$ . In particular,  $\mathscr{B}(w)$  is invariant under  $\tilde{e}_i, \tilde{f}_i$ .

Saito [31, Cor. 3.4.8] introduced a bijection

$$\Lambda_i \colon \{ b \in \mathscr{B}(\infty) \mid \varepsilon_i^*(b) = 0 \} \to \{ b \in \mathscr{B}(\infty) \mid \varepsilon_i(b) = 0 \}$$

by

$$\Lambda_i(b) = (\tilde{f}_i^*)^{\varphi_i(b)}(\tilde{e}_i)^{\varepsilon_i(b)}b, \qquad \Lambda_i^{-1}(b) = (\tilde{f}_i)^{\varphi_i^*(b)}(\tilde{e}_i^*)^{\varepsilon_i^*(b)}b.$$

This is related to the braid group operator as follows. Let us consider  $w = s_{i_1}s_{i_2}\cdots s_{i_\ell}$  and  $w' = s_{i_2}\cdots s_{i_\ell}s_{i_1} = s_{i_1}ws_{i_1}$  and corresponding PBW base elements

$$L = f_{i_1}^{(c_1)} T_{i_1}(f_{i_2}^{(c_2)}) \cdots (T_{i_1} \cdots T_{i_{\ell-1}})(f_{i_{\ell}}^{(c_{\ell})}),$$
  
$$L' = f_{i_2}^{(c'_2)} T_{i_2}(f_{i_3}^{(c'_3)}) \cdots (T_{i_2} \cdots T_{i_{\ell}})(f_{i_1}^{(c'_1)}),$$

for  $(c_1, \ldots, c_\ell) \in \mathbb{Z}_{\geq 0}^\ell$ ,  $(c'_2, \ldots, c'_\ell, c'_1) \in \mathbb{Z}_{\geq 0}^\ell$ . If  $c_1 = 0, c_2 = c'_2, \ldots, c_\ell = c'_\ell, 0 = c'_1$ , we have  $L = T_{i_1}L'$ . Let b, b' be the canonical base elements corresponding to L and L' respectively. Then Saito [31, Prop. 3.4.7] proved that the corresponding canonical base elements are related by  $b = \Lambda_{i_1}b'$ . As a corollary of this result, we have a bijection

$$\{b \in \mathscr{B}(w') \mid \varepsilon_i^*(b) = 0\} \xrightarrow{\Lambda_i} \{b \in \mathscr{B}(w) \mid \varepsilon_i(b) = 0\}.$$

This together with the invariance of  $\mathscr{B}(w)$  under  $\tilde{e}_{i_1}$ ,  $f_{i_1}$  gives a characterization of  $\mathscr{B}(w)$  inductively in the length of w, starting from  $\mathscr{B}(1) = \{u_\infty\}$ .

2(iii). A subcategory  $C_w$ . In view of §1(iv) it is natural to look for a characterization of  $\mathscr{B}(w)$  in terms of Lusztig's lagrangian subvarieties  $\Lambda_V$ , or the representation theory of the preprojective algebra. It turns out to be related to the subcategory introduced by Buan-Iyama-Reiten-Scott [4], and further studied by Geiss-Leclerc-Schröer [8] and Baumann-Kamnitzer-Tingley [1].

We do not recall the definition of the subcategory of the category of finite-dimensional nilpotent representations of the preprojective algebra, denoted by  $C_w$  following [8], here. This is because there are many equivalent definitions, and the author does not know what is the best for our purpose. See the above papers.

Let  $\Lambda_V^w = \{B \in \Lambda_V \mid B \in \mathcal{C}_w\}$ , where we identify B with the corresponding representation of the preprojective algebra. This is an open subvariety in  $\Lambda_V$  (see [10, Lem. 7.2]). We set

$$C_0^w(\Lambda_V) = \{ f \in C_0(\Lambda_V) \mid f(X) = 0 \text{ for } X \in \mathcal{C}_w \}.$$

Therefore  $C_0(\Lambda_V)/C_0^w(\Lambda_V)$  consists of constructible functions on  $\Lambda_V^w$ . Let us consider

$$C_0^w(\Lambda_V)^{\perp} = \{\xi \in C_0(\Lambda_V)^* \mid \langle f, \xi \rangle = 0 \text{ for } f \in C_0^w(\Lambda_V) \}.$$

This space is spanned by an evaluation at a point  $X \in \Lambda_V^w$ . As  $\mathcal{C}_w$  is an additive category (in fact, it is closed under extensions), we have

$$C_0^w(\Lambda_T)^{\perp} \cdot C_0^w(\Lambda_W)^{\perp} \subset C_0^w(\Lambda_V)^{\perp},$$

where the multiplication is given by the transpose of r, the natural cocommutative coproduct on  $\mathbf{U}(\mathbf{n})$ . This follows from the interpretation of r explained in §1(v).

Therefore  $\bigoplus C_0^w(\Lambda_V)^{\perp} \subset (C_0)_{\text{gr}}^* \cong \mathbf{U}(\mathfrak{n})_{\text{gr}}^*$  is a subalgebra. By [8, Th. 3.3] it is the  $\mathbb{C}[N(w)]$ , which is the q = 1 limit of  $\mathbf{U}_q^-(w)$  ([19, Th. 4.44]).

