Cancellativization of dimer models

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Abstract

We show that any dimer model can be made cancellative without changing the characteristic polygon.

1 Introduction

A *dimer model* is a bicolored graph on a real 2-torus T giving a polygon division of T. It is originally introduced in 1930s as a model in statistical mechanics [FR37], and has been actively studied since then. See e.g. a review by Kenyon [Ken04] and references therein for dimer models as statistical mechanical models.

More recently, a new connection between dimer models and quivers has been discovered by string theorists (cf. e.g. [Ken07]). A dimer model encodes the information of a quiver Γ with relations, and the resulting path algebra $\mathbb{C}\Gamma$ is a *Calabi-Yau algebra* of dimension three in the sense of Ginzburg [Gin06] if and only if $\mathbb{C}\Gamma$ is *cancellative* (i.e., $ab = ac \neq 0$ for an arrow a and a pair (b, c) of paths implies $b = c$, and similarly for $ba = ca$) [Bro12, MR10, Dav11]. One can also give a purely combinatorial condition on a dimer model, called the *consistency condition*, which is equivalent to the cancellation property of the path algebra if the dimer model is *non-degenerate* [IU11, Boc12]. We say that a dimer model is *cancellative* if it satisfies one (and hence all) of these equivalent conditions.

With a dimer model, one can associate two convex lattice polygons called the *characteristic polygon* and the *zigzag polygon*. Here, a *lattice polygon* is the convex hull of a finite lattice points on $\mathbb{R}^2$. Although these two polygons are different in general, they coincide if the dimer model is cancellative [Gul08, IU]. We say that a polygon is *non-degenerate* if it has an interior point.

It is easy to make a dimer model cancellative without changing the zigzag polygon:

Theorem 1.1. *If the zigzag polygon of a dimer model G is non-degenerate, then one can remove some edges and nodes from G to obtain another dimer model G', which is cancellative with the same zigzag polygon as G.*

As a corollary, one obtains the following:

Corollary 1.2. *For any dimer model, the zigzag polygon is contained in the characteristic polygon.*

It is more difficult to make a dimer model consistent without changing the characteristic polygon. The main result in this paper states that this is always possible:

Theorem 1.3. *If the characteristic polygon of a dimer model G is non-degenerate, then one can remove some edges from G to obtain a cancellative dimer model G' with the same characteristic polygon as G.*

A dimer model is *strongly non-degenerate* if every edge is contained in at least one corner perfect matching. Any dimer model can be made strongly non-degenerate without changing the characteristic polygon, simply by removing edges not contained in any corner perfect matching. The proof of Theorem 1.3 also shows the following:

Corollary 1.4. *If every corner perfect matching in a strongly non-degenerate dimer model is multiplicity-free, then the zigzag polygon coincides with the characteristic polygon.*

Although cancellativity is a strong condition and there are many examples of non-cancellative dimer models (see e.g. [DHP10]; in fact, we suspect that almost all dimer models (in some random graph theoretic sense) are non-cancellative), Theorems 1.1 and 1.3 shows the abundance of cancellative dimer models among all dimer models.

This paper is organized as follows: In Section 2, we recall basic definitions on dimer models. In Section 3, we recall the definition of a zigzag polygon and prove Theorem 1.1. In Section 4, we discuss an operation of cancellativization which keeps the characteristic polygon fixed. We first remove suitable edges from the dimer model G to obtain another dimer model G' satisfying the following conditions:

- The characteristic polygon of G' coincides with that of G.
- For any pair $(\mathfrak{c}_1, \mathfrak{c}_2)$ of adjacent corners of the characteristic polygon of G', there are perfect matchings D_1 and D_2 of G' such that
  - (i) the height changes of D_1 and D_2 give these corners; $h(D_1) = \mathfrak{c}_1$ and $h(D_2) = \mathfrak{c}_2$, and
  - (ii) every connected component of the symmetric difference $D_1 \triangle D_2$ is a zigzag path on G'.

Then G' has sufficiently many zigzag paths to ensure that the zigzag polygon contains the characteristic polygon. On the other hand, the characteristic polygon always contains the zigzag polygon by Corollary 1.2, and hence they must coincide. Now one can perform the operation in Theorem 1.1 and obtain a cancellative dimer model G'' with the same characteristic polygon as G.

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2 Dimer models and quivers

Let $T = \mathbb{R}^2/\mathbb{Z}^2$ be a real two-torus equipped with an orientation. A *bicolored graph* on T consists of

- a finite set $B \subset T$ of black nodes,
- a finite set $W \subset T$ of white nodes, and
- a finite set E of edges, consisting of embedded closed intervals e on T such that one boundary of e belongs to B and the other boundary belongs to W. We assume that two edges intersect only at the boundaries.

A *face* of a graph is a connected component of $T \setminus \cup_{e \in E} e$. The set of faces will be denoted by F. A bicolored graph G on T is called a *dimer model* if G contains no univalent node and every face $f \in F$ is simply-connected.

A *quiver* consists of

- a set V of vertices,
- a set A of arrows, and
- two maps $s, t : A \to V$ from A to V.

