Stability of analytical and numerical solutions of nonlinear stochastic delay differential equations^{*}

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Abstract

This paper concerns the stability of analytical and numerical solutions of nonlinear stochastic delay differential equations (SDDEs). We derive sufficient conditions for the stability, contractivity and asymptotic contractivity in mean square of the solutions for nonlinear SDDEs. The results provide a unified theoretical treatment for SDDEs with constant delay and variable delay (including bounded and unbounded variable delays). Then the stability, contractivity and asymptotic contractivity in mean square are investigated for the backward Euler method. It is shown that the backward Euler method preserves the properties of the underlying SDDEs. The main results obtained in this work are different from those of Razumikhin-type theorems. Indeed, our results hold without the necessity of constructing of finding an appropriate Lyapunov functional.

AMS subject classification: 34K50, 65C20, 65C30.

Key Words: Nonlinear stochastic delay differential equation, stability in mean square, contractivity in mean square, asymptotic contractivity in mean square, backward Euler method

1 Introduction

Many physical, engineering and economic processes can be modeled by stochastic differential equations (SDEs). The rate of change of such a system depends only on its present state and some noisy input. However, in many practical situations the rate of change of the state depends not only on the present but also on the past states of the system. Stochastic functional differential equations (SFDEs) give a mathematical formulation for

^{*}This work was partially supported by NSF of China (No.11171352, 11271311) and State Key Laboratory of High Performance Complex Manufacturing.

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such system. For more details on SFDEs, we refer to [11, 12, 13] and the references therein.

SFDEs also can be regarded as a generalization of deterministic functional differential equations when stochastic effects are taken into account. For deterministic Volterra functional differential equations (VFDEs) in Banach spaces, Li [8] discussed the stability, contractivity and asymptotic stability of the solutions. In [8], the author introduced a so-called $\frac{1}{n}$ -perturbed problem and constructed an auxiliary function Q(t) for the corresponding study. The $\frac{1}{n}$ -perturbed problem can be used to deal with a wide variety of delay arguments and the auxiliary function Q(t) is the crux to establish the main results in [8]. The work [8] provides a unified framework for stability analysis of nonlinear stiff problems in ordinary differential equations, delay differential equations, integro-differential equations and VFDEs of other types. The theory in [8] was further extended to nonlinear Volterra neutral functional differential equations (VNFDEs) [17]. Moreover, in [18], it is proved that the implicit Euler method preserves the stability of VFDEs and VNFDEs.

It is natural to ask whether the solutions of SFDEs possess similar properties to those presented in [8] and which methods can reproduce the properties. Due to the unique features of stochastic calculus, the numerical analyses of SFDEs significantly differ from those developed for the numerical analyses of their deterministic counterparts. In the literature, much attention on numerical stability has been focused on a special class of SFDEs, namely, stochastic delay differential equations (SDDEs); see [1, 10, 16, 19, 20, 21]. The results mainly concern the mean-square stability, asymptotic stability and exponential mean-square stability for SDDEs with bounded lags. Very recently, Fan, Song and Liu [3] discussed the mean-square stability of semi-implicit Euler methods for linear stochastic pantograph equations. Far less is known for long-run behavior of nonlinear SDDEs with unbounded lags. Moreover, to our best knowledge, there is no work on the contractivity analysis of numerical methods for SDDEs. Our aims in this paper are to investigate the stability and contractivity of nonlinear SDDEs with bounded and unbounded lags and to study the numerical preservation of those the properties. The main results of this paper could be summarized as follows.

- (i) Sufficient conditions for the stability, contractivity and asymptotic contractivity in mean square of the solutions for nonlinear SDDEs are derived. The results provide a unified theoretical treatment for SDDEs with constant delay and variable delay (including bounded and unbounded variable delays). Applicability of the theory is illustrated by linear and nonlinear SDDEs with a wide variety of delay arguments such as constant delays, piecewise constant arguments, proportional delays and so on. The theorems established in this paper work for some SDDEs to which the existing theories cannot be applied. Our main results of analytic solutions can be regarded as a generalization of those in [8] restricted in finite-dimensional Hilbert spaces and finitely many delays to the stochastic version.
- (ii) It is proved that the backward Euler method preserves the stability, contractivity and asymptotic contractivity in mean square of the underlying systems. In particular, Theorem 4.2 and Theorem 4.4 show that the backward Euler method preserves the contractivity and asymptotic contractivity without any constraint on the numerical stepsize.

We point out that the main theorems in the present paper are different from the Razumikhin-type theorems established in [1, 11]. Our theorems can be directly applied to establish the stability without the necessity of constructing and finding an appropriate

Lyapunov functional, as required by the Razumikhin-type theorems. In this sense, our theorems are more convenient for stability analysis than the Razumikhin-type theorems.

The rest of the paper is organized as follows. In section 2, we introduce some notations and assumptions, which will be used throughout the rest of the paper. In section 3, some criteria for the stability, contractivity and asymptotic contractivity in mean square of solutions for nonlinear SDDEs are established. The main results obtained in this section are applied to SDDEs with bounded and unbounded lags, respectively. In section 4, sufficient conditions for the stability, contractivity and asymptotic contractivity in mean square for the backward Euler method are derived. Stability of analytical and numerical solutions of SDDEs with several delays is discussed in section 5.

2 Stochastic delay differential equations

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq a}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \geq a}$ satisfying the usual conditions (i.e., it is right continuous and \mathcal{F}_a contains all the \mathbb{P} -null sets). Let $w(t) = (w_1(t), ..., w_m(t))^T$ be an *m*-dimensional Wiener process defined on the probability space. Let $\langle \cdot \rangle$ be inner product in \mathbb{C}^d and $|\cdot|$ corresponding norm. In this paper, $|\cdot|$ also denotes the trace norm (F-norm) in $\mathbb{C}^{d \times m}$. Also, $C([t_1, t_2]; \mathbb{C}^d)$ is used to represent the family of continuous mappings ψ from $[t_1, t_2]$ to \mathbb{C}^d . Let p > 2 and denote by $L^p_{\mathcal{F}_t}([t_1, t_2]; \mathbb{C}^d)$ the family of \mathcal{F}_t -measurable $C([t_1, t_2]; \mathbb{C}^d)$ -valued random variables $\psi =$ $\{\psi(u): t_1 \leq u \leq t_2\}$ such that $\|\psi\|_{\mathbb{E}}^p = \sup_{t_1 \leq u \leq t_2} \mathbb{E}|\psi(u)|^p < \infty$. \mathbb{E} denotes mathematical expectation with respect to \mathbb{P} .

Consider the following initial value problems of SDDEs in the sense of Itô

$$\begin{cases} dx(t) = f(t, x(t), x(t - \tau(t)))dt + g(t, x(t), x(t - \tau(t)))dw(t), \ t \in [a, b], \\ (2.1a) \\ (2.1a)$$

$$x(t) = \xi(t), \quad t \in [a - \tau_0, a], \quad \xi \in L^p_{\mathcal{F}_a}([a - \tau_0, a]; \mathbb{C}^d),$$
(2.1b)

where a, b, τ_0 are constants with $-\infty < a < b < +\infty$ and $\tau_0 \ge 0, \tau(t) \ge 0, \inf_{a \le t \le b} (t - \tau(t)) \ge a - \tau_0, f : [a, b] \times \mathbb{C}^d \times \mathbb{C}^d \to \mathbb{C}^d, g : [a, b] \times \mathbb{C}^d \times \mathbb{C}^d \to \mathbb{C}^{d \times m}$ are given continuous mappings. We assume that the drift coefficient f and the diffusion coefficient g satisfy the following conditions.

For each R > 0 there exists a constant C_R , depending only on R, such that

$$|f(t, x_1, y) - f(t, x_2, y)| \le C_R |x_1 - x_2|, \quad |x_1| \lor |x_2| \lor |y| \le R,$$

$$\Re \langle x_1 - x_2, f(t, x_1, y) - f(t, x_2, y) \rangle \le \alpha(t) |x_1 - x_2|^2,$$
(2.3)

$$|f(t, x, y_1) - f(t, x, y_2)| \le \beta(t)|y_1 - y_2|,$$
(2.4)

$$|g(t, x_1, y_1) - g(t, x_2, y_2)| \le \gamma_1(t) |x_1 - x_2| + \gamma_2(t) |y_1 - y_2|,$$
(2.5)

for all $t \in [a, b], x, x_1, x_2, y, y_1, y_2 \in \mathbb{C}^d$, where $\Re a$ denotes the real part of the complex number a. Here $\alpha(t), \beta(t), \gamma_1(t)$ and $\gamma_2(t)$ are continuous real-valued functions. We introduce the following notations

$$\mu_1^{(0)} = \inf_{a \le t \le b} \tau(t) \ge 0, \qquad \mu_2^{(0)}(t_1, t_2) = \inf_{\substack{t_1 \le t \le t_2}} (t - \tau(t)) \ge a - \tau_0,$$
$$\forall t_1, t_2 : a \le t_1 \le t_2 \le b.$$

For convenience, we denote by $\mathcal{SD}(\alpha, \beta, \gamma_1, \gamma_2)$ the all problems (2.1) which satisfy the conditions (2.2)-(2.5). Such problems will be introduced in the next section (see Example 3.16 and Example 3.20).

In order to deal with a wide variety of delay arguments, we introduce the so-called $\frac{1}{n}$ -perturbed problem of (2.1), which was first introduced by Li [8] for VFDEs. We call the initial value problem

$$\begin{cases} dx(t) = f(t, x(t), x^{(n,t)}(t - \tau(t)))dt + g(t, x(t), x^{(n,t)}(t - \tau(t)))dw(t), t \in [a, b], (2.6a) \\ x(t) = \xi(t), \quad t \in [a - \tau_0, a], \ \xi \in L^p_{\mathcal{F}_a}([a - \tau_0, a]; \mathbb{C}^d), \end{cases}$$
(2.6b)

an $\frac{1}{n}$ -perturbed problem of the problem (2.1), where

$$x^{(n,t)}(t-\tau(t)) = \begin{cases} x(t-\tau(t)), & \tau(t) \ge \frac{1}{n}, \\ x(t-\frac{1}{n}), & \tau(t) < \frac{1}{n}. \end{cases}$$
(2.7)

Here the natural number $n > \frac{1}{\tau_0}$ can be arbitrarily given. Without lose of generality, we always assume $\tau_0 > 0$. In fact, in the case of $\tau_0 = 0$, we can replace τ_0 by some positive number $\tilde{\tau}_0$ and define $\xi(u) = \xi(a)$ for $u \in [a - \tilde{\tau}_0, a]$.

It is easy to verify that, if problem $(2.1) \in SD(\alpha, \beta, \gamma_1, \gamma_2)$, then its $\frac{1}{n}$ -perturbed problem $(2.6) \in SD(\alpha, \beta, \gamma_1, \gamma_2)$. It is clear that $\tilde{\tau}(t) = \max{\{\tau(t), \frac{1}{n}\}}$ is the time-lag argument of the $\frac{1}{n}$ -perturbed problem and

$$\tilde{\mu}_{1}^{(0)} = \inf_{a \le t \le b} \tilde{\tau}(t) \ge \frac{1}{n}.$$
(2.8)

3 Stability analysis of SDDEs

We discuss the following types of stability of SDDEs.