By [8, Th. 3.2]  $\mathbb{C}[N(w)]$  is compatible with the dual semicanonical basis, i.e., the intersection  $\{\rho_{\Lambda_b} \mid b \in \mathscr{B}(\infty)\} \cap \mathbb{C}[N(w)]$  is a base of  $\mathbb{C}[N(w)]$ . This base consists of  $\rho_{\Lambda_b}$  such that  $\Lambda_b$  intersects with the open subvariety  $\Lambda_V^w$ . The intersection  $\Lambda_b \cap \Lambda_V^w$  is an open dense subset of  $\Lambda_b$ .

Finally we have

$$\{\rho_{\Lambda_b} \mid b \in \mathscr{B}(\infty)\} \cap \mathbb{C}[N(w)] = \{\rho_{\Lambda_b} \mid b \in \mathscr{B}(w)\}.$$

This follows from [1, §5.5] and the characterization of  $\mathscr{B}(w)$  in §2(ii).

2(iv). Cluster algebras and  $C_w$ . Geiss-Leclerc-Schröer [8] have introduced a structure of the cluster algebra, in the sense of Fomin-Zelevinsky [6], on  $\mathbb{C}[N(w)]$ . One of their main results says that dual semicanonical base of  $\mathbb{C}[N(w)]$  contains *cluster monomials*. We review their theory only briefly here. See the original paper for more detail. The construction is based on  $C_w$  in §2(iii).

A  $\Lambda$ -module T is rigid if  $\operatorname{Ext}_{\Lambda}^{1}(T,T) = 0$ . It is easy to see from the formula dim  $\operatorname{Ext}_{\Lambda}^{1}(T,T) = 2 \dim \operatorname{Hom}_{\Lambda}(T,T) - (\dim V, \dim V)$  (see e.g., [8, Lem. 2.1]) that this is equivalent to say that the orbit through

T is open in  $\Lambda_V$ , where V is the I-graded vector space underlying T, and (,) is the Cartan matrix for the graph  $\mathcal{G}$ . We say T is  $\mathcal{C}_w$ maximal rigid if  $\operatorname{Ext}^1_{\Lambda}(T \oplus X, X) = 0$  with  $X \in \mathcal{C}_w$  implies X is in add(T), the subcategory of modules which are isomorphic to finite direct sums of direct summands of T. In  $\mathcal{C}_w$ , there is a distinguished  $\mathcal{C}_w$ -maximal module, denoted by  $V_i$  in [8], where  $\mathbf{i} = (i_1, \ldots, i_\ell)$  is a reduced expression of w. It is conjectured that every  $\mathcal{C}_w$ -maximal rigid module T is reachable, i.e., it is obtained from  $V_i$  using a sequence of operations, called mutations. This will be recalled below.

Let T be a reachable  $C_w$ -maximal rigid module, and  $T = T_1 \oplus \cdots \oplus T_\ell$ be the decomposition into indecomposables. We assume that T is *basic*, which means  $T_i$  are pairwise non-isomorphic. In this case the number of summands  $\ell$  is known to be equal to the length of w. If  $R \in \text{add}(T)$ , it is a rigid  $\Lambda$ -module, and hence the closure of the corresponding orbit is an irreducible component of  $\Lambda_V$  for an appropriate choice of V. We denote it by  $\rho_R \in \mathbb{C}[N(w)]$  the corresponding dual semicanonical base elements.

From the identification of the coproduct r on  $\mathbf{U}(\mathfrak{n})^*_{\mathrm{gr}}$ , we see that

(2.1) 
$$\rho_R = \rho_{T_1}^{c_1} \cdots \rho_{T_\ell}^{c_\ell},$$

where  $R = T_1^{\oplus c_1} \oplus \cdots \oplus T_{\ell}^{c_{\ell}}$  with  $c_i \in \mathbb{Z}_{\geq 0}$ . In the context of the cluster algebra theory,  $\rho_{T_i}$   $(1 \leq i \leq \ell)$  are called *cluster variables* and  $\rho_R$  is a *cluster monomial*.

Let T be a basic  $C_w$ -maximal rigid module, and  $T_k$  be a non-projective indecomposable direct summand of T. The mutation  $\mu_k(T)$  is a new basic  $C_w$ -maximal rigid module of the form  $(T/T_k) \oplus T_k^*$ , where  $T_k^*$  is another indecomposable module. Such a  $\mu_k(T)$  exists and is uniquely determined from T and  $T_k$  (see [8, Prop. 2.19] and the reference therein). We have dim  $\operatorname{Ext}^1_{\Lambda}(T_k, T_k^*) = \dim \operatorname{Ext}^1_{\Lambda}(T_k^*, T_k) = 1$  and

(2.2) 
$$\rho_{T_k} \rho_{T_k^*} = \rho_{T'} + \rho_{T''},$$

where  $T', T'' \in \operatorname{add}(T/T_k)$  are given by short exact sequences

$$0 \to T_k \to T' \to T_k^* \to 0, \qquad 0 \to T_k^* \to T'' \to T_k \to 0$$

respectively.

