EXTENDING STRUCTURES FOR LIE ALGEBRAS

A. L. AGORE AND G. MILITARU

ABSTRACT. Let $\mathfrak g$ be a Lie algebra, E a vector space containing $\mathfrak g$ as a subspace. The paper is devoted to the extending structures problem which asks for the classification of all Lie algebra structures on E such that \frak{g} is a Lie subalgebra of E. A general product, called the unified product, is introduced as a tool for our approach. Let V be a complement of g in E: the unified product g $\sharp V$ is associated to a system $(\vartriangleleft, \vartriangleright, f, \{-, -\})$ consisting of two actions \triangleleft and \triangleright , a generalized cocycle f and a twisted Jacobi bracket $\{-, -\}$ on V. There exists a Lie algebra structure $[-, -]$ on E containing g as a Lie subalgebra if and only if there exists an isomorphism of Lie algebras $(E, [-,-]) \cong \mathfrak{g} \mathfrak{h} V$. All such Lie algebra structures on E are classified by two cohomological type objects which are explicitly constructed. The first one $\mathcal{H}^2_{\mathfrak{g}}(V, \mathfrak{g})$ will classify all Lie algebra structures on E up to an isomorphism that stabilizes g while the second object $\mathcal{H}^2(V,\mathfrak{g})$ provides the classification from the view point of the extension problem. Several examples that compute both classifying objects $\mathcal{H}^2_{\mathfrak{g}}(V,\mathfrak{g})$ and $\mathcal{H}^2(V,\mathfrak{g})$ are worked out in detail in the case of flag extending structures.

INTRODUCTION

Lie algebras are studied in different fields such as differential geometry, classical/quantum mechanics or the theory of particle physics. In differential geometry, Lie algebras arise naturally on the tangent space of symmetry (Lie) groups on manifolds. In Hamiltonian mechanics the phase space is an example of a Lie algebra while in quantum mechanics Heisenberg postulated the existence of an infinite-dimensional Lie algebra of operators: the theory of quantum mechanics follows more or less from properties of Lie algebras. In the theory of particle physics Lie algebras play a key role. For instance, bosonic string theory uses a Lie algebra to formulate operators and the state space. Beyond the remarkable applications in the above mentioned fields, Lie algebras are objects of study in their own right. In this context a natural question arises (throughout this paper, by 'an isomorphism of Lie algebras $\varphi : E \to E$ that stabilizes g' we mean an isomorphism of Lie algebras that acts as the identity on the subspace g):

Extending structures problem. Let g be a Lie algebra and E a vector space containing g as a subspace. Describe and classify up to an isomorphism of Lie algebras that stabilizes g the set of all Lie algebra structures $[-,-]$ that can be defined on E such that $\mathfrak g$ is a Lie subalgebra of $(E, [-,-])$.

²⁰¹⁰ Mathematics Subject Classification. 17B05, 17B55, 17B56.

Key words and phrases. The extension and the factorization problem, unified products, relative (nonabelian) cohomology for Lie algebras.

A.L. Agore is research fellow "Aspirant" of FWO-Vlaanderen. This work was supported by a grant of the Romanian National Authority for Scientific Research, CNCS-UEFISCDI, grant no. 88/05.10.2011.

We formulated and studied the same problem at the level of groups in [\[1\]](#page-22-0) and in a more general setting for Hopf algebras in [\[2\]](#page-22-1). Even if the statement of the problem is elementary, the problem turns out to be a difficult one. For instance, if $\mathfrak{g} = \{0\}$ then the ES problem asks for the classification of all Lie algebra structures on a given vector space E , which is of course a wild problem. For this reason, from now on we will assume that $\mathfrak{g} \neq \{0\}$. Although the ES problem is very difficult, we can provide a detailed answer to it in the case of what we call flag extending structures of $\mathfrak g$ to E in the sense of Definition [4.1.](#page-15-0) To start with, we will explain what we mean by an answer to the classification part of the ES problem. Having in mind that we want to extend the Lie algebra structure on $\mathfrak g$ to a bigger vector space, by classification we will always mean classification up to an isomorphism of Lie algebras $\varphi : E \to E$ that stabilizes g, i.e. $\varphi(q) = q$, for all $q \in \mathfrak{g}$. Therefore, the problem comes down to actually constructing the classifying object from this first point of view: it will be a relative cohomological 'group'. On the other hand, as we shall explain below, the ES problem generalizes the extension problem. Thus, we can also ask for the classification from this point of view, i.e. up to an isomorphism of Lie algebras that simultaneously stabilizes $\mathfrak g$ and co-stabilizes V, its complement in E . This will be the second classifying object which will generalize the classical second cohomology group $H^2(V, \mathfrak{g})$.

The ES problem generalizes and unifies two famous problems in the theory of Lie algebras: the extension problem which goes back to Chevalley and Eilenberg [\[9\]](#page-22-2) and the factorization problem who's roots descend to the classical results of Levi and Malcev [\[8,](#page-22-3) Theorem 5. We will explain this briefly. Let $\mathfrak g$ and $\mathfrak h$ be two given Lie algebras. The extension problem asks for the classification of all Lie algebras \mathfrak{E} which contain \mathfrak{g} as an ideal such that $\mathfrak{E}/\mathfrak{g} \cong \mathfrak{h}$. Equivalently, the extension problem asks for the classification of all Lie algebras $\mathfrak E$ that fit into an exact sequence of Lie algebras

$$
0 \longrightarrow \mathfrak{g} \xrightarrow{i} \mathfrak{E} \xrightarrow{\pi} \mathfrak{h} \longrightarrow 0 \tag{1}
$$

Now, if in the ES problem we replace the condition " $\mathfrak g$ is a Lie subalgebra of $(E, [-,-])$ " by a more restrictive one, namely " $\mathfrak g$ is an ideal of E ", then what we obtain is in fact a reformulation of the extension problem: any Lie algebra structure on E containing $\mathfrak g$ as an ideal is of course an extension of g through the Lie algebra $\mathfrak{h} := E/\mathfrak{g}$. In this case, let $\pi: E \to \mathfrak{h}$ be the canonical projection and $s: \mathfrak{h} \to E$ a linear section of π , i.e. $\pi \circ s = \text{Id}_{\mathfrak{h}}$. We define the action ⊳ and the cocycle f by the usual formulas:

$$
\triangleright : \mathfrak{h} \times \mathfrak{g} \to \mathfrak{g}, \qquad x \triangleright g \ := \ [s(x), g] \tag{2}
$$

$$
f: \mathfrak{h} \times \mathfrak{h} \to \mathfrak{g}, \qquad f(x, y) := [s(x), s(y)] - s([x, y]) \tag{3}
$$

for all x, $y \in \mathfrak{h}$ and $g \in \mathfrak{g}$. In geometrical language the action ⊳ is called *connection*, while the cocycle f is called *curvature*: for more details about the importance of the extension problem in differential geometry we refer to [\[4,](#page-22-4) Section 4] and [\[16\]](#page-22-5). Then the system $(\mathfrak{g}, \mathfrak{h}, \triangleright, f)$ is a crossed system of Lie algebras and the map

$$
\psi: \mathfrak{g} \#_q^f \mathfrak{h} \to E, \qquad \psi(g, x) := g + s(x)
$$

is an isomorphism of Lie algebras (see Corollary [3.1](#page-13-0) for details). In this classical reconstruction of a Lie algebra E from an ideal and the corresponding quotient, the fact that

 $\mathfrak g$ is an ideal of E plays a crucial role from the very beginning: namely it is the main ingredient in proving that the action ⊳ and the cocycle f take values in $\mathfrak{g} = \text{Ker}(\pi)$. If we drop the assumption that $\mathfrak g$ is an ideal of E and we only ask for $\mathfrak g$ to be a Lie subalgebra of E, as we formulated the ES problem, then the above construction can not be performed anymore and we have to come up with a new method of reconstructing the Lie algebra E from a given Lie subalgebra and another set of data. This is what we do in Section [2.](#page-5-0) For further reference on the extension problem for Lie algebras, in the abelian or non-abelian case, we refer to $[3]$, $[4]$, $[9]$, $[11]$, $[13]$, $[24]$.

The factorization problem is the dual of the extension problem. It consists of describing and classifying up to an isomorphism all Lie algebras E that factorize through two given Lie algebras g and h: i.e. E contains g and h as Lie subalgabras such that $E = \mathfrak{g} + \mathfrak{h}$ and $\mathfrak{g} \cap \mathfrak{h} = \{0\}$. The factorization problem is also a special case of the ES problem, if we impose the following additional assumption: we consider the complement V of $\mathfrak g$ in E to be also a Lie subalgebra of E isomorphic to $\mathfrak h$. Dual to the extension problem, it was independently proven in [\[19\]](#page-22-10) and [\[17\]](#page-22-11), that a Lie algebra E factorizes through $\mathfrak g$ and h if and only if $E \cong \mathfrak{g} \bowtie \mathfrak{h}$, where $\mathfrak{g} \bowtie \mathfrak{h}$ is the bicrossed product associated to a matched pair of Lie algebras $(\mathfrak{g}, \mathfrak{h}, \triangleleft, \triangleright)$. The details are given in Section [3.](#page-12-0)

The paper is organized as follows: in Section [2](#page-5-0) we will perform the abstract construction of the unified product $\mathfrak{gl}(V)$: it is associated to a Lie algebra g, a vector space V and a system of data $\Omega(\mathfrak{g}, V) = (\triangleleft, \triangleright, f, \{-, - \})$ called an extending datum of \mathfrak{g} through V. Theorem [2.2](#page-5-1) establishes the set of axioms that has to be satisfied by $\Omega(\mathfrak{g}, V)$ such that $\mathfrak{gl} \nmid V$ with a given canonical bracket becomes a Lie algebra, i.e. is a unified product. In this case, $\Omega(\mathfrak{g}, V) = (\triangleleft, \triangleright, f, \{-, -\})$ will be called a Lie extending structure of \mathfrak{g} through V. Now let $\mathfrak g$ be a Lie algebra, E a vector space containing $\mathfrak g$ as a subspace and V a given complement of $\mathfrak g$ in E. Theorem [2.4](#page-8-0) provides the answer to the description part of the ES problem: there exists a Lie algebra structure $[-,-]$ on E such that $\mathfrak g$ is a subalgebra of $(E, [-,-])$ if and only if there exists an isomorphism of Lie algebras $(E, [-,-]) \cong \mathfrak{g} \sharp V$, for some Lie extending structure $\Omega(\mathfrak{g}, V) = (\triangleleft, \triangleright, f, \{-, - \})$ of \mathfrak{g} through V. The answer to the classification part of the ES problem is given in Theorem [2.7:](#page-11-0) we will construct explicitly a relative cohomology group, denoted by $\mathcal{H}^2_{\mathfrak{g}}(V, \mathfrak{g})$, which will be the classifying object of all extending structures of the Lie algebra $\mathfrak g$ to E - the classification is given up to an isomorphism of Lie algebras which stabilizes g. Moreover, we also indicate the bijection between the elements of $\mathcal{H}^2_{\mathfrak{g}}(V, \mathfrak{g})$ and the isomorphism classes of all extending structures of $\mathfrak g$. The construction of the second classifying object, denoted by $\mathcal H^2(V,\mathfrak g)$, is performed in Remark [2.8:](#page-11-1) it parameterizes all extending structures of g to a Lie algebra on E up to an isomorphism which simultaneously stabilizes $\mathfrak g$ and co-stabilizes V - i.e. this classification is given from the point of view of the extension problem. There exists a canonical projection $\mathcal{H}^2(V, \mathfrak{g}) \to \mathcal{H}^2_{\mathfrak{g}}(V, \mathfrak{g})$ between these two classifying objects. We point out that $\mathcal{H}^2(V, \mathfrak{g})$ generalizes the classical cohomology group $H^2(V, \mathfrak{g})$: the latter is obtained as a special case of $\mathcal{H}^2(V, \mathfrak{g})$ if we let the right action \triangleleft to be the trivial one and the extending structures of g to be 'abelian' that is, if we ask that the Lie algebra $\mathfrak g$ is contained in the center of the unified products $\mathfrak g \mathfrak h V$. One of the special cases that we introduce in Example [2.3](#page-7-0) is called *twisted product*, the terminology being borrowed from Hopf algebra theory. The two Lie algebras $\mathfrak g$ and V involved in the construction of the twisted product are connected by a classical 2-cocycle $f: V \times V \rightarrow \mathfrak{g}$ and plays a key role in the classification of all 6-dimensional nilpotent Lie algebras [\[12\]](#page-22-12). Apart from the twisted product, we show in Section [3](#page-12-0) that both the classical crossed product and bicrossed product of Lie algebras appear as special cases of the unified product.

