OPEN–CONSTRUCTIBLE FUNCTIONS (CORRECTED VERSION, JANUARY 2014)

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ABSTRACT. Let f be a continuous function between subspaces X, Y of the Cantor set $\mathbf{C}$. We prove that:

if f is one-to-one and maps open sets into resolvable, then f is a piecewise homeomorphism and

if f maps discrete subsets into resolvable, then f is piecewise open.

1. Introduction

The present paper continues the series of publications about decomposibility of Borel functions [6], [8] - see also [2], [3] where functions of such type are the main subject.

A subset E of a topological space X is *resolvable* if for each nonempty closed in X subset F we have

$$cl_X(F \cap E) \cap cl_X(F \setminus E) \neq F$$

Recall that a function f is open if it maps open sets into open ones. More generally, a function f is said to be *open–resolvable* (resolvable in [1]) if f maps open sets into resolvable ones.

A function $f : X \to Y$ for which $X \subset \mathbf{C}$ admits a countable, closed and disjoint cover $\mathcal{C}$, such that for each $C \in \mathcal{C}$ the restriction $f|C$ is open, is called *piecewise open*.

Theorem 1. *Let f be a continuous, one-to-one and open–resolvable function between $X, Y \subset \boldsymbol{C}$. Then f is piecewise open and hence is piecewise homeomorphism.*

Note, that using standard sets H, we obtain the proof for open-resolvable functions simpler than for open-constructible.

Analogously, using H-sets we can easy extend the proof for closed-constructible functions in [6] to the case of closed-resolvable.

Standard set H will be used in the proof of the following theorem:

Theorem 2. *If a continuous function $f : X \to Y$ between $X, Y \subset \boldsymbol{C}$ maps discrete subsets in X onto resolvable, then f is piecewise open.*

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1.1. **Standard set H.**

The set H was introduced by W. Hurewicz [4] and has a lot of applications. H (called standard in [7],[5]) is a countable set without isolated points:

$$H = \{p, ..., p_{i_1,...i_k} : k \in N^+; i_1, ..., i_k \in N^+\}$$

such that

$$p_{i_1} \to p, \text{ as } i_{i_1} \to \infty$$

$$p_{i_1,...i_k,i_{k+1}} \to p_{i_1,...i_k}, \text{ as } i_{k+1} \to \infty$$

Obviously, H is homeomorphic to the space of rational $\mathbf{Q}$.

Using the metric in $X \supset H$ we can suppose additionally that there are decreasing bases $U^i(p)$, and $U^i(p_{i_1,...i_k})$ at points p and $p_{i_1,...i_k}$ satisfying conditions a), b) and c) below:

a) $U^{i_k}(p_{i_1,...i_k}) \supset U^1(p_{i_1,...i_{k+1}}), i_k \in N^+$

b) for $i'_k \neq i''_k$ we have $U^1(p_{i_1,...i'_k}) \cap U^1(p_{i_1,...i_{i''_k}}) = \emptyset$.

c) $\mathrm{diam}(U^{i_1}(p_{i_1,...i_k})) < 1/(i_1 + ... + i_k)$

2. Construction of $Z \subset X$ for which $f|Z$ is nowhere open

Given a function $f : X \to Y$, we shall construct in the next Lemma 1 a subset $Z \subset X$ on which the restriction $f|Z$ is *nowhere open* on Z; i.e. for every clopen in X subset U the restriction $f|(U \cap Z)$ is not open.

Lemma 1. *Let $f : X \to Y$ be a continuous function from a subspace X of the Cantor set $\boldsymbol{C}$ onto a metrizable space Y. Then there is a closed subset $Z \subset X$ such that the restriction $f|Z$ is nowhere open on Z and the restriction $f|(X \setminus Z)$ is piecewise open.*

Proof of Lemma 1. Let us begin by proving the first part of the assertion from lemma stating that for some Z the restriction $f|Z$ is nowhere open on Z. Indeed, if for some nonempty clopen set $V \subset X$ the restriction $f|V$ is open, then we could construct the closed set

$$X_1 = X \setminus V$$

and the corresponding restriction

$$f|X_1 : X_1 \to f(X_1).$$

Repeating this process, we could also construct a chain of closed sets ($X_\gamma = \bigcap_{\beta<\gamma} X^\beta$ for a limit γ)

$$X \supset X_1 \supset ... \supset X_\gamma \supset ...$$

which, as we know, stabilizes at some $\gamma_0 < \omega_1$. Therefore, there exists a subspace Z for which holds true

$$Z = X_{\gamma_0} = X_{\gamma_0+1} = ...$$

and the restriction $f|Z$ is nowhere open on Z.