Definition 3.1 The solution of problem (2.1) is said to be stable in mean square if

$$\mathbb{E}|x(t) - y(t)|^2 \le C \sup_{a - \tau_0 \le \theta \le a} \mathbb{E}|\xi(\theta) - \eta(\theta)|^2,$$
(3.1)

where y(t) is the solution of the perturbed problem

$$\int dy(t) = f(t, y(t), y(t - \tau(t)))dt + g(t, y(t), y(t - \tau(t)))dw(t), \ t \in [a, b],$$
(3.2a)

$$y(t) = \eta(t), t \in [a - \tau_0, a], \quad \eta \in L^p_{\mathcal{F}_a}([a - \tau_0, a]; \mathbb{C}^d).$$
(3.2b)

Definition 3.2 The solution of problem (2.1) is said to be contractive in mean square if (3.1) with $C \leq 1$ holds.

Definition 3.3 The solution of problem (2.1) is said to be asymptotically contractive in mean square if

$$\lim_{t \to +\infty} \mathbb{E}|x(t) - y(t)|^2 = 0,$$

for which [a, b] is replaced by $[a, +\infty)$ in (2.1a) and (3.2a).

Remark 3.4 In the strict sense, (3.1) with $C \leq 1$ means generalized contraction. For brevity, we simply call the solution is contractive in mean square.

There exist well-known stability definitions in literatures which are closely related to those presented in this paper, but there are differences among them. The existing notions of stability include mean-square stability for SDEs, that is, $\lim_{t\to+\infty} \mathbb{E}|x(t)|^2 = 0$ (cf. [14]); exponential mean-square contraction of trajectories for SDEs with jumps (cf. [6]). The contractivity in mean square is weaker than that in [6]. The continuity of f and g implies that

$$|f(t,0,0)| \le C, \quad |g(t,0,0)| \le C, \quad t \in [a,b],$$
(3.3)

where C only depends on f, g and the interval [a, b]. We note that condition (2.5) implies that the diffusion coefficient g satisfies the local Lipschitz condition

$$|g(t, x_1, y_1) - g(t, x_2, y_2)| \le C_R(|x_1 - x_2| + |y_1 - y_2|),$$

$$t \in [a, b], x_1, x_2, y_1, y_2 \in \mathbb{C}^d, \ |x_1| \lor |x_2| \lor |y_1| \lor |y_2| \le R.$$
(3.4)

In fact, we can choose any C_R with $C_R \ge \max\{\max_{a \le t \le b} \gamma_1(t), \max_{a \le t \le b} \gamma_2(t)\}$. Using (2.3), (2.4) and (3.3), we have

$$\begin{aligned} \Re\langle x, f(t, x, y) \rangle &= \Re \Big\langle x - 0, f(t, x, y) - f(t, 0, y) + f(t, 0, y) - f(t, 0, 0) + f(t, 0, 0) \Big\rangle \\ &\leq \alpha(t) |x|^2 + \beta(t) |x| |y| + C |x| \leq C_1 \Big(1 + |x|^2 + |y|^2 \Big), \end{aligned}$$
(3.5)

where C_1 only depends on C, $\max_{a \le t \le b} \alpha(t)$ and $\max_{a \le t \le b} \beta(t)$. By (2.5), we have

$$|g(t,x,y)|^{2} \leq 2|g(t,x,y) - g(t,0,0)|^{2} + 2|g(t,0,0)|^{2} \leq C_{1}\left(1 + |x|^{2} + |y|^{2}\right), \quad (3.6)$$

where C_1 only depends on C, $\max_{a \le t \le b} \gamma_1^2(t)$ and $\max_{a \le t \le b} \gamma_2^2(t)$.

3.1 Finite interval

In order to prove the main theorems in this section, we prepare the following lemmas.

Lemma 3.5 Assume that problem $(2.1) \in SD(\alpha, \beta, \gamma_1, \gamma_2)$. Then for each $p \ge 2$ there is $\overline{C} = \overline{C}(p, a, b, \alpha, \beta, \gamma_1, \gamma_2)$ such that

$$\mathbb{E}(\sup_{a \le t \le b} |x(t)|^p) \le \bar{C}(1 + \mathbb{E}(\sup_{a - \tau_0 \le t \le a} |\xi(t)|^p)) = A.$$
(3.7)

Proof. For every integer $k \ge 1$, define the stopping time

$$\rho_k = \inf\{t \in [a,b] : \sup_{a-\tau_0 \le \theta \le t} |x(\theta)| \ge k\},\tag{3.8}$$

where we use the convention $\rho_k = b$ if the set is empty in the right-hand side. Clearly, $\rho_k \uparrow b$ almost surely as $k \to +\infty$. Let $x^k(t) = x(t \land \rho_k)$. Using the Itô formula, we have for $a \leq t \leq b$

$$\begin{split} & \left(1+|x^{k}(t)|^{2}\right)^{\frac{p}{2}} = \left(1+|\xi(a)|^{2}\right)^{\frac{p}{2}} \\ & +p\int_{a}^{t}\left(1+|x^{k}(s)|^{2}\right)^{\frac{p-2}{2}} \Re\langle x^{k}(s), f(s, x^{k}(s), x^{k}(s-\tau(s)))\rangle I_{[[a,\rho_{k}]]}(s)ds \\ & +\frac{p}{2}\int_{a}^{t}\left(1+|x^{k}(s)|^{2}\right)^{\frac{p-2}{2}}|g(s, x^{k}(s), x^{k}(s-\tau(s)))|^{2}I_{[[a,\rho_{k}]]}(s)ds \\ & +\frac{p(p-2)}{2}\int_{a}^{t}\left(1+|x^{k}(s)|^{2}\right)^{\frac{p-4}{2}}|(x^{k}(s))^{T}g(s, x^{k}(s), x^{k}(s-\tau(s)))|^{2}I_{[[a,\rho_{k}]]}(s)ds \\ & +p\int_{a}^{t}\left(1+|x^{k}(s)|^{2}\right)^{\frac{p-2}{2}}\Re\langle x^{k}(s), g(s, x^{k}(s), x^{k}(s-\tau(s)))\rangle I_{[[a,\rho_{k}]]}(s)dw(s), \end{split}$$

by (3.5) and (3.6), and hence

$$\begin{split} & \left(1+|x^{k}(t)|^{2}\right)^{\frac{p}{2}} \\ \leq & 2^{\frac{p-2}{2}} \Big(1+|\xi(a)|^{p}\Big)+2pC_{1} \int_{a}^{t} \Big(1+|x^{k}(s)|^{2}\Big)^{\frac{p-2}{2}} \Big(1+\sup_{s-\tau(s)\leq u\leq s}|x^{k}(u)|^{2}\Big) ds \\ & +pC_{1} \int_{a}^{t} \Big(1+|x^{k}(s)|^{2}\Big)^{\frac{p-2}{2}} \Big(1+\sup_{s-\tau(s)\leq u\leq s}|x^{k}(u)|^{2}\Big) ds \\ & +p(p-2)C_{1} \int_{a}^{t} \Big(1+|x^{k}(s)|^{2}\Big)^{\frac{p-2}{2}} \Re\langle x^{k}(s),g(s,x^{k}(s),x^{k}(s-\tau(s)))\rangle I_{[[a,\rho_{k}]]}(s) dw(s) \\ & \leq & 2^{\frac{p-2}{2}} \Big(1+|\xi(a)|^{p}\Big) + p(p+1)C_{1} \int_{a}^{t} \Big(1+\sup_{s-\tau(s)\leq u\leq s}|x^{k}(u)|^{2}\Big)^{\frac{p}{2}} ds \\ & +p \int_{a}^{t} \Big(1+|x^{k}(s)|^{2}\Big)^{\frac{p-2}{2}} \Re\langle x^{k}(s),g(s,x^{k}(s),x^{k}(s-\tau(s)))\rangle I_{[[a,\rho_{k}]]}(s) dw(s), \end{split}$$

which yields

$$\mathbb{E} \sup_{a \le s \le t} (1 + |x^{k}(s)|^{2})^{\frac{p}{2}}$$

$$\le 2^{\frac{p-2}{2}} \Big(1 + \mathbb{E} \sup_{a-\tau_{0} \le s \le a} |\xi(s)|^{p} \Big) + C_{2} \mathbb{E} \int_{a}^{t} \Big(1 + \sup_{a-\tau_{0} \le u \le s} |x^{k}(u)|^{2} \Big)^{\frac{p}{2}} ds$$

$$+ p \mathbb{E} \Big(\sup_{a \le s \le t} \int_{a}^{s} (1 + |x^{k}(u)|^{2})^{\frac{p-2}{2}} \Re \langle x^{k}(u), g(u, x^{k}(u), x^{k}(u - \tau(u))) \rangle I_{[[a, \rho_{k}]]}(u) dw(u) \Big).$$

Applying the Burkholder-Davis-Gundy inequality to the third term on the right-hand side of the above inequality, we obtain the bound

$$p\mathbb{E}\Big(\sup_{a\leq s\leq t}\int_{a}^{s}(1+|x^{k}(u)|^{2})^{\frac{p-2}{2}}\Re\langle x^{k}(u),g(u,x^{k}(u),x^{k}(u-\tau(u)))\rangle I_{[[a,\rho_{k}]]}(u)dw(u)\Big)$$

$$\leq C_{p}\mathbb{E}\Big(\int_{a}^{t}(1+|x^{k}(u)|^{2})^{p-2}|x^{k}(u)|^{2}|g(u,x^{k}(u),x^{k}(u-\tau(u)))|^{2}du\Big)^{\frac{1}{2}}$$

$$\leq C_{p}\mathbb{E}\Big(\sup_{a\leq u\leq t}(1+|x^{k}(u)|^{2})^{\frac{p}{2}}\int_{a}^{t}(1+|x^{k}(u)|^{2})^{\frac{p-4}{2}}|x^{k}(u)|^{2}|g(u,x^{k}(u),x^{k}(u-\tau(u)))|^{2}du\Big)^{\frac{1}{2}}$$

$$\leq \frac{1}{2}\mathbb{E}\sup_{a\leq u\leq t}(1+|x^{k}(u)|^{2})^{\frac{p}{2}}+\frac{C_{p}^{2}}{2}2C_{1}\mathbb{E}\int_{a}^{t}\left(1+\sup_{a-\tau_{0}\leq s\leq u}|x^{k}(s)|^{2}\right)^{\frac{p}{2}}du.$$

Consequently,

$$\mathbb{E} \sup_{a \le s \le t} (1 + |x^k(s)|^2)^{\frac{p}{2}} \le 2^{\frac{p}{2}} (1 + \mathbb{E} \sup_{a - \tau_0 \le s \le a} |\xi(s)|^p) + C_3 \mathbb{E} \int_a^t \left(1 + \sup_{a - \tau_0 \le s \le u} |x^k(s)|^2 \right)^{\frac{p}{2}} du$$

= $2^{\frac{p}{2}} (1 + \mathbb{E} \sup_{a - \tau_0 \le s \le a} |\xi(s)|^p) + C_3 \mathbb{E} \int_a^t \sup_{a - \tau_0 \le s \le u} \left(1 + |x^k(s)|^2 \right)^{\frac{p}{2}} du.$

Further, we notice that

$$\mathbb{E} \sup_{a-\tau_0 \le s \le t} (1+|x^k(s)|^2)^{\frac{p}{2}} \le \mathbb{E} \sup_{a-\tau_0 \le s \le a} (1+|\xi(s)|^2)^{\frac{p}{2}} + \mathbb{E} \sup_{a \le s \le t} (1+|x^k(s)|^2)^{\frac{p}{2}}
\le 2^{\frac{p-2}{2}} (1+\mathbb{E} \sup_{a-\tau_0 \le s \le a} |\xi(s)|^p) + \mathbb{E} \sup_{a \le s \le t} (1+|x^k(s)|^2)^{\frac{p}{2}}.$$

Therefore,

$$\mathbb{E}\sup_{a-\tau_0 \le s \le t} (1+|x^k(s)|^2)^{\frac{p}{2}} \le \frac{3}{2} 2^{\frac{p}{2}} (1+\mathbb{E}\sup_{a-\tau_0 \le s \le a} |\xi(s)|^p) + C_3 \int_a^t \mathbb{E}\sup_{a-\tau_0 \le s \le u} \left(1+|x^k(s)|^2\right)^{\frac{p}{2}} du.$$

Now the Gronwall's inequality yields that

$$\mathbb{E}(\sup_{a-\tau_0 \le s \le t} |x^k(s)|^p) \le \mathbb{E}(\sup_{a-\tau_0 \le s \le t} (1+|x^k(s)|^2)^{\frac{p}{2}}) \le \frac{3}{2} 2^{\frac{p}{2}} (1+\mathbb{E}\sup_{a-\tau_0 \le s \le a} |\xi(s)|^p) e^{C_3(t-a)}.$$

Letting $k \to +\infty$ and applying the Fatou's lemma, we obtain the desired result.