Theorem [2.7](#page-11-0) offers the theoretical answer to the extending structures problem. The challenge we are left to deal with is a purely computational one: for a given Lie algebra $\mathfrak g$ that is a subspace in a vector space E with a given complement V we have to compute explicitly the classifying object $\mathcal{H}^2_{\mathfrak{g}}(V, \mathfrak{g})$ and then to list the set of types of all Lie algebra structures on E which extend the Lie algebra structure on \mathfrak{g} . This is highly nontrivial considering that the construction of $\mathcal{H}^2_{\mathfrak{g}}(V, \mathfrak{g})$ is very laborious. In Section [4](#page-15-1) we shall identify a way of computing $\mathcal{H}^2_{\mathfrak{g}}(V, \mathfrak{g})$ for the case when the complement V is finite dimensional: namely for those that are flag extending structures of $\mathfrak g$ to E in the sense of Definition [4.1.](#page-15-0) All flag extending structures of $\mathfrak g$ to E can be completely described by a recursive reasoning where the key step is the case when $\mathfrak g$ has codimension 1 as a subspace of E. This case is completely solved in Theorem [4.7](#page-17-0) where $\mathcal{H}^2_{\mathfrak{g}}(V,\mathfrak{g})$ and $\mathcal{H}^2(V,\mathfrak{g})$ are completely described: both objects are quotient pointed sets of the set TwDer(g) of all twisted derivations of g introduced in Definition [4.2.](#page-15-2) The set TwDer(g) contains the usual space of derivations $Der(g)$ via the canonical embedding which is an isomorphism in the case when $\mathfrak g$ is a perfect Lie algebra. Finally, two explicit examples are given in Example [4.11](#page-19-0) and Example [4.12:](#page-19-1) in the first case all extending structures of a 5-dimensional perfect Lie algebra to a space of dimension 6 are classified while in the second one we list all types of extending structures of the non-perfect Lie algebra $gl(2, k)$ to a space of dimension 5.

1. Preliminaries

Throughout this paper k will be a field. All vector spaces, Lie algebras, linear or bilinear maps are over k. A map $f: V \to W$ between two vector spaces is called the *trivial map* if $f(v) = 0$, for all $v \in V$. Let $\mathfrak{g} \leq E$ be a subspace in a vector space E; a subspace V of E such that $E = \mathfrak{g} + V$ and $V \cap \mathfrak{g} = 0$ is called a complement of \mathfrak{g} in E. Such a complement is unique up to an isomorphism and its dimension is called the codimension of $\mathfrak g$ in E. We recall briefly the basic concepts related to Lie algebras; for all unexplained notations or definitions we refer the reader to $[8]$, $[10]$ or $[14]$. A Lie algebra is a vector space $\mathfrak g$, together with a bilinear map $[-, -] : \mathfrak g \times \mathfrak g \to \mathfrak g$ called bracket satisfying the following two properties:

$$
[g, g] = 0,
$$
 $[g, [h, l]] + [h, [l, g]] + [l, [g, h]] = 0$

for all g, $h, l \in \mathfrak{g}$. The second condition is called the Jacobi identity. Let \mathfrak{g} be a Lie algebra and $g' := [g, g]$ be the derived algebra of g ; g is called perfect if $g' = g$ and abelian if $g' = 0$. Representations of a Lie algebra g will be viewed as modules over g; moreover, we shall work with both concepts of right and left $\mathfrak{g}\text{-modules.}$ Explicitly, a right g-module is a vector space V together with a bilinear map $\triangleleft : V \times \mathfrak{g} \to V$, called a right action of $\mathfrak g$ on V , satisfying the following compatibility

$$
x \triangleleft [g, h] = (x \triangleleft g) \triangleleft h - (x \triangleleft h) \triangleleft g \tag{4}
$$

for all $x \in V$ and g, $h \in \mathfrak{g}$. A left $\mathfrak{g}\text{-}module$ is a vector space V together with a bilinear map $\triangleright : \mathfrak{g} \times V \to V$, called a left action of \mathfrak{g} on V such that:

$$
[g, h] \triangleright x = g \triangleright (h \triangleright x) - h \triangleright (g \triangleright x) \tag{5}
$$

for all g, $h \in \mathfrak{g}$ and $x \in V$. Any right g-module is a left g-module via $q \triangleright x := -x \triangleleft q$ and viceversa, that is the category of right g-modules is isomorphic to the category of left g-modules and both of them are isomorphic to the category of representations of g. Der(g) denotes the Lie algebra of all derivations of \mathfrak{g} , that is all linear maps $D : \mathfrak{g} \to \mathfrak{g}$ such that

$$
D([g, h]) = [D(g), h] + [g, D(h)]
$$

for all g, $h \in \mathfrak{g}$. Der(\mathfrak{g}) is a Lie algebra with the bracket $[D_1, D_2] := D_1 \circ D_2 - D_2 \circ D_1$ and the map

$$
ad: \mathfrak{g} \to \mathrm{Der}(\mathfrak{g}), \quad ad(g) := [g, -] : \mathfrak{g} \to \mathfrak{g}, \quad h \mapsto [g, h]
$$

is called the adjoint representation of $\mathfrak g$. Then, Ker(ad) = $Z(\mathfrak g)$, the center of $\mathfrak g$, and Im(ad) is called the space of inner derivation of g and will be denoted by $\text{Inn}(g)$. Inn(g) is a Lie ideal in $Der(\mathfrak{g})$ and

$$
Out(\mathfrak{g}) := Der(\mathfrak{g})/Inn(\mathfrak{g})
$$

is the Lie algebra of outer derivations of g. If g is semisimple, then g is perfect, $\text{Inn}(g)$ = Der(g) and $Z(g) = 0$ ([\[14\]](#page-22-14)).

In order to answer the classification part of the extending structures problem we need to introduce the following:

Definition 1.1. Let \mathfrak{g} be a Lie algebra, E a vector space such that \mathfrak{g} is a subspace of E and V a complement of $\mathfrak g$ in E. For a linear map $\varphi : E \to E$ we consider the diagram:

$$
\mathfrak{g} \xrightarrow{i} E \xrightarrow{\pi} V
$$
\n
$$
Id \qquad \downarrow \varphi \qquad \downarrow Id
$$
\n
$$
\mathfrak{g} \xrightarrow{i} E \xrightarrow{\pi} V
$$
\n
$$
(6)
$$

where $\pi : E \to V$ is the canonical projection of $E = \mathfrak{g} + V$ on V and $i : \mathfrak{g} \to E$ is the inclusion map. We say that $\varphi : E \to E$ stabilizes g (resp. co-stabilizes V) if the left square (resp. the right square) of the diagram (6) is commutative.

Let $\{-,-\}$ and $\{-,-\}'$ be two Lie algebra structures on E both containing g as a Lie subalgebra. $\{-, -\}$ and $\{-, -\}'$ are called *equivalent*, and we denote this by $(E, \{-, -\}) \equiv (E, \{-, -\})$, if there exists a Lie algebra isomorphism $\varphi : (E, \{-, -\}) \rightarrow$ $(E, \{-, -\})'$ which stabilizes g.

 ${-, -}$ and ${-, -}'$ are called *cohomologous*, and we denote this by $(E, {{-, -}}) \approx$ $(E, \{-, -\})$, if there exists a Lie algebra isomorphism $\varphi : (E, \{-, -\}) \to (E, \{-, -\})$ which stabilizes $\mathfrak g$ and co-stabilizes V, i.e. the diagram (6) is commutative.

 \equiv and \approx are both equivalence relations on the set of all Lie algebras structures on E containing **g** as a Lie subalgebra and we denote by Extd (E, \mathfrak{g}) (resp. Extd' (E, \mathfrak{g})) the set of all equivalence classes via \equiv (resp. \approx). Thus, Extd (E, \mathfrak{g}) is the classifying object of the extending structures problem: by explicitly computing $\text{Extd}(E, \mathfrak{g})$ we obtain a parametrization of the set of all isomorphism classes of Lie algebra structures on E that stabilizes $\mathfrak g$. Extd' $(E, \mathfrak g)$ gives a classification of the ES problem from the point of view of the extension problem. Any two cohomologous brackets on E are of course equivalent, hence there exists a canonical projection

$$
Extd'(E, \mathfrak{g}) \twoheadrightarrow Extd(E, \mathfrak{g})
$$

The classification part of the extending structures problem will be solved by computing explicitly both classifying objects. Borrowing the terminology from Lie algebra cohomology, we will see that $\operatorname{Extd}^{\prime}(E, \mathfrak{g})$ is parameterized by a cohomological object denoted by $\mathcal{H}^2(V, \mathfrak{g})$, which will be explicitly constructed and which generalizes the classical sec-ond cohomology group for Lie algebras [\[9\]](#page-22-2), while Extd (E, \mathfrak{g}) will be parameterized by a relative cohomological object, denoted by $\mathcal{H}^2_{\mathfrak{g}}(V, \mathfrak{g})$ which turns out to be a quotient of $\mathcal{H}^2(V, \mathfrak{g}).$

2. Unified products for Lie algebras

Definition 2.1. Let \mathfrak{g} be a Lie algebra and V a vector space. An extending datum of \mathfrak{g} through V is a system $\Omega(\mathfrak{g}, V) = \{ \triangleleft, \triangleright, f, \{-, -\} \}$ consisting of four bilinear maps

$$
\vartriangleleft: V \times \mathfrak{g} \to V, \quad \vartriangleright: V \times \mathfrak{g} \to \mathfrak{g}, \quad f: V \times V \to \mathfrak{g}, \quad \{-, -\}: V \times V \to V
$$

Let $\Omega(\mathfrak{g}, V) = (\triangleleft, \triangleright, f, \{-, -\})$ be an extending datum. We denote by $\mathfrak{g} \nmid_{\Omega(\mathfrak{g}, V)} V =$ $\mathfrak{g} \nmid V$ the vector space $\mathfrak{g} \times V$ with the bilinear map $[-, -] : (\mathfrak{g} \times V) \times (\mathfrak{g} \times V) \rightarrow \mathfrak{g} \times V$ defined by:

$$
[(g, x), (h, y)] := ([g, h] + x \triangleright h - y \triangleright g + f(x, y), \{x, y\} + x \triangleleft h - y \triangleleft g)
$$
(7)

for all g, $h \in \mathfrak{g}$ and $x, y \in V$. The object $\mathfrak{g} \not\downarrow V$ is called the *unified product* of \mathfrak{g} and $\Omega(\mathfrak{g}, V)$ if it is a Lie algebra with the bracket given by [\(7\)](#page-5-2). In this case the extending datum $\Omega(\mathfrak{g}, V) = (\triangleleft, \triangleright, f, \{-, -\})$ is called a Lie extending structure of $\mathfrak g$ through V. The maps \triangleleft and \triangleright are called the *actions* of $\Omega(\mathfrak{g}, V)$ and f is called the *cocycle* of $\Omega(\mathfrak{g}, V)$.

The extending datum $\Omega(\mathfrak{g}, V) = \{ \varphi, \varphi, f, \{-, -\} \}$, for which $\{ \varphi, \varphi, f, \{-, -\} \}$ are all the trivial maps is an example of a Lie extending structure, called the trivial extending structure of g through V. Let $\Omega(\mathfrak{g}, V)$ be an extending datum of g through V. Then, the following relations, very useful in computations, hold in $\mathfrak{gl} V$:

$$
[(g,0), (h,y)] = ([g,h]-y \triangleright g, -y \triangleleft g)
$$
\n(8)

$$
[(0, x), (h, y)] = (x \triangleright h + f(x, y), x \triangleleft h + \{x, y\})
$$
\n(9)

for all $q, h \in \mathfrak{g}$ and $x, y \in V$.

Theorem 2.2. Let $\mathfrak g$ be a Lie algebra, V a k-vector space and $\Omega(\mathfrak g, V)$ an extending datum of $\mathfrak g$ by V. The following statements are equivalent:

(1) $\mathfrak{gl} \, V$ is a unified product;

(2) The following compatibilities hold for any g, $h \in \mathfrak{g}, x, y, z \in V$:

\n- (LE1)
$$
f(x, x) = 0
$$
, $\{x, x\} = 0$;
\n- (LE2) (V, \triangleleft) is a right $\mathfrak{g}\text{-module}$;
\n- (LE3) $x \triangleright [g, h] = [x \triangleright g, h] + [g, x \triangleright h] + (x \triangleleft g) \triangleright h - (x \triangleleft h) \triangleright g$;
\n- (LE4) $\{x, y\} \triangleleft g = \{x, y \triangleleft g\} + \{x \triangleleft g, y\} + x \triangleleft (y \triangleright g) - y \triangleleft (x \triangleright g\}$;
\n- (LE5) $\{x, y\} \triangleright g = x \triangleright (y \triangleright g) - y \triangleright (x \triangleright g) + [g, f(x, y)] + f(x, y \triangleleft g) + f(x \triangleleft g, y)$;
\n- (LE6) $f(x, \{y, z\}) + f(y, \{z, x\}) + f(z, \{x, y\}) + x \triangleright f(y, z) + y \triangleright f(z, x) + z \triangleright f(x, y) = 0$;
\n- (LE7) $\{x, \{y, z\}\} + \{y, \{z, x\}\} + \{z, \{x, y\}\} + x \triangleleft f(y, z) + y \triangleleft f(z, x) + z \triangleleft f(x, y) = 0$.
\n

Before going into the proof of the theorem, we make a few remarks on the compatibilities in Theorem [2.2.](#page-5-1) Aside from the fact that V is not a Lie algebra, $(LE3)$ and $(LE4)$ are exactly the compatibilities defining a matched pair of Lie algebras [\[21,](#page-22-15) Definition 8.3.1. The compatibility condition (*LE5*) is called the *twisted module condition* for the action ⊳; in the case when V is a Lie algebra it measures how far $(\mathfrak{g}, \triangleright)$ is from being a left V-module. (LE6) is called the *twisted cocycle condition*: if ⊳ is the trivial action and $(V, \{-, -\})$ is a Lie algebra then the compatibility condition (LE6) is exactly the classical 2-cocycle condition for Lie algebras. (LE7) is called the twisted Jacobi condition: it measures how far $\{-, -\}$ is from being a Lie structure on V. If either \triangleleft or f is the trivial map, then $(LE7)$ is equivalent to $\{-, -\}$ being a Lie bracket on V.