For an arrow $a \in A$, the vertices $s(a)$ and $t(a)$ are said to be the *source* and the *target* of a respectively. A *path* on a quiver is an ordered set of arrows $(a_n, a_{n-1}, \dots, a_1)$ such that $s(a_{i+1}) = t(a_i)$ for $i = 1, \dots, n-1$. We also allow for a path of length zero, starting and ending at the same vertex. The *path algebra* $\mathbb{C}Q$ of a quiver $Q = (V, A, s, t)$ is the algebra spanned by the set of paths as a vector space, and the multiplication is defined by the concatenation of paths;

$$(b_m, \dots, b_1) \cdot (a_n, \dots, a_1) = \begin{cases} (b_m, \dots, b_1, a_n, \dots, a_1) & s(b_1) = t(a_n), \\ 0 & \text{otherwise.} \end{cases}$$

A *quiver with relations* is a pair of a quiver and a two-sided ideal $\mathcal{I}$ of its path algebra. For a quiver $\Gamma = (Q, \mathcal{I})$ with relations, its path algebra $\mathbb{C}\Gamma$ is defined as the quotient algebra $\mathbb{C}Q/\mathcal{I}$. Two paths a and b are said to be *equivalent* if they give the same element in $\mathbb{C}\Gamma$.

A dimer model (B, W, E) encodes the information of a quiver $\Gamma = (V, A, s, t, \mathcal{I})$ with relations in the following way: The set V of vertices is the set of connected components of the complement $T \setminus (\bigcup_{e \in E} e)$, and the set A of arrows is the set E of edges of the graph. The directions of the arrows are determined by the colors of the nodes of the graph, so that the white node $w \in W$ is on the right of the arrow. In other words, the quiver is the dual graph of the dimer model equipped with an orientation given by rotating the white-to-black flow on the edges of the dimer model by minus 90 degrees. The relations of the quiver are described as follows: For an arrow $a \in A$, there exist two paths $p_+(a)$ and $p_-(a)$ from $t(a)$ to $s(a)$, the former going around the white node connected to $a \in E = A$ clockwise and the latter going around the black node connected to a counterclockwise. Then the ideal $\mathcal{I}$ of the path algebra is generated by $p_+(a) - p_-(a)$ for all $a \in A$.

A *perfect matching* on a dimer model $G = (B, W, E)$ is a subset D of E such that for any node $v \in B \cup W$, there is a unique edge $e \in D$ connected to v. A dimer model is *non-degenerate* if for any edge $e \in E$, there is a perfect matching D such that $e \in D$.

A dimer model $G = (B, W, E)$ gives a chain complex

$$0 \to \mathbb{Z}^F \to \mathbb{Z}^E \to \mathbb{Z}^{B \sqcup W} \to 0$$

computing the homology of T. The orientation on a face comes from the standard orientation of $T = \mathbb{R}^2/\mathbb{Z}^2$, and the orientation on an edge is such that $\partial e = w - b$, where w and b are the white and the black node adjacent to the edge e. A perfect matching $D \subset E$ gives a 1-chain $\sum_{e \in D} e \in \mathbb{Z}^E$ in this complex, which will often be written as D by abuse of notation. By the definition of a perfect matching, the difference of 1-chains associated with a pair (D, D') of perfect matchings is a 1-cycle, whose class in $H_1(T; \mathbb{Z})$ will be denoted by $[D - D']$. This class is equivalent to the class $[D \triangle D']$ of a 1-cycle supported on the symmetric difference $D \triangle D' = (D \cup D') \setminus (D \cap D')$. We have $D \triangle D' = -D' \triangle D$ as 1-cycles, although the underlying sets are identical.

Let $\langle -, - \rangle : H_1(T; \mathbb{Z}) \otimes H_1(T; \mathbb{Z}) \to \mathbb{Z}$ be the intersection pairing. The Poincaré dual of $[D \triangle D'] \in H_1(T; \mathbb{Z})$ is written as $h(D, D') \in H^1(T; \mathbb{Z})$, and called the *height change* of D with respect to the *reference matching* D';

$$h(D, D')(C) = \langle C, [D \triangle D'] \rangle, \quad \forall C \in H_1(T; \mathbb{Z}).$$

We often suppress the reference matching from the notation and write $h(D) = h(D, D')$. We will use the isomorphism $H^1(T; \mathbb{Z}) \cong \mathbb{Z}^2$ coming from the identification $T = \mathbb{R}^2/\mathbb{Z}^2$ to think of a height change as an element of $\mathbb{Z}^2$; $h(D) = (h_x(D), h_y(D)) \in \mathbb{Z}^2$. The *characteristic polynomial* of G is the generating function

$$Z(x, y) = \sum_{D \in \operatorname{Perf}(G)} x^{h_x(D)} y^{h_y(D)}$$

for the height change, which is a Laurent polynomial in two variables. Its Newton polygon

$$\operatorname{Conv}\{(h_x(D), h_y(D)) \in \mathbb{Z}^2 \mid D \text{ is a perfect matching}\}$$

is called the *characteristic polygon*. One clearly has $h(D, D'') = h(D, D') - h(D'', D')$, so that the characteristic polygon will be translated if one changes the reference matching. A perfect matching D is said to be a *corner perfect matching* if the height change $h(D)$ is at a corner of the characteristic polygon. The *multiplicity* of a perfect matching D is the number of perfect matchings whose height change is the same as D.

A perfect matching D can be considered as a set of walls which block some of the arrows. A path p on the quiver is said to be *allowed* by D if p does not contain any arrow contained in $D \subset E = A$.

With a perfect matching, one can associate a representation of the quiver with dimension vector $(1, \ldots, 1)$ by sending any allowed path to 1 and other paths to 0. One can easily check that this satisfies the relation of the quiver. A perfect matching is said to be *simple* if the associated quiver representation is simple, i.e., has no non-trivial subrepresentation. This is equivalent to the condition that there is an allowed path starting and ending at any given pair of vertices.