Lemma 3.6 Assume that problem $(2.1) \in SD(\alpha, \beta, \gamma_1, \gamma_2)$. Then there exists a unique solution x(t) to equation (2.1).

Proof. Using (2.2), (2.4), (3.4), (3.5) and (3.6), we are able to complete the proof along the lines of the ones for Theorem 5.2.7 and Theorem 2.3.5 in [11].

Lemma 3.7 Assume that problem $(2.1) \in SD(\alpha, \beta, \gamma_1, \gamma_2)$. Then we have

$$\lim_{v \to 0} \sup_{|t-s|=v} \mathbb{E} |x(t) - x(s)|^2 = 0, \quad a \le s, t \le b.$$
(3.9)

Proof. The proof is mainly based on the techniques employed in the proof of Theorem 2.2 in [5]. Integrating (2.1) gives for $a \le s < t \le b$

$$x(t) - x(s) = \int_{s}^{t} f(u, x(u), x(u - \tau(u))) du + \int_{s}^{t} g(u, x(u), x(u - \tau(u))) dw(u).$$
(3.10)

Let e(s,t) = x(t) - x(s),

$$\rho_R = \inf\{t \in [a, b] : \sup_{a - \tau_0 \le \theta \le t} |x(\theta)| \ge R\},\$$

where $\rho_R = b$ if the set is empty in the right-hand side. Using the Young inequality: for $r^{-1} + q^{-1} = 1$

$$ab \leq \frac{\delta}{r}a^r + \frac{1}{q\delta^{q/r}}b^q, \ \forall a, b, \delta > 0$$

and letting $r = \frac{p}{2}, q = \frac{p}{p-2}$, we thus have for any $\delta > 0$

$$\mathbb{E}(|e(s,t)|^{2}) = \mathbb{E}(|e(s,t)|^{2}I_{\{\rho_{R} \ge b\}}) + \mathbb{E}(|e(s,t)|^{2}I_{\{\rho_{R} < b\}})$$

$$\leq \mathbb{E}(|e(s,t)|^{2}I_{\{\rho_{R} \ge b\}}) + \frac{2\delta}{p}\mathbb{E}(|e(s,t)|^{p}) + \frac{1-\frac{2}{p}}{\delta^{2/(p-2)}}\mathbb{P}(\rho_{R} < b),$$
(3.11)

where p > 2. It follows from (3.7) that

$$\mathbb{P}(\rho_R < b) = \mathbb{E}\left(I_{\{\rho_R < b\}} \frac{|x(\rho_R)|^p}{R^p}\right) \le \frac{1}{R^p} \mathbb{E}(\sup_{a \le t \le b} |x(t)|^p) \le \frac{A}{R^p},\tag{3.12}$$

$$\mathbb{E}(|e(s,t)|^p) \le 2^{p-1} \mathbb{E}(\sup_{a \le s \le b} |x(s)|^p + \sup_{a \le t \le b} |x(t)|^p)) \le 2^p A.$$
(3.13)

We then have

$$\mathbb{E}(|e(s,t)|^2) \le \mathbb{E}(|e(s,t)|^2 I_{\{\rho_R \ge b\}}) + \frac{2^{p+1}\delta A}{p} + \frac{(p-2)A}{p\delta^{2/(p-2)}R^p}.$$
(3.14)

Further, using the Hölder's inequality and the Itô isometry, we obtain

$$\begin{split} & \mathbb{E}(|e(s,t)|^{2}I_{\{\rho_{R}\geq b\}}) \\ &= \mathbb{E}\Big(\Big|\int_{s}^{t}f(u,x(u),x(u-\tau(u)))du + \int_{s}^{t}g(u,x(u),x(u-\tau(u)))dw(u)\Big|^{2}I_{\{\rho_{R}\geq b\}}\Big) \\ &\leq 2\mathbb{E}\Big(\Big(\big|\int_{s}^{t}f(u,x(u),x(u-\tau(u)))du\big|^{2} + \big|\int_{s}^{t}g(u,x(u),x(u-\tau(u)))dw(u)\big|^{2}\Big)I_{\{\rho_{R}\geq b\}}\Big) \\ &\leq 2(t-s)\mathbb{E}\Big(I_{\{\rho_{R}\geq b\}}\int_{s}^{t}|f(u,x(u),x(u-\tau(u)))|^{2}du\Big) + 2\mathbb{E}\int_{s}^{t}|g(u,x(u),x(u-\tau(u)))|^{2}du \\ &\leq 2(t-s)\mathbb{E}\Big(I_{\{\rho_{R}\geq b\}}\int_{s}^{t}|f(u,x(u),x(u-\tau(u))) - f(u,0,0) + f(u,0,0)|^{2}du\Big) \\ &+ 2\mathbb{E}\int_{s}^{t}|g(u,x(u),x(u-\tau(u)))|^{2}du. \end{split}$$

By (2.2), (2.4), (3.3), (3.6) and Lemma 3.5, we have

$$\mathbb{E}(|e(s,t)|^{2}I_{\{\rho_{R}\geq b\}})$$

$$\leq 8C_{R}^{2}(t-s)\int_{s}^{t} \left(\mathbb{E}|x(u)|^{2} + \mathbb{E}|x(u-\tau(u))|^{2}\right)du + 4(t-s)\mathbb{E}\int_{s}^{t} |f(u,0,0)|^{2}du$$

$$+ 2C_{1}\int_{s}^{t} \left(1 + \mathbb{E}|x(u)|^{2} + \mathbb{E}|x(u-\tau(u))|^{2}\right)du \leq C_{4}(t-s),$$

where C_4 is independent of s and t. A combination of this expression and (3.14) leads to

$$\mathbb{E}(|e(s,t)|^2) \le C_4(t-s) + \frac{2^{p+1}\delta A}{p} + \frac{(p-2)A}{p\delta^{2/(p-2)}R^p}.$$
(3.15)

Therefore, for any given $\epsilon > 0$, we can choose δ and R such that

$$\frac{2^{p+1}\delta A}{p} < \frac{1}{3}\epsilon, \quad \frac{(p-2)A}{p\delta^{2/(p-2)}R^p} < \frac{1}{3}\epsilon, \tag{3.16}$$

and then choose t-s sufficiently small such that $C_4(t-s) < \frac{1}{3}\epsilon$. Hence, we have

$$\lim_{v \to 0} \sup_{|t-s|=v} \mathbb{E}(|x(t) - x(s)|^2) = \lim_{v \to 0} \sup_{|t-s|=v} \mathbb{E}(|e(s,t)|^2) = 0.$$

The proof is complete.

Lemma 3.8 Assume that problem $(2.1) \in SD(\alpha, \beta, \gamma_1, \gamma_2)$. Then for any $t_1, t_2 : a \le t_1 \le t_2 \le b$,

$$\mathbb{E}|x(t_2) - y(t_2)|^2 \le e^{\int_{t_1}^{t_2} \sigma(t)dt} \mathbb{E}|x(t_1) - y(t_1)|^2 + \int_{t_1}^{t_2} \rho(s) e^{\int_s^{t_2} \sigma(u)du} ds \sup_{\mu_2^{(0)}(t_1, t_2) \le \theta \le t_2 - \mu_1^{(0)}} \mathbb{E}|x(\theta) - y(\theta)|^2,$$
(3.17)

where y(t) is the solution of the perturbed problem (3.2), and $\sigma(t), \varrho(t)$ are defined by

$$\sigma(t) = 2\alpha(t) + \beta(t) + \gamma_1(t)\gamma_2(t) + \gamma_1^2(t),
\varrho(t) = \beta(t) + \gamma_1(t)\gamma_2(t) + \gamma_2^2(t).$$
(3.18)

 $\mathbf{Proof.}\ \mathrm{Let}$

$$V(t, x(t)) = p(t)(|x(t)|^2 + \delta q(t)), \quad t_1 \le t \le t_2,$$
(3.19)

where

$$p(t) = e^{-\int_a^t \sigma(u)du}, \quad q(t) = -(p(t))^{-1} \int_a^t \varrho(u)p(u)du, \quad (3.20)$$

 δ is a constant to be determined. Then we have $p'(t) = -\sigma(t)p(t), (p(t)q(t))' = -p(t)\varrho(t)$. By (3.19),(3.20) and the Itô formula, one can derive that, for $a \le t_1 \le t_2 \le b$,

$$\begin{split} & \mathbb{E}V(t_{2}, x(t_{2}) - y(t_{2})) = \mathbb{E}V(t_{1}, x(t_{1}) - y(t_{1})) \\ & + \int_{t_{1}}^{t_{2}} \bigg\{ -\sigma(t)p(t)\mathbb{E}|x(t) - y(t)|^{2} - \delta p(t)\varrho(t) \\ & + 2p(t)\mathbb{E}\Re\langle x(t) - y(t), f(t, x(t), x(t - \tau(t))) - f(t, y(t), y(t - \tau(t)))\rangle \\ & + p(t)\mathbb{E}|g(t, x(t), x(t - \tau(t))) - g(t, y(t), y(t - \tau(t)))|^{2} \bigg\} dt \\ & \leq \mathbb{E}V(t_{1}, x(t_{1}) - y(t_{1})) + \int_{t_{1}}^{t_{2}} \bigg\{ -\sigma(t)p(t)\mathbb{E}|x(t) - y(t)|^{2} - \delta p(t)\varrho(t) \\ & + 2p(t)\mathbb{E}\Re\langle x(t) - y(t), f(t, x(t), x(t - \tau(t))) - f(t, y(t), x(t - \tau(t)))\rangle \\ & + 2p(t)\mathbb{E}\Re\langle x(t) - y(t), f(t, y(t), x(t - \tau(t))) - f(t, y(t), y(t - \tau(t)))\rangle \\ & + p(t)\mathbb{E}|g(t, x(t), x(t - \tau(t))) - g(t, y(t), y(t - \tau(t)))|^{2} \bigg\} dt. \end{split}$$