Geiss-Leclerc-Schröer [9] have obtained a q-analog of the result explained above, namely they have introduced a structure of the quantum cluster algebra, in the sense of Berenstein-Zelevinsky [3], on the quantum unipotent subgroup  $\mathbf{U}_q^-(w)$ . (This result was conjectured in [19].) The construction is again based on  $\mathcal{C}_w$ . Let T be a reachable  $\mathcal{C}_w$ -maximal rigid module. For  $R \in \operatorname{add}(T)$ , there is an element  $Y_R \in \mathbf{U}_q^-(w)$ , which satisfies the q-analog of (2.1, 2.2):

(2.3) 
$$Y_R = q^{-\alpha_R} Y_{T_1}^{c_1} \cdots Y_{T_\ell}^{c_\ell},$$
$$Y_{T_k}^* Y_{T_k} = q^{[T_k^*, T_k]} (q^{-1} Y_{T'} + Y_{T''})$$

for appropriate explicit  $\alpha_R$ ,  $[T_k^*, T_k] \in \mathbb{Z}$ . See [9, (10.17) and Prop. 10.5]. One of the main conjectures in this theory is

**Conjecture 2.4.** All quantum cluster monomials  $Y_R$  are contained in  $\mathscr{B}^{up}(w)$ .

This is closely related to their earlier open orbit conjecture  $[8, \S 18.3]$ :

**Conjecture 2.5.** Suppose that an irreducible component  $\Lambda_b$  of  $\Lambda_V$  contains an open  $G_V$ -orbit, then the corresponding dual semicanonical base element  $\rho_{\Lambda_b}$  is equal to the specialization of the corresponding canonical base element b at q = 1.

In fact, this is implied by Conj. 2.4 for reachable rigid modules.

# 3. SINGULAR SUPPORTS UNDER THE RESTRICTION

Let SS(L) denote the singular support of a complex  $L \in \mathcal{Q}_V$ . See [17] for the definition. Lusztig proved that  $SS(L) \subset \Lambda_V$  [22, 13.6]. In fact, we have finer estimates

(3.1) 
$$\Lambda_b \subset SS(L_b) \subset \bigcup_{\substack{b' \in \mathscr{B}(\infty) \\ \forall i \ \varepsilon_i(b') \ge \varepsilon_i(b)}} \Lambda_{b'}$$

See [16, Thm. 6.2.2], but note that there is a misprint. See [16, Lem. 8.2.1] for the correct statement.

These estimates give us some relation between the canonical base and  $\bigsqcup_V \operatorname{Irr} \Lambda_V$  via singular supports. We study the behavior of singular supports under the functor Res in this section.

3(i). The statement. In order to state the result, we prepare notation.

Let  $f: Y \to X$  be a morphism. Let TX, TY (resp.  $T^*X$ ,  $T^*Y$ ) be tangent (resp. cotangent) bundles of X, Y respectively. Let  $f^{-1}TX =$  $Y \times_X TX$  (resp.  $f^{-1}T^*X = Y \times_X T^*X$ ) be the pull-back of TX (resp.  $T^*X$ ) by f. We have associated morphisms

$$T^*Y \xleftarrow{^{t}f'} f^{-1}T^*X \xrightarrow{f_\pi} T^*X,$$

where  ${}^{t}f'$  is the transpose of the differential  $f': TY \to f^{-1}TX = Y \times_X TX$ .

We apply this construction for the morphisms  $\iota$ ,  $\kappa$  to get morphisms

$$T^*E(W) \xleftarrow{\iota_{\iota'}} \iota^{-1}T^*\mathbf{E}_V \xrightarrow{\iota_{\pi}} T^*\mathbf{E}_V,$$
$$T^*E(W) \xleftarrow{\iota_{\kappa'}} \kappa^{-1}T^*(\mathbf{E}_T \times \mathbf{E}_W) \xrightarrow{\kappa_{\pi}} T^*(\mathbf{E}_T \times \mathbf{E}_W).$$

Theorem 3.2. We have

$$SS(\operatorname{Res}(L)) \subset \kappa_{\pi}({}^{t}\kappa'^{-1}({}^{t}\iota'(\iota_{\pi}^{-1}(SS(L))))).$$

Let us remark that the proof shows the following statement. Choose a complementary subspace of W in V, and identify V with  $W \oplus T$ . Then we have the induced embedding  $T^*(\mathbf{E}_T \times \mathbf{E}_W) \subset T^*\mathbf{E}_V$ . Then we have

(3.3) 
$$SS(\operatorname{Res}(L)) \subset T^*(\mathbf{E}_T \times \mathbf{E}_W) \cap SS(L).$$

The proof occupies the rest of this section.

3(ii). **Inverse image.** If  $\iota$  were smooth, we would have  $SS(\iota^*L) \subset {}^{t}\iota'(\iota_{\pi}^{-1}(SS(L)))$  by [17, Prop. 5.4.5]. And if  $\kappa$  were proper, we would have  $SS(\kappa_{!}\iota^*L) \subset [3]\kappa_{\pi}({}^{t}\kappa'^{-1}(SS(\iota^*L)))$  by [17, Prop. 5.4.4]. Therefore the assertion follows. However neither are true, so we need more refined versions of these estimates.

In order to study the behavior of the singular support under the pull-back by a non-smooth morphism, we need several more notions related to cotangent manifolds from [17, Ch. VI].