Proof. For any $q \in \mathfrak{g}$ and $x \in V$ we have:

$$
[(g, x), (g, x)] = ([g, g] + x \triangleright g - x \triangleright g + f(x, x), \{x, x\} + x \triangleleft g - x \triangleleft g)
$$

= $(f(x, x), \{x, x\})$

Therefore, $[(g, x), (g, x)] = 0$ if and only if $(LE1)$ holds. From now on we will assume that (LE1) holds. In particular, we have that $f(x, y) = -f(y, x)$ and $\{x, y\} = -\{y, x\}$, for all $x, y \in V$ since f and $\{-,-\}$ are bilinear maps. Thus $\mathfrak{gl} V$ is a Lie algebra if and only if Jacobi's identity holds, i.e.:

$$
[(g, x), [(h, y), (l, z)]] + [(h, y), [(l, z), (g, x)]] + [(l, z), [(g, x), (h, y)]] = 0 \qquad (10)
$$

for all g, h, $l \in \mathfrak{g}$ and x, y, $z \in V$. Since in $\mathfrak{gl} V$ we have $(g, x) = (g, 0) + (0, x)$ it follows that [\(10\)](#page-6-0) holds if and only if it holds for all generators of $\mathfrak{gl}(V)$, i.e. the set $\{(g, 0) \mid g \in \mathfrak{g}\} \cup \{(0, x) \mid x \in V\}$. Since [\(10\)](#page-6-0) is invariant under circular permutations we are left with only three cases to study. First, we should notice that [\(10\)](#page-6-0) holds for the triple $(g, 0), (h, 0), (l, 0)$ as we have:

$$
[(g, 0), [(h, 0), (l, 0)]] + [(h, 0), [(l, 0), (g, 0)]] + [(l, 0), [(g, 0), (h, 0)]] =
$$

= ([g, [h, l]] + [h, [l, g]] + [l, [g, h]], 0) = (0, 0)

8 A. L. AGORE AND G. MILITARU

Next, we prove that [\(10\)](#page-6-0) holds for $(g, 0)$, $(h, 0)$, $(0, x)$ if and only if (LE2) and (LE3) hold. Indeed, we have:

$$
[(g, 0), [(h, 0), (0, x)]] + [(h, 0), [(0, x), (g, 0)]] + [(0, x), [(g, 0), (h, 0)]] =
$$

\n
$$
\stackrel{(8),(9)}{=} [(g, 0), (-x \triangleright h, -x \triangleleft h)] + [(h, 0), (x \triangleright g, x \triangleleft g)] + [(0, x), ([g, h], 0)]
$$

\n
$$
\stackrel{(8),(9)}{=} (-[g, x \triangleright h] + (x \triangleleft h) \triangleright g + [h, x \triangleright g] - (x \triangleleft g) \triangleright h + x \triangleright [g, h],
$$

\n
$$
(x \triangleleft h) \triangleleft g - (x \triangleleft g) \triangleleft h + x \triangleleft [g, h] \right)
$$

Thus we proved that [\(10\)](#page-6-0) holds for $(q, 0)$, $(h, 0)$, $(0, x)$ if and only if (LE2) and (LE3) hold. Now, we prove that [\(10\)](#page-6-0) holds for $(g, 0)$, $(0, x)$, $(0, y)$ if and only if (LE4) and $(LE5)$ hold. Indeed, we have:

$$
[(g, 0), [(0, x), (0, y)]] + [(0, x), [(0, y), (g, 0)]] + [(0, y), [(g, 0), (0, x)]] =
$$

\n
$$
(\frac{8)(9)}{2} \quad [(g, 0), (f(x, y), \{x, y\})] + [(0, x), (y \triangleright g, y \triangleleft g)] + [(0, y), (-x \triangleright g, -x \triangleleft g)]
$$

\n
$$
(\frac{8)(9)}{2} \quad ((g, f(x, y)) - \{x, y\} \triangleright g, -\{x, y\} \triangleleft g) + (x \triangleright (y \triangleright g) + f(x, y \triangleleft g), x \triangleleft (y \triangleright g) + (x, y \triangleleft g)\} + (y \triangleright (-x \triangleright g) + f(y, -x \triangleleft g), y \triangleleft (-x \triangleright g) + \{y, -x \triangleleft g\})
$$

\n
$$
= ((g, f(x, y)) - \{x, y\} \triangleright g + x \triangleright (y \triangleright g) + f(x, y \triangleleft g) - y \triangleleft (x \triangleright g) - f(y, x \triangleleft g), -\{x, y\} \triangleleft g + x \triangleleft (y \triangleright g) + \{x, y \triangleleft g\} - y \triangleleft (x \triangleright g) - \{y, x \triangleleft g\} \}
$$

Therefore, having in mind that for all $x, y \in V$ we have $f(x, y) = -f(y, x)$ and $\{x, y\} =$ $-\{y, x\}$ it follows that [\(10\)](#page-6-0) holds for $(g, 0), (0, x)$ (0, y) if and only if (LE4) and (LE5) hold. Finally, we will prove that (10) holds for $(0, x)$, $(0, y)$, $(0, z)$ if and only if $(LE6)$ and (LE7) hold. Indeed, we have:

$$
[(0, x), [(0, y), (0, z)]] + [(0, y), [(0, z), (0, x)]] + [(0, z), [(0, x), (0, y)]] =
$$

\n
$$
= [(0, x), (f(y, z), \{y, z\})] + [(0, y), (f(z, x), \{z, x\})] + [(0, z), (f(x, y), \{x, y\})]
$$

\n
$$
\stackrel{(9)}{=} (x \triangleright f(y, z) + f(x, \{y, z\}), x \triangleleft f(y, z) + \{x, \{y, z\}\}) + (y \triangleright f(z, x) + f(y, \{z, x\}), y \triangleleft f(z, x) + \{y, \{z, x\}\}) + (z \triangleright f(x, y) + f(z, \{x, y\}), z \triangleleft f(x, y) + \{z, \{x, y\}\})
$$

Thus, [\(10\)](#page-6-0) holds for $(0, x)$, $(0, y)$, $(0, z)$ if and only if (LE6) and (LE7) hold and the proof is finished. \Box

From now on, in light of Theorem [2.2,](#page-5-1) a Lie extending structure of $\mathfrak g$ through V will be viewed as a system $\Omega(\mathfrak{g}, V) = (4, 6, 6, 6, 6, -1)$ satisfying the compatibility conditions $(LE1)-(LE7)$. We denote by $\mathcal{L}(\mathfrak{g}, V)$ the set of all Lie extending structures of g through V .

Example 2.3. We provide the first example of a Lie extending structure and the corresponding unified product. More examples will be given in Section [3](#page-12-0) and Section [4.](#page-15-1) Let $\Omega(\mathfrak{g}, V) = (1, 1, 1, 1, \ldots, V)$ be an extending datum of a Lie algebra g through a

vector space V such that \triangleleft and \triangleright are both trivial maps, i.e. $x \triangleleft g = x \triangleright g = 0$, for all x, $y \in V$ and $g \in \mathfrak{g}$. Then, $\Omega(\mathfrak{g}, V) = (f, \{-, -\})$ is a Lie extending structure of \mathfrak{g} through V if and only if $(V, \{-,-\})$ is a Lie algebra and $f: V \times V \rightarrow \mathfrak{g}$ is a classical 2-cocycle, that is:

$$
f(x,x) = 0, \t [g, f(x,y)] = 0, \t f(x,\{y,z\}) + f(y,\{z,x\}) + f(z,\{x,y\}) = 0
$$

for all $g \in \mathfrak{g}$, $x, y, z \in V$. In this case, the associated unified product $\mathfrak{gl}_{\Omega(\mathfrak{g}, V)}V$ will be denoted by $\mathfrak{g} \#^f V$ and we shall call it the *twisted product* of the Lie algebras g and V. Hence, the twisted product associated to a given 2-cocycle $f: V \times V \rightarrow \mathfrak{g}$ between Lie algebras is the vector space $g \times V$ with the bracket given for any g, $h \in g$ and $x, y \in V$ by:

$$
[(g, x), (h, y)] := ([g, h] + f(x, y), \{x, y\})
$$
\n(11)

The twisted product of two Lie algebras plays the crucial role in the classification of all 6-dimensional nilpotent Lie algebras given in [\[12\]](#page-22-12).

Let $\Omega(\mathfrak{g}, V) = (1, 1) \circ f, \{-, -\} \circ f \in \mathcal{L}(\mathfrak{g}, V)$ be a Lie extending structure and $\mathfrak{gl} V$ the associated unified product. Then the canonical inclusion

$$
i_{\mathfrak{g}} : \mathfrak{g} \to \mathfrak{g} \, \natural \, V, \qquad i_{\mathfrak{g}}(g) = (g, 0)
$$

is an injective Lie algebra map. Therefore, we can see g as a Lie subalgebra of $\mathfrak{gl} V$ through the identification $\mathfrak{g} \cong i_{\mathfrak{g}}(\mathfrak{g}) \cong \mathfrak{g} \times \{0\}$. Conversely, we will prove that any Lie algebra structure on a vector space E containing $\mathfrak g$ as a Lie subalgebra is isomorphic to a unified product. In this way, we obtain the answer to the description part of the extending structures problem:

Theorem 2.4. Let $\mathfrak g$ be a Lie algebra, E a vector space containing $\mathfrak g$ as a subspace and $[-, -]$ a Lie algebra structure on E such that \frak{g} is a Lie subalgebra in $(E, [-, -])$. Then there exists a Lie extending structure $\Omega(\mathfrak{g}, V) = (\triangleleft, \triangleright, f, \{-, -\})$ of \mathfrak{g} trough a subspace V of E and an isomorphism of Lie algebras $(E, [-, -]) \cong \mathfrak{gl} V$ that stabilizes g and co-stabilizes V .

Proof. As k is a field, there exists a linear map $p : E \to \mathfrak{g}$ such that $p(g) = g$, for all $g \in \mathfrak{g}$. Then $V := \ker(p)$ is a subspace of E and a complement of \mathfrak{g} in E. We define the extending datum of $\mathfrak g$ through V by the following formulas:

$$
\triangleright = \triangleright_p : V \times \mathfrak{g} \to \mathfrak{g}, \qquad x \triangleright g \ := \ p([x, g])
$$

\n
$$
\triangleleft = \triangleleft_p : V \times \mathfrak{g} \to V, \qquad x \triangleleft g \ := \ [x, g] - p([x, g])
$$

\n
$$
f = f_p : V \times V \to \mathfrak{g}, \qquad f(x, y) \ := \ p([x, y])
$$

\n
$$
\{ \, , \, \} = \{ \, , \, \}_p : V \times V \to V, \qquad \{x, y\} \ := \ [x, y] - p([x, y])
$$

for any $q \in \mathfrak{g}$ and $x, y \in V$. First of all, we observe that the above maps are all well defined bilinear maps: $x \triangleleft g \in V$ and $\{x, y\} \in V$, for all $x, y \in V$ and $g \in \mathfrak{g}$. We shall prove that $\Omega(\mathfrak{g}, V) = (\triangleleft, \triangleright, f, \{-, -\})$ is a Lie extending structure of \mathfrak{g} trough V and

$$
\varphi: \mathfrak{gl}\, V \to E, \qquad \varphi(g, x) := g + x
$$

is an isomorphism of Lie algebras that stabilizes $\mathfrak g$ and co-stabilizes V. Instead of proving the seven compatibility conditions $(LE1) - (LE7)$, which requires a long and laborious computation, we use the following trick combined with Theorem [2.2:](#page-5-1) $\varphi : \mathfrak{g} \times V \to E$, $\varphi(g, x) := g+x$ is a linear isomorphism between the Lie algebra E and the direct product

of vector spaces $\mathfrak{g} \times V$ with the inverse given by $\varphi^{-1}(y) := (p(y), y - p(y))$, for all $y \in E$. Thus, there exists a unique Lie algebra structure on $\mathfrak{g} \times V$ such that φ is an isomorphism of Lie algebras and this unique bracket on $\mathfrak{g} \times V$ is given by

$$
[(g, x), (h, y)] := \varphi^{-1}([\varphi(g, x), \varphi(h, y)])
$$

for all $q, h \in \mathfrak{g}$ and $x, y \in V$. The proof is completely finished if we prove that this bracket coincides with the one defined by [\(7\)](#page-5-2) associated to the system $(\varphi_p, \varphi_p, f_p, \{-, -\}_p)$. Indeed, for any g, $h \in \mathfrak{g}$ and $x, y \in V$ we have:

$$
[(g, x), (h, y)] = \varphi^{-1}([\varphi(g, x), \varphi(h, y)]) = \varphi^{-1}([g, h] + [g, y] + [x, h] + [x, y])
$$

\n
$$
= (p([g, h]), [g, h] - p([g, h])) + (p([g, y]), [g, y] - p([g, y]))
$$

\n
$$
+ (p([x, h]), [x, h] - p([x, h])) + (p([x, y]), [x, y] - p([x,]))
$$

\n
$$
= (p([g, h]) + p([g, y]) + p([x, h]) + p([x, y]), [g, h] + [g, y]
$$

\n
$$
+ [x, h] + [x, y] - p([g, h]) - p([g, y]) - p([x, h]) - p([x, y])
$$

\n
$$
= ((g, h] - y \triangleright g + x \triangleright h + f(x, y), \{x, y\} + x \triangle h - y \triangle g)
$$

as needed. Moreover, the following diagram is commutative

where $\pi : E \to V$ is the projection of $E = \mathfrak{g} + V$ on the vector space V and $q : \mathfrak{g} \natural V \to V$, $q(g, x) := x$ is the canonical projection. The proof is now finished.