The main theorem of [IU08] states that when a dimer model is non-degenerate, then the moduli space $\mathcal{M}_\theta$ of θ-stable representations of $\mathbb{C}\Gamma$ of dimension vector $(1, \ldots, 1)$ is a smooth toric Calabi-Yau 3-fold for a generic stability parameter θ in the sense of King [Kin94]. A toric divisor in $\mathcal{M}_\theta$ gives a perfect matching so that the stabilizer group of the divisor is determined by the height change of the perfect matching.

Although the following results are stated in [IU, Proposition 8.2] for cancellative dimer models, the proof works for any non-degenerate dimer model.

Proposition 2.1. *The following hold for a non-degenerate dimer model:*

(i) A perfect matching D is simple if and only if it is multiplicity-free.

(ii) A multiplicity-free perfect matching is a corner perfect matching.

The dimer model G_1 in Figure 2.1 shows that the converse to Proposition 2.1.(ii) does not hold in general. The corresponding quiver is shown in Figure 2.2. The set of perfect matchings and the characteristic polygon are shown in Figures 2.3 and 2.4 respectively, where the perfect matching D_1 is chosen as the reference matching. This example also shows that one cannot obtain a cancellative dimer model with the same characteristic polygon simply by removing all arrows not contained in any simple matchings; if we perform this operation on the dimer model G_1, then the resulting dimer model G_2 shown in Figure 2.5 has a smaller characteristic polygon, which coincides with the convex hull of height changes of simple perfect matchings.

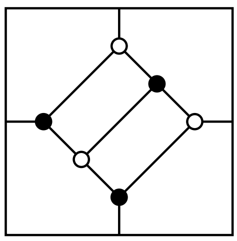


Figure 2.1: The dimer model G_1

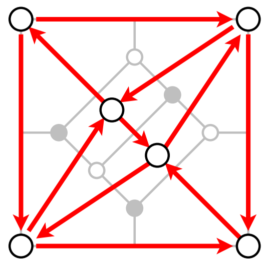
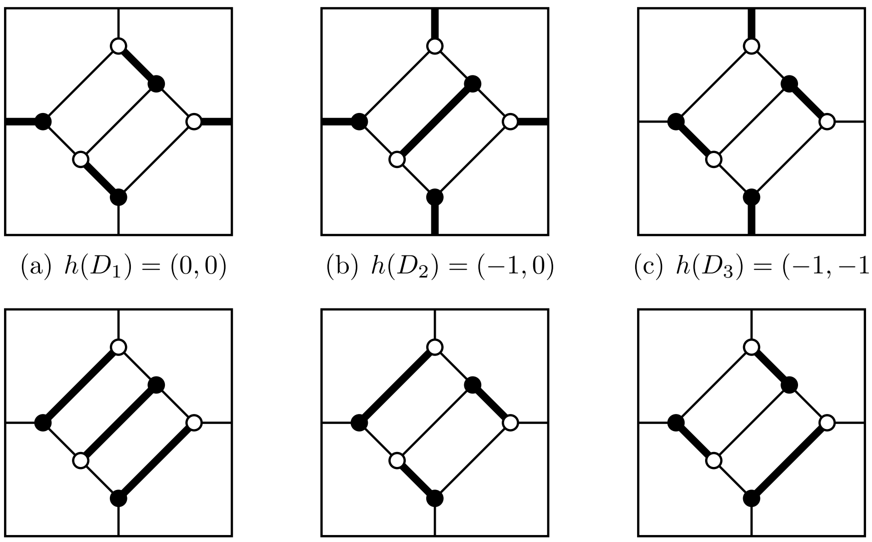


Figure 2.2: The quiver Γ_1



(a) $h(D_1) = (0, 0)$ (b) $h(D_2) = (-1, 0)$ (c) $h(D_3) = (-1, -1)$

(d) $h(D_4) = (0, -1)$ (e) $h(D_5) = (0, -1)$ (f) $h(D_6) = (0, -1)$

Figure 2.3: Perfect matchings on G_1

3 Zigzag polygon and cancellativity

A *zigzag path* is a path on a dimer model which makes a maximum turn to the right on a white node and to the left on a black node. Note that it is not a path on a quiver.

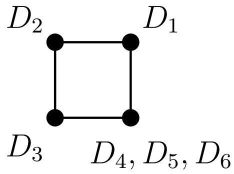


Figure 2.4: The characteristic polygon

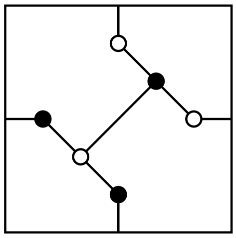


Figure 2.5: The union of simple matchings

We assume that a zigzag path does not have an endpoint, so that we can regard a zigzag path as a sequence $(e_i)_{i\in\mathbb{Z}}$ of edges e_i parameterized by $i \in \mathbb{Z}$, up to translations of i. The homology class $[z]$ of a zigzag path considered as an element of $\mathbb{Z}^2$ will be called its *slope*.

Let k be the number of zigzag paths. Fix a zigzag path z_1, and let $\{z_i\}_{i=1}^k$ be the set of zigzag paths, so that their slopes $([z_i])_{i=1}^k$ are cyclically ordered starting from $[z_1]$. Note that some of the slopes may coincide in general. Define another sequence $(w_i)_{i=1}^r$ in $\mathbb{Z}^2$ by $w_0 = 0$ and

$$w_{i+1} = w_i + [z_{i+1}]', \qquad i = 0, \dots, k-1,$$

where $[z_{i+1}]'$ is obtained from $[z_{i+1}]$ by rotating 90 degrees counter-clockwise. Note that one has $w_r = 0$ since every edge is contained in exactly two zigzag paths with different directions and hence the homology classes of the zigzag paths add up to zero. The convex hull of $(w_i)_{i=1}^r$ is called the *zigzag polygon*.