We first recall the normal cone to S along M briefly. Suppose that M is a closed submanifold of a manifold X. Let  $T_M X$  denote the normal bundle of M in X. Then one can define a new manifold  $\tilde{X}_M$ , which connects X and  $T_M X$  in the following way: there are two maps

$$p\colon \tilde{X}_M \to X, \quad t\colon \tilde{X}_M \to \mathbb{R}$$

such that  $p^{-1}(X \setminus M)$ ,  $t^{-1}(\mathbb{R} \setminus \{0\})$  and  $t^{-1}(0)$  are isomorphic to  $(X \setminus M) \times (\mathbb{R} \setminus \{0\})$ ,  $X \times (\mathbb{R} \setminus \{0\})$  and  $T_M X$  respectively. In our application, M is the zero section of a vector bundle X, and hence the normal bundle  $T_M X$  is X itself. In this case,  $\tilde{X}_M$  is  $X \times \mathbb{R}$  and p, t are the first and second projections. A general definition is in [17, §4.1] for an interested reader.

Let  $\Omega$  be the inverse image of  $\mathbb{R}^+$  under t, and  $\tilde{p}$  the restriction of p to  $\Omega$ .

Let S be a subset of X. The normal cone to S along M, denoted by  $C_M(S)$  is defined by

$$C_M(S) \stackrel{\text{def.}}{=} T_M X \cap \overline{\tilde{p}^{-1}(S)}.$$

If M is the zero section of a vector bundle X as before, we have  $C_M(S)$  is identified with S itself under  $T_M X \cong X$ .

Now we return back to a morphism  $f: Y \to X$ . We assume that f is a closed embedding for simplicity. We consider the conormal bundle  $T_Y^*X$ . We denote the projection  $T_Y^*X \to Y$  by p. We have morphisms  $T^*(T_Y^*X) \xleftarrow{}^{t_{p'}} p^{-1}T^*Y \xrightarrow{p_{\pi}} T^*Y$  as before.

We consider  $T^*Y$  as a submanifold of  $T^*(T^*_YX)$  via the composition

$$T^*Y \hookrightarrow p^{-1}T^*Y \xrightarrow{tp'} T^*(T^*_YX).$$

See [17, (5.5.10)].

We consider  $T_Y^*X$  as a closed submanifold of  $T^*X$ . So we can define the normal cone to a subset of  $T^*X$  along  $T_Y^*X$ .

If  $f: Y \to X$  is the embedding of the zero section to a vector bundle X, we can identify  $T_Y^*X$  with the dual vector bundle  $X^*$ . Then  $T_Y^*X \to T^*X$  is also the embedding of the zero section to a vector bundle. In fact, we have a natural identification  $T^*X \cong T^*X^*$ , therefore  $T_Y^*X \to T^*X$  is identified with  $X^* \to T^*X^*$ . Therefore  $C_{T_Y^*X}(A)$  is identified with A itself under the isomorphism  $T_{T_Y^*X}(T^*X) \cong T^*X$ .

Note also that  $T^*(T^*_YX)$  is identified with  $T^*X^* \cong T^*X$ . Under this identification, we have an isomorphism  $p^{-1}T^*Y \cong f^{-1}T^*X$  which gives a commutative diagram

Let A be a conic subset of  $T^*X$ , i.e., invariant under the  $\mathbb{R}^+$ -action, the multiplication on fibers. We define

$$f^{\#}(A) \stackrel{\text{def.}}{=} T^*Y \cap C_{T^*_Y X}(A).$$

Here we identify  $T_{T_Y^*X}T^*X$  with  $T^*(T_Y^*X)$ . See [17, (6.2.3)]. Moreover we also have

$$f^{\#}(A) = p_{\pi}{}^{t} p'^{-1}(C_{T_{Y}^{*}X}(A)).$$

See [17, Lem. 6.2.1].

If  $f: Y \to X$  is the embedding of the zero section to a vector bundle X, we have

(3.5) 
$$f^{\#}(A) = T^*Y \cap A = {}^t f' f_{\pi}^{-1}(A)$$

by the commutative diagram (3.4).

3(iii). **Spaces.** Let us describe relevant spaces explicitly.

As E(W) is a linear subspace of  $\mathbf{E}_V$ , we have  $T^*E(W) = E(W) \times E(W)^*$  and  $E(W)^*$  is identified with  $\mathbf{E}_V^*/E(W)^{\perp}$ , where

$$E(W)^{\perp} = \{ B' \in \mathbf{E}_V^* \mid B'(W) = 0, \operatorname{Im} B' \subset W \}.$$

Taking a complementary subspace of W in V, we identify T as an I-graded subspace of V. We then have the direct sum decomposition  $V \cong W \oplus T$  and the induced projection  $V \to W$ . Then we have matrix notations of B and B':

$$B = \begin{pmatrix} B_{TT} & 0\\ B_{WT} & B_{WW} \end{pmatrix}, \quad B' = \begin{pmatrix} B'_{TT} & B'_{TW}\\ 0 & B'_{WW} \end{pmatrix}$$

Similarly the space  $\iota^{-1}(T^*\mathbf{E}_V)$  is nothing but  $E(W) \times \mathbf{E}_V^*$ , and identified with the space of linear maps B, B' of the forms

$$B = \begin{pmatrix} B_{TT} & 0\\ B_{WT} & B_{WW} \end{pmatrix}, \quad B' = \begin{pmatrix} B'_{TT} & B'_{TW}\\ B'_{WT} & B'_{WW} \end{pmatrix}$$

The morphism  ${}^{t}\iota': \iota^{-1}(T^*\mathbf{E}_V) \to T^*E(W)$  is induced by the projection  $\mathbf{E}_V^* \to E(W)^*$ . In the matrix notation, it is given by forgetting the component  $B'_{WT}$ .