Using the conditions (2.3)-(2.5), we obtain

$$\begin{split} \mathbb{E}V(t_{2},x(t_{2})-y(t_{2})) &\leq \mathbb{E}V(t_{1},x(t_{1})-y(t_{1})) \\ + \int_{t_{1}}^{t_{2}} \left\{ -\sigma(t)p(t)\mathbb{E}|x(t)-y(t)|^{2} - \delta p(t)\varrho(t) \\ + 2p(t)\alpha(t)\mathbb{E}|x(t)-y(t)|^{2} + 2p(t)\beta(t)\mathbb{E}(|x(t)-y(t)||x(t-\tau(t))-y(t-\tau(t))|) \\ + p(t)\mathbb{E}\left(\gamma_{1}(t)|x(t)-y(t)| + \gamma_{2}(t)|x(t-\tau(t))-y(t-\tau(t))|\right)^{2} \right\} dt \\ &\leq \mathbb{E}V(t_{1},x(t_{1})-y(t_{1})) + \int_{t_{1}}^{t_{2}} \left\{ -\sigma(t)p(t)\mathbb{E}|x(t)-y(t)|^{2} - \delta p(t)\varrho(t) \\ + 2p(t)\alpha(t)\mathbb{E}|x(t)-y(t)|^{2} + p(t)\beta(t)\left(\mathbb{E}|x(t)-y(t)|^{2} + \mathbb{E}|x(t-\tau(t))-y(t-\tau(t))|^{2} \right) \\ + p(t)\gamma_{1}^{2}(t)\mathbb{E}|x(t)-y(t)|^{2} + p(t)\gamma_{2}^{2}(t)\mathbb{E}|x(t-\tau(t))-y(t-\tau(t))|^{2} \\ + p(t)\gamma_{1}(t)\gamma_{2}(t)\left(\mathbb{E}|x(t)-y(t)|^{2} + \mathbb{E}|x(t-\tau(t))-y(t-\tau(t))|^{2} \right) \right\} dt \\ &\leq \mathbb{E}V(t_{1},x(t_{1})-y(t_{1})) - \delta \int_{t_{1}}^{t_{2}} p(t)\varrho(t) dt \\ + \int_{t_{1}}^{t_{2}} p(t)\left(-\sigma(t)+2\alpha(t)+\beta(t)+\gamma_{1}^{2}(t)+\gamma_{1}(t)\gamma_{2}(t)\right)\mathbb{E}|x(t)-y(t)|^{2} dt \\ + \int_{t_{1}}^{t_{2}} p(t)\left(\beta(t)+\gamma_{2}^{2}(t)+\gamma_{1}(t)\gamma_{2}(t)\right)\mathbb{E}|x(t-\tau(t))-y(t-\tau(t))|^{2} dt. \end{split}$$

The substitution of (3.18) into this gives

$$\mathbb{E}V(t_{2}, x(t_{2}) - y(t_{2})) \leq \mathbb{E}V(t_{1}, x(t_{1}) - y(t_{1})) + \int_{t_{1}}^{t_{2}} p(t)\beta(t) \Big(\mathbb{E}|x(t - \tau(t)) - y(t - \tau(t))|^{2} - \delta\Big) dt + \int_{t_{1}}^{t_{2}} p(t)\gamma_{2}^{2}(t) \Big(\mathbb{E}|x(t - \tau(t)) - y(t - \tau(t))|^{2} - \delta\Big) dt + \int_{t_{1}}^{t_{2}} p(t)\gamma_{1}(t)\gamma_{2}(t) \Big(\mathbb{E}|x(t - \tau(t)) - y(t - \tau(t))|^{2} - \delta\Big) dt.$$

$$(3.21)$$

Lemma 3.5 implies that $\mathbb{E} \sup_{\substack{a-\tau_0 \leq t \leq b \\ a-\tau_0 \leq t \leq b}} |x(t)|^2 < +\infty$, $\mathbb{E} \sup_{\substack{a-\tau_0 \leq t \leq b \\ \mu_2^{(0)}(t_1,t_2) \leq t \leq t_2 - \mu_1^{(0)}}} |y(t)|^2 < +\infty$. Consequently, sup $\mathbb{E}|x(t) - y(t)|^2$. It follows from $\mu_2^{(0)}(t_1,t_2) \leq t \leq t_2 - \mu_1^{(0)}$ (3.21) that

$$\mathbb{E}V(t_2, x(t_2) - y(t_2)) \le \mathbb{E}V(t_1, x(t_1) - y(t_1)).$$
(3.22)

The required estimate (3.17) now follows from (3.22).

Lemma 3.9 Assume that problem $(2.1) \in SD(\alpha, \beta, \gamma_1, \gamma_2)$. Then

$$\lim_{n \to \infty} \sup_{a - \tau_0 \le t \le b} \mathbb{E} |x(t) - x_n(t)|^2 = 0,$$
(3.23)

where $x_n(t)$ is the solution of the $\frac{1}{n}$ -perturbed problem (2.6).

Proof. For any given natural number $n > \frac{1}{\tau_0}$, we can choose a natural number q sufficiently large such that $\mu = (b-a)/q < \frac{1}{n}$. Let

$$\begin{split} t_1 &= a + (i-1)\mu, \quad t_2 = a + i\mu, \quad i = 1, 2, \cdots, q, \\ \alpha_0 &= \max \Big\{ \max_{a \le t \le b} \sigma(t), 0 \Big\}, \ \beta_0 = \max_{a \le t \le b} \varrho(t), \ \gamma_0 &= \max \Big\{ \max_{a \le t \le b} (\sigma(t) + \varrho(t)), 1 \Big\}, \\ V(t, x(t)) &= p(t)(|x(t)|^2 + \delta q(t)), \quad t_1 \le t \le t_2, \end{split}$$

where

$$\delta = 2(\varepsilon_n + \sup_{a-\tau_0 \le \theta \le t_1} \mathbb{E}|x(\theta) - x_n(\theta)|^2), \quad \varepsilon_n = \sup_{a \le t \le b} (\sup_{t-\frac{1}{n} \le \theta \le t} \mathbb{E}|x(\theta) - x(t-\frac{1}{n})|^2), \quad (3.24)$$

p(t) and q(t) are defined by (3.20), $\sigma(t)$ and $\varrho(t)$ are defined by (3.18). For $t_1 \le t \le t_2$, we can obtain the following estimate in the same way as (3.21)

$$\begin{split} & \mathbb{E}V(t, x(t) - x_n(t)) \leq \mathbb{E}V(t_1, x(t_1) - x_n(t_1)) \\ & + \int_{t_1}^t p(s)\beta(s) \Big(\mathbb{E}|x(s - \tau(s)) - x_n^{(n,s)}(s - \tau(s))|^2 - \delta \Big) ds \\ & + \int_{t_1}^t p(s)\gamma_2^2(s) \Big(\mathbb{E}|x(s - \tau(s)) - x_n^{(n,s)}(s - \tau(s))|^2 - \delta \Big) ds \\ & + \int_{t_1}^t p(s)\gamma_1(s)\gamma_2(s) \Big(\mathbb{E}|x(s - \tau(s)) - x_n^{(n,s)}(s - \tau(s))|^2 - \delta \Big) ds \\ & \leq \mathbb{E}V(t_1, x(t_1) - x_n(t_1)) \\ & + \Big(\sup_{a \leq u \leq t} \mathbb{E}|x(u - \tau(u)) - x_n^{(n,u)}(u - \tau(u))|^2 - \delta \Big) \int_{t_1}^t p(s)\varrho(s) ds. \end{split}$$
(3.25)

Moreover, by (2.7) we find that

$$\sup_{a \le u \le t} \mathbb{E} |x(u - \tau(u)) - x_n^{(n,u)}(u - \tau(u))|^2$$

$$= \max \left\{ \sup_{\substack{a \le u \le t \\ \tau(u) \ge \frac{1}{n}}} \mathbb{E} |x(u - \tau(u)) - x_n^{(n,u)}(u - \tau(u))|^2, \\ \sup_{\substack{a \le u \le t \\ 0 \le \tau(u) < \frac{1}{n}}} \mathbb{E} |x(u - \tau(u)) - x_n^{(n,u)}(u - \tau(u))|^2 \right\}$$

$$\leq \max \left\{ \sup_{a - \tau_0 \le u \le t - \frac{1}{n}} \mathbb{E} |x(u) - x_n(u)|^2, \sup_{a \le u \le t} \left(\sup_{u - \frac{1}{n} \le s \le u} \mathbb{E} |x(s) - x_n(u - \frac{1}{n})|^2 \right) \right\}$$

$$\leq \max \left\{ \sup_{a - \tau_0 \le u \le t - \frac{1}{n}} \mathbb{E} |x(u) - x_n(u)|^2, \sup_{a \le u \le t} \left((\sup_{u - \frac{1}{n} \le s \le u} \mathbb{E} |x(s) - x_n(u - \frac{1}{n})|^2 \right) \right\}$$

$$\leq 2\max \left\{ \sup_{a - \tau_0 \le u \le t - \frac{1}{n}} \mathbb{E} |x(u) - x_n(u)|^2, \sup_{a \le u \le t} \left((\sup_{u - \frac{1}{n} \le s \le u} \mathbb{E} |x(s) - x(u - \frac{1}{n})|^2 \right) \right\}$$

$$\leq 2\max \left\{ \sup_{a - \tau_0 \le u \le t - \frac{1}{n}} \mathbb{E} |x(u) - x_n(u)|^2, \sup_{a \le u \le t} \left((\sup_{u - \frac{1}{n} \le s \le u} \mathbb{E} |x(s) - x(u - \frac{1}{n})|^2 \right) \right\}$$

$$\leq 2\left\{ \sup_{a - \tau_0 \le u \le t_1} \mathbb{E} |x(u) - x_n(u)|^2 + \sup_{a \le u \le t} \left((\sup_{u - \frac{1}{n} \le s \le u} \mathbb{E} |x(s) - x(u - \frac{1}{n})|^2 \right) \right\} = \delta,$$

where δ is defined by (3.24). Thus, (3.25) shows that

$$\mathbb{E}V(t, x(t) - x_n(t)) \le \mathbb{E}V(t_1, x(t_1) - x_n(t_1)),$$

that is,

$$\begin{split} \mathbb{E}|x(t) - x_{n}(t)|^{2} &\leq e^{\int_{t_{1}}^{t} \sigma(u)du} \mathbb{E}|x(t_{1}) - x_{n}(t_{1})|^{2} \\ &+ \int_{t_{1}}^{t} \varrho(u)e^{\int_{u}^{t} \sigma(s)ds} du \Big(\varepsilon_{n} + \sup_{a-\tau_{0} \leq \theta \leq t_{1}} \mathbb{E}|x(\theta) - x_{n}(\theta)|^{2}\Big) \\ &\leq \left(e^{\int_{t_{1}}^{t} \sigma(u)du} + \int_{t_{1}}^{t} \varrho(u)e^{\int_{u}^{t} \sigma(s)ds} du\right) \sup_{a-\tau_{0} \leq \theta \leq t_{1}} \mathbb{E}|x(\theta) - x_{n}(\theta)|^{2} + \left(\int_{t_{1}}^{t} \varrho(u)e^{\int_{u}^{t} \sigma(s)ds} du\right)\varepsilon_{n} \\ &= \left(1 + \int_{t_{1}}^{t} (\sigma(u) + \varrho(u))e^{\int_{u}^{t} \sigma(s)ds} du\right) \sup_{a-\tau_{0} \leq \theta \leq t_{1}} \mathbb{E}|x(\theta) - x_{n}(\theta)|^{2} + \left(\int_{t_{1}}^{t} \varrho(u)e^{\int_{u}^{t} \sigma(s)ds} du\right)\varepsilon_{n} \\ &\leq (1 + \gamma_{0}\mu e^{\alpha_{0}(b-a)}) \sup_{a-\tau_{0} \leq \theta \leq a+(i-1)\mu} \mathbb{E}|x(\theta) - x_{n}(\theta)|^{2} + \beta_{0}\mu e^{\alpha_{0}(b-a)}\varepsilon_{n} \end{split}$$

for all $t \in [a + (i - 1)\mu, a + i\mu]$, $i = 1, 2, \cdots, q$. Consequently,

$$\sup_{a-\tau_0 \le \theta \le a+i\mu} \mathbb{E}|x(\theta) - x_n(\theta)|^2$$

$$= \max\left\{\sup_{a-\tau_0 \le \theta \le a+(i-1)\mu} \mathbb{E}|x(\theta) - x_n(\theta)|^2, \sup_{a+(i-1)\mu \le \theta \le a+i\mu} \mathbb{E}|x(\theta) - x_n(\theta)|^2\right\}$$