Now we recall the definition of the consistency condition for dimer models:

Definition 3.1 ([IU, Definition 5.2]). A dimer model is said to be *consistent* if

- there is no homologically trivial zigzag path,
- no zigzag path on the universal cover has a self-intersection, and
- no pair of zigzag paths on the universal cover intersect each other in the same direction more than once.

See [IU11, Boc12] for more on consistency conditions for dimer models. The consistency condition is equivalent to cancellativity:

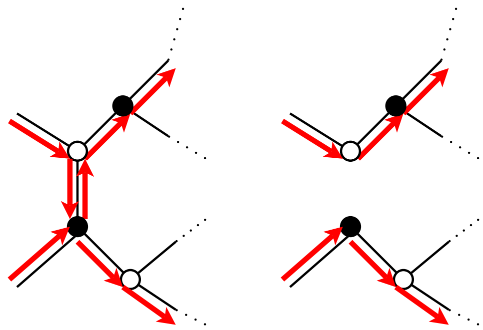
Theorem 3.2 ([IU11, Theorem 1.1], [Boc12, Theorem 6.2]). *A non-degenerate dimer model is consistent if and only if the path algebra of the associated quiver with relations is cancellative.*

The characteristic polygon and the zigzag polygon coincides for cancellative dimer models:

Theorem 3.3 ([Gul08, Theorem 3.3], cf. also [IU, Corollary 8.3]). *For a consistent dimer model, the characteristic polygon Δ coincides with the zigzag polygon up to translation.*

Now we prove Theorem 1.1:

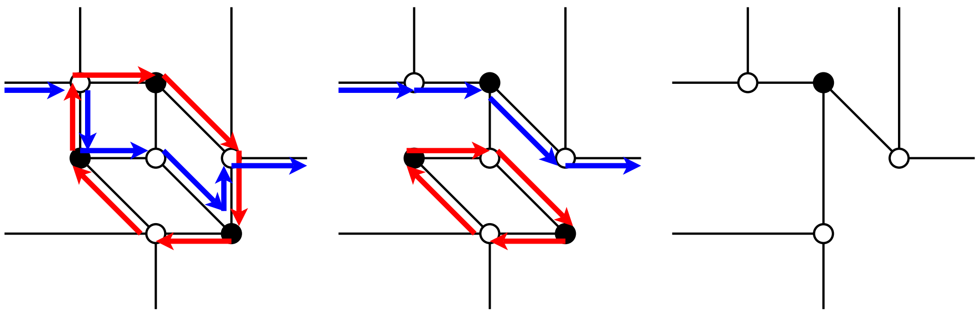
Proof of Theorem 1.1. If some zigzag path on the universal cover has a self-intersection, then by removing all the edges at the self-intersection, one obtains another bicolored graph on T^2 with the same zigzag polygon as the original dimer model. Figure 3.1 shows an example of this operation. If there is a connected component of the resulting graph which is contained in a simply-connected domain in T^2, then one can remove this connected component without changing the zigzag polygon. By removing all such components, one obtains a dimer model which has no zigzag path on the universal cover with a self-intersection.



(a) A self-intersection (b) Removing edges

Figure 3.1: A self-intersecting zigzag path on the universal cover

If there is a homologically trivial zigzag path z, then there are two cases; either there is at least one edge inside the zigzag path z, or there is no such edge. If there is an edge inside the zigzag path, take any zigzag path w which intersects z. Then z and w intersect in the same direction more than once, and one can remove edges at the intersections to obtain another dimer model. If there are no edge inside the zigzag path z, then every other node in z is divalent, and one can remove all these divalent nodes and contract all other nodes to a single node. Figure 3.2 shows an example of these operations.



(a) A homologically-trivial zigzag path (b) Removing edges (c) Removing edges and contracting nodes

Figure 3.2: Homologically trivial zigzag paths

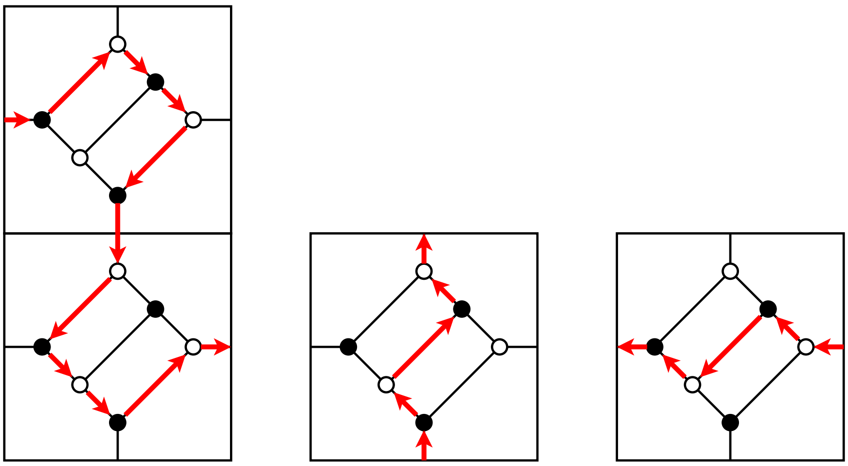
If there is a pair of zigzag paths on the universal cover which intersect each other more than once in the same direction, choose any such pair of zigzag paths and remove the edges at a pair of consecutive intersections of this pair of zigzag paths. The resulting graph on the torus T has the same set of slopes of zigzag paths, and the non-degeneracy of the zigzag polygon implies that this graph is still a dimer model (i.e., there are no univalent node and all the faces are simply-connected).