Using Theorem [2.4,](#page-8-0) the classification of all Lie algebra structures on E that contains $\mathfrak g$ as a Lie subalgebra, reduces to the classification of all unified products $\mathfrak{g} \sharp V$, associated to all Lie extending structures $\Omega(\mathfrak{g}, V) = \{ \triangleleft, \triangleright, f, \{-, -\} \}$, for a given complement V of $\mathfrak g$ in E. In order to construct the cohomological objects $\mathcal H^2_{\mathfrak g}(V,\mathfrak g)$ and $\mathcal H^2(V,\mathfrak g)$ which will parameterize the classifying sets Extd (E, \mathfrak{g}) and respectively Extd' (E, \mathfrak{g}) defined in Definition [1.1,](#page-4-1) we need the following technical lemma:

Lemma 2.5. Let $\Omega(\mathfrak{g}, V) = (\triangleleft, \triangleright, f, \{-, -\})$ and $\Omega'(\mathfrak{g}, V) = (\triangleleft', \triangleright', f', \{-, -\})$ be two Lie algebra extending structures of $\mathfrak g$ trough V and $\mathfrak g \natural V$, $\mathfrak g \natural'V$ the associated unified products. Then there exists a bijection between the set of all morphisms of Lie algebras ψ : $\mathfrak{gl}V \to \mathfrak{gl}V$ which stabilizes $\mathfrak g$ and the set of pairs (r, v) , where $r : V \to \mathfrak g$, $v: V \to V$ are two linear maps satisfying the following compatibility conditions for any $q \in \mathfrak{g}, x, y \in V$:

(ML1) $v(x) \triangleleft' g = v(x \triangleleft g);$ (ML2) $r(x \triangleleft g) = [r(x), g] - x \triangleright g + v(x) \triangleright' g;$ $(ML3) \ v(\lbrace x, y \rbrace) = \{v(x), v(y)\}' + v(x) \ \preceq \ r(y) - v(y) \ \preceq \ r(x);$ (ML4) $r(\lbrace x,y \rbrace) = [r(x), r(y)] + v(x) \triangleright' r(y) - v(y) \triangleright' r(x) + f'(v(x), v(y)) - f(x, y)$ Under the above bijection the morphism of Lie algebras $\psi = \psi_{(r,\nu)} : \mathfrak{gl} \times \mathfrak{gl} \times V$ corresponding to (r, v) is given for any $g \in \mathfrak{g}$ and $x \in V$ by:

$$
\psi(g, x) = (g + r(x), v(x))
$$

Moreover, $\psi = \psi_{(r,v)}$ is an isomorphism if and only if $v : V \to V$ is an isomorphism and $\psi = \psi_{(r,v)}$ co-stabilizes V if and only if $v = \text{Id}_V$.

Proof. A linear map $\psi : \mathfrak{gl} \nmid V \to \mathfrak{gl}'V$ which makes the following diagram commutative:

is uniquely determined by two linear maps $r : V \to \mathfrak{g}, v : V \to V$ such that $\psi(g, x) = (g +$ $r(x), v(x)$, for all $g \in \mathfrak{g}$, and $x \in V$. Indeed, if we denote $\psi(0, x) = (r(x), v(x)) \in \mathfrak{g} \times V$ for all $x \in V$, we have:

$$
\psi(g, x) = \psi((g, 0) + \psi(0, x)) = \psi(g, 0) + \psi(0, x)
$$

= $(g, 0) + (r(x), v(x)) = (g + r(x), v(x))$

Let $\psi = \psi_{(r,v)}$ be such a linear map, i.e. $\psi(g, x) = (g + r(x), v(x))$, for some linear maps $r: V \to \mathfrak{g}, v: V \to V$. We will prove that ψ is a morpshism of Lie algebras if and only if the compatibility conditions $(ML1)-(ML4)$ hold. It is enough to prove that the compatibility

$$
\psi\big([(g,x),(h,y)]\big) = [\psi(g,x),\,\psi(h,y)]\tag{12}
$$

holds for all generators of $\mathfrak{gl}(V)$. First of all, it is easy to see that [\(12\)](#page-10-0) holds for the pair $(g, 0), (h, 0),$ for all g, $h \in \mathfrak{g}$. Now we prove that [\(12\)](#page-10-0) holds for the pair $(g, 0), (0, x)$ if and only if $(ML1)$ and $(ML2)$ hold. Indeed, $\psi([(g, 0), (0, x)]) = [\psi(g, 0), \psi(0, x)]$ it is equivalent to $\psi(-x \triangleright g, -x \triangleleft g) = [(g, 0), (r(x), v(x))]$ and hence to $(-x \triangleright g + r(-x \triangleleft$ $g(x)$, $v(-x\triangleleft g)$ = $([g, r(x)] - v(x)v'g, -v(x)\triangleleft g)$, i.e. to the fact that $(ML1)$ and $(ML2)$ hold. In the same manner it can be proved that (12) holds for the pair $(0, x)$, $(g, 0)$ if and only if $(ML1)$ and $(ML2)$ hold.

Next, we prove that [\(12\)](#page-10-0) holds for the pair $(0, x)$, $(0, y)$ if and only if $(ML3)$ and $(ML4)$ hold. Indeed, $\psi([0, x), (0, y)]) = [\psi(0, x), \psi(0, y)]$ it is equivalent to $\psi(f(x, y), \{x, y\}) =$ $[(r(x), v(x)), (r(y), v(y))]$; therefore it is equivalent to: $(f(x, y) + r(\lbrace x, y \rbrace), v(\lbrace x, y \rbrace))$ = $(r(x), r(y)|+v(x) \triangleright' r(y)-v(y) \triangleright' r(x)+f'(v(x), v(y)), \{v(x), v(y)\}'+v(x) \triangleleft' r(y)-v(y) \triangleleft' r(y)$ $r(x)$, i.e. to the fact that $(ML3)$ and $(ML4)$ hold.

Assume now that $v: V \to V$ is bijective. Then $\psi_{(r,v)}$ is an isomorphism of Lie algebras with the inverse given for any $h \in \mathfrak{g}$ and $y \in V$ by:

$$
\psi^{-1}_{(r,v)}(h,y) = \left(h - r(v^{-1}(y)), v^{-1}(y)\right)
$$

Conversely, assume that $\psi_{(r,v)}$ is bijective. It follows easily that v is surjective. Thus, we are left to prove that v is injective. Indeed, let $x \in V$ such that $v(x) = 0$. We have $\psi_{(r,v)}(0,0) = (0,0) = (0,v(x)) = \psi_{(r,v)}(-r(x),x)$, and hence we obtain $x = 0$, i.e. v is a bijection. The last assertion is trivial and the proof is now finished. \Box **Definition 2.6.** Let \mathfrak{g} be a Lie algebra and V a k-vector space. Two Lie algebra extending structures of $\mathfrak g$ by V , $\Omega(\mathfrak g, V) = (\triangleleft, \triangleright, f, \{-, -\})$ and $\Omega'(\mathfrak g, V) = (\triangleleft', \triangleright', f', \{-, -\})$ are called *equivalent*, and we denote this by $\Omega(\mathfrak{g}, V) \equiv \Omega'(\mathfrak{g}, V)$, if there exists a pair (r, v) of linear maps, where $r : V \to \mathfrak{g}$ and $v \in Aut_k(V)$ such that $(\prec', \rhd', f', \{-, -\}')$ is implemented from $(\triangle, \triangleright, f, \{-, -\})$ using (r, v) via:

$$
x \triangleleft' g = v(v^{-1}(x) \triangleleft g)
$$

\n
$$
x \triangleright' g = r(v^{-1}(x) \triangleleft g) + v^{-1}(x) \triangleright g + [g, r(v^{-1}(x))]
$$

\n
$$
f'(x, y) = f(v^{-1}(x), v^{-1}(y)) + r(\{v^{-1}(x), v^{-1}(y)\}) + [r(v^{-1}(x)), r(v^{-1}(y))]
$$

\n
$$
-r(v^{-1}(x) \triangleleft r(v^{-1}(y))) - v^{-1}(x) \triangleright r(v^{-1}(y))) + r(v^{-1}(y) \triangleleft r(v^{-1}(x)))
$$

\n
$$
\{x, y\}' = v(\{v^{-1}(x), v^{-1}(y)\}) - v(v^{-1}(x) \triangleleft r(v^{-1}(y))) + v(v^{-1}(y) \triangleleft r(v^{-1}(x)))
$$

for all $g \in \mathfrak{g}, x, y \in V$.

We recall from Definition [1.1](#page-4-1) that $\text{Extd}(E, \mathfrak{g})$ denotes the set of all equivalence classes of Lie algebra structures on E which stabilizes \mathfrak{g} . As a conclusion of this section the main result of this paper, which gives the theoretical answer to the extending structure problem, follows:

Theorem 2.7. Let \mathfrak{g} be a Lie algebra, E a vector space that contains \mathfrak{g} as a subspace and V a complement of $\mathfrak g$ in E. Then:

(1) \equiv is an equivalence relation on the set $\mathcal{L}(\mathfrak{g}, V)$ of all Lie extending structures of \mathfrak{g} through V. We denote by $\mathcal{H}^2_{\mathfrak{g}}(V, \mathfrak{g}) := \mathcal{L}(\mathfrak{g}, V)/\equiv$, the pointed quotient set.

(2) The map

$$
\mathcal{H}^{2}_{\mathfrak{g}}(V, \mathfrak{g}) \to \text{Extd}(E, \mathfrak{g}), \qquad \overline{(\triangleleft, \triangleright, f, \{-, -\})} \to (\mathfrak{g} \, \natural \, V, \ [-, -])
$$

is bijective, where $\overline{(\triangleleft, \triangleright, f, \{-, -\})}$ is the equivalence class of $(\triangleleft, \triangleright, f, \{-, -\})$ via \equiv .

Proof. The proof follows from Theorem [2.2,](#page-5-1) Theorem [2.4](#page-8-0) and Lemma [2.5](#page-9-0) once we observe that $\Omega(\mathfrak{g}, V) \equiv \Omega'(\mathfrak{g}, V)$ in the sense of Definition [2.6](#page-11-2) if and only if there exists an isomorphism of Lie algebras $\psi : \mathfrak{gl} \nabla V \to \mathfrak{gl}'V$ which stabilizes g. Therefore, \equiv is an equivalence relation on the set $\mathcal{L}(\mathfrak{g}, V)$ of all Lie algebra extending structures $\Omega(\mathfrak{g}, V)$ and the conclusion follows from Theorem [2.4](#page-8-0) and Lemma [2.5.](#page-9-0) \Box

Remark 2.8. The second cohomological object $\mathcal{H}^2(V, \mathfrak{g})$ that parameterizes Extd' (E, \mathfrak{g}) is constructed in a simple manner as follows: two Lie algebra extending structures $\Omega(\mathfrak{g}, V) = (\mathfrak{q}, \mathfrak{d}, f, \{-, -\})$ and $\Omega'(\mathfrak{g}, V) = (\mathfrak{q}', \mathfrak{d}', f', \{-, -\}'')$ are called *cohomolo*gous, and we denote this by $\Omega(\mathfrak{g}, V) \approx \Omega'(\mathfrak{g}, V)$ if and only if $\alpha' = \alpha$ and there exists a linear map $r: V \to \mathfrak{g}$ such that

$$
x \triangleright' g = x \triangleright g + r(x \triangleleft g) - [r(x), g]
$$

\n
$$
f'(x, y) = f(x, y) + r(\{x, y\}) + [r(x), r(y)] +
$$

\n
$$
+ y \triangleright r(x) - x \triangleright r(y) + r(y \triangleleft r(x)) - r(x \triangleleft r(y))
$$

\n
$$
\{x, y\}' = \{x, y\} - x \triangleleft r(y) + y \triangleleft r(x)
$$

for all $g \in \mathfrak{g}, x, y \in V$.