The morphism  $\iota_{\pi}: \iota^{-1}(T^*\mathbf{E}_V) \to T^*\mathbf{E}_V$  is the embedding  $E(W) \times \mathbf{E}_V^* \to \mathbf{E}_V \times \mathbf{E}_V^*$ .

For  $(B, B') \in T^*(\mathbf{E}_T \times \mathbf{E}_W)$ , we have

$$B = \begin{pmatrix} B_{TT} & 0\\ 0 & B_{WW} \end{pmatrix}, \quad B' = \begin{pmatrix} B'_{TT} & 0\\ 0 & B'_{WW} \end{pmatrix}$$

For  $(B, B') \in \kappa^{-1}(T^*(\mathbf{E}_T \times \mathbf{E}_W))$ , we have

$$B = \begin{pmatrix} B_{TT} & 0\\ B_{WT} & B_{WW} \end{pmatrix}, \quad B' = \begin{pmatrix} B'_{TT} & 0\\ 0 & B'_{WW} \end{pmatrix}.$$

The morphisms

$$T^*E(W) \xleftarrow{^{\iota_{\kappa'}}} \kappa^{-1}(T^*(\mathbf{E}_T \times \mathbf{E}_W)) \xrightarrow{\kappa_{\pi}} T^*(\mathbf{E}_T \times \mathbf{E}_W)$$

are given by taking appropriate matrix entries of B, B'.

Since we have chosen an isomorphism  $V \cong W \oplus T$ , we have the projection  $p: \mathbf{E}_V \to E(W)$  which gives a structure of a vector bundle so that  $\iota$  is the embedding of the zero section. Therefore we have the commutative diagram (3.4) for  $f = \iota$ , and hence

(3.6) 
$$\iota^{\#}(A) = T^* E(W) \cap A = {}^t \iota' \iota_{\pi}^{-1}(A)$$

by (3.5).

3(iv). **Proof.** We first study the behavior of the singular support under the functor  $\iota^*$ . Let  $L \in \mathcal{Q}_V$ . By [17, Cor. 6.4.4] we have

$$SS(\iota^*L) \subset \iota^\#(SS(L)).$$

In our situation, we have  $\iota^{\#}(SS(L)) = {}^{t}\iota'(\iota_{\pi}^{-1}(SS(L)))$  by (3.6).

Next study the functor  $\kappa_1$ . Note that  $\kappa \colon E(W) \to \mathbf{E}_T \times \mathbf{E}_W$  is the projection of a vector bundle. Therefore the results in [17, §5.5] are applicable. A complex F in  $\mathscr{D}(E(W))$  is *conic* if  $H^j(F)$  is locally constant on the orbits of the  $\mathbb{R}^+$ -action for all j. In our situation,  $F = i^*L$  satisfies this condition. Then we have

$$SS(\kappa_!(i^*L)) \subset T^*(\mathbf{E}_T \times \mathbf{E}_W) \cap SS(i^*L) = \kappa_\pi^{\ t} \kappa'^{-1} SS(i^*L).$$

See [17, Prop. 5.5.4] for the first inclusion and [17, (5.5.11)] for the second equality. Combining two estimates, we complete the proof of Theorem 3.2. The estimate (3.3) has been given during the proof.

# 4. Conjectures

4(i). Quantum unipotent subgroup and singular supports. Let w be a Weyl group element as before. Motivated by §2(iii), we introduce a subset  $\mathscr{B}'(w)$  in  $\mathscr{B}(\infty)$  by

$$\mathscr{B}'(w) \stackrel{\text{def.}}{=} \{ b \in \mathscr{B}(\infty) \mid SS(L_b) \cap \Lambda^w_V \neq \emptyset \},\$$

where we suppose  $L_b \in \mathcal{P}_V$  in the equation  $SS(L_b) \cap \Lambda_V^w \neq \emptyset$ . Equivalently  $b \notin \mathscr{B}'(w)$  if and only if  $SS(L_b)$  is contained in the closed subvariety  $\Lambda_V \setminus \Lambda_V^w$ .

By (3.1), the condition  $\Lambda_b \cap \Lambda_V^w \neq \emptyset$  implies  $b \in \mathscr{B}'(w)$ . Therefore  $\mathscr{B}(w) \subset \mathscr{B}'(w)$  by §2(iii).

Let  $b \notin \mathscr{B}'(w)$ . We have

$$SS(\operatorname{Res}(L_b)) \cap (\Lambda_T^w \times \Lambda_W^w) \subset SS(L_b) \cap \Lambda_V^w = \emptyset$$

by (3.3) and the fact that  $\mathcal{C}_w$  is an additive category. Writing

$$\operatorname{Res}(L_b) = \bigoplus \left( L_{b_1} \boxtimes L_{b_2} \right) [n]^{\oplus r_{b;n}^{b_1, b_2}},$$

we get

$$SS(L_{b_1} \boxtimes L_{b_2}) \cap (\Lambda_T^w \times \Lambda_W^w) = \emptyset$$

if  $r_{b;n}^{b_1,b_2} \neq 0$  for some *n*. This is because  $SS(L \oplus L') = SS(L) \cup SS(L')$ and SS(L[1]) = SS(L) (see [16, Chap. V]). In the notation in §1(iii) we have  $r_b^{b_1,b_2} = \sum_n r_{b;n}^{b_1,b_2} q^n$ .