One can iterate these operations finitely many times until the dimer model becomes cancellative. □

For example, the dimer model G_1 in Figure 2.1 has three zigzag paths as shown in Figure 3.3. The corresponding zigzag polygon is shown in Figure 3.4. A pair of lifts of the zigzag path shown in Figure 3.3.3(a) intersects in the same direction twice on the universal cover as shown in Figure 3.5. Under the operation of 'cancellativization' in Theorem 1.1, the pair of edges at these intersections will be removed, and one obtains the dimer model shown in Figure 2.5.

Corollary 1.2 is an immediate consequence of Theorem 1.1:

Proof of Corollary 1.2. The operation of cancellativization in the proof of Theorem 1.1 does not change the zigzag polygon, but makes the characteristic polygon smaller in general. Since characteristic polygon and the zigzag polygon coincide for a cancellative dimer model, the zigzag polygon is smaller than the characteristic polygon in general. □



(a) A zigzag path with homology class $(1, -1)$

(b) A zigzag path with homology class $(0, 1)$

(c) A zigzag path with homology class $(-1, 0)$

Figure 3.3: Zigzag paths on G_1

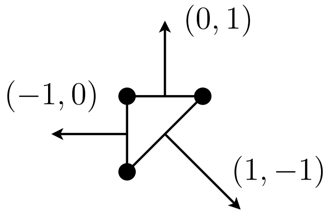


Figure 3.4: The zigzag polygon of G_1

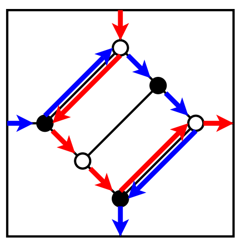


Figure 3.5: Intersections of zigzag paths on G_1

Remark 3.4. The dimer model G_1 gives an example where one can not obtain a cancellative dimer model by the following simple operation:

- Take any generic stability parameter θ and contract all arrows which does not vanish in any θ-stable representations.

Proof. Note that the height change $(0, -1)$ has multiplicity three. Take a generic stability parameter which makes the perfect matching D_4 stable. Three other corner perfect matchings D_1, D_2 and D_3 are simple, so that they are stable for any stability parameter. Now one can see that every arrow of Q goes to zero in at least one θ-stable representation of dimension vector $(1, \dots, 1)$. □

4 Characteristic polygon and cancellativity

We can always assume that a dimer model is non-degenerate without changing the characteristic polygon:

Proposition 4.1. *Let G be a dimer model with a non-degenerate characteristic polygon. Then one can remove some nodes and edges from G to obtain a non-degenerate dimer model G' with the same characteristic polygon as G.*

Proof. Let G'' be the bicolored graph on T whose set E'' of edges consists of edges of G contained in at least one perfect matching of G, and whose set of nodes consists of nodes of G incident to at least one edge in E''. Then G'' is clearly a non-degenerate graph. In order to make G'' into a dimer model, one removes all connected components of G'' having a simply-connected neighborhood in T. The resulting graph G' is a dimer model (i.e. no node is univalent and every connected component of $T \setminus G'$ is simply-connected) having the same characteristic polygon as G. □

Let G be a non-degenerate dimer model, and consider a pair (D_1, D_2) of perfect matchings. Recall from Section 2 that the homology class $[D_1 \triangle D_2]$ is Poincaré dual to the height change $h(D_1, D_2)$.

Lemma 4.2. *Let D_1 and D_2 be perfect matchings with $v := [D_1 \triangle D_2] \neq 0 \in H_1(T, \mathbb{Z})$. If the homology class of a connected component of $D_1 \triangle D_2$ is non-zero, it is one of the two primitive elements in $\mathbb{Q}v \cap H_1(T, \mathbb{Z})$. Moreover, if either D_1 or D_2 is a corner perfect matching, then it is the primitive element in $\mathbb{Q}_+ v \cap H_1(T, \mathbb{Z})$.*

Proof. Note that two cycles on a torus can be disjoint only if their homology classes are proportional to each other. Since $D_1 \triangle D_2$ is homeomorphic to the disjoint union of copies of S^1, we obtain the first assertion. Assume there is a connected component w of $D_1 \triangle D_2$ whose homology class is in $\mathbb{Q}_- v$. Then we can construct another perfect matching D_3 with $D_1 \triangle D_3 = w$. The height change $h(D_1)$ of D_1 lies on the line segment connecting $h(D_2)$ and $h(D_3)$, so that D_1 is not a corner perfect matching. By the same reasoning, D_2 is not a corner perfect matching either. □

Fix a pair (D_1, D_2) of corner perfect matchings whose height changes are adjacent in the counter-clockwise order.

Lemma 4.3. *For any perfect matching D_3 whose height change is not on the line segment connecting D_1 and D_2, one has*

$$\langle [D_1 \triangle D_2], [D_2 \triangle D_3] \rangle > 0,$$

where $\langle -, - \rangle$ denotes the intersection pairing on $H_1(T, \mathbb{Z})$.