$$\le (1+\gamma_0\mu e^{\alpha_0(b-a)}) \sup_{a-\tau_0 \le \theta \le a+(i-1)\mu} \mathbb{E}|x(\theta) - x_n(\theta)|^2 + \beta_0\mu e^{\alpha_0(b-a)}\varepsilon_n$$

for $i = 1, 2, \dots, q$. Therefore,

$$\sup_{a-\tau_0 \le \theta \le b} \mathbb{E} |x(\theta) - x_n(\theta)|^2 = \sup_{a-\tau_0 \le \theta \le a+q\mu} \mathbb{E} |x(\theta) - x_n(\theta)|^2$$
$$\le C_{\mu}^q \sup_{a-\tau_0 \le \theta \le a} \mathbb{E} |x(\theta) - x_n(\theta)|^2 + \frac{C_{\mu}^q - 1}{\gamma_0 \mu e^{\alpha_0(b-a)}} \beta_0 \mu e^{\alpha_0(b-a)} \varepsilon_n \qquad (3.26)$$
$$= \frac{\beta_0}{\gamma_0} (C_{\mu}^q - 1) \varepsilon_n,$$

where $C_{\mu} = 1 + \gamma_0 \mu e^{\alpha_0(b-a)}$. By Lemma 3.7, we have $\varepsilon_n = \sup_{a \le t \le b} \sup_{t-\frac{1}{n} \le \theta \le t} \mathbb{E}|x(\theta) - x(t - \frac{1}{n})|^2 \to 0$, as $n \to \infty$. Let $n \to \infty$ and take into account that

$$\lim_{q \to \infty} C^q_{\mu} = \lim_{q \to \infty} \left(1 + \frac{\gamma_0(b-a)e^{\alpha_0(b-a)}}{q} \right)^q = e^{\gamma_0(b-a)e^{\alpha_0(b-a)}}.$$

Then (3.26) leads to the relation (3.23). The proof is complete.

Theorem 3.10 Assume that problem $(2.1) \in SD(\alpha, \beta, \gamma_1, \gamma_2)$. Let $c = \max_{a \le t \le b} (2\alpha(t) + 2\beta(t) + \gamma_1^2(t) + 2\gamma_1(t)\gamma_2(t) + \gamma_2^2(t))$. Then $\forall t \in [a, b]$,

$$\mathbb{E}|x(t) - y(t)|^2 \le e^{c(t-a)} \sup_{a-\tau_0 \le \theta \le a} \mathbb{E}|\xi(\theta) - \eta(\theta)|^2, \quad \text{if } c \ge 0,$$
(3.27)

$$\mathbb{E}|x(t) - y(t)|^2 \le \sup_{a - \tau_0 \le \theta \le a} \mathbb{E}|\xi(\theta) - \eta(\theta)|^2, \qquad \text{if } c \le 0,$$
(3.28)

where y(t) is the solution of the perturbed problem (3.2).

The inequalities (3.27) and (3.28) mean that problem (2.1) is stable in mean square and contractive in mean square, respectively.

Proof. We divide the proof into two cases: $\mu_1^{(0)} > 0$ and $\mu_1^{(0)} = 0$.

Case A: $\mu_1^{(0)} > 0$. In this case, we can obtain the desired result in a similar manner as in the proof of Theorem 2.1 in [8]. In fact, replacing $\alpha(t), \beta(t), ||y(t) - z(t)||$ in [8] with $2\alpha(t) + \beta(t) + \gamma_1(t)\gamma_2(t) + \gamma_1^2(t), \beta(t) + \gamma_1(t)\gamma_2(t) + \gamma_2^2(t), \mathbb{E}|x(t) - y(t)|^2$, respectively, using Lemma 3.8 and following the proof of Theorem 2.1 in [8], we can obtain either (3.27) or (3.28) immediately.

Case B: $\mu_1^{(0)} = 0$. Note that

$$\mathbb{E}|x(t) - y(t)|^2 \leq \mathbb{E}(|x(t) - x_n(t)| + |x_n(t) - y_n(t)| + |y(t) - y_n(t)|)^2 \\ \leq 3 \left(\mathbb{E}|x(t) - x_n(t)|^2 + \mathbb{E}|x_n(t) - y_n(t)|^2 + \mathbb{E}|y(t) - y_n(t)|^2 \right),$$

where $y_n(t)$ is the solution of

$$\begin{cases} dy(t) = f(t, y(t), y^{(n,t)}(t - \tau(t)))dt + g(t, y(t), y^{(n,t)}(t - \tau(t)))dw(t), t \in [a, b], (3.29a) \\ y(t) = \eta(t), t \in [a - \tau_0, a], \quad \eta \in L^p_{\mathcal{F}_a}([a - \tau_0, a]; \mathbb{C}^d), \end{cases}$$
(3.29b)

which is the $\frac{1}{n}$ -perturbed problem of (3.2), and $y^{(n,t)}(t-\tau(t))$ is defined by

$$y^{(n,t)}(t-\tau(t)) = \begin{cases} y(t-\tau(t)), & \tau(t) \ge \frac{1}{n}, \\ y(t-\frac{1}{n}), & \tau(t) < \frac{1}{n}. \end{cases}$$

It is known that, problem $(2.1) \in SD(\alpha, \beta, \gamma_1, \gamma_2)$ implies that problem $(2.6) \in SD(\alpha, \beta, \gamma_1, \gamma_2)$. It follows from (2.8) that $\tilde{\mu}_1^{(0)} \geq \frac{1}{n} > 0$. Therefore, by case A, for $\mathbb{E}|x_n(t)-y_n(t)|^2$, either (3.27) holds if c > 0 or (3.28) holds if $c \leq 0$. Letting $n \to +\infty$ and using Lemma 3.9, we can obtain the desired estimate of $\mathbb{E}|x(t) - y(t)|^2$ in this case.

Corollary 3.11 Under the assumptions of Theorem 3.10. Suppose f(t,0,0) = 0 and g(t,0,0) = 0, then $\forall t \in [a,b]$,

$$\mathbb{E}|x(t)|^2 \le e^{c(t-a)} \sup_{a-\tau_0 \le \theta \le a} \mathbb{E}|\xi(\theta)|^2, \quad \text{if } c > 0,$$
$$\mathbb{E}|x(t)|^2 \le \sup_{a-\tau_0 \le \theta \le a} \mathbb{E}|\xi(\theta)|^2, \quad \text{if } c \le 0.$$

Lemma 3.12 Suppose problem $(2.1) \in SD(\alpha, \beta, \gamma_1, \gamma_2)$, and that

$$\frac{2\alpha(t) + \beta(t) + \gamma_1(t)\gamma_2(t) + \gamma_1^2(t) \le \alpha_0 < 0,}{\frac{\beta(t) + \gamma_1(t)\gamma_2(t) + \gamma_2^2(t)}{|2\alpha(t) + \beta(t) + \gamma_1(t)\gamma_2(t) + \gamma_1^2(t)|} \le \nu < 1,} \quad \forall t \in [a, b],$$
(3.30)

where α_0 and ν are constants. Then for any given $c_1, c_2, c_3 : a \leq c_1 < c_2 < c_3 \leq b$, we have

$$\mathbb{E}|x(t) - y(t)|^{2} \leq \left(\nu + (1 - \nu)e^{\alpha_{0}(c_{2} - c_{1})}\right) \sup_{\substack{\mu_{2}^{(0)}(c_{1}, c_{3}) \leq \theta \leq c_{2}}} \mathbb{E}|x(\theta) - y(\theta)|^{2}, \\ \forall t \in [c_{2}, c_{3}]. \tag{3.31}$$

Proof. We divide the proof into two cases: $\mu_1^{(0)} > 0$ and $\mu_1^{(0)} = 0$.

Case A: $\mu_1^{(0)} > 0$. In this case, replacing $\alpha(t), \beta(t), ||y(t) - z(t)||$ in [8] with $2\alpha(t) + \beta(t) + \gamma_1(t)\gamma_2(t) + \gamma_1^2(t), \beta(t) + \gamma_1(t)\gamma_2(t) + \gamma_2^2(t), \mathbb{E}|x(t) - y(t)|^2$, respectively, using Lemma 3.8 and following the proof of Lemma 2.3 in [8], we can obtain (3.31) immediately.

Case B: $\mu_1^{(0)} = 0$. In this case, we can obtain the estimate (3.31) in a similar manner as in the proof of Case B of Theorem 3.10.

3.2 Infinite interval

Let us now proceed to discuss the equation (2.1) which satisfies conditions (2.2)-(2.5) but the integration interval [a, b] replaced by $[a, +\infty)$. Accordingly, interval $[a - \tau_0, b]$ is replaced by $[a - \tau_0, +\infty)$, and the symbol $\mathcal{SD}(\alpha, \beta, \gamma_1, \gamma_2)$ is replaced by $\overline{\mathcal{SD}}(\alpha, \beta, \gamma_1, \gamma_2)$.

Theorem 3.13 Assume that problem $(2.1) \in \overline{SD}(\alpha, \beta, \gamma_1, \gamma_2)$, and

$$\lim_{t \to +\infty} (t - \tau(t)) = +\infty, \quad \sup_{a \le t < +\infty} (2\alpha(t) + \beta(t) + \gamma_1(t)\gamma_2(t) + \gamma_1^2(t)) = \alpha_0 < 0,$$
$$\sup_{a \le t < +\infty} \frac{\beta(t) + \gamma_1(t)\gamma_2(t) + \gamma_2^2(t)}{|2\alpha(t) + \beta(t) + \gamma_1(t)\gamma_2(t) + \gamma_1^2(t)|} = \nu < 1.$$

Then, for any given constant $\mu > 0$, there exists a strictly increased sequence $\{t_k\}$ which diverges to $+\infty$ as $k \to +\infty$, where $t_0 = a$, such that

$$\sup_{t_k \le t \le t_{k+1}} \mathbb{E} |x(t) - y(t)|^2 \le C_{\mu}^{k+1} \sup_{a - \tau_0 \le t \le a} \mathbb{E} |\xi(t) - \eta(t)|^2, \quad k = 0, 1, 2, \cdots,$$
(3.32)

where $C_{\mu} = \nu + (1 - \nu)e^{\alpha_0 \mu} \in (0, 1)$. Hence,

$$\lim_{t \to +\infty} \mathbb{E} |x(t) - y(t)|^2 = 0.$$
(3.33)

Proof. It is obvious that (3.32) implies (3.33). So, only the proof of (3.32) is required. First we construct a sequence $\{t_k\}$ by induction. Let $t_0 = a$. Suppose that t_k is chosen appropriately, where $k \ge 0$. Because $\lim_{t\to+\infty} (t-\tau(t)) = +\infty$, there exists a M such that for all $t \ge M$, we have $t - \tau(t) \ge t_k$ and therefore $\mu_2^{(0)}(M, +\infty) \ge t_k$. So we can choose $t_{k+1} = M + \mu$ and have the relation

$$t_k \le \mu_2^{(0)}(t_{k+1} - \mu, +\infty) \le t_{k+1} - \mu < t_{k+1}.$$
(3.34)

Using (3.34) and Lemma 3.12, we get

$$\sup_{t_k \le t \le t_{k+1}} \mathbb{E} |x(t) - y(t)|^2 \le (\nu + (1 - \nu)e^{\alpha_0 \mu}) \sup_{\mu_2^{(0)}(t_k - \mu, t_{k+1}) \le t \le t_k} \mathbb{E} |x(t) - y(t)|^2$$

$$\le C_{\mu} \sup_{t_{k-1} \le t \le t_k} \mathbb{E} |x(t) - y(t)|^2 \le \ldots \le C_{\mu}^{k+1} \sup_{a - \tau_0 \le t \le a} \mathbb{E} |\xi(t) - \eta(t)|^2.$$

The proof is complete.