We have an estimate  $SS(L_{b_1} \boxtimes L_{b_2}) \subset SS(L_{b_1}) \times SS(L_{b_2})$  [16, Prop. 5.4.1]. However this does not imply  $SS(L_{b_1}) \times SS(L_{b_2}) \cap (\Lambda_T^w \times \Lambda_W^w) = \emptyset$ , so we need a finer estimate. Since  $L_{b_a}$  (a = 1, 2) is a perverse sheaf, it corresponds to a regular holonomic *D*-module under the Riemann-Hilbert

correspondence (see e.g., [14, Th. 7.2.5]). Then the singular support of  $L_{b_a}$  is the same as the characteristic variety of the corresponding *D*-module [17, Th. 11.3.3], [14, Th. 4.4.5]. As the characteristic variety of the exterior product is the product of the characteristic varieties [17, (11.2.22)], we deduce  $SS(L_{b_1} \boxtimes L_{b_2}) = SS(L_{b_1}) \times SS(L_{b_2})$ . Therefore we have

 $(SS(L_{b_1}) \cap \Lambda_T^w) \times (SS(L_{b_2}) \cap \Lambda_W^w) = \emptyset.$ 

Therefore either  $b_1 \notin \mathscr{B}'(w)$  or  $b_2 \notin \mathscr{B}'(w)$ . In other words,  $b_1, b_2 \in \mathscr{B}'(w)$  and  $r_b^{b_1,b_2} \neq 0$  implies  $b \in \mathscr{B}'(w)$ . Therefore  $\bigoplus_{b \in \mathscr{B}'(w)} \mathbb{Q}(q) b^{\mathrm{up}}$  is a subalgebra of  $\mathbf{U}_q^-$  by §1(iii).

Our first conjecture is the following.

**Conjecture 4.1.**  $\mathscr{B}'(w) = \mathscr{B}(w)$ . In other words, if  $b \notin \mathscr{B}(w)$ , then  $SS(L_b) \subset \Lambda_V \setminus \Lambda_V^w$ .

This conjecture is also equivalent to say  $\bigoplus_{b \in \mathscr{B}'(w)} \mathbb{Q}(q) b^{\mathrm{up}} = \mathbf{U}_q^-(w).$ 

4(ii). Cluster algebra and singular supports. Recall that Geiss-Leclerc-Schröer [9] have introduced the structure of a quantum cluster algebra on  $\mathbf{U}_q^-(w)$  and conjectured that quantum cluster monomials are contained in  $\mathscr{B}^{\mathrm{up}}(w)$ . If this is true, we should have two formulas (2.3) for dual canonical base elements corresponding to  $Y_R$ ,  $Y_{T_k}$ , etc. Conversely (2.3) implies that  $Y_R$ ,  $Y_{T_k^*}$  are dual canonical base elements by induction on the number of mutations.

Let us speculate why these formulas hold in terms of the corresponding perverse sheaves.

The proposal here is the following conjecture:

**Conjecture 4.2.** Let T be a reachable  $C_w$ -maximal rigid module and  $R \in \operatorname{add}(T)$ . Let  $\Lambda_R$  be the closure of the orbit through R and  $b_R$  the corresponding canonical base element.

If another canonical base element  $b \in \mathscr{B}(w)$  satisfies

$$\Lambda_R \subset SS(L_b),$$

we should have  $b = b_R$ .

If Conj. 4.1 is true,  $b \in \mathscr{B}(\infty)$  with  $\Lambda_{b_R} \subset SS(L_b)$  is contained in  $\mathscr{B}(w)$ . Therefore the above conjecture holds for any  $b \in \mathscr{B}(\infty)$ .

This conjecture is true for a special case when  $\Lambda_R$  is the zero section  $\mathbf{E}_V$  of  $T^*\mathbf{E}_V$  and  $G_V$  has an open orbit in  $\mathbf{E}_V$ . In fact, if  $SS(L_b) \supset \mathbf{E}_V$ , we have  $\operatorname{supp}(L_b) = \mathbf{E}_V$ . Then  $L_b$  is  $G_V$ -equivariant and gives an irreducible  $G_V$ -equivariant local system on the open orbit in  $\mathbf{E}_V$ . As the stabilizer of a point is connected from a general property from quiver representations, it must be the trivial rank 1 local system. Thus  $L_b$  is

the constant sheaf on  $\mathbf{E}_V$ . In fact, the observation that  $\operatorname{supp}(L_b) = \mathbf{E}_V$ implies  $L_b$  = the constant sheaf was used in a crucial way to prove the cluster character formula in [27].

If  $SS(L_b)$  is irreducible for all b, Conj. 4.2 is obviously true. This condition is satisfied for  $\mathfrak{g}$  of type  $A_4$ , but not for  $A_5$  [16].