Proof. This follows from the fact that $[D_1 \triangle D_2]$ is the Poincare dual of the relative height change $h(D_1, D_2)$ and the definition of the characteristic polygon. □

Example 4.4. Consider the dimer model G_1 given in Section 2. The cycles $[D_1 \triangle D_2]$ and $[D_2 \triangle D_3]$ are shown in Figures 4.1 and 4.2 respectively, which indeed satisfies

$$\langle [D_1 \triangle D_2], [D_2 \triangle D_3] \rangle > 0.$$

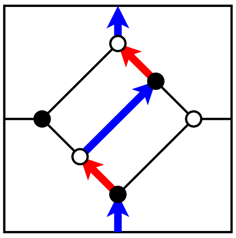


Figure 4.1: The cycle $[D_1 \triangle D_2]$

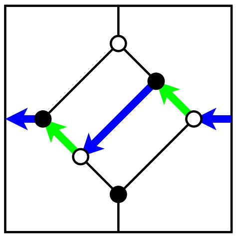


Figure 4.2: The cycle $[D_2 \triangle D_3]$

Lemma 4.3 can be rephrased as follows:

Corollary 4.5. *If one goes along $D_1 \triangle D_2$ and count the number of edges in D_3 connected to $D_1 \triangle D_2$ from the left, then the number of edges of D_3 connected to white nodes is larger than the number of those connected to black nodes. The opposite inequality holds if we count the number of edges of D_3 connected to $D_1 \triangle D_2$ from the right.*

If some connected component c of $D_1 \triangle D_2$ is homologically trivial, then we can replace D_1 by another perfect matching D_1' satisfying $D_1 \triangle D_1' = c$. Then one has $h(D_1) = h(D_1') + [c] = h(D_1')$. By continuing this process, we may assume that $D_1 \triangle D_2$ does not have any homologically trivial components.

Let n be the number of homologically non-trivial connected components of $D_1 \triangle D_2$. We label these connected components as $\{z_i\}_{i \in \mathbb{Z}/n\mathbb{Z}}$ in such a way that z_i is right next to z_{i-1} on the right as shown in Figure 4.3.

Lemma 4.6. *There is a connected component z_i with the following property:*

- *There is no path p consisting of edges of G satisfying the following conditions:*
  - *(i) The path p is homeomorphic to the interval* $[0, 1]$.
  - *(ii) Every other edge of p belongs to $D_1 \cap D_2$.*
  - *(iii) The path p connects a white node on z_{i-1} to a black node on z_i.*

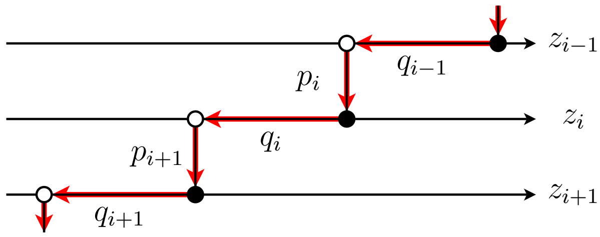


Figure 4.3: The paths z_i, p_i and q_i

(iv) The edge containing the white node $p \cap z_{i-1}$ is on the right of z_{i-1} and the edge containing the black node $p \cap z_i$ is on the left of z_i.

Proof. Note that the condition (ii) implies that p can not cross z_j for any $j \in \mathbb{Z}/n\mathbb{Z}$. Assume for contradiction that the assertion of Lemma 4.6 is false. Then for each $i \in \mathbb{Z}/n\mathbb{Z}$, there is a path p_i satisfying the conditions above as shown in Figure 4.3. Let q_i be the part of z_i which starts at the black node $z_i \cap p_i$ and goes backward (with respect to the orientation of z_i) to the white node $z_i \cap p_{i+1}$. We can consider the path $y = \bigcup_i (p_i \cup q_i)$ which starts at $p_1 \cap z_0$, goes along p_1 to $p_1 \cap z_1$, then goes along q_1 to $p_2 \cap z_1$, then goes along p_2 to $p_2 \cap z_2$, and so on. Then every other edge of y belongs to D_2, so that we can construct a perfect matching D_3 with $D_2 \triangle D_3 = y$. Then every edge of D_3 connected to $D_1 \triangle D_2$ from the left is connected to a black node. This contradicts Corollary 4.5, and Lemma 4.6 is proved. □

We say that a path z is *zigzag at white nodes* if there is no edge of G connected to a white node on z from the right.

Lemma 4.7. *Let z_i be a connected component of $D_1 \triangle D_2$ with the property in Lemma 4.6. Then there are perfect matchings $\overline{D}_1$ and $\overline{D}_2$ with the same height changes as D_1 and D_2 respectively such that*

$$\overline{D}_1 \triangle \overline{D}_2 = \overline{z}_i \cup \bigcup_{j \neq i} z_j,$$

where $\overline{z}_i$ is zigzag at white nodes.

Proof. We may assume $i = 0$ without loss of generality. Assume that an edge e is connected to a white node w on z_0 from the right, and take a perfect matching D containing e. Note that D_1 and D_2 coincide on the strip between z_0 and z_1. The connected component q of $D \triangle (D_1 \cap D_2)$ containing e forms an arc starting from the white node w and ends at either z_0 or z_1. The node at the intersection of q with z_0 or z_1 other than w must be a black, and we will call it b. Then the property in Lemma 4.6 implies that b must be on z_0. For one of the two connected components of $z_0 \setminus \{b, w\}$, which we will call q', the union $q \cup q'$ forms a homologically trivial cycle.

Let us first consider the case when q' goes from w to b along the direction of z_0 as shown in Figure 4.4. Then we can take a perfect matching D_2' with $D_2 \triangle D_2' = q \cup q'$. The resulting perfect matching D_2' has the same height change as D_2, and the connected components of $D_1 \triangle D_2'$ are z_i $(i \neq 0)$ and $z_0' := (z_0 \setminus q') \cup q$.