The second part of Lemma 1 stating that $f|(X \setminus Z)$ is piecewise open, obviously, satisfied.

In what follows, it is convenient in the case when Z is empty to regard a piecewise open function $f|X = f|(X \setminus Z)$ onto Y.

□

2.1. **Proof of Theorems 1 and 2.**

On the step 1 take a point $x \in Z$ and a base of clopen in X neighborhoods $U^k(x) \subset X$ with diametr less than $1/k$.

Since f is nowhere open on Z, there are

$$U^l(x_k) \subset Z \cap (U^k(x) \setminus (U^{k+1}(x))$$

such that $x_k \to x$, $f(x_k) = p_k \to p = f(x)$ and

$$f^{-1}(p_{k,l}) \cap U^1(x_k) = \emptyset$$

Take $x_{k,l} \in U(x_{k,l}) \subset \cap(U^l(x_k) \setminus (U^{l+1}(x_k))$.

For the basic induction step m we suppose that the points $p = f(x), ...p_{i_1,...i_k} = f(x_{i_1,...i_k})$ satisfying the conditions a), b),c) of definition the set H are constructed.

Analogously pick in some clopen sets $U^1(x_{i_1,...i_k})$ the points $x_k \to x$ with pairwise disjunct clopen neighborhoods $U^1(x_k)$ such that for some sequence $p_{k,l} \to p_k = f(x_k)$ we have

$$f^{-1}(p_{k,l}) \cap U^1(x_k) = \emptyset$$

Take the points

$$x_{k,l} \in f^{-1}(p_{k,l}) \setminus U^1(x_k)$$

and clopen sets

$$U^1(x_{k,l}) \subset X \setminus U^1(x_k).$$

Since f is continuous we can suppose that $U^1(x_{k,l})$ are disjunct with all $U^1(x_k)$.

Since f is nowhere open on $Z \cap U^1(x_{k,l})$ we can repeat the construction for $U^1(x_{k,l})$ etc. Analogously we obtain

$$p_{i_1,...i_m,k,l} \to p_{i_1,...i_m,k} \text{ as } l \to \infty$$

$$x_{i_1,...i_m,k,l} \in f^{-1}(p_{i_1,...i_m,k,l}) \setminus U^1(x_{i_1,...i_m,k}).$$

$$U^1(x_{i_1,...i_m,k}) \cap f^{-1}(p_{i_1,...i_m,k,l}) = \emptyset, l \in N^+$$

We can suppose that $U^1(p_{i_1,...i_m,k})$ are clopen and there is a clopen neighborhood $U^1(x_{i_1,...i_m,k,l})$ of point $x_{i_1,...i_m,k,l}$, disjunct with all $U^1(x_{i_1,...i_m,k})$.

Denote

$$D = \{x_{i_1}, x_{i_1,i_2,i_3} \dots x_{i_1,...i_{2k+1}} : k \in N^+; i_1, ..., i_{2k+1} \in N^+\}$$

By our construction D is discrete and $f(D)$ is dense and codense in

$$H = \{p_{i_1,...i_k} : k \in N^+; i_1, ..., i_k \in N^+\}$$

that proves Theorem 2.

To prove Theorem 1 we note, that in case of one-to-one functions the open set $O(D) = \{U^1(x_{i_1}), \dots U^1(x_{i_1,\dots i_{2k+1}}) : k \in N^+; i_1, \dots, i_{2k+1} \in N^+\}$ has the same image as D.

□

Question 1. *Are the continuous open–Δ_2^0 functions (even for Polish or analytic spaces $X \subset \boldsymbol{C}$) piecewise or countably open?*

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