Let us remark a relation between the above conjecture and a conjecture in [7, §1.5]. This is pointed out by the referee to the author. Let us define the *semicanonical base*  $\{f_Y\}$  of  $\mathbf{U}(\mathbf{n})$  as the dual base of the dual semicanonical base  $\{\rho_Y\}$ . In [7, §1.5] it is conjectured that the specialization of b is a linear combination  $\sum m_Y f_Y$  ( $m_Y \in \mathbb{Z}$ ), where the summation runs over irreducible components Y of  $SS(L_b)$ . (More precisely it is probably given by the characteristic cycle (see [14, 2.2.2] for the definition) of  $L_b$ .) Dually, an irreducible component  $Y = \Lambda_b$ cannot be contained in other  $SS(L_{b'})$  ( $b' \neq b$ ) if  $b^{up}|_{q=1} = \rho_{\Lambda_b}$ . This is nothing but our conjecture. Thus under the conjecture in [7, §1.5], our conjecture is equivalent to Conj. 2.4 for reachable rigid modules.

Let us explain how the first formula in (2.3) is related to Conj. 4.2. We assume  $R = T_1 \oplus T_2$  for brevity. From the assumption  $\Lambda_R$  contains the product  $\Lambda_{T_1}^{\circ} \times \Lambda_{T_2}^{\circ}$  as an open dense subset. Here  $\Lambda_{T_i}^{\circ}$  denote the open orbit through  $T_i$ . Its closure is  $\Lambda_{T_i}$ . Suppose that  $b^{\text{up}}$  appears in the product  $b_{T_1}^{\text{up}} b_{T_2}^{\text{up}}$ . Then  $r_b^{b_{T_1}, b_{T_2}} \neq 0$ . We have

$$SS(L_b) \cap (\Lambda_{T_1}^{\circ} \times \Lambda_{T_2}^{\circ}) \supset SS(\operatorname{Res}(L_b)) \cap (\Lambda_{T_1}^{\circ} \times \Lambda_{T_2}^{\circ})$$
$$\supset SS(L_{b_{T_1}} \boxtimes L_{b_{T_2}}) \cap (\Lambda_{T_1}^{\circ} \times \Lambda_{T_2}^{\circ})$$
$$= (SS(L_{b_{T_1}}) \cap \Lambda_{T_1}^{\circ}) \times (SS(L_{b_{T_2}}) \cap \Lambda_{T_2}^{\circ}),$$

where the first inclusion is by (3.3), the second as a shift of  $L_{b_{T_1}} \boxtimes L_{b_{T_2}}$ is a direct summand of Res  $L_b$ , and the third equality was observed above. The last expression is nonempty thanks to (3.1). Therefore we have  $SS(L_b) \supset \Lambda_R$ . Then Conj. 4.2 implies that  $b = b_R$ . Therefore  $b_{T_1}^{up} b_{T_2}^{up}$  is a multiple of  $b_R^{up}$ . A refinement of this argument probably proves that  $b_{T_1}^{up} b_{T_2}^{up}$  is equal to  $b_R^{up}$  up to a power of q.

Let us turn to the second formula in (2.3). The same argument above implies that

$$SS(L_b) \cap (\Lambda_{T_k^*}^\circ \times \Lambda_{T_k}^\circ) \neq \emptyset$$

if  $r_b^{b_{T_k^*}, b_{T_k}} \neq 0$ . From what we have explained in §2(iv), there are two irreducible components  $\Lambda_{T'}, \Lambda_{T''}$ , where T' and T'' are  $\Lambda$ -modules given by non-trivial extensions of  $T_k^*$  and  $T_k$ . As a non-trivial extension can degenerate to the trivial one, both  $\Lambda_{T'}$  and  $\Lambda_{T''}$  contain  $\Lambda_{T_k^*}^\circ \times \Lambda_{T_k}^\circ$ . It is also easy to check that dim  $\Lambda_{T_k^*}^\circ \times \Lambda_{T_k}^\circ = \dim \Lambda_V - 1$ , where V is the underlying vector space of  $T_k \oplus T_k^*$ . **Lemma 4.3.** If an irreducible component Y of  $\Lambda_V$  contains  $\Lambda_{T_k}^{\circ} \times \Lambda_{T_k}^{\circ}$ , we have either  $Y = \Lambda_{T'}$  or  $= \Lambda_{T''}$ .

*Proof.* Take a sequence  $Z_n$  of points of Y converging to the module  $T_k^* \oplus T_k$ , regarded as a point of Y. We may assume  $Z_n \not\cong T_k^* \oplus T_k$ .

Then we have dim  $\operatorname{Hom}(Z_n, T_k^* \oplus T_k) \leq \dim \operatorname{Hom}(T_k^* \oplus T_k, T_k^* \oplus T_k)$  for sufficiently large *n* by the upper semicontinuity of the dimension of cohomology groups. If the equality holds, we can take  $\xi_n \in \operatorname{Hom}(Z_n, T_k^* \oplus T_k)$  converging to the identity of  $\operatorname{Hom}(T_k^* \oplus T_k, T_k^* \oplus T_k)$  for  $n \to \infty$ . In particular,  $\xi_n$  is invertible, hence  $Z_n \cong T_k^* \oplus T_k$ . This contradicts with our assumption. Therefore we have the strict inequality dim  $\operatorname{Hom}(Z_n, T_k^* \oplus T_k) < \dim \operatorname{Hom}(T_k^* \oplus T_k, T_k^* \oplus T_k)$ . The same argument gives dim  $\operatorname{Hom}(T_k^* \oplus T_k, Z_n) < \dim \operatorname{Hom}(T_k^* \oplus T_k, T_k^* \oplus T_k)$ .