We claim that z_0' also has the property in Lemma 4.6. Assume for contradiction that there is a path p satisfying the conditions in Lemma 4.6 for z_0' as shown in Figure 4.5.

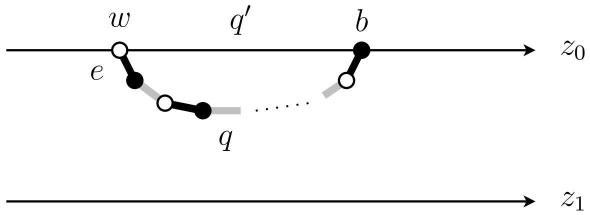


Figure 4.4: The edge e and the paths q and q'

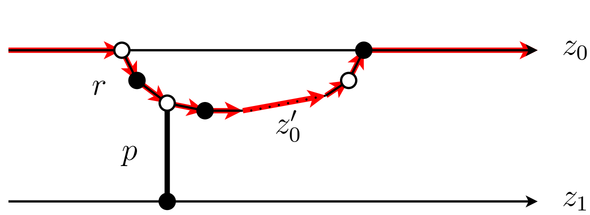


Figure 4.5: The paths p, r, and z_0'

Let r be the part of q starting from the white node w and goes along z_0' until it meets the white node at $p \cap z_0'$. Let further p' be the path obtained by concatenating r and p. Then p' satisfies the conditions in Lemma 4.6 for z_0, which is a contradiction. Hence z_0' has the property in Lemma 4.6.

In the case where q' goes from b to w, we can replace D_1 by D_1' with $D_1 \triangle D_1' = q \cup q'$. By the same argument as above, one can show that $D_1' \triangle D_2 = z_0' \cup \bigcup_{i \neq 0} z_i$ and z_0' has the property in Lemma 4.6.

Note that z_0' is 'closer' to z_1 than z_0. If z_0' is not zigzag at a white node, then we can repeat the same operation. Since there are only finitely many edges between z_0 and z_1, this process terminates in finitely many steps, and one obtains desired perfect matchings $\overline{D}_1$ and $\overline{D}_2$. □

So far, we have shown the existence of a connected component z_i in $D_1 \triangle D_2$ which is zigzag at white nodes. In Lemma 4.8 below, we show that if z_i is not zigzag at a black node by some edge e of G, then we can remove the edge e without changing the characteristic polygon.

Lemma 4.8. *Assume that z_i is zigzag at white nodes. If e is an edge connected to a black node b on z_i from the left of z_i, then one can remove e without changing the characteristic polygon.*

Proof. It suffices to show that for any perfect matching D containing e, there is another perfect matching D' with the same height change as D not containing e. One may assume that the height change $h(D)$ of D is not on the line segment between $h(D_1)$ and $h(D_2)$.

Let y be the connected component of $D_2 \triangle D$ containing e. If y is homologically trivial, then take the perfect matching D' such that $D' \triangle D = y$. The matching D' has the same height change as D and does not contain e. Hence we may assume that y is homologically non-trivial.

Choose a lift $\widetilde{b}$ of the node b to the universal cover $\mathbb{R}^2 \to T$ and let $\widetilde{z}_i$ and $\widetilde{y}$ be the lifts of z_i and y containing $\widetilde{b}$ respectively. Lemmas 4.2 and 4.3 imply $\langle z_i, y \rangle > 0$, so that $\widetilde{y}$ first comes from the right of $\widetilde{z}_i$, intersects $\widetilde{z}_i$ several times, and goes away to the left of $\widetilde{z}_i$. Hence there must be a white node $\widetilde{w} \in \widetilde{z}_i \cap \widetilde{y}$ such that the direction of $\widetilde{y}$ is from $\widetilde{w}$ to $\widetilde{b}$. We assume $\widetilde{w}$ is the nearest to $\widetilde{b}$ in the part of $\widetilde{y}$ before $\widetilde{b}$. Let b and w be the images on the torus T of $\widetilde{b}$ and $\widetilde{w}$ respectively.

First we discuss the case when $\widetilde{z}_i$ goes from $\widetilde{w}$ to $\widetilde{b}$. Figure 4.6 shows the paths y and z_i on the torus T. When we travel from b along $y \subset D_2 \triangle D$, the next edge e_1 is on $z_i \subset D_1 \triangle D_2$, and the direction of y is opposite to that of z_i on that edge. Then the next

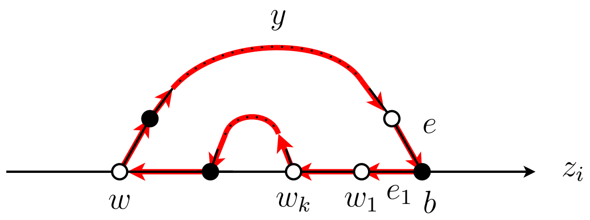


Figure 4.6: The case when $\widetilde{z}_i$ goes from $\widetilde{w}$ to $\widetilde{b}$

node w_1 is a white node on $y \cap z_i$. Since z_i is zigzag at white nodes, the path y cannot escape to the right of z_i, and the next edge e_2 in y either goes to the left of z_i or on the path z_i.

If y goes to the left of z_i, then y must eventually intersect z_i again since y is an embedded circle in T. If e_2 is on the path z_i, then $e_2 \in D_1 \cap D$ and the next edge e_3 on the path y is in D_2. By continuing in this way, one sees that y must be contained in the simply connected open subset U of T bounded by the parts of y and z_i between b and w. This implies that y is homologically trivial, which contradicts our assumption.

