Virtual Genus of Satellite Links

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Abstract

The virtual genus of a virtual satellite link is equal to that of its companion.

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1 Introduction

The notion of virtual knots and links was introduced by L. Kauffman [6]. It is a nontrivial extension of the classical theory. Virtual links can be defined as link diagrams in the plane with "virtual crossings" as well as crossings of the usual kind and an extended set of Reidemeister moves, or as combinatorial Gauss diagrams. It is shown in [5] that virtual links correspond bijectively to abstract link diagrams, introduced by N. Kamada in [4].

Alternatively, a virtual link ℓ can be defined as an equivalence class of link diagrams $\mathcal{D}$ in a surface S. The surface is required to be closed and orientable; it need not be connected, but we require that each component contain at least one link component. The equivalence relation is generated by Reidemister moves on $\mathcal{D}$, orientation-preserving homeomorphisms of S and adding or deleting hollow 1-handles in the complement of the diagram.

Adding a handle (*stabilization*) is the surgery that removes two open disks disjoint from $\mathcal{D}$, and then joins the resulting boundary components by an annulus. Deleting a handle (*destabilization*) is the surgery that removes

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the interior of a neighborhood of a simple closed curve that misses $\mathcal{D}$, and then attaches a pair of disks to the resulting boundary. Destabilization might produce a diagram for the link in a surface that has smaller (total) genus than S.

Following [3], we define the *virtual genus* of ℓ, denoted here by $\mathrm{vg}(\ell)$, to be the minimal genus of a surface that contains a diagram representing the link. The virtual genus was first studied in [4], where it is called *supporting genus*. Methods for estimating virtual genus are found in [3], [7], [2].

By [6], [1], a virtual link can also be regarded as an equivalence class of embedded links in thickened surfaces. The equivalence relation is generated by isotopy as well as stabilization/destabilization. Destabilization in this context consists of surgery along an embedded annulus A that is *vertical* in the sense that $A = p^{-1}(p(A))$, where p is first-coordinate projection $S \times I \to S$. The reverse operation, stabilization, is a parametrized connected-sum operation with a thickened torus.

Lemma 3.4 of [9] implies the following.

Lemma 1.1. *A properly embedded annulus* $(A; \partial_1 A, \partial_0 A) \subset (S \times I; S \times \{1\}, S \times \{0\})$ *is isotopic to a vertical annulus provided* $\partial_i A$ *is essential in* $S \times \{i\},\ i = 0, 1$.

Theorem 1 of [8] implies that if $\mathrm{vg}(\ell)$ is less than the genus of S, then after isotopy a vertical annulus $A \subset S \times I \setminus \ell$ can be found such that surgery along it produces an embedding of the link in a surface of strictly smaller genus than that of S. Note that surgery on such an annulus will reduce genus if and only if each of its boundary components represents a nontrivial element of $H_1(S; \mathbb{Z})$.

Assume that $\ell = \ell_1 \cup \cdots \cup \ell_d \subset S \times I$ is a link in a thickened surface. Let $N = N_1 \cup \cdots \cup N_d$ be a regular neighborhood of ℓ with boundary $\partial N = \partial N_1 \cup \cdots \partial N_d$ consisting of mutually disjoint tori. For each $i = 1, \ldots, d$, let $\tilde\ell_i \subset \mathrm{int}\ N_i$ be a link that is not contained in any 3-ball neighborhood in N_i. Then $\tilde\ell = \tilde\ell_1 \cup \cdots \cup \tilde\ell_d$ is a *satellite link* of ℓ with *companion* ℓ.

It is clear that the virtual genus of $\tilde\ell$ is not greater than that of ℓ. The following theorem asserts the virtual genus of the links are in fact equal.

Theorem 1.2. *If* $\tilde\ell$ *is any satellite link with companion* ℓ*, then the virtual genus of* $\tilde\ell$ *is equal to that of* ℓ.

A virtual link ℓ is *classical* if $\mathrm{vg}(\ell) = 0$ (equivalently, if it can be represented by a planar diagram).