Therefore

 $\dim \operatorname{Ext}^1(T_k^* \oplus T_k, Z_n)$ 

 $= -(\dim V, \dim V) + \dim \operatorname{Hom}(T_k^* \oplus T_k, Z_n) + \dim \operatorname{Hom}(Z_n, T_k^* \oplus T_k)$ 

 $\leq -(\dim V, \dim V) + 2\dim \operatorname{Hom}(T_k^* \oplus T_k, T_k^* \oplus T_k) - 2$ 

$$= \operatorname{Ext}^{1}(T_{k}^{*} \oplus T_{k}, T_{k}^{*} \oplus T_{k}) - 2 = 0,$$

where we have used the formula in [8, Lem. 2.1]. The upper semicontinuity also shows  $\operatorname{Ext}^1(T/T_k, Z_n) = 0$ , hence  $Z_n \in \operatorname{add}(T/T_k)$ .

As the inequality above must be an equality, we get

$$\dim \operatorname{Hom}(T_k^* \oplus T_k, Z_n) = \dim \operatorname{Hom}(Z_n, T_k^* \oplus T_k)$$
$$= \dim \operatorname{Hom}(T_k^* \oplus T_k, T_k^* \oplus T_k) - 1.$$

Therefore

 $\dim \operatorname{Hom}(T_k^*, Z_n) + \dim \operatorname{Hom}(T_k, Z_n)$ 

 $= \dim \operatorname{Hom}(T_k^*, T_k^* \oplus T_k) + \dim \operatorname{Hom}(T_k, T_k^* \oplus T_k) - 1.$ 

Note that dim Hom $(T_k^*, Z_n) \leq \dim \operatorname{Hom}(T_k^*, T_k^* \oplus T_k)$  and dim Hom $(T_k, Z_n) \leq \dim \operatorname{Hom}(T_k, T_k^* \oplus T_k)$  by the semicontinuity. Hence the above implies that one of inequalities must be an equality. Suppose that the first one is an equality. Then we have

$$\dim \operatorname{Hom}(T_k^*, Z_n) = \dim \operatorname{Hom}(T_k^*, T_k^* \oplus T_k),$$
$$\dim \operatorname{Hom}(T_k, Z_n) = \dim \operatorname{Hom}(T_k, T_k^* \oplus T_k) - 1.$$

The same argument shows that  $\dim \operatorname{Hom}(Z_n, T_k^*) = \dim \operatorname{Hom}(T_k^* \oplus T_k, T_k^*)$  or  $\dim \operatorname{Hom}(Z_n, T_k) = \dim \operatorname{Hom}(T_k^* \oplus T_k, T_k)$ . The first equality is impossible, as  $0 = \dim \operatorname{Ext}^1(T_k^*, Z_n) \neq \dim \operatorname{Ext}^1(T_k^*, T_k \oplus T_k) = 1$  and the above dimension formula. Therefore we have

 $\dim \operatorname{Hom}(Z_n, T_k^*) = \dim \operatorname{Hom}(T_k^* \oplus T_k, T_k^*) - 1,$  $\dim \operatorname{Hom}(Z_n, T_k) = \dim \operatorname{Hom}(T_k^* \oplus T_k, T_k).$ 

We take  $\eta_n \in \text{Hom}(T_k^*, Z_n)$  converging to  $\text{id}_{T_k^*} \oplus 0$  in  $\text{Hom}(T_k^*, T_k^* \oplus T_k)$ . In particular,  $\eta_n$  is injective for sufficiently large n. We consider an exact sequence

 $0 \to \operatorname{Hom}(Z_n / \operatorname{Im} \eta_n, T_k) \to \operatorname{Hom}(Z_n, T_k) \to \operatorname{Hom}(\operatorname{Im} \eta_n, T_k).$ 

The next term  $\operatorname{Ext}^1(Z_n/\operatorname{Im} \eta_n, T_k)$  vanishes, as we have dim  $\operatorname{Ext}^1(Z_n/\operatorname{Im} \eta_n, T_k) \leq \dim \operatorname{Ext}^1(T_k, T_k) = 0$  by the upper semicontinuity. Therefore we have

 $\dim \operatorname{Hom}(Z_n / \operatorname{Im} \eta_n, T_k) = \dim \operatorname{Hom}(T_k, T_k).$ 

We take  $\zeta_n \in \text{Hom}(Z_n/\text{Im}\eta_n, T_k)$  converging to  $\text{id}_{T_k}$ . Then  $\zeta_n$  is an isomorphism for sufficiently large n. Composing the projection  $p: Z_n \to Z_n/\text{Im}\eta_n$  with  $\zeta_n$ , we have an exact sequence

$$0 \to T_k^* \xrightarrow{\eta_n} Z_n \xrightarrow{\zeta_n \circ p} T_k \to 0.$$

This shows that  $Z_n \cong T''$ .

When dim Hom $(T_k, Z_n)$  = dim Hom $(T_k, T_k^* \oplus T_k)$ , the same argument shows that  $Z_n \cong T'$ .

Now Conj. 4.2 implies that  $b_{T_k}^{up} b_{T_k}^{up}$  is a linear combination of  $b_{T'}^{up}$  and  $b_{T''}^{up}$ . A refinement of the argument hopefully gives the second formula in (2.3).

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