Similar to the proof of Theorem [2.7](#page-11-0) we can easily see that $\Omega(\mathfrak{g}, V) \approx \Omega'(\mathfrak{g}, V)$ if and only if there exists an isomorphism of Lie algebras $\psi : \mathfrak{gl} \nabla V \to \mathfrak{gl} \nabla V$ which stabilizes g and co-stabilizes V. Thus, \approx is an equivalence relation on the set $\mathcal{L}(\mathfrak{g}, V)$ of all Lie extending structures of $\mathfrak g$ through V. If we denote $\mathcal H^2(V,\mathfrak g):=\mathcal L(\mathfrak g,V)/\approx,$ the map

$$
\mathcal{H}^{2} \left(V, \, \mathfrak{g} \right) \rightarrow \mathrm{Extd}^{\prime} \left(E, \mathfrak{g} \right), \qquad \overline{\left(\triangleleft, \triangleright, f, \{-, \,- \} \right)} \rightarrow \left(\mathfrak{g} \mathop{\natural} V, \, [-, \, -] \right)
$$

is a bijection between $\mathcal{H}^2(V,\mathfrak{g})$ and the isomorphism classes of all Lie algebra structures on E which stabilizes $\mathfrak g$ and co-stabilizes V .

3. Special cases of unified products

In this section we show that crossed products and bicrossed products of two Lie algebras are both special cases of unified products. We make the following convention: if one of the maps $\triangleleft, \triangleright, f$ or $\{-, -\}$ of an extending datum $\Omega(\mathfrak{g}, V) = (\triangleleft, \triangleright, f, \{-, -\})$ is trivial then we will omit it from the quadruple $(\triangle, \triangleright, f, \{-, -\})$.

Crossed products and the extension problem. Let $\Omega(\mathfrak{g}, V) = (\triangleleft, \triangleright, f, \{-, -\})$ be an extending datum of g through V such that \triangleleft is the trivial map, i.e. $x \triangleleft q = 0$, for all x, $y \in V$ and $g \in \mathfrak{g}$. Then, $\Omega(\mathfrak{g}, V) = (4, 6, 6, 6, 6, 6, 6) = (6, 6, 6, 6, 6, 6, 6)$ is a Lie extending structure of $\frak g$ through V if and only if $(V, \{-,-\})$ is a Lie algebra and the following compatibilities hold for any g, $h \in \mathfrak{g}$ and $x, y, z \in V$:

- $f(x, x) = 0$
- $x \triangleright [q, h] = [x \triangleright q, h] + [q, x \triangleright h]$
- $\{x, y\} \triangleright q = x \triangleright (y \triangleright q) y \triangleright (x \triangleright q) + [q, f(x, y)]$
- $f(x, \{y, z\}) + f(y, \{z, x\}) + f(z, \{x, y\}) + x \triangleright f(y, z) + y \triangleright f(z, x) + z \triangleright f(x, y) = 0$

In this case, the associated unified product $\mathfrak{g} \, \natural_{\Omega(\mathfrak{g}, V)} V = \mathfrak{g} \#_{\triangleleft}^f V$ is the crossed product of the Lie algebras g and V. A system $(g, V, \triangleright, f)$ consisting of two Lie algebras g, V and two bilinear maps $\triangleright : V \times \mathfrak{g} \to \mathfrak{g}$, $f : V \times V \to \mathfrak{g}$ satisfying the above four compatibility conditions will be called a crossed system of Lie algebras. The crossed product associated to the crossed system $(\mathfrak{g}, V, \triangleright, f)$ is the Lie algebra defined as follow: $\mathfrak{g} \#_{\triangleleft}^f V = \mathfrak{g} \times V$ with the bracket given for any g, $h \in \mathfrak{g}$ and $x, y \in V$ by:

$$
[(g, x), (h, y)] := ([g, h] + x \triangleright h - y \triangleright g + f(x, y), \{x, y\})
$$
\n(13)

The crossed product of Lie algebras provides the answer to the following restricted version of the extending structures problem: Let $\mathfrak g$ be a Lie algebra, E a vector space containing $\mathfrak g$ as a subspace. Describe and classify all Lie algebra structures on E such that $\mathfrak g$ is an ideal of E.

Indeed, let $(\mathfrak{g}, V, \triangleright, f)$ be a crossed system of two Lie algebras. Then, $\mathfrak{g} \cong \mathfrak{g} \times \{0\}$ is an ideal in the Lie algebra $\mathfrak{g} \#_{\triangleleft}^f V$ since $[(g, 0), (h, y)] := ([g, h] - y \triangleright g, 0)$. Conversely, crossed products describe all Lie algebra structures on a vector space E such that a given Lie algebra $\mathfrak g$ is an ideal of E .

Corollary 3.1. Let $\mathfrak g$ be a Lie algebra, E a vector space containing $\mathfrak g$ as a subspace. Then any Lie algebra structure on E that contains $\mathfrak g$ as an ideal is isomorphic to a crossed product of Lie algebras $\mathfrak{g} \#_{\triangleleft}^f V$.

Proof. Let $[-, -]$ be a Lie algebra structure on E such that g is an ideal in E. In particular, $\mathfrak g$ is a subalgebra of E and hence we can apply Theorem [2.4.](#page-8-0) In this case the action $\triangleleft = \triangleleft_p$ of the Lie extending structure $\Omega(\mathfrak{g}, V) = (\triangleleft_p, \triangleright_p, f_p, \{-, -\}_p)$ constructed in the proof of Theorem [2.4](#page-8-0) is the trivial map since for any $x \in V$ and $g \in \mathfrak{g}$ we have that $[x, g] \in \mathfrak{g}$ and hence $p([x, g]) = [x, g]$. Thus, $x \triangleleft_p g = 0$, i.e. the unified product $\mathfrak{gl}\, \mathfrak{g}_{\mathfrak{Q}(\mathfrak{g},V)}V = \mathfrak{g} \#_{\mathfrak{q}}^f V$ is the crossed product of the Lie algebras \mathfrak{g} and $V := \text{Ker}(p)$. \Box

Remark 3.2. We have proved Corollary [3.1](#page-13-0) based on Theorem [2.4](#page-8-0) by taking a linear retraction p of the canonical inclusion $i : \mathfrak{g} \hookrightarrow E$ and the crossed system arising from p. The classical proof of Corollary [3.1,](#page-13-0) given in the extension theory of Lie algebras, is completely different by the way the crossed system is constructed: since $\mathfrak g$ is an ideal of the Lie algebra E we can consider the quotient Lie algebra $\mathfrak{h} := E/\mathfrak{g}$. Let $\pi : E \to \mathfrak{h}$ be the canonical projection and $s : \mathfrak{h} \to E$ be a linear section of π . We define the action $\rho = \rho_s$ and the cocycle $f = f_s$ associated to s by the formulas [\(2\)](#page-1-0) and [\(3\)](#page-1-0) from the introduction. Then, $(\mathfrak{g}, \mathfrak{h}, \mathfrak{d}) = \mathfrak{d}_s$, $f = f_s$ is a crossed system of Lie algebras and the map $\psi : \mathfrak{g} \#_{\preceq}^f \mathfrak{h} \to E$, $\psi(g, x) := g + s(x)$ is an isomorphism of Lie algebras with the inverse $\psi^{-1}(z) := (z - s(\pi(z)), \pi(z))$, for all $z \in E$.

The restricted version of the extending structures problem is in fact an equivalent reformulation of the (non-abelian) extension problem. Indeed, first of all we remark that any crossed product $\mathfrak{g} \#_{\mathfrak{q}}^f V$ is an extension of $\mathfrak g$ by V via the following sequence:

$$
0 \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{g} \nrightarrow i \mathfrak{g} \nrightarrow f V \longrightarrow V \longrightarrow 0
$$

where $i : \mathfrak{g} \to \mathfrak{g} \#_{\triangleleft}^f V$, $i(g) := (g, 0)$ and $\pi : \mathfrak{g} \#_{\triangleleft}^f V \to V$, $\pi(g, x) := x$. Conversely, let E be an extension of g by h , that is there exists an exact sequence of Lie algebras of the form [\(1\)](#page-1-1). By identifying $\mathfrak{g} \cong \text{Im}(\mathfrak{i}) = \text{Ker}(\pi)$ we view \mathfrak{g} as an ideal of \mathfrak{E} . Then, it follows from Corollary [3.1](#page-13-0) that there exists a crossed system $(\mathfrak{g}, \mathfrak{h}, \mathfrak{b}, f)$ of Lie algebras such that $\mathfrak{g} \#_{\triangleleft}^f \mathfrak{h} \cong \mathfrak{E}$, an isomorphism of Lie algebras. Furthermore, using Theorem [2.4,](#page-8-0) the isomorphism can be chosen such that it stabilizes g and co-stabilizes h.

The abelian case: cotangent extending structures. Let E be a vector space, g a subspace of E with the abelian Lie algebra structure. A Lie algebra structure on E containing $\mathfrak g$ as an ideal is called a *cotangent extending structure*. The terminology is a generalization of the concept introduced in $[18, \text{ Section } 3.1]$. Let V be a given complement of $\mathfrak g$ in E: it follows from Corollary [3.1](#page-13-0) and Theorem [2.4](#page-8-0) that the set of all cotangent extending structures of the abelian Lie algebra $\mathfrak g$ to the vector space E is parameterized by the set of all triples (\triangleright , $f, \{-, -\}$), such that $(V, \{-, -\})$ is a Lie algebra, $(\mathfrak{g}, \triangleright)$ is a left V-module and $f : V \times V \to \mathfrak{g}$ is a bilinear map such that $f(x, x) = 0$ and

$$
f(x, \{y, z\}) + f(y, \{z, x\}) + f(z, \{x, y\}) + x \triangleright f(y, z) + y \triangleright f(z, x) + z \triangleright f(x, y) = 0
$$

for all x, y, z ∈ V. For such a triple $(\triangleright, f, \{-, -\})$, the bracket of the cotangent extending structure on $E \cong \mathfrak{g} \times V$ is given by:

$$
[(g, x), (h, y)] := (x \triangleright h - y \triangleright g + f(x, y), \{x, y\})
$$
\n(14)

for all g, $h \in \mathfrak{g}$ and $x, y \in V$. Moreover, any cotangent bracket on E has the form [\(14\)](#page-14-0).

Bicrossed products and the factorization problem. Let $\Omega(\mathfrak{g}, V) = (4, 5, 6, 6, 6, 6)$ be an extending datum of g through V such that f is the trivial map, i.e. $f(x, y) = 0$, for all $x, y \in V$. Then, $\Omega(\mathfrak{g}, V) = (\triangleleft, \triangleright, \{-, -\})$ is a Lie extending structure of \mathfrak{g} through V if and only if $(V, \{-,-\})$ is a Lie algebra and $(\mathfrak{g}, V, \leq, \infty)$ is a matched pair of Lie algebras as defined in $[19,$ Theorem 4.1] and independently in $[17,$ Theorem 3.9]: i.e. \mathfrak{g} is a left V-module under \rhd : $V \otimes \mathfrak{g} \to \mathfrak{g}$, V is a right \mathfrak{g} -module under $\triangleleft : V \otimes \mathfrak{g} \to V$ and the following compatibilities hold for all $g, h \in \mathfrak{g}, x, y \in V$:

$$
x \triangleright [g, h] = [x \triangleright g, h] + [g, x \triangleright h] + (x \triangleleft g) \triangleright h - (x \triangleleft h) \triangleright g \tag{15}
$$

$$
\{x, y\} \triangleleft g = \{x, y \triangleleft g\} + \{x \triangleleft g, y\} + x \triangleleft (y \triangleright g) - y \triangleleft (x \triangleright g) \tag{16}
$$

In this case, the associated unified product $\mathfrak{gl}_{\Omega(\mathfrak{g},V)}V = \mathfrak{g} \bowtie V$ is precisely the bicrossed product of the matched pair $(\mathfrak{g}, V, \leq, \geq)$ of Lie algebras. In this paper we adopt the name bicrossed product established in group theory [\[23\]](#page-22-17) and Hopf algebra theory [\[15\]](#page-22-18). Other names used in the literature for the above product are: bicrossproduct in [\[19,](#page-22-10) Theorem 4.1], double cross sum in [\[21,](#page-22-15) Proposition 8.3.2], double Lie algebra [\[17,](#page-22-11) Definition 3.3] or knit product in [\[22\]](#page-22-19). Important examples of bicrossed products of Lie algebras are Manin's triples [\[17,](#page-22-11) Definition 1.13 and Theorem 1.12]. The bicrossed product of two Lie algebras is the construction which provides the answer for the so-called factorization problem, the dual of the extension problem and is also a special case of the extending structures problem:

Let $\mathfrak g$ and $\mathfrak h$ be two given Lie algebras. Describe and classify all Lie algebras Ξ that factorize trough g and h, i.e. Ξ contains g and h as Lie subalgebras such that $\Xi = \mathfrak{g} + \mathfrak{h}$ and $\mathfrak{g} \cap \mathfrak{h} = \{0\}.$

Now, a Lie algebra Ξ factorizes through $\mathfrak g$ and $\mathfrak h$ if and only if there exists a matched pair of Lie algebras $(\mathfrak{g}, \mathfrak{h}, \triangleleft, \triangleright)$ such that $\Xi \cong \mathfrak{g} \bowtie \mathfrak{h}$ [\[21,](#page-22-15) Proposition 8.3.2] or [\[17,](#page-22-11) Theorem 3.9]. Thus, the factorization problem can be restated in a purely computational manner: Let g and h be two given Lie algebras. Describe the set of all matched pairs $(g, \mathfrak{h}, \langle \zeta, \mathfrak{b} \rangle)$ and classify up to an isomorphism all bicrossed products $\mathfrak{g} \bowtie \mathfrak{h}$.