Corollary 3.14 Under the same conditions as Theorem 3.13. Furthermore, suppose that f(t,0,0) = 0, g(t,0,0) = 0, then

$$\sup_{t_k \le t \le t_{k+1}} \mathbb{E}|x(t)|^2 \le C_{\mu}^{k+1} \sup_{a-\tau_0 \le t \le a} \mathbb{E}|\xi(t)|^2, \quad k = 0, 1, 2, \cdots,$$
(3.35)

$$\lim_{t \to +\infty} \mathbb{E}|x(t)|^2 = 0. \tag{3.36}$$

Remark 3.15 Li [8] discussed the stability of nonlinear stiff Volterra functional differential equations in Banach spaces. Theorem 3.10 and Theorem 3.13 can be regarded as generalizations of Theorem 2.1 and Theorem 2.2 of [8] restricted in finite-dimensional Hilbert spaces \mathbb{C}^d and finitely many delays to the stochastic version, respectively. It should be pointed out that the drift coefficient f is required to be locally Lipschitz continuous in this paper, whereas the condition is not required in [8]. It is known that local Lipschitz continuity is not a strong restriction.

3.3 Examples

System (2.1) includes the following three classes of SDDEs as special cases

- SDDEs with constant delays: $\tau(t) \equiv \tau$.
- Stochastic pantograph equations: $t \tau(t) = qt$, where 0 < q < 1 is a constant.
- SDDEs with piecewise constant arguments: $t \tau(t) = \lfloor t i \rfloor$, where $\lfloor t \rfloor$ denotes the largest integer number less than or equal to t, i is a nonnegative integer.

Therefore, Theorem 3.10, Theorem 3.13, Corollary 3.11 and Corollary 3.14 are valid for the three classes of SDDEs mentioned above.

Example 3.16 Consider the linear SDDEs

$$dx(t) = (A_1(t)x(t) + A_2(t)x(t - \tau(t)) + F(t))dt + (B_1(t)x(t) + B_2(t)x(t - \tau(t)) + G(t))dw(t),$$
(3.37)

where $A_1(t), A_2(t), B_1(t), B_2(t) \in \mathbb{C}^{d \times d}, F(t), G(t) \in \mathbb{C}^d$ are continuous with respect to t, w(t) is an 1-dimensional Wiener process. For the problems (3.37), it is easy to verify the conditions (2.2)-(2.5) with

$$\alpha(t) = \lambda_{\max}^{\frac{A_1^*(t) + A_1(t)}{2}}, \ \beta(t) = |A_2(t)|, \ \gamma_1(t) = |B_1(t)|, \ \gamma_2(t) = |B_2(t)|,$$

where $\lambda_{\max}^{\frac{A_1^*(t)+A_1(t)}{2}}$ denotes the largest eigenvalue of the Hermite matrix $\frac{A_1^*(t)+A_1(t)}{2}$.

Applying Theorem 3.10 and Corollary 3.11 to (3.37), we have the following corollary.

Corollary 3.17 The solutions of (3.37) satisfy

$$\mathbb{E}|x(t) - y(t)|^2 \le e^{c(t-a)} \sup_{a-\tau_0 \le \theta \le a} \mathbb{E}|\xi(\theta) - \eta(\theta)|^2, \quad t \in [a,b], \text{ if } c \ge 0,$$
$$\mathbb{E}|x(t) - y(t)|^2 \le \sup_{a-\tau_0 \le \theta \le a} \mathbb{E}|\xi(\theta) - \eta(\theta)|^2, \qquad t \in [a,b], \text{ if } c \le 0,$$

where x(t), y(t) are the solutions of (3.37) corresponding to the initial functions $\xi(t)$ and $\eta(t)$, respectively,

$$c = \max_{a \le t \le b} \left(2\lambda_{\max}^{\frac{A_1^*(t) + A_1(t)}{2}} + 2|A_2(t)| + \left(|B_1(t)| + |B_2(t)|\right)^2 \right).$$

Furthermore, if F(t) = 0 and G(t) = 0, then the solutions of (3.37) satisfy

$$\mathbb{E}|x(t)|^2 \le e^{c(t-a)} \sup_{a-\tau_0 \le \theta \le a} \mathbb{E}|\xi(\theta)|^2, \quad t \in [a,b], \text{ if } c > 0,$$
$$\mathbb{E}|x(t)|^2 \le \sup_{a-\tau_0 \le \theta \le a} \mathbb{E}|\xi(\theta)|^2, \quad t \in [a,b], \text{ if } c \le 0.$$

Applying Theorem 3.13 and Corollary 3.14 to (3.37) leads to the following

Corollary 3.18 If $\lim_{t \to +\infty} (t - \tau(t)) = +\infty$,

$$\sup_{a \le t < +\infty} \left(2\lambda_{\max}^{\frac{A_1^*(t) + A_1(t)}{2}} + |A_2(t)| + |B_1(t)||B_2(t)| + |B_1(t)|^2 \right) < 0,$$

$$\sup_{a \le t < +\infty} \frac{|A_2(t)| + |B_1(t)||B_2(t)| + |B_2(t)|^2}{|2\lambda_{\max}^{\frac{A_1^*(t) + A_1(t)}{2}} + |A_2(t)| + |B_1(t)||B_2(t)| + |B_1(t)|^2|} < 1,$$

then the solutions of (3.37) satisfy $\lim_{t \to +\infty} \mathbb{E}|x(t) - y(t)|^2 = 0$. Furthermore, if F(t) = 0and G(t) = 0, then the solutions of (3.37) satisfy $\lim_{t \to +\infty} \mathbb{E}|x(t)|^2 = 0$.

In particular, if d = 1, $A_1(t)$, $A_2(t)$, $B_1(t)$, $B_2(t)$ are constants, that is, $A_1(t) = A_1$, $A_2(t) = A_2$, $B_1(t) = B_1$, $B_2(t) = B_2$, then the solutions of (3.37) satisfy $\lim_{t \to +\infty} \mathbb{E}|x(t)|^2 = 0$ if

$$\Re A_1 + |A_2| + \frac{1}{2} \Big(|B_1| + |B_2| \Big)^2 < 0.$$
 (3.38)

Remark 3.19 For linear scalar SDDEs with constant delay and linear scalar stochastic pantograph equations, the condition (3.38) is stated in [11, 10] and [3], respectively. Therefore, Corollary 3.14 includes as special cases the related results in [11, 10, 3].

Example 3.20 Consider the nonlinear equation

$$dx(t) = \left(A_1(t)x + A_2(t)x^3 + A_3(t)\sqrt{x^2(t - \tau(t)) + 1} + F(t)\right)dt + \left(B_1(t)\sin x(t) + B_2(t)\arctan x(t - \tau(t)) + G(t)\right)dw(t),$$
(3.39)

where $A_1(t), A_2(t), A_3(t), B_1(t), B_2(t), F(t), G(t)$ are continuous real-valued functions in t and $A_2(t) < 0$. It is easy to verify that (3.39) satisfies the conditions (2.2)-(2.5) with

$$\alpha(t) = A_1(t), \quad \beta(t) = |A_3(t)|, \quad \gamma_1(t) = |B_1(t)|, \quad \gamma_2(t) = |B_2(t)|.$$

Applying Theorem 3.10, Corollary 3.11, Theorem 3.13 and Corollary 3.14 to (3.39), we can derive the results of the solutions of (3.39). For the sake of brevity, we do not present them here.

Remark 3.21 The drift coefficient f of (3.39) satisfies local Lipschitz condition and onesided Lipschitz condition with respect to x but global Lipschitz condition. The stability analysis in this work is based on the local Lipschitz condition and the one-sided Lipschitz condition, rather than a more restrictive global Lipschitz condition.

4 Stability of backward Euler method

In this section, we investigate whether numerical methods can reproduce the contractivity in mean square. For the deterministic differential equations, it is known that the contractivity of numerical methods is too strong [2, 7]. The existing theories [2, 15] show that only the backward Euler method and the two-stage Lobatto IIIC method can preserve the contractivity of nonlinear delay differential equations. Therefore, in the stochastic setting, we only focus on the backward Euler method instead of other methods.

For simplicity, from now on, we assume that

$$\alpha(t) \equiv \alpha, \ \beta(t) \equiv \beta, \ \gamma_1(t) \equiv \gamma_1, \ \gamma_2(t) \equiv \gamma_2, \ t \in [a, b].$$

On a finite time interval [a, b], a uniformly partition is defined by

$$t_i = a + ih, \quad i = 0, 1, \dots, h = \frac{b - a}{N}$$

The backward Euler method applied to (2.1) yields

$$\begin{cases} X_{n+1} = X_n + hf(t_{n+1}, X_{n+1}, X^h(t_{n+1} - \tau(t_{n+1}))) \\ +g(t_n, X_n, X^h(t_n - \tau(t_n)))\Delta w_n, \quad n = 0, 1, \dots, N-1, \end{cases}$$
(4.1a)

$$X^{h}(t) = \pi^{h}(t,\xi,X_{1},X_{2},\ldots,X_{n}), \quad a - \tau_{0} \le t \le t_{n},$$
(4.1b)

where π^h is an appropriate interpolation operator which approximates to the exact solution x(t) on the interval $[a - \tau_0, b]$, X_n is an approximation to the exact solution $x(t_n)$, $\Delta w_n = w(t_{n+1}) - w(t_n)$. It is well known that the backward Euler method is convergent with strong order only 1/2 for stochastic differential equations. So, interpolation operator π^h could be chosen as the follows

$$X^{h}(t) = \begin{cases} \frac{1}{h} [(t_{i+1} - t)X_{i} + (t - t_{i})X_{i+1}], t_{i} \le t \le t_{i+1}, i = 0, 1, 2, \dots, N-1, \\ \xi(t), \quad a - \tau_{0} \le t \le a. \end{cases}$$
(4.2)

Applying the backward Euler method to the perturbed problem (3.2) we can obtain the corresponding scheme

$$\begin{cases} Y_{n+1} = Y_n + hf(t_{n+1}, Y_{n+1}, Y^h(t_{n+1} - \tau(t_{n+1}))) \\ +g(t_n, Y_n, Y^h(t_n - \tau(t_n)))\Delta w_n, & n = 0, 1, \dots, N-1, \\ Y^h(t) = \pi^h(t, \eta, Y_1, Y_2, \dots, Y_n), & a - \tau_0 \le t \le t_n. \end{cases}$$
(4.3)

For simplicity, for any given nonnegative integer n, we write

$$P_{n} = X_{n} - Y_{n}, \quad Q_{n} = \max\{\max_{1 \le i \le n} \mathbb{E}|P_{i}|^{2}, \sup_{a - \tau_{0} \le t \le a} \mathbb{E}|\xi(t) - \eta(t)|^{2}\}, \quad n \ge 1,$$

$$Q_{0} = \sup_{a - \tau_{0} \le t \le a} \mathbb{E}|\xi(t) - \eta(t)|^{2}.$$
(4.4)

Moreover, for convenience, we introduce notations to denote the values of drift and diffusion coefficients at specific points.

$$\begin{aligned}
f^{xx}(n+1) &= f(t_{n+1}, X_{n+1}, X^{h}(t_{n+1} - \tau(t_{n+1}))), \\
f^{yy}(n+1) &= f(t_{n+1}, Y_{n+1}, Y^{h}(t_{n+1} - \tau(t_{n+1}))), \\
f^{yx}(n+1) &= f(t_{n+1}, Y_{n+1}, X^{h}(t_{n+1} - \tau(t_{n+1}))), \\
g^{xx}(n) &= g(t_{n}, X_{n}, X^{h}(t_{n} - \tau(t_{n}))), g^{yy}(n) = g(t_{n}, Y_{n}, Y^{h}(t_{n} - \tau(t_{n}))).
\end{aligned}$$
(4.5)

Lemma 4.1 Under the conditions (2.3) and (2.4), if $(\alpha + \beta)h < 1$, the implicit equation (4.1a) admits a unique solution.