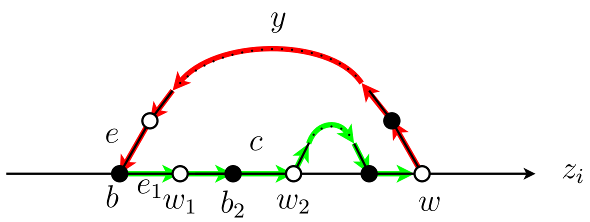


Figure 4.7: The case when $\widetilde{z}_i$ goes from $\widetilde{w}$ to $\widetilde{b}$

Hence the path $\widetilde{z}_i$ goes from $\widetilde{b}$ to $\widetilde{w}$ as shown in Figure 4.7. Let e_1 be the edge in D_1 incident to the node b. By the definition of z_i, the other node w_1 of e_1 is on the path z_i. Take the edge e_2 in D incident to w_1. Since z_i is zigzag on white nodes, e_2 is either on z_i or goes to the left of z_i. If e_2 is on z_i, then let e_3 be the edge in D_1 incident to the other node b_2 of e_2. If e_2 goes to the left of z_i, then continue e_2 along the connected component y_1 of $D_2 \triangle D$ containing e_2. Then y_1 must eventually intersect z_i at a black node, which we will call b_2. Let e_3 be the edge of D_1 incident to b_2.

By continuing in this way, one can find a path c from b to w which consists of parts of z_i or $D_2 \triangle D$. By concatenating y with c, one obtains a homologically trivial path on G such that every other edge belongs to D. Then the perfect matching D' such that $D \triangle D' = y \cup c$ has the same height change as D and does not contain e. This concludes the proof of Lemma 4.8. $\square$

Lemma 4.9. *Let $\mathfrak{c}_1$ and $\mathfrak{c}_2$ be adjacent corners of the characteristic polygon of G. We can remove some edges from G to obtain a dimer model G' such that*

- *the characteristic polygon of G' coincides with that of G, and*
- *there are perfect matchings D_1 and D_2 of G' such that $h(D_1) = \mathfrak{c}_1$, $h(D_2) = \mathfrak{c}_2$ and $D_1 \triangle D_2$ consists of zigzag paths.*

Proof. First choose arbitrary perfect matchings with height changes $\mathfrak{c}_1$ and $\mathfrak{c}_2$ respectively. Take a path z_i satisfying the property in Lemma 4.6. We may assume $i = 0$ without loss of generality. By Lemma 4.7, we can assume that z_0 is zigzag at white nodes by replacing D_1 and D_2 if necessary. If z_0 is not zigzag at some black node, then one can use Lemma 4.8 to remove the edge e which makes z_0 not zigzag at that node. After iterating this operation finitely many times, we can turn z_0 into a zigzag path.

Now the property in Lemma 4.6 holds for z_{-1}, since a path p satisfying the conditions should be connected to a black node on z_0 from the left of z_0, which is impossible since z_0 is a zigzag path. Then we can repeat the same process to turn z_{-1} into a zigzag path.

By successively performing this operation, we can turn all z_i into zigzag paths. □

Now we can prove Theorem 1.3:

Proof of Theorem 1.3. We can use Lemma 4.9 repeatedly to obtain another dimer model G'' such that

- the characteristic polygon of G'' coincides with that of G, and
- for any pair $(\mathfrak{c}_1, \mathfrak{c}_2)$ of adjacent corners of the characteristic polygon, there are perfect matchings D_1 and D_2 of G'' such that $h(D_1) = \mathfrak{c}_1$, $h(D_2) = \mathfrak{c}_2$ and $D_1 \triangle D_2$ consists of zigzag paths.

Zigzag paths constituting $D_1 \triangle D_2$ for pairs (D_1, D_2) of adjacent corner perfect matchings ensure that the zigzag polygon is at least as large as the characteristic polygon. Then Corollary 1.2 shows that the zigzag polygon and the characteristic polygon of G'' coincide. Now we can apply Theorem 1.1 to G'' to obtain a cancellative dimer model G', whose zigzag polygon is the same as that of G''. Since G' is cancellative, the characteristic polygon of G' coincides with its zigzag polygon, which is the same as the characteristic polygon of G. □

Corollary 1.4 is an immediate consequence of the proof of Theorem 1.3:

Proof of Corollary 1.4. The proof of Lemma 4.7 shows that if the connected component z_i of $D_1 \triangle D_2$ with the property in Lemma 4.6 is not zigzag at a white node, then at least one of D_1 or D_2 has multiplicity. In other words, z_i is zigzag at white nodes, if both D_1 and D_2 are multiplicity-free.

On the other hand, the proof of Lemma 4.8 shows that if a component z of $D_1 \triangle D_2$ is zigzag at white nodes but not zigzag at a black node by an edge e, then any perfect matching containing e has a multiplicity. This cannot be the case if the dimer model is strongly non-degenerate and all the corner perfect matchings are multiplicity-free.

It follows by the argument in Lemma 4.9 that for a strongly non-degenerate dimer model, the symmetric difference $D_1 \triangle D_2$ of a pair (D_1, D_2) of perfect matchings, whose height changes are adjacent corners of the characteristic polygon, consists of zigzag paths. This implies that the zigzag polygon is at least as large as (and hence coincides with) the characteristic polygon, and Corollary 1.4 is proved. □

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