In the case that k is algebraically closed of characteristic zero and Ξ is a finite dimensional Lie algebra then the famous Levi-Malcev theorem [\[8,](#page-22-3) Theorem 5] proves that there exists a Lie subalgebra h of Ξ, called a Levi subalgebra, such that Ξ factorizes through $Rad(\Xi)$ and h, where $\text{Rad}(\Xi)$ is the radical of Ξ . Thus, any finite dimensional Lie algebra Ξ is isomorphic to a bicrossed product between $Rad(\Xi)$ and a semi-simple Lie algebra $\mathfrak{h} \cong \Xi/\mathrm{Rad}(\Xi).$

Remark 3.3. An interesting equivalent description for the factorization of a Lie algebra Ξ through two Lie subalgebras is proved in [\[5,](#page-22-20) Proposition 2.2]. A linear map $f : \Xi \to \Xi$ is called a *complex product structure* on Ξ [\[6,](#page-22-21) Definition 2.1] if $f \neq \pm Id$, $f^2 = f$ and f is

integrable, that is for any $x, y \in \Xi$ we have:

$$
f([x, y]) = [f(x), y] + [x, f(y)] - f([f(x), f(y)])
$$

If the characteristic of k is $\neq 2$, then the linear map $f : \mathfrak{g} \bowtie \mathfrak{h} \rightarrow \mathfrak{g} \bowtie \mathfrak{h}$, $f(g, h) :=$ $(q, -h)$, for all $q \in \mathfrak{g}$ and $h \in \mathfrak{h}$ is a complex product structure on any bicrossed product $\mathfrak{g} \bowtie \mathfrak{h}$ of Lie algebras. Conversely, if f is a complex product structure on Ξ , then Ξ factorizes through two Lie subalgebras $\Xi = \Xi_+ + \Xi_+$, where Ξ_{\pm} denote the eigenspaces corresponding to the eigenvalue ± 1 of f [\[5,](#page-22-20) Proposition 2.2].

4. Flag extending structures. Examples

Theorem [2.7](#page-11-0) offers the theoretical answer to the extending structures problem. The next challenge is a computational one: for a given Lie algebra g that is a subspace in a vector space E with a given complement V to compute explicitly the classifying object $\mathcal{H}^2_{\mathfrak{g}}(V, \mathfrak{g})$ and then to list all Lie algebra structures on E which extend the Lie algebra structure on g. In what follows we provide a way of answering this problem for a large class of such structures.

Definition 4.1. Let \mathfrak{g} be a Lie algebra and E a vector space containing \mathfrak{g} as a subspace. A Lie algebra structure on E such that $\mathfrak g$ is a Lie subalgebra is called a flag extending structure of $\frak g$ to E if there exists a finite chain of Lie subalgebras of E

$$
\mathfrak{g} = E_0 \subset E_1 \subset \cdots \subset E_m = E \tag{17}
$$

such that E_i has codimension 1 in E_{i+1} , for all $i = 0, \dots, m-1$.

In the context of Definition [4.1](#page-15-0) we have that $\dim_k(V) = m$, where V is the complement of $\frak g$ in E. The existence of such a chain of Lie subalgebras is quite common in the theory of solvable Lie algebras [\[8,](#page-22-3) Proposition 2], [\[14,](#page-22-14) Lie Theorem]. All flag extending structures of $\mathfrak g$ to E can be completely described by a recursive reasoning where the key step is $m = 1$. More precisely, this key step describes and classifies all unified products $\mathfrak{gl}(V_1)$, for a 1-dimensional vector space V_1 . We will prove that they are parameterized by the space $TwDer(g)$ of all twisted derivations of g. Then, by replacing the initial Lie algebra g with such a unified product $\mathfrak{gl} V$ which can be described in terms of g only, we can iterate the process: in this way, on our second step we describe and classify all unified products of the form $(\mathfrak{g} \mathfrak{h} V_1) \mathfrak{h} V_2$, where V_1 and V_2 are vector spaces of dimension 1. Of course, after $m = \dim_k(V)$ steps, we obtain the description of all flag extending structures of $\mathfrak g$ to E. We start by introducing the following concept which generalizes the notion of derivation of a Lie algebra:

Definition 4.2. A twisted derivation of a Lie algebra g is a pair (λ, D) consisting of two linear maps $\lambda : \mathfrak{g} \to k$ and $D : \mathfrak{g} \to \mathfrak{g}$ such that for any $g, h \in \mathfrak{g}$:

$$
\lambda([g, h]) = 0 \tag{18}
$$

$$
D([g, h]) = [D(g), h] + [g, D(h)] + \lambda(g)D(h) - \lambda(h)D(g)
$$
\n(19)

The set of all twisted derivations of $\mathfrak g$ will be denoted by TwDer $(\mathfrak g)$. The compatibility condition [\(18\)](#page-15-3) is equivalent to $\mathfrak{g}' \subseteq \text{Ker}(\lambda)$, where \mathfrak{g}' is the derived algebra of \mathfrak{g} .

Examples 4.3. 1. TwDer(q) contains the usual space of derivations Der(q) via the canonical embedding

$$
Der(\mathfrak{g}) \hookrightarrow TwDer(\mathfrak{g}), \qquad D \mapsto (0, D)
$$

which is an isomorphism if $\mathfrak g$ is a perfect Lie algebra.

2. Let $g_0 \in \mathfrak{g}$ and $\lambda : \mathfrak{g} \to k$ be a k-linear map such that $\mathfrak{g}' \subseteq \text{Ker}(\lambda)$. We define the map

$$
D_{g_0,\lambda} : \mathfrak{g} \to \mathfrak{g}, \quad D_{g_0,\lambda}(h) := [g_0, h] - \lambda(h)g_0
$$

for all $h \in \mathfrak{g}$. Then $(\lambda, D_{g_0,\lambda})$ is a twisted derivation called a *inner twisted derivation*.

We shall prove now that the set of all Lie extending structures $\mathcal{L}(\mathfrak{g}, V)$ of a Lie algebra $\mathfrak g$ through a 1-dimensional vector space V is parameterized by TwDer $(\mathfrak g)$.

Proposition 4.4. Let \mathfrak{g} be a Lie algebra and V a vector space of dimension 1 with a basis $\{x\}$. Then there exists a bijection between the set $\mathcal{L}(\mathfrak{g}, V)$ of all Lie extending structures of $\mathfrak g$ trough V and the space $TwDer(\mathfrak g)$ of all twisted derivations of $\mathfrak g$. Through the above bijection, the Lie extending structure $\Omega(\mathfrak{g}, V) = (\triangleleft, \triangleright, f, \{-, -\})$ corresponding to $(\lambda, D) \in \text{TwDer}(\mathfrak{g})$ is given by:

$$
x \triangleleft g = \lambda(g)x, \quad x \triangleright g = D(g), \quad f = 0, \quad \{-, -\} = 0 \tag{20}
$$

for all $g \in \mathfrak{g}$. The unified product associated to the Lie extending structure [\(20\)](#page-16-0) will be denoted by $\mathfrak{gl}_{(\lambda,D)}V$ and has the bracket given for any g, $h \in \mathfrak{g}$ by:^{[1](#page-16-1)}

$$
[(g,0), (h,0)] = ([g,h], 0), \quad [(g,0), (0,x)] = -(D(g), \lambda(g)x)
$$
 (21)

Proof. We have to compute all bilinear maps $\triangleleft : V \times \mathfrak{g} \to \mathfrak{g}, \triangleright : V \times \mathfrak{g} \to V, f : V \times V \to \mathfrak{g}$ and $\{-,-\} : V \times V \to V$ satisfying the compatibility conditions $(LE1) - (LE7)$ of Theorem [2.2.](#page-5-1) Since, $f: V \times V \to \mathfrak{g}$ and $\{-, -\}: V \times V \to V$ are bilinear maps and V has dimension 1, the axiom (LE1) is equivalent to the fact that both maps are trivial, i.e. $f = 0$ and $\{-, -\} = 0$. Again by the fact that $\dim_k(V) = 1$, we obtain a bijection between the set of all bilinear maps $\triangle: V \times \mathfrak{g} \to V$ and the set of all linear maps $\lambda: \mathfrak{g} \to k$ and the bijection is given such that the action $\triangleleft : V \times \mathfrak{g} \to V$ associated to λ is given by the formula: $x \triangleleft g := \lambda(g)x$, for all $g \in \mathfrak{g}$. Analogous, any bilinear map $\triangleright : V \times \mathfrak{g} \to \mathfrak{g}$ is uniquely implemented by a linear map $D = D_{\triangleright} : \mathfrak{g} \to \mathfrak{g}$ via the formula: $x \triangleright g := D(g)$, for all $g \in \mathfrak{g}$.

Finally, we are left to deal with the compatibilities $(LE2) - (LE7)$. Since, $\dim_k(V) = 1$, $f = 0, \{-, -\} = 0$, we observe that the compatibilities $(LE4) - (LE7)$ are automatically satisfied. Now, the compatibility condition (LE2) is equivalent to $\lambda([g, h]) = 0$, for all g, $h \in \mathfrak{g}$. Finally, the compatibility condition (LE3) is equivalent to the fact that [\(19\)](#page-15-3) holds and the proof is now finished. \square

Remark 4.5. Since the cocycle f in [\(20\)](#page-16-0) is trivial we obtain that $\mathfrak{gl}_{(\lambda, D)}V = \mathfrak{g} \bowtie V$, where $\mathfrak{g} \bowtie V$ is a bicrossed product between \mathfrak{g} and an abelian Lie algebra of dimension 1. Hence, Proposition [4.4](#page-16-2) shows in fact that any unified product $\mathfrak{gl}_{(\lambda, D)}V$ between an arbitrary Lie algebra $\mathfrak g$ and a 1-dimensional vector space V is isomorphic to a bicrossed product $\mathfrak{g} \bowtie V$ between \mathfrak{g} and the abelian Lie algebra of dimension 1.

¹As usually, we define the bracket only in the points where the values are non-zero.

Next, we classify all Lie algebras $\mathfrak{gl}_{(\lambda,D)}V$ by computing the cohomological objects $\mathcal{H}^2_{\mathfrak{g}}(V,\mathfrak{g})$ and $\mathcal{H}^2(V,\mathfrak{g})$. First we need the following:

Definition 4.6. Two twisted derivations (λ, D) and $(\lambda', D') \in \text{TwDer}(\mathfrak{g})$ are called *equivalent* and we denote this by $(\lambda, D) \equiv (\lambda', D')$ if $\lambda = \lambda'$ and there exists a pair $(g_0, q) \in \mathfrak{g} \times k^*$ such that for any $h \in \mathfrak{g}$ we have:

$$
D(h) = qD'(h) + [g_0, h] - \lambda(h)g_0
$$
\n(22)

Hence, $(\lambda, D) \equiv (\lambda', D')$ if and only if $\lambda = \lambda'$ and there exists a non-zero scalar $q \in k$ such that $D - qD'$ is a inner twisted derivation. We provide below the first explicit classification result of the extending structures problem for Lie algebras. This is also the key step in the classification of all flag extending structures.

Theorem 4.7. Let $\mathfrak g$ be a Lie algebra of codimension 1 in the vector space E and V a complement of $\mathfrak g$ in E. Then:

(1) \equiv is an equivalence relation on the set TwDer(\frak{g}) of all twisted derivations of \frak{g} .

(2) Extd $(E, \mathfrak{g}) \cong H_{\mathfrak{g}}^2(V, \mathfrak{g}) \cong \text{TwDer}(\mathfrak{g})/\equiv$. The bijection between $\text{TwDer}(\mathfrak{g})/\equiv$ and Extd (E, \mathfrak{g}) , the isomorphisms classes of all Lie algebras structures on E that stabilizes g, is given by:

$$
(\lambda, D) \mapsto \mathfrak{gl}_{(\lambda, D)}V
$$

where $\overline{(\lambda, D)}$ is the equivalence class of (λ, D) via the relation \equiv and $\mathfrak{gl}_{(\lambda, D)}V$ is the Lie algebra constructed in [\(21\)](#page-16-3).