Proof. Let $\tilde{f}(z) = f(\cdot, z, z^h(\cdot))$, then implicit equation (4.1a) can be rewritten as

$$z = h\tilde{f}(z) + b = hf(\cdot, z, z^h(\cdot)) + b, \qquad (4.6)$$

where z is unknown whereas b and h are known. Inserting the interpolation operator (4.2) into (4.6), we have

$$z = h\tilde{f}(z) + b = hf(\cdot, z, lz + b_0) + b,$$
(4.7)

where $0 \le l \le 1$, l and b_0 are also known. It follows from (2.3), (2.4) and (4.7) that

$$\begin{aligned} \Re \langle z_1 - z_2, \tilde{f}(z_1) - \tilde{f}(z_2) \rangle &= \Re \langle z_1 - z_2, f(\cdot, z_1, lz_1 + b_0) - f(\cdot, z_2, lz_2 + b_0) \rangle \\ &= \Re \langle z_1 - z_2, f(\cdot, z_1, lz_1 + b_0) - f(\cdot, z_2, lz_1 + b_0) \rangle \\ &+ \Re \langle z_1 - z_2, f(\cdot, z_2, lz_1 + b_0) - f(\cdot, z_2, lz_2 + b_0) \rangle &\leq \alpha |z_1 - z_2|^2 + \beta |z_1 - z_2|^2. \end{aligned}$$

The assertion follows immediately from Theorem 5.6.1 in [4].

Theorem 4.2 Assume that problem $(2.1) \in SD(\alpha, \beta, \gamma_1, \gamma_2)$. Let $\{X_n\}$ and $\{Y_n\}$ be two sequences of numerical solutions obtained by the backward Euler schemes (4.1) and (4.3), respectively. Write $c = 2\alpha + 2\beta + \gamma_1^2 + 2\gamma_1\gamma_2 + \gamma_2^2$.

(i) If c > 0, for any given $c_0 \in (0, 1)$, then we have for $hc \leq c_0$

$$\mathbb{E}|X_n - Y_n|^2 \le e^{\tilde{c}(t_n - a)} \sup_{a - \tau_0 \le t \le a} \mathbb{E}|\xi(t) - \eta(t)|^2, \quad n = 1, 2, \cdots, N,$$
(4.8)

where $\tilde{c} = \frac{c_1}{h}$,

$$c_{1} = \max\left\{\frac{1+h\gamma_{1}^{2}+2h\gamma_{1}\gamma_{2}+h\gamma_{2}^{2}}{1-2h\alpha-2h\beta}, \frac{1+h\beta+h\gamma_{1}^{2}+2h\gamma_{1}\gamma_{2}+h\gamma_{2}^{2}}{1-2h\alpha-h\beta}\right\}$$
$$= \frac{1+h\gamma_{1}^{2}+2h\gamma_{1}\gamma_{2}+h\gamma_{2}^{2}}{1-2h\alpha-2h\beta} > 1.$$

(ii) If $c \leq 0$, then we have for any h > 0

$$\mathbb{E}|X_n - Y_n|^2 \le \sup_{a - \tau_0 \le t \le a} \mathbb{E}|\xi(t) - \eta(t)|^2, \quad n = 1, 2, \cdots, N.$$
(4.9)

Note that (4.8) and (4.9) can be regarded as numerical analogs of (3.27) and (3.28), respectively.

Proof. (i) By (4.1) and (4.3), we have

$$P_{n+1} - h(f^{xx}(n+1) - f^{yy}(n+1)) = P_n + (g^{xx}(n) - g^{yy}(n))\Delta w_n$$

which yields

$$|P_{n+1}|^2 - 2h\Re \langle P_{n+1}, f^{xx}(n+1) - f^{yy}(n+1) \rangle + h^2 |f^{xx}(n+1) - f^{yy}(n+1)|^2$$

= $|P_n|^2 + 2\Re \langle P_n, (g^{xx}(n) - g^{yy}(n))\Delta w_n \rangle + |(g^{xx}(n) - g^{yy}(n))\Delta w_n|^2.$

Taking expectation and using (2.3)-(2.5) and (4.2), we get

$$\begin{split} \mathbb{E}|P_{n+1}|^{2} &\leq \mathbb{E}|P_{n}|^{2} + 2h\mathbb{E}\Re\langle P_{n+1}, f^{xx}(n+1) - f^{yy}(n+1)\rangle + h\mathbb{E}|g^{xx}(n) - g^{yy}(n)|^{2} \\ &\leq \mathbb{E}|P_{n}|^{2} + 2h\mathbb{E}\Re\langle P_{n+1}, f^{yx}(n+1) - f^{yy}(n+1)\rangle + h\mathbb{E}|g^{xx}(n) - g^{yy}(n)|^{2} \\ &\leq \mathbb{E}|P_{n}|^{2} + 2h\alpha\mathbb{E}|P_{n+1}|^{2} \\ &\quad + 2h\beta\mathbb{E}(|P_{n+1}||X^{h}(t_{n+1} - \tau(t_{n+1})) - Y^{h}(t_{n+1} - \tau(t_{n+1}))|) \\ &\quad + h\mathbb{E}(\gamma_{1}|P_{n}| + \gamma_{2}|X^{h}(t_{n} - \tau(t_{n})) - Y^{h}(t_{n} - \tau(t_{n}))|)^{2} \\ &\leq \mathbb{E}|P_{n}|^{2} + 2h\alpha\mathbb{E}|P_{n+1}|^{2} \\ &\quad + h\beta\mathbb{E}|P_{n+1}|^{2} + h\beta\mathbb{E}|X^{h}(t_{n+1} - \tau(t_{n+1})) - Y^{h}(t_{n+1} - \tau(t_{n+1}))|^{2} \\ &\quad + h(\gamma_{1}^{2} + \gamma_{1}\gamma_{2})\mathbb{E}|P_{n}|^{2} + h(\gamma_{2}^{2} + \gamma_{1}\gamma_{2})\mathbb{E}|X^{h}(t_{n} - \tau(t_{n})) - Y^{h}(t_{n} - \tau(t_{n}))|^{2} \\ &\leq \mathbb{E}|P_{n}|^{2} + 2h\alpha\mathbb{E}|P_{n+1}|^{2} \\ &\quad + h\beta\max\{\max_{1\leq i\leq n+1}\mathbb{E}|P_{i}|^{2}, \sup_{a-\tau_{0}\leq i\leq a}\mathbb{E}|\xi(t) - \eta(t)|^{2}\} \\ &\quad + h(\gamma_{1}^{2} + \gamma_{1}\gamma_{2})\mathbb{E}|P_{n}|^{2} + h(\gamma_{2}^{2} + \gamma_{1}\gamma_{2})Q_{n}, \end{aligned}$$

$$(4.10)$$

where we used the piecewise linear interpolation (4.2) and the following inequality

$$\mathbb{E}|(1-\delta)P_i + \delta P_{i+1}|^2 \le \max\{\mathbb{E}|P_i|^2, \mathbb{E}|P_{i+1}|^2\}, \quad 0 \le \delta \le 1.$$
(4.11)

It is clear from (4.10) that

$$(1 - 2h\alpha - h\beta)\mathbb{E}|P_{n+1}|^2 \le (1 + h\gamma_1^2 + h\gamma_1\gamma_2)\mathbb{E}|P_n|^2 +h\beta \max\{\max_{1\le i\le n+1} \mathbb{E}|P_i|^2, \sup_{a-\tau_0\le t\le a} \mathbb{E}|\xi(t) - \eta(t)|^2\} + h(\gamma_2^2 + \gamma_1\gamma_2)Q_n,$$
(4.12)

We now consider two cases:

(a)
$$\max\{\max_{1 \le i \le n+1} \mathbb{E}|P_i|^2, \sup_{a-\tau_0 \le t \le a} \mathbb{E}|\xi(t) - \eta(t)|^2\} = \mathbb{E}|P_{n+1}|^2,$$

(b)
$$\max\{\max_{1 \le i \le n+1} \mathbb{E}|P_i|^2, \sup_{a-\tau_0 \le t \le a} \mathbb{E}|\xi(t) - \eta(t)|^2\} \neq \mathbb{E}|P_{n+1}|^2.$$

In the case of (a), it follows from (4.12) that

$$(1 - 2h\alpha - 2h\beta)\mathbb{E}|P_{n+1}|^2 \le (1 + h\gamma_1^2 + 2h\gamma_1\gamma_2 + h\gamma_2^2)Q_n,$$
(4.13)

which yields

$$\mathbb{E}|P_{n+1}|^2 \le \frac{1+h\gamma_1^2+2h\gamma_1\gamma_2+h\gamma_2^2}{1-2h\alpha-2h\beta}Q_n \le c_1Q_n.$$
(4.14)

In the case of (b), (4.12) implies that

$$(1 - 2h\alpha - h\beta)\mathbb{E}|P_{n+1}|^2 \le (1 + h\gamma_1^2 + h\gamma_1\gamma_2)\mathbb{E}|P_n|^2 + h\beta Q_n + h(\gamma_2^2 + \gamma_1\gamma_2)Q_n \le (1 + h\beta + h\gamma_1^2 + 2h\gamma_1\gamma_2 + h\gamma_2^2)Q_n,$$

which yields

$$\mathbb{E}|P_{n+1}|^2 \le \frac{1+h\beta+h\gamma_1^2+2h\gamma_1\gamma_2+h\gamma_2^2}{1-2h\alpha-h\beta}Q_n \le c_1Q_n.$$
(4.15)

To summarize, both in the cases we have shown that $\mathbb{E}|P_{n+1}|^2 \leq c_1 Q_n$, which yields

$$Q_n \le Q_{n-1} + \mathbb{E}|P_n|^2 \le (1+c_1)Q_{n-1}.$$
(4.16)

By induction, we further obtain

$$\mathbb{E}|X_n - Y_n|^2 = \mathbb{E}|P_n|^2 \le Q_n \le (1 + c_1)Q_{n-1} \le \dots \le (1 + c_1)^n Q_0 \le e^{c_1 n} Q_0 = e^{\tilde{c}(t_n - a)} \sup_{a - \tau_0 \le t \le a} \mathbb{E}|\xi(t) - \eta(t)|^2.$$

(ii) When $c \leq 0$, noting that (4.14), (4.15) and

$$\frac{1+h\gamma_1^2+2h\gamma_1\gamma_2+h\gamma_2^2}{1-2h\alpha-2h\beta} \leq 1, \quad \frac{1+h\beta+h\gamma_1^2+2h\gamma_1\gamma_2+h\gamma_2^2}{1-2h\alpha-h\beta} \leq 1,$$

we have for any h > 0

$$\mathbb{E}|X_n - Y_n|^2 \le Q_{n-1} \le Q_{n-2} \le \dots \le Q_0 = \sup_{a - \tau_0 \le t \le a} \mathbb{E}|\xi(t) - \eta(t)|^2.$$
(4.17)

Therefore we have completed the proof of the theorem.