Corollary 1.3. *If a satellite virtual link is classical, then its companion is classical.*

Remark 1.4. The main result of [8] is that every virtual knot has a unique representative $\ell \subset S \times I$ for which the genus of S is equal to $\mathrm{vg}(\ell)$ and the number of components of S is maximal. (Uniqueness is up to isotopy and orientation-preserving self-homeomorphism of $(S \times I, S \times \{1\}, S \times \{0\})$.) Let $\tilde{\ell} \subset S \times I$ be a satellite with companion ℓ, and assume that S has no genus-0 components and no embedded 2-sphere in N_i separates $\tilde{\ell}_i$. We see easily from the proof of Theorem 1.2 that S has both minimal genus and maximal number of components for $\tilde{\ell}$. Hence topological invariants of $S \times I \setminus \tilde{\ell}$ (e.g. fundamental group, homology groups of abelian covers) are also invariants of the virtual link $\tilde{\ell}$.

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2 Proof of Theorem 1.2.

Consider a link $\ell = \ell_1 \cup \cdots \cup \ell_d \subset S \times I$ such that the genus of S is equal to $\mathrm{vg}(\ell)$. Let $\tilde{\ell} = \tilde{\ell}_1 \cup \cdots \cup \ell_d \subset S \times \tilde{I}$ be any satellite link with companion ℓ, as above.

Suppose that some embedded 2-sphere in $S \times I$ separates ℓ into nonempty sublinks. Since one of the sublinks must be contained in a 3-ball, it suffices to prove Theorem 1.2 for the other sublink. By an induction argument, we can assume without loss of generality that no 2-sphere in $S \times I$ separates ℓ.

Similarly, we may assume that no embedded 2-sphere or properly embedded annulus in N_i separates $\tilde{\ell}_i$. Otherwise, at least one of the two sublinks of $\tilde{\ell}_i$ is not contained in any 3-ball, and it suffices prove the result for the link obtained by deleting the other sublink.

Assume that $\mathrm{vg}(\tilde{\ell}) < \mathrm{vg}(\ell)$. By the argument of [8], there exists a vertical annulus

$$(A; \partial_0 A, \partial_1 A) \subset (S \times I \setminus \tilde{\ell}; S \times \{0\}, S \times \{1\})$$

such that $\partial_i A$ represents a nontrivial element of $H_1(S \times \{i\}; \mathbb{Z})$ for $i = 0, 1$. We will derive a contradiction.

Deform A so that it meets ∂N transversely. Regard the intersection as a closed 1-submanifold of A.

If some component of $A \cap \partial N$ is null-homotopic in A, then let C be such a component that is innermost in the sense that no other component of $A \cap \partial N$ is contained in the 2-disk $D \subset A$ bounded by C. Let ∂N_i be the component of ∂N that contains C.

The assumption that $\tilde{\ell}_i$ is not contained in a 3-ball in N_i, implies that ∂N_i is incompressible in $N_i \setminus \tilde{\ell}_i$. Hence C also bounds a 2-disk D' in ∂N_i. The union $D \cup_C D'$ is an embedded 2-sphere bounding a ball in N_i that does not meet $\tilde{\ell}_i$, by our assumption. We can deform A in this ball to remove the circle C of intersection with ∂N_i. By repeating this procedure, we can assume that every component of $A \cap \partial N$ is essential in A.

Now $A \cap \partial N$ consists of finitely many pairwise disjoint simple closed curves that divide A into successive annular regions $A_1, B_1, \ldots, A_n, B_n, A_{n+1}$ such that each A_j is contained in the exterior $X = S \times I \setminus \mathrm{int}\ N$ while each B_j is contained in some $N_{i_j} \setminus \tilde{\ell}_{i_j}$.

Consider any annular region B_j. Its boundary components $\partial_\pm B_j$ are homologous in $N_{i_j} \setminus \tilde{\ell}_{i_j}$, and each is homologous in $S \times I \setminus \ell$ to $\partial_1 A$. They separate ∂N_{i_j} into two annuli A', A''. Both $B_j \cup_\partial A'$ and $B_j \cup_\partial A''$ are tori; by our assumption, one of them contains the link $\tilde{\ell}_{i_j}$ while the other bounds a solid torus in $S \times I \setminus \tilde{\ell}$. The latter can be used to deform A and remove the points $\partial_\pm B_j$ of intersection with ∂N_{i_j}. By repeating the procedure, we can deform A away from ∂N.

The link ℓ is the core of N. Since $A \cap N$ is empty, we can use the vertical annulus A to reduce the genus of S, contradicting the assumption that the genus of S is equal to $\mathrm{vg}(\ell)$.

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