(3) $\mathcal{H}^2(V,\mathfrak{g}) \cong \text{TwDer}(\mathfrak{g})/\approx$, where \approx is the following relation: $(\lambda, D) \approx (\lambda', D')$ if and only if $\lambda = \lambda'$ and $D - D'$ is a inner twisted derivation of g.

Proof. Let $(\lambda, D), (\lambda', D') \in \text{TwDer}(\mathfrak{g})$ be two twisted derivations of \mathfrak{g} and $\Omega(\mathfrak{g}, V) =$ $(\triangleleft, \triangleright, f, \{-, -\})$ respectively $\Omega'(\mathfrak{g}, V) = (\triangleleft', \triangleright', f', \{-, -\}'')$ the corresponding Lie ex-tending structure constructed in [\(20\)](#page-16-0). We will prove that $(\lambda, D) \equiv (\lambda', D')$ if and only if there exists an isomorphism of Lie algebras $\mathfrak{gl}_{(\lambda,D)}V \cong \mathfrak{gl}_{(\lambda',D')}V$ that stabilizes g. This observation together with Proposition [4.4](#page-16-2) and Theorem [2.7](#page-11-0) will finish the proof.

Indeed, using Lemma [2.5](#page-9-0) we obtain that there exists an isomorphism $\mathfrak{gl}_{(\lambda,D)}V \cong$ $\mathfrak{gl}_{(\lambda',D')}V$ of Lie algebras which stabilizes $\mathfrak g$ if and only if there exists a pair (r,v) , where $r: V \to \mathfrak{g}$ and $v: V \to V$ are linear maps satisfying the compatibility conditions $(ML1) - (ML4)$ and v is bijective. Since $\dim_k(V) = 1$, any linear map $r : V \to \mathfrak{g}$ is uniquely determined by an element $g_0 \in \mathfrak{g}$ such that $r(x) = g_0$, where $\{x\}$ is a basis in V . On the other hand, any automorphism v of V is uniquely determined by a non-zero scalar $q \in k^*$ such such $v(x) = qx$. It remains to check the compatibility conditions $(ML1) - (ML4)$, for this pair of maps $(r = r_{g_0}, v = v_q)$. Since $f = f' = 0$ and $\{-,-\} = \{-,-\}' = 0$ in the corresponding Lie extending structure, we obtain that the compatibility conditions $(ML3)$ and $(ML4)$ are trivially fulfilled. Now, the compatibility condition (ML1) is equivalent to $q\lambda'(g)x = q\lambda(g)x$, for all $g \in \mathfrak{g}$, i.e. to the fact that $\lambda = \lambda'$, since $q \neq 0$. Finally, the compatibility condition $(ML1)$ takes the following equivalent form:

$$
\lambda(g)g_0 = [g_0, g] - D(g) + qD'(g)
$$

for all $g \in \mathfrak{g}$, which is precisely [\(22\)](#page-17-1) from Definition [4.6.](#page-17-2) Thus, using Lemma [2.5](#page-9-0) we have proved that $\mathfrak{gl}_{(\lambda,D)}V \cong \mathfrak{gl}_{(\lambda',D')}V$ (an isomorphism of Lie algebras that stabilizes g) if and only if $(\lambda, D) \equiv (\lambda', D')$ and the proof is finished.

Theorem [4.7](#page-17-0) takes the following simplified form in the case of perfect Lie algebras.

Corollary 4.8. Let $\mathfrak g$ be a perfect Lie algebra of codimension 1 in the vector space E and V a complement of $\mathfrak g$ in E. Then:

(1) Extd $(E, \mathfrak{g}) \cong \mathcal{H}^2_{\mathfrak{g}}(V, \mathfrak{g}) \cong \mathrm{Der}(\mathfrak{g})/\approx$, where \approx is the equivalence relation on $\mathrm{Der}(\mathfrak{g})$ defined by: $D \approx D^{\prime}$ if and only if there exists $q \in k^*$ such that $D - qD^{\prime}$ is an inner derivation of g. The bijection between $\text{Der}(\mathfrak{g})/\approx$ and $\text{Extd}(E,\mathfrak{g})$ is given by

 $\overline{D} \mapsto \mathfrak{a} \mathfrak{b} \mathfrak{b} \mathfrak{d} V$

where \overline{D} is the equivalence class of D via the relation \approx and $\mathfrak{gl}_D V$ is the Lie algebra constructed in [\(21\)](#page-16-3) for $\lambda = 0$.

(2) $\mathcal{H}^2(V,\mathfrak{g}) \cong \mathrm{Out}(\mathfrak{g}).$

Proof. Follows directly from Theorem [4.7](#page-17-0) using Example [4.3:](#page-16-4) for a perfect Lie algebra \mathfrak{g} , we have that $TwDer(\mathfrak{g}) = \{0\} \times Der(\mathfrak{g})$.

Remark 4.9. We recall that the space of outer derivations of a Lie algebra g is the quotient vector space $Out(g) := Der(g)/Inn(g)$. Thus, by definition, two derivations D, $D' \in \text{Der}(\mathfrak{g})$ are congruent and we denote this by $D \sim D'$ if $D - D' \in \text{Inn}(\mathfrak{g})$ and then the space Out(\mathfrak{g}) is defined as the quotient via this congruence relation, i.e. Der(\mathfrak{g}/\sim . Now, two congruent derivations are equivalent in the sense of Corollary [4.8](#page-18-0) but the converse does not hold. This means that there exists a canonical surjection

$$
\mathrm{Out}(\mathfrak{g})\twoheadrightarrow \mathrm{Der}(\mathfrak{g})/\approx
$$

We also note that the classifying object $\text{Der}(\mathfrak{g})/\approx$ is just a pointed set as it does not carry a group structure: it is straightforward to see that the following possible group structure $\overline{D} + \overline{D'} = \overline{D + D'}$ is not well defined on $\text{Der}(\mathfrak{g})/\approx$. However, this is not at all surprising from the the point of view of non-abelian extension theory: the classifying object is not a group anymore but a pointed set.

Now, we indicate a class of Lie algebras $\mathfrak g$ such that the classifying objects of Corollary [4.8](#page-18-0) are both singletons. This class contains the semisimple Lie algebras or, more generally, the sympathetic Lie algebras [\[7\]](#page-22-22). In particular, it shows that for a semisimple Lie algebra $\mathfrak g$ there exists, up to an isomorphism that stabilizes $\mathfrak g$, a unique Lie algebra structure on a vector space of dimension $1 + \dim_k(\mathfrak{g})$ which extends the one on \mathfrak{g} : more precisely, this unique Lie algebra structure is given by the direct product $g \times V$, between g and an abelian Lie algebra of dimension 1.

Corollary 4.10. Let $\mathfrak g$ be a Lie algebra of codimension 1 in the vector space E and V a complement of $\mathfrak g$ in E. Assume that $\mathfrak g$ is perfect and $\text{Der}(\mathfrak g) = \text{Inn}(\mathfrak g)$. Then $\mathcal{H}^2_{\mathfrak{g}}(V,\mathfrak{g}) = \mathcal{H}^2(V,\mathfrak{g}) = 0.$

Proof. We apply Corollary [4.8:](#page-18-0) since $Der(g) = Inn(g)$ we obtain that the space of outer derivations $Out(g) = 0$, hence, so is $Der(g)/ \approx$, being a quotient of a null space.

Next we provide two explicit examples for the above results by computing $\mathcal{H}^2_{\mathfrak{g}}(V,\mathfrak{g})$ and then describing all Lie algebra structures which extend the Lie algebra structure from g to a vector space of dimension $1 + \dim_k(\mathfrak{g})$. The detailed computations are rather long but straightforward and can be provided upon request. We start with the case when $\mathfrak g$ is a perfect Lie algebra which is not semisimple.

Example 4.11. Let k be a field of characteristic zero and \boldsymbol{g} be the perfect 5-dimensional Lie algebra with a basis $\{e_1, e_2, e_3, e_4, e_5\}$ and bracket given by:

$$
[e_1, e_2] = e_3, \quad [e_1, e_3] = -2e_1, \quad [e_1, e_5] = [e_3, e_4] = e_4
$$

$$
[e_2, e_3] = 2e_2, \quad [e_2, e_4] = e_5, \quad [e_3, e_5] = -e_5
$$

We shall compute the classifying object Extd (k^6, \mathfrak{g}) by proving that Extd $(k^6, \mathfrak{g}) \cong k^7 / \equiv$, where \equiv is the equivalence relation on k^7 defined by: $(a_1, \dots, a_7) \equiv (a'_1, \dots, a'_r)$ $\lambda'_1, \cdots, \lambda'_7$ if and only if there exists $q \in k^*$ such that $a_2 = qa_2'$ a'_2 and $2a_7 - a_1 = q(2a'_7 - a'_1)$ $_{1}^{\prime}).$

Indeed, by a rather long but straightforward computation it can be proved that the space of derivations $Der(\mathfrak{g})$ coincides with the space of all matrices from $\mathcal{M}_5(k)$ of the form:

$$
A = \begin{pmatrix} a_1 & 0 & a_6 & 0 & 0 \\ 0 & -a_1 & -2a_2 & 0 & 0 \\ a_2 & a_4 & 0 & 0 & 0 \\ a_3 & 0 & a_5 & a_7 & a_4 \\ 0 & a_5 & -a_3 & a_2 & (a_7 - a_1) \end{pmatrix}
$$

for all $a_1, \dots, a_7 \in k$. Thus, any 6-dimensional Lie algebra that contains g as a Lie subalgebra is isomorphic to one of the following seven parameter Lie algebra denoted by $\mathfrak{g}_{(a_1,\dots,a_7)}(x) := \mathfrak{g} \mathfrak{h} V$, which has the basis $\{e_1, e_2, e_3, e_4, e_5, x\}$ and bracket given by:

$$
[e_1, x] = -a_1e_1 - a_2e_3 - a_3e_4, \quad [e_2, x] = a_1e_2 - a_4e_3 - a_5e_5
$$

 $[e_3, x] = -a_6e_1+2a_2e_2-a_5e_4+a_3e_5, [e_4, x] = -a_7e_4-a_2e_5, [e_5, x] = -a_4e_4+(a_1-a_7)e_5$ for some scalars $a_1, \dots, a_7 \in k$. Two such Lie algebras $\mathfrak{g}_{(a_1, \dots, a_7)}(x)$ and $\mathfrak{g}_{(a'_1, \dots, a'_7)}(x)$ are equivalent in the sense of Definition [1.1](#page-4-1) if and only if there exists $q \in k^*$ such that $a_2 = qa'_2$ a'_2 and $2a_7 - a_1 = q(2a'_7 - a'_1)$ $_{1}^{\prime}$), as needed.

Finally, we give an example in the case when g is not perfect.

Example 4.12. Let k be a field of characteristic zero and $g(2, k)$ the Lie algebra of all 2×2 matrices over k with the usual Lie bracket defined by:

$$
[e_{ij}, e_{kl}] = \delta_{jk} e_{il} - \delta_{il} e_{kj}
$$
\n
$$
(23)
$$

for all $1 \leq i, j \leq 2$, where δ is the Kronecker delta and e_{ij} the matrix units. We will compute the twisted derivations for $\mathsf{g}(\{2, k\})$. The k-linear maps $\lambda : \mathsf{g}(\{2, k\}) \to k$ satisfying [\(18\)](#page-15-3) are given as follows:

$$
\lambda(e_{11}) = \lambda(e_{22}) := q \in k, \qquad \lambda(e_{12}) = \lambda(e_{21}) = 0
$$

Now depending on the values of q we obtain, by a rather long but straightforward computation, the following k-linear maps $D : \mathsf{gl}(2,k) \to \mathsf{gl}(2,k)$ satisfying [\(19\)](#page-15-3):

Case 1: Suppose first that $q \notin \{0, 1, -1, 2\}$. Then the space of k-linear maps D : $\mathsf{gl}(2,k) \to \mathsf{gl}(2,k)$ satisfying [\(19\)](#page-15-3) coincide with the space of all matrices from $\mathcal{M}_4(k)$ of the form:

$$
A = \begin{pmatrix} a_1 & -(1-q)^{-1}a_3 & -(1+q)a_2 & a_1 \\ a_2 & q^{-1}(a_4-a_1) & 0 & (q-1)(q+1)^{-1}a_2 \\ a_3 & 0 & q^{-1}(a_1-a_4) & (q+1)(q-1)^{-1}a_3 \\ a_4 & (1-q)^{-1}a_3 & (1+q)^{-1}a_2 & a_4 \end{pmatrix}
$$

for all $a_1, \dots, a_4 \in k$. Thus, in this case any 5-dimensional Lie algebra that contains $\mathsf{gl}(2,k)$ as a Lie subalgebra is isomorphic to one of the following five parameter Lie algebra denoted by $\mathsf{gl}(2,k)_{(q,a_1,\cdots,a_4)}(x)$, which has the basis $\{e_{11},e_{12},e_{21},e_{22},x\}$ and bracket given by:

$$
[e_{11}, x] = -a_1e_{11} - a_2e_{12} - a_3e_{21} - a_4e_{22} - qx
$$

\n
$$
[e_{12}, x] = (1-q)^{-1}a_3e_{11} + q^{-1}(a_1 - a_4)e_{12} - (1-q)^{-1}a_3e_{22}
$$