Corollary 4.3 Assume that problem $(2.1) \in SD(\alpha, \beta, \gamma_1, \gamma_2)$. Let $\{X_n\}$ be a sequence of numerical solutions obtained by the backward Euler method (4.1). Furthermore, if f(t, 0, 0) = 0, g(t, 0, 0) = 0, and

(i) if c > 0, for any given $c_0 \in (0, 1)$, then we have for $hc \leq c_0$

$$\mathbb{E}|X_n|^2 \le e^{\tilde{c}(t_n-a)} \sup_{a-\tau_0 \le t \le a} \mathbb{E}|\xi(t)|^2, \quad n = 1, 2, \cdots, N;$$
(4.18)

(ii) if $c \leq 0$, then we have for any h > 0

$$\mathbb{E}|X_n|^2 \le \sup_{a-\tau_0 \le t \le a} \mathbb{E}|\xi(t)|^2, \quad n = 1, 2, \cdots, N.$$
(4.19)

Theorem 4.4 Assume that problem $(2.1) \in \overline{SD}(\alpha, \beta, \gamma_1, \gamma_2)$, and

$$\lim_{t \to +\infty} (t - \tau(t)) = +\infty, \quad c = 2\alpha + 2\beta + \gamma_1^2 + 2\gamma_1\gamma_2 + \gamma_2^2 < 0.$$
(4.20)

Let $\{X_n\}$ and $\{Y_n\}$ be two sequences of numerical solutions obtained by the backward Euler schemes (4.1) and (4.3). Then,

(i) there exists a strictly increased positive integer sequence $\{n_k\}$ which diverges to $+\infty$ as $k \to +\infty$, where $n_0 = 0$, such that for any given h > 0,

$$\max_{n_k < i \le n_{k+1}} \mathbb{E}|X_i - Y_i|^2 \le c_2^{k+1} \sup_{a - \tau_0 \le t \le a} \mathbb{E}|\xi(t) - \eta(t)|^2, \quad k = 0, 1, 2, \dots,$$
(4.21)

where

$$c_{2} = \max\left\{\frac{1+h\gamma_{1}^{2}+2h\gamma_{1}\gamma_{2}+h\gamma_{2}^{2}}{1-2h\alpha-2h\beta}, \frac{1+h\beta+h\gamma_{1}^{2}+2h\gamma_{1}\gamma_{2}+h\gamma_{2}^{2}}{1-2h\alpha-h\beta}\right\}$$
$$= \frac{1+h\beta+h\gamma_{1}^{2}+2h\gamma_{1}\gamma_{2}+h\gamma_{2}^{2}}{1-2h\alpha-h\beta} < 1;$$

(ii) for any given h > 0,

$$\lim_{n \to +\infty} \mathbb{E}|X_n - Y_n|^2 = 0.$$
(4.22)

Note that (4.21) and (4.22) can be regarded as numerical analogs of (3.32) and (3.33), respectively.

Proof. It is obvious that (4.21) implies (4.22), and we only need to prove (4.21). By (4.20), we have $2\alpha + \beta + \gamma_1\gamma_2 + \gamma_1^2 < 0$, $(\beta + \gamma_1\gamma_2 + \gamma_2^2)/|2\alpha + \beta + \gamma_1\gamma_2 + \gamma_1^2| < 1$ and $c_2 < 1$.

First, as done in [9, 18], we can construct a strictly increased sequence of integers $\{n_k\}$ which diverges to $+\infty$ as $k \to +\infty$, such that

$$t - \tau(t) > t_{n_k+1}, \quad \forall t \ge t_{n_{k+1}},$$

where $n_0 = 0$. In fact, suppose that $n_k (k \ge 0)$ has been chosen appropriately. Then there exists a constant $M > t_{n_k}$ such that for all $t \ge M$ we have $t - \tau(t) > t_{n_k} + h$ since $\lim_{t\to+\infty} (t - \tau(t)) = +\infty$. If M is a node, we let $t_{n_{k+1}} = M$, otherwise there exists natural number j such that $t_j < M < t_{j+1}$, then we let $n_{k+1} = j + 1$ and $t_{n_{k+1}} = t_{j+1}$. Thus we obtain the required sequence $\{n_k\}$ which satisfies

$$t_0 < t_1 < \dots < t_{n_1} < t_{n_1+1} < \dots < t_{n_2} \cdots < t_{n_k} < \dots$$

For $n_k < n+1 \le n_{k+1}$, by the second inequality of (4.10) and conditions (2.3)-(2.5), we have

$$\begin{split} \mathbb{E}|P_{n+1}|^2 &\leq \mathbb{E}|P_n|^2 + 2h\mathbb{E}\Re\langle P_{n+1}, f^{xx}(n+1) - f^{yx}(n+1)\rangle \\ &+ 2h\mathbb{E}\Re\langle P_{n+1}, f^{yx}(n+1) - f^{yy}(n+1)\rangle + h\mathbb{E}|g^{xx}(n) - g^{yy}(n)|^2 \\ &\leq \mathbb{E}|P_n|^2 + 2h\alpha\mathbb{E}|P_{n+1}|^2 + 2h\beta\mathbb{E}(|P_{n+1}||X^h(t_{n+1} - \tau(t_{n+1})) - Y^h(t_{n+1} - \tau(t_{n+1}))|) \\ &+ h\mathbb{E}\Big(\gamma_1|P_n| + \gamma_2|X^h(t_n - \tau(t_n)) - Y^h(t_n - \tau(t_n))|\Big)^2 \\ &\leq \mathbb{E}|P_n|^2 + 2h\alpha\mathbb{E}|P_{n+1}|^2 + h\beta\mathbb{E}|P_{n+1}|^2 + h\beta\mathbb{E}|X^h(t_{n+1} - \tau(t_{n+1})) - Y^h(t_{n+1} - \tau(t_{n+1}))|^2 \\ &+ h(\gamma_1^2 + \gamma_1\gamma_2)\mathbb{E}|P_n|^2 + h(\gamma_2^2 + \gamma_1\gamma_2)\mathbb{E}|X^h(t_n - \tau(t_n)) - Y^h(t_n - \tau(t_n))|^2, \end{split}$$

which yields

$$(1 - 2h\alpha - h\beta)\mathbb{E}|P_{n+1}|^{2}$$

$$\leq (1 + h\gamma_{1}^{2} + h\gamma_{1}\gamma_{2})\mathbb{E}|P_{n}|^{2} + h\beta\mathbb{E}|X^{h}(t_{n+1} - \tau(t_{n+1})) - Y^{h}(t_{n+1} - \tau(t_{n+1}))|^{2}$$

$$+ h(\gamma_{2}^{2} + \gamma_{1}\gamma_{2})\mathbb{E}|X^{h}(t_{n} - \tau(t_{n})) - Y^{h}(t_{n} - \tau(t_{n}))|^{2}$$

$$\leq (1 + h\gamma_{1}^{2} + h\gamma_{1}\gamma_{2})\mathbb{E}|P_{n}|^{2} + h\beta \max_{n_{k-1} < i \le n+1} \mathbb{E}|P_{i}|^{2} + h(\gamma_{2}^{2} + \gamma_{1}\gamma_{2}) \max_{n_{k-1} < i \le n} \mathbb{E}|P_{i}|^{2},$$

where we used the piecewise linear interpolation operator (4.2) and the inequality (4.11). We now consider the following two cases.

If $\max_{n_{k-1} < i \le n+1} \mathbb{E}|P_i|^2 = \mathbb{E}|P_{n+1}|^2$, we have

$$\mathbb{E}|P_{n+1}|^2 \le \frac{1+h\gamma_1^2+2h\gamma_1\gamma_2+h\gamma_2^2}{1-2h\alpha-2h\beta} \max_{n_{k-1}< i\le n} \mathbb{E}|P_i|^2 \le c_2 \max_{n_{k-1}< i\le n} \mathbb{E}|P_i|^2.$$

If $\max_{n_{k-1} < i \le n+1} \mathbb{E} |P_i|^2 \neq \mathbb{E} |P_{n+1}|^2$, we have

$$\mathbb{E}|P_{n+1}|^2 \le \frac{1+h\beta+h\gamma_1^2+2h\gamma_1\gamma_2+h\gamma_2^2}{1-2h\alpha-h\beta} \max_{n_{k-1}< i \le n} \mathbb{E}|P_i|^2 \le c_2 \max_{n_{k-1}< i \le n} \mathbb{E}|P_i|^2.$$

In both cases, we have

$$\mathbb{E}|P_{n+1}|^2 \le c_2 \max_{n_{k-1} < i \le n} \mathbb{E}|P_i|^2, \quad n_k < n+1 \le n_{k+1}.$$
(4.23)

(4.23) with $n = n_k$ reduces to $\mathbb{E}|P_{n_k+1}|^2 \leq c_2 \max_{n_{k-1} < i \leq n_k} \mathbb{E}|P_i|^2$. By induction, we have

$$\max_{n_k < i \le n_{k+1}} \mathbb{E}|X_i - Y_i|^2 = \max_{n_k < i \le n_{k+1}} \mathbb{E}|P_i|^2 \le c_2 \max_{n_{k-1} < i \le n_k} \mathbb{E}|P_i|^2$$
$$\le \cdots \le c_2^{k+1} \max_{a - \tau_0 \le t \le a} \mathbb{E}|\xi(t) - \eta(t)|^2.$$

The proof is complete.

Corollary 4.5 Under the same assumptions of Theorem 4.4. Let $\{X_n\}$ be a sequence of numerical solution obtained by the backward Euler method (4.1). Furthermore, if f(t,0,0) = 0 and g(t,0,0) = 0, then,

(i) there exists a strictly increased positive integer sequence $\{n_k\}$ which diverges to $+\infty$ as $k \to +\infty$, where $n_0 = 0$, such that for any given h > 0,

$$\max_{n_k < i \le n_{k+1}} \mathbb{E}|X_i|^2 \le c_2^{k+1} \sup_{a - \tau_0 \le t \le a} \mathbb{E}|\xi(t)|^2, \quad k = 0, 1, 2, \dots$$

(*ii*) for any given h > 0, $\lim_{n \to +\infty} \mathbb{E}|X_n|^2 = 0$.

5 SDDEs with several delays

Consider the following SDDEs with several delays

$$\begin{cases} dx(t) = f(t, x(t), x(t - \tau_1(t)), \cdots, x(t - \tau_r(t)))dt \\ +g(t, x(t), x(t - \tau_1(t)), \cdots, x(t - \tau_r(t)))dw(t), & t \ge a, \\ x(t) = \xi(t), & t \in [a - \tau_0, a], \end{cases}$$
(5.1)

where $\tau_i(t) \ge 0, i = 1, 2, \dots, r$ and $\max_{1 \le i \le r} \inf_{t \ge a} (t - \tau_i(t)) \ge a - \tau_0$. All results given in this paper can be extended easily to the case of several delays. For the sake of brevity, we do not present the corresponding results for (5.1).

6 Conclusions and future work

In this paper, we investigate the stability of analytical and numerical solutions of nonlinear SDDEs. We derive sufficient conditions for the stability, contractivity and asymptotic contractivity in mean square of the solutions for nonlinear SDDEs. The results provide a unified theoretical treatment for SDEs, SDDEs with constant delay and variable delay (including bounded and unbounded variable delays). Then, it is proved that the backward Euler method can preserve the properties of the underlying system. The main results of analytic solution in this paper can be regarded as a generalization of those in [8] restricted in finite-dimensional Hilbert spaces and finitely many delays to the stochastic version. We have encountered problems when we tried to obtain a unified framework for general SFDEs. It is worth noting that whether the results in [8] can be extended to general SFDEs or not. One area for the future work is to give a positive or negative answer for the question. Neutral stochastic delay differential equation (NSDDE) is more general than stochastic delay differential equation. It is interesting to investigate whether the theory of this paper can be extended to NSDDEs and corresponding numerical methods. It will also be our future work.

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