\n
$$
[e_{21}, x] = (1+q)^{-1}a_2e_{11} + q^{-1}(a_4 - a_1)e_{21} - (1+q)^{-1}a_2e_{22}
$$

\n
$$
[e_{22}, x] = -a_1e_{11} - (q-1)(q+1)^{-1}a_2e_{12} - (q+1)(q-1)^{-1}a_3e_{21} - a_4e_{22} - qx
$$

Two such Lie algebras $\mathsf{gl}(2,k)_{(q,a_1,\cdots,a_4)}(x)$ and $\mathsf{gl}(2,k)_{(q,a'_1,\cdots,a'_4)}(x)$ are equivalent in the sense of Definition [1.1](#page-4-1) if and only if there exists $p \in k^*$ such that $a_2 = pa'_2$ $y'_2, a_3 = pa'_3$ $_{3}^{\prime}$ and $a_1 - a_4 = p(a'_1 - a'_4)$ $_{4}^{\prime}).$

Case 2: Assume that $q = 0$. Then the space of k-linear maps $D : \mathsf{gl}(2, k) \to \mathsf{gl}(2, k)$ satisfying [\(19\)](#page-15-3) coincide with the space of all matrices from $\mathcal{M}_4(k)$ of the form:

$$
A = \begin{pmatrix} a_1 & -a_3 & -a_2 & a_1 \\ a_2 & a_4 & 0 & -a_2 \\ a_3 & 0 & -a_4 & -a_3 \\ a_1 & a_3 & a_2 & a_1 \end{pmatrix}
$$

for all $a_1, \dots, a_5 \in k$. Thus, in this case any 5-dimensional Lie algebra that contains $\mathsf{gl}(2,k)$ as a Lie subalgebra has the basis $\{e_{11}, e_{12}, e_{21}, e_{22}, x\}$ and bracket given by:

$$
[e_{11}, x] = -a_1e_{11} - a_2e_{12} - a_3e_{21} - a_1e_{22}, \quad [e_{12}, x] = a_3e_{11} - a_4e_{12} - a_3e_{22}
$$

$$
[e_{21}, x] = a_2e_{11} + a_4e_{21} - a_2e_{22}, \quad [e_{22}, x] = -a_1e_{11} + a_2e_{12} + a_3e_{21} - a_1e_{22}
$$

Two such Lie algebras are equivalent in the sense of Definition [1.1](#page-4-1) if and only if there exists $p \in k^*$ such that $a_1 = pa'_1$ $\frac{1}{1}$.

Case 3: Assume that $q = 1$. Then the space of k-linear maps $D : \mathsf{gl}(2, k) \to \mathsf{gl}(2, k)$ satisfying [\(19\)](#page-15-3) coincide with the space of all matrices from $\mathcal{M}_4(k)$ of the form:

$$
A = \begin{pmatrix} a_1 & a_3 & -2^{-1}a_2 & a_1 \\ a_2 & a_4 - a_1 & 0 & 0 \\ 0 & 0 & a_1 - a_4 & 2a_3 \\ a_4 & -a_3 & 2^{-1}a_2 & a_4 \end{pmatrix}
$$

for all $a_1, \dots, a_4 \in k$. Thus, in this case any 5-dimensional Lie algebra that contains $gl(2, k)$ as a Lie subalgebra has the basis $\{e_{11}, e_{12}, e_{21}, e_{22}, x\}$ and bracket given by:

$$
[e_{11}, x] = -a_1e_{11} - a_2e_{12} - a_4e_{22} - x, \quad [e_{12}, x] = -a_3e_{11} + (a_1 - a_4)e_{12} + a_3e_{22}
$$

$$
[e_{21}, x] = 2^{-1}a_2e_{11} + (a_2 - a_1)e_{21} - 2^{-1}a_2e_{22}, \quad [e_{22}, x] = -a_1e_{11} - 2a_3e_{21} - a_4e_{22} - x
$$

Two such Lie algebras are equivalent in the sense of Definition [1.1](#page-4-1) if and only if there exists $p \in k^*$ such that $a_2 = pa'_2$ $'_{2}$, $a_{3} = pa'_{3}$ a'_3 and $a_1 - a_4 = p(a'_1 - a'_4)$ $_{4}^{\prime}).$

Case 4: Assume that $q = -1$. Then the space of k-linear maps $D : \mathsf{gl}(2, k) \to \mathsf{gl}(2, k)$ satisfying [\(19\)](#page-15-3) coincide with the space of all matrices from $\mathcal{M}_4(k)$ of the form:

$$
A = \begin{pmatrix} a_1 & -2^{-1}a_3 & a_2 & a_1 \\ 0 & a_1 - a_4 & 0 & 2a_2 \\ a_3 & 0 & a_4 - a_1 & 0 \\ a_4 & 2^{-1}a_3 & -a_2 & a_4 \end{pmatrix}
$$

for all $a_1, \dots, a_4 \in k$. Thus, in this case any 5-dimensional Lie algebra that contains $\mathsf{gl}(2,k)$ as a Lie subalgebra has the basis $\{e_{11}, e_{12}, e_{21}, e_{22}, x\}$ and bracket given by:

$$
[e_{11}, x] = -a_1e_{11} - a_3e_{21} - a_4e_{22} + x, \quad [e_{12}, x] = 2^{-1}a_3e_{11} + (a_4 - a_1)e_{12} - 2^{-1}a_3e_{22}
$$

$$
[e_{21}, x] = -a_2e_{11} + (a_1 - a_4)e_{21} + a_2e_{22}, \quad [e_{22}, x] = -a_1e_{11} - 2a_2e_{12} - a_4e_{22} + x
$$

Two such Lie algebras are equivalent in the sense of Definition [1.1](#page-4-1) if and only if there exists $p \in k^*$ such that $a_3 = pa'_3$ $a'_3, a_2 = pa'_2$ a'_2 and $a_1 - a_4 = p(a'_1 - a'_4)$ $_{4}^{\prime}).$

Case 5: Assume that $q = 2$. Then the space of k-linear maps $D : \mathsf{gl}(2,k) \to \mathsf{gl}(2,k)$ satisfying [\(19\)](#page-15-3) coincide with the space of all matrices from $\mathcal{M}_4(k)$ of the form:

$$
A = \begin{pmatrix} a_1 & a_3 & -3^{-1}a_2 & a_1 \\ a_2 & 2^{-1}(a_4 - a_1) & a_5 & 3^{-1}a_2 \\ a_3 & 0 & 2^{-1}(a_1 - a_4) & 3a_3 \\ a_4 & -a_3 & 3^{-1}a_2 & a_4 \end{pmatrix}
$$

for all $a_1, \dots, a_5 \in k$. Thus, in this case any 5-dimensional Lie algebra that contains $gl(2, k)$ as a Lie subalgebra has the basis $\{e_{11}, e_{12}, e_{21}, e_{22}, x\}$ and bracket given by:

$$
[e_{11}, x] = -a_1e_{11} - a_2e_{12} - a_3e_{21} - a_4e_{22} - 2x
$$

\n
$$
[e_{12}, x] = -a_3e_{11} + 2^{-1}(a_1 - a_4)e_{12} + a_3e_{22}
$$

\n
$$
[e_{21}, x] = 3^{-1}a_2e_{11} - a_5e_{12} + 2^{-1}(a_4 - a_1)e_{21} - 3^{-1}a_2e_{22}
$$

\n
$$
[e_{22}, x] = -a_1e_{11} - 3^{-1}a_2e_{12} - 3a_3e_{21} - a_4e_{22} - 2x
$$

Two such Lie algebras are equivalent in the sense of Definition [1.1](#page-4-1) if and only if there exists $p \in k^*$ such that $a_2 = pa'_2$ $y'_2, a_3 = pa'_3$ $a_3, a_5 = pa'_5$ a'_5 and $a_1 - a_4 = p(a'_1 - a'_4)$ $_{4}^{\prime}).$ Thus we have described the classifying object $\text{Extd}(k^5, \mathsf{gl}(2, k)) \cong \mathcal{H}^2_{\mathsf{gl}(2,k)}(V, \mathsf{gl}(2, k))$: it is equal to the disjoint union of the five quotient spaces described above. This is easy to see having in mind that if two twisted derivations (λ_1, D_1) are equivalent in the sense

of Corollary [4.8](#page-18-0) then $\lambda_1 = \lambda_2$.

REFERENCES

- [1] Agore, A.L. and Militaru, G. - Extending structures I: the level of groups, Algebr. Represent. Theory, DOI: 10.1007/s10468-013-9420-4, arXiv:1011.1633.
- [2] Agore, A.L. and Militaru, G. - Extending structures II: the quantum version, J. Algebra 336 (2011), 321–341.
- [3] Alekseevsky, D., Michor, P. W. and Ruppert, W. Extensions of Lie algebras. Unpublished. ESI Preprint 881. arXiv:math.DG/0005042.
- [4] Alekseevsky, D., Michor, P. W. and Ruppert, W. - Extensions of super Lie algebras, J. Lie Theory 15 (2005), 125–134.
- [5] Andrada, A. and Salamon, S. - Complex product structures on Lie algebras, Forum Mathematicum., 17 (2005), 261-295.
- [6] Andrada, A., Barberis, M.L., Dotti, I.G. and Ovando, G.P. Product structures on four dimensional solvable Lie algebras, Homology, Homotopy and Applications, 7(2005), 9–37.
- [7] Benayadi, S. - Structure of perfect Lie algebras without center and outer derivation, Annales de la faculte de Science de Toulouse, 5(1996), 203–231.
- [8] Bourbaki, N. - Lie groups and Lie algebres, Chap. 1-3, Springer, Paris, 1989.
- [9] Chevalley, C. and Eilenberg S. Cohomology theory of Lie groups and Lie algebras, Trans. Amer. Math. Soc., **63**(1948), 85–124.
- [10] Erdmann, K. and Wildon, M.J. - Introduction to Lie algebras, Springer, 2006.
- [11] Farnsteiner, R. - On the cohomology of associative algebras and Lie algebras, Proc. AMS, 99(1987), 415-420.
- [12] de Graaf, W.A. Classification of 6-dimensional nilpotent Lie algebras over fields of characteristic not 2, J. Algebra, 309(2007), 640-653
- [13] Hu, N., Pei, Y. and Liu, D. - A cohomological characterization of Leibniz central extensions of Lie algebras, Proc. AMS, 136(2008), 437-447
- [14] Humphreys, J.E. - Introducution to Lie algebras and Representation theory, Spinger, 1972.
- [15] Kassel, C. - Quantum groups, Graduate Texts in Mathematics 155. Springer-Verlag, New York, 1995.
- [16] Lecomte, P. Sur la suite exacte canonique asociée à un fibrè principal, Bul. Soc. Math. France, 13(1985), 259–271.
- [17] Lu, J.H. and Weinstein, A. - Poisson Lie groups, dressing transformations and Bruhat decompositions, J. Differential Geom., $31(1990)$, $501-526$.
- [18] Ovando, G. - Four Dimensional Symplectic Lie Algebras, Beiträge zur Algebra und Geometrie, 47(2006), 419–434.
- [19] Majid, S. - Physics for algebraists: non-commutative and non-cocommutative Hopf algebras by a bicrossproduct construction, J. Algebra, 130 (1990), 17–64.
- [20] Majid, S. Matched Pairs of Lie Groups and Hopf Algebra Bicrossproducts, Nuclear Physics B, Proc. Supl. 6 (1989), 422–424.
- [21] Majid, S. - Foundations of quantum groups theory, Cambridge University Press, 1995.
- [22] Michor, P. W. Knit products of graded Lie algebras and groups, Proceedings of the Winter School on Geometry and Physics (Srn, 1989). Rend. Circ. Mat. Palermo (2) Suppl. No. 22 (1990), 171–175 - arXiv:math.GR/9204220.
- [23] Takeuchi, M. - Matched pairs of groups and bismash products of Hopf algebras, Comm. Algebra 9(1981), 841–882.
- [24] Zusmanovich, P. - Central extensions of current algebras, Trans. AMS, 334(1992), 143–152.

Faculty of Engineering, Vrije Universiteit Brussel, Pleinlaan 2, B-1050 Brussels, Belgium E-mail address: ana.agore@vub.ac.be and ana.agore@gmail.com

FACULTY OF MATHEMATICS AND COMPUTER SCIENCE, UNIVERSITY OF BUCHAREST, STR. ACADEMIEI 14, RO-010014 Bucharest 1, Romania

 $\it E\mbox{-}mail\,\,address:$ gigel.militaru@fmi.unibuc.ro and gigel.militaru@